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# Analytically Continued Hypergeometric Expression of the Incomplete Beta Function

Jack C. Straton

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**Abstract** The Incomplete Beta Function is rewritten as a Hypergeometric Function that is the analytic continuation of the conventional form, a generalization of the finite series, which simplifies the Stieltjes transform of powers of a monomial divided by powers of a binomial.

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The finite hypergeometric series expression for the Incomplete Beta Function, [1]

$${}_2F_1(-n, 1; c; z) = (1-c)z^{1-c}(z-1)^{n+c-1}B_{1-1/z}(1-c-n, n+1), \quad (1)$$

may be generalized to

## Theorem

$${}_2F_1(-\nu, 1; \gamma; z) = (1-\gamma)z^{1-\gamma}(z-1)^{\nu+\gamma-1} \left[ B_{1-1/z}(1-\gamma-\nu, \nu+1) - B(1-\gamma-\nu, \nu+1) \left( 1 - \frac{(-1)^{-\nu} \sin[\pi(\gamma+\nu)]}{\sin(\pi\gamma)} \right) \right]. \quad (2)$$

The Incomplete Beta Function [2] is conventionally defined [3] with real parameters for statistical problems,

$$B_x(p, q) = \int_0^x t^{p-1}(1-t)^{q-1} dt \quad (0 \leq x \leq 1, \quad p, q > 0), \quad (3)$$

but is a smooth function of  $p, q$  or  $x$  when any or all are taken off the real axis (though it diverges as  $x$  takes on large, real values). Its hypergeometric expression [4] is likewise well-behaved for complex parameters, so we rewrite this expression in its more general form

$$\begin{aligned} {}_2F_1(\alpha, \beta; \beta+1; w) &= \beta w^{-\beta} B_w(\beta, 1-\alpha) = \beta w^{-\beta} B(\beta, 1-\alpha) (1 - I_{1-w}(1-\alpha, \beta)) \\ &= \beta w^{-\beta} [B(1-\alpha, \beta) - B_{1-w}(1-\alpha, \beta)]. \end{aligned} \quad (4)$$

One may analytically continue the left-hand side to [5]

$$\begin{aligned} {}_2F_1(\alpha, \beta; \beta+1; w) &= (-1)^{-\alpha} (w)^{-\alpha} \frac{\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta+1-\alpha)} {}_2F_1(\alpha, \alpha-\beta; \alpha+1-\beta; 1/w) \\ &\quad + (-1)^{-\beta} (w)^{-\beta} \frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1)} {}_2F_1(\beta, 0; \beta+1-\alpha; 1/w), \end{aligned} \quad (5)$$

Then equating right-hand sides of (4) and (5) and transforming the nontrivial hypergeometric function again [6] gives

$$(B(1-\alpha, \beta) - B_{1-w}(1-\alpha, \beta)) = (-1)^{-\alpha} w^{-\alpha+\beta} \frac{1}{(\beta-\alpha)} \left(1 - \frac{1}{w}\right)^{1-\alpha} {}_2F_1(1-\beta, 1; \alpha+1-\beta; 1/w) \\ + (-1)^{-\beta} B(1-\alpha, \beta) \frac{\Gamma[1-(\alpha-\beta)]\Gamma(\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(\alpha)}, \quad (6)$$

Letting  $z = 1/w$  this simplifies [7] to

$$B_{1-1/z}(1-\alpha, \beta) = z^{\alpha-\beta} \frac{1}{(\beta-\alpha)} (z-1)^{1-\alpha} {}_2F_1(1-\beta, 1; \alpha+1-\beta; z) \\ + B(1-\alpha, \beta) \left(1 + (-1)^{1-\beta} \frac{\sin[\pi\alpha]}{\sin[\pi(\alpha-\beta)]}\right), \quad (7)$$

Finally one substitutes  $\beta = \nu + 1$  and  $\alpha = \gamma + \beta - 1$  and rearranges sides to obtain Eq. (2).

In addition, if one substitutes  $\beta = 1 - \nu$ ,  $\alpha = 2 - \mu$ , and  $z = \frac{\beta}{\gamma}$  and analytically continues the Gauss function, [8] one may obtain a more useful form for the known [9] Stieltjes transform [10] of powers of a monomial divided by powers of a binomial,

### Corollary

$$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x} dx = 2 \int_0^\infty \frac{x^{\nu-1/2}(\beta+x^2)^{1-\mu}}{\gamma+x^2} dx = \pi \gamma^{\nu-1} (\beta-\gamma)^{1-\mu} \csc(\nu\pi) I_{1-\frac{\gamma}{\beta}}(\mu-1, 1-\nu) \\ = \pi \gamma^{\nu-1} (\beta-\gamma)^{1-\mu} \csc(\nu\pi) \left(1 + (-1)^\nu \frac{\sin[\pi(2-\mu)]}{\sin[\pi(1+\nu-\mu)]}\right) \\ - \frac{\pi \csc(\nu\pi) \beta^{\nu+1-\mu}}{(\mu-1-\nu)(\beta-\gamma)B(\mu-1, 1-\nu)} {}_2F_1(2-\mu, 1; 2-\mu+\nu; \frac{\beta}{\beta-\gamma}), \quad (8)$$

( $|\arg\gamma| < \pi$ ,  $|\arg\beta| < \pi$ ,  $0 < \operatorname{Re} \nu < \operatorname{Re} \mu$ ) which is a finite series for integer  $\mu > 1$ .

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