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Derivation of the Hellinger-Reissner Variational Form of the Linear Elasticity Equations, and a Finite Element Discretization

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Abstract

In this paper we are going to derive the linear elasticity equations in the Strong Form to the Hellinger Reissner Form. We find a suitable solution to solve our stress tensor. Then we will use finite element discretization from [1]. We will run tests on a unit cube and multiple other shapes, which are described at the end. We view the different magnitudes of the displacement vector of each shape.

1 Derivation of the Hellinger-Reissner Form

1.1 The Linear Elasticity Equations

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, with boundary $\partial\Omega$. The first-order (strong) form of the linear elasticity problem is to find a displacement vector $\mathbf{u} : \Omega \to \mathbb{R}^3$ and symmetric stress tensor $\sigma : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}}$ such that:

$$\begin{cases} A\sigma = \epsilon(\mathbf{u}) & \text{in } \Omega \\ -(\nabla \cdot \sigma) = \mathbf{f} & \text{in } \Omega \end{cases}, \tag{1}$$

with appropriate conditions prescribed on the boundary $\partial\Omega$. Here and below, $\mathbb{R}^{3\times3}_{\text{sym}}$ are the symmetric 3×3 matrices. The function $\mathbf{f} : \Omega \to \mathbb{R}^3$ is given, and the *symmetric gradient* $\epsilon(\mathbf{u}) : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}}$ and *compliance tensor* $\mathcal{A}\sigma : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}}$ are defined by

$$\epsilon(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right) \tag{2}$$

$$\mathcal{A}\sigma = \frac{1}{2\mu}\left(\sigma - \frac{\lambda}{3\lambda + 2\mu}\text{tr}(\sigma)I\right) , \tag{3}$$

where the *Lamé parameters* λ and μ are given. We note that the gradient of a vector field $\mathbf{v} : \Omega \to \mathbb{R}^3$ is defined by

$$\nabla\mathbf{v} = \begin{pmatrix} \frac{\partial v_1}{\partial x} & \frac{\partial v_1}{\partial y} & \frac{\partial v_1}{\partial z} \\ \frac{\partial v_2}{\partial x} & \frac{\partial v_2}{\partial y} & \frac{\partial v_2}{\partial z} \\ \frac{\partial v_3}{\partial x} & \frac{\partial v_3}{\partial y} & \frac{\partial v_3}{\partial z} \end{pmatrix} \tag{4}$$

Furthermore, the divergence of a tensor $\tau : \Omega \to \mathbb{R}^{3\times3}$ is defined by

$$\nabla \cdot \tau = \begin{pmatrix} \frac{\partial \tau_{11}}{\partial x} + \frac{\partial \tau_{21}}{\partial y} + \frac{\partial \tau_{31}}{\partial z} \\ \frac{\partial \tau_{12}}{\partial x} + \frac{\partial \tau_{22}}{\partial y} + \frac{\partial \tau_{32}}{\partial z} \\ \frac{\partial \tau_{13}}{\partial x} + \frac{\partial \tau_{23}}{\partial y} + \frac{\partial \tau_{33}}{\partial z} \end{pmatrix} = \begin{pmatrix} \nabla \cdot \boldsymbol{\tau}_1 \\ \nabla \cdot \boldsymbol{\tau}_2 \\ \nabla \cdot \boldsymbol{\tau}_3 \end{pmatrix} , \tag{5}$$

where $\boldsymbol{\tau}_j$ is the j^{th} column of τ.

The main goal of this section is to derive the *Hellinger-Reissner* variational form of (1), and describe the appropriate spaces of functions in which $\mathbf{u}$ and σ are sought. We will also briefly discuss an alternate variational formulation. The first step in the derivation of these variational forms is to multiply the equations (1) by suitable *test functions*, and integrate over the domain. For $\mathbf{v} : \Omega \to \mathbb{R}^3$ and $\tau : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$, we obtain

$$\begin{cases} \int_\Omega A\sigma : \tau \, dx = \int_\Omega \epsilon(u) : \tau \, dx \\ -\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, dx = \int_\Omega f \cdot \mathbf{v} \, dx \end{cases} \tag{6}$$

Here, the *double dot product* of two matrices $A, B \in \mathbb{R}^{3\times 3}$ is defined by

$$A : B = \sum_{i,j=1}^{3} a_{ij} b_{ji} \ . \tag{7}$$

We note that $A : B = \mathrm{tr}(AB) = \mathrm{tr}(BA) = B : A$. Let $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^3$ denote the i^{th} columns of A and B, respectively. If A or B is symmetric, which they are in our case, then $A : B = \sum_{i=1}^3 \mathbf{a}_i \cdot \mathbf{b}_i$ as well.

1.2 Integration-By-Parts

At this stage, the key step in deriving the variational forms is integration-by-parts, and we now provide the appropriate formula, as well as its justification.

For a scalar field $v : \Omega \to \mathbb{R}$ and a vector field $\boldsymbol{\tau} : \Omega \to \mathbb{R}^3$, we have the following *product rule* formula,

$$\nabla \cdot (v\boldsymbol{\tau}) = \nabla v \cdot \boldsymbol{\tau} + v(\nabla \cdot \boldsymbol{\tau}) \ , \tag{8}$$

which follows directly from the standard product rule for scalar fields. Integrating both sides of (8) over Ω, and using the Divergence Theorem on the left-hand side, we obtain our basic integration-by-parts formula,

$$\int_{\partial\Omega} v(\boldsymbol{\tau} \cdot \mathbf{n}) \, ds = \int_\Omega \nabla v \cdot \boldsymbol{\tau} \, dx + \int_\Omega v(\nabla \cdot \boldsymbol{\tau}) \, dx \ , \tag{9}$$

where $\mathbf{n} : \partial\Omega \to \mathbb{R}^3$ is the outward unit normal vector.

Now suppose that $\mathbf{v} : \Omega \to \mathbb{R}^3$ and $\tau : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}}$. We want a natural analogue of (9). Since

$$\begin{aligned} \mathbf{v} \cdot (\tau\mathbf{n}) &= v_1(\boldsymbol{\tau}_1 \cdot \mathbf{n}) + v_2(\boldsymbol{\tau}_2 \cdot \mathbf{n}) + v_3(\boldsymbol{\tau}_3 \cdot \mathbf{n}) \ , \\ \nabla\mathbf{v} : \tau &= \nabla v_1 \cdot \boldsymbol{\tau}_1 + \nabla v_2 \cdot \boldsymbol{\tau}_2 + \nabla v_3 \cdot \boldsymbol{\tau}_3 \ , \\ \mathbf{v} \cdot (\nabla \cdot \tau) &= v_1(\nabla \cdot \boldsymbol{\tau}_1) + v_2(\nabla \cdot \boldsymbol{\tau}_2) + v_3(\nabla \cdot \boldsymbol{\tau}_3) \ , \end{aligned}$$

and

$$\int_{\partial\Omega} v_j(\boldsymbol{\tau}_j \cdot \mathbf{n}) \, ds = \int_\Omega \nabla v_j \cdot \boldsymbol{\tau}_j \, dx + \int_\Omega v_j(\nabla \cdot \boldsymbol{\tau}_j) \, dx \ ,$$

for each j, we sum these equations to obtain our main integration-by-parts formula,

$$\int_{\partial\Omega} \mathbf{v} \cdot (\tau\mathbf{n}) \, ds = \int_\Omega \nabla\mathbf{v} : \tau \, dx + \int_\Omega \mathbf{v} \cdot (\nabla \cdot \tau) \, dx \ , \tag{10}$$

1.3 The Hellinger-Reissner Form

The Hellinger-Reissner form is derived by applying the integration-by-parts formula (10) to the term $\int_\Omega \epsilon(u) : \tau \, dx$ in (6) . Since τ is symmetric, $\nabla\mathbf{u} : \tau = (\nabla\mathbf{u})^T : \tau$, and we deduce that $\epsilon(\mathbf{u}) : \tau = \nabla\mathbf{u} : \tau$. So the integration-by-parts formula reads

$$\int_{\partial\Omega} \mathbf{u} \cdot (\tau\mathbf{n}) \, ds = \int_\Omega \epsilon(\mathbf{u}) : \tau \, dx + \int_\Omega \mathbf{u} \cdot (\nabla \cdot \tau) \, dx \ .$$

Substituting this into (6), making the assumption

$$\mathbf{u} = \mathbf{0} \text{ on } \partial\Omega \ , \tag{11}$$

and rearranging terms, we obtain,

$$\begin{aligned} \int_\Omega A\sigma : \tau \, dx + \int_\Omega (\nabla \cdot \tau) \cdot \mathbf{u} \, dx &= 0 \\ -\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, dx \qquad\qquad &= \int_\Omega f \cdot \mathbf{v} \, dx \end{aligned} \tag{12}$$

At this stage, we have shown that, if $\mathbf{u}, \sigma$ satisfy (1), then they satisfy (12). The Hellinger-Reissner variational formulation is obtained by choosing appropriate spaces of functions in which to search for $\mathbf{u}, \sigma$, and requiring that (12) holds for all $\mathbf{v}, \tau$ in these spaces. More specifically: Find $\mathbf{u} \in L^2(\Omega; \mathbb{R}^3)$ and $\sigma \in H(\text{div}, \Omega; \mathbb{R}^{3\times3}_{\text{sym}})$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, \mathbf{u}) &= 0 \\ -b(\sigma, \mathbf{v}) \qquad &= \int_\Omega f \cdot \mathbf{v} \, dx \end{aligned} \tag{13}$$

for all $\mathbf{v} \in L^2(\Omega; \mathbb{R}^3)$ and $\tau \in H(\text{div}, \Omega; \mathbb{R}^{3\times3}_{\text{sym}})$, where

$$a(\sigma, \tau) = \int_\Omega A\sigma : \tau \, dx \quad , \quad b(\tau, \mathbf{v}) = \int_\Omega (\nabla \cdot \tau) \cdot \mathbf{v} \, dx \ .$$

To complete this description, we must describe the function spaces, which are

$$\begin{aligned} L^2(\Omega; \mathbb{R}^3) &= \left\{ \mathbf{v} : \Omega \to \mathbb{R}^3 : \int_\Omega \mathbf{v} \cdot \mathbf{v} \, dx < \infty \right\} \ , \\ H(\mathbf{div}, \Omega; \mathbb{R}^{3\times3}_{\text{sym}}) &= \left\{ \tau : \Omega \to \mathbb{R}^{3\times3}_{\text{sym}} : \int_\Omega \tau : \tau \, dx < \infty \text{ and } \int_\Omega (\nabla \cdot \tau) \cdot (\nabla \cdot \tau) \, dx < \infty \right\} \ . \end{aligned}$$

1.4 An Alternate Variational Formulation

$-\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, dx$ will be manipulated so we can come to a solution of what σ needs to be for the integral around the boundary to be zero. We apply the integration by parts formula to it, and we get

$$\int_{d\Omega} \mathbf{v} : (\sigma \mathbf{n}) \, ds = \int_\Omega \epsilon(\mathbf{v}) : \sigma \, dx + \int_\Omega v \cdot (\nabla \cdot \sigma) \, dx$$

then we rearrange it to

$$-\int_\Omega (\nabla \cdot \sigma) \cdot \mathbf{v} \, dx = \int_\Omega \epsilon(\mathbf{v}) : \sigma \, dx - \int_{d\Omega} v \cdot (\sigma \mathbf{n}) \, dx$$

We will make the assumption that

$$\sigma \mathbf{n} = \mathbf{0} \text{ on } \partial\Omega,$$

and when we rearrange the terms and we substitute it into the two equations in (6), we obtain,

$$\int_\Omega A\sigma : \tau \, dx - \int_\Omega \epsilon(\mathbf{u}) : \tau \, dx = 0 \tag{14a}$$

$$\int_\Omega \epsilon(\mathbf{v}) : \sigma \, dx = \int_\Omega f \cdot \mathbf{v} \, dx \tag{14b}$$

After choosing appropriate function subspaces the stress tensor and displacement vector, variational form like (13) can be obtained, but we will not pursue it further.

2 A Discretization of the Hellinger-Reissner Form

2.1 A Stabilized Mixed Finite Element Method

We will be using a stabilized mixed finite element method of the Hellinger-Reissner form introduced in [1]. Let $\Omega \subset \mathbb{R}^3$ be a polyhedron, $\mathcal{T} = \{K\}$ be a partition of Ω into tetrahedral cells (a mesh), and $\mathcal{F} = \{F\}$ be the collection of all triangular faces in this mesh. We wish to find $(\sigma_h, \mathbf{u}_h) \in \Sigma_h \times V_h$ such that

$$a(\sigma_h, \tau_h) + b(\tau_h, \mathbf{u}_h) = 0 \tag{15}$$

$$-b(\sigma_h, \mathbf{v}_h) + c(\mathbf{u}_h, \mathbf{v}_h) = \int_\Omega \mathbf{f} \cdot \mathbf{v}_h \, dx \tag{16}$$

for all $\tau_h \in \Sigma_h$ and for all $\mathbf{v}_h \in V_h$. To complete this definition we must define the finite dimensional spaces,

$$\Sigma_h \subset H(\mathbf{div}, \Omega; \mathbb{R}^{3\times 3}_{\text{sym}}) \quad , \quad V_h \subset L^2(\Omega; \mathbb{R}^3) \ , \tag{17}$$

which we do in the next subsection, as well as the function c, which we do now. The jump stabilization term for displacement is,

$$c(\mathbf{u}_h, \mathbf{v}_h) := \sum_{F \in \mathcal{F}} h_F \int_F [\![\mathbf{u}_h]\!] : [\![\mathbf{v}_h]\!] \, ds, \tag{18}$$

where h_F is the longest edge on the face F. For $\mathbf{w} : \Omega \to \mathbb{R}^3$ that is continuous on each $K \in \mathcal{T}$, we define $[\![\mathbf{w}]\!]$ as follows. On a face F laying on the boundary $\partial\Omega$,

$$[\![\mathbf{w}]\!] := \frac{1}{2}(\mathbf{w}\mathbf{n}^T + \mathbf{n}\mathbf{w}^T),$$

where $\mathbf{n}$ is the unit outward normal to the boundary. For an interior face F let K^+ and K^- be the adjacent tetrahedra, and let $\mathbf{n}^+$ and $\mathbf{n}^-$ be the unit outward normals to K^+ and K^- on F (Note: $\mathbf{n}^+ = -\mathbf{n}^-$). On such an F,

$$[\![\mathbf{w}]\!] := \frac{1}{2}\left(\mathbf{w}^+(\mathbf{n}^+)^T + \mathbf{n}^+(\mathbf{w}^+)^T + \mathbf{w}^-(\mathbf{n}^-)^T + \mathbf{n}^-(\mathbf{w}^-)^T\right)$$

2.2 Defining Σ_h and V_h

The spaces Σ_h and V_h in which we compute σ_h and $\mathbf{u}_h$ are given by,

$$\Sigma_h = \{\tau \in C(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}}) : \tau\big|_K \in \mathbb{P}_1(K; \mathbb{R}^{3\times 3}_{\text{sym}}) \text{ for all } K \in \mathcal{T}\} \tag{19}$$

$$V_h = \{\mathbf{v} : \Omega \to \mathbb{R}^3 : \mathbf{v}\big|_K \in \mathbb{R}^3 \text{ for all } K \in \mathcal{T}\} \tag{20}$$

A basis for V_h is,

$$\{\phi_{K,j} : K \in \mathcal{T} \text{ and } 1 \le j \le 3\} \text{ where, } \phi_{K,j}(x) = \begin{cases} \mathbf{e}_j, \ x \in K \\ \mathbf{0}, \ x \notin K \end{cases}$$

and our total dimension of V_h is $\dim(V_h) = 3NT$, where NT is the number of tetrahedra. The two spaces, $C(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}})$ and $\mathbb{P}_1(K; \mathbb{R}^{3\times 3}_{\text{sym}})$, in the definition of Σ_h are

$$C(\Omega; \mathbb{R}^{3\times 3}_{\text{sym}}) = \{\alpha : \Omega \to \mathbb{R}^{3\times 3}_{\text{sym}} : \alpha_{ij} \in C(\Omega; \mathbb{R})\} \ ,$$

$$\mathbb{P}_1(K; \mathbb{R}^{3\times 3}_{\text{sym}}) = \{\alpha : K \to \mathbb{R}^{3\times 3}_{\text{sym}} : \alpha_{ij} \in \mathbb{P}_1(K; \mathbb{R})\} \ .$$

We will give basis for $\mathbb{P}_1(K;\mathbb{R}^{3\times3}_{\text{sym}})$, and to do so we first need a basis of $\mathbb{R}^{3\times3}_{\text{sym}}$, and a basis of $\mathbb{P}_1(K,\mathbb{R})$. The space $\mathbb{R}^{3\times3}_{\text{sym}}$ of 3×3 symmetric matrices is six-dimensional, and a basis for it, $\{S_1,\ldots,S_6\}$, is given by

$$S_1=\begin{pmatrix}1&0&0\\0&0&0\\0&0&0\end{pmatrix},\ S_2=\begin{pmatrix}0&0&0\\0&1&0\\0&0&0\end{pmatrix},\ S_3=\begin{pmatrix}0&0&0\\0&0&0\\0&0&1\end{pmatrix}\ ,$$
$$S_4=\begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix},\ S_5=\begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix},\ S_6=\begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix}\ .$$

For example, a generic $\alpha\in\mathbb{R}^{3\times3}_{\text{sym}}$ has the expansion

$$\alpha=\begin{pmatrix}\alpha_{11}&\alpha_{12}&\alpha_{13}\\\alpha_{21}&\alpha_{22}&\alpha_{23}\\\alpha_{31}&\alpha_{32}&\alpha_{33}\end{pmatrix}=\alpha_{11}S_1+\alpha_{22}S_2+\alpha_{33}S_3+\alpha_{12}S_4+\alpha_{13}S_5+\alpha_{23}S_6\ .$$

For $\alpha\in\Sigma_h$, each component $\alpha_{ij}\in\mathbb{P}_1(K;\mathbb{R})$ is a (scalar-valued) linear function on K, so we now describe a basis for $\mathbb{P}_1(K;\mathbb{R})$. We denote the vertices of the tetrahedron K by $\{z_1,\ldots,z_4\}$. Since a linear function in three variables is uniquely determined by its values at the vertices, we define four linear functions $\ell_j\in\mathbb{P}_1(K;\mathbb{R})$, $1\le j\le4$, related to these vertices, by

$$\ell_j(z_i)=\begin{cases}1,\ i=j\\0,\ i\neq j\end{cases}\qquad,\quad 1\le i,j\le4\ .$$

Let h_j be the perpendicular distance between z_j and the plane containing the opposite face of K, and $\mathbf{n}_j$ be the outward normal unit vector to this face. A "point-slope" formula for ℓ_j is

$$\ell_j(x)=1-\frac{(x-z_j)\cdot\mathbf{n}_j}{h_j}$$

One can verify directly that $\ell_j(z_j)=1$ (the point) and $\nabla\ell_j=\frac{-\mathbf{n}_j}{h_j}$ (the "slope"). So $\{\ell_1,\ldots,\ell_4\}$ is a basis for $\mathbb{P}_1(K;\mathbb{R})$. In fact, $v\in\mathbb{P}_1(K;\mathbb{R})$ is given by

$$v(x)=v(z_1)\ell_1(x)+v(z_2)\ell_2(x)+v(z_3)\ell_3(x)+v(z_4)\ell_4(x)\ .$$

Finally, a basis for $\mathbb{P}_1(K;\mathbb{R}^{3\times3}_{\text{sym}})$ is $\{\ell_iS_j:1\le i\le4,\ 1\le j\le6\}$, so $\dim(\mathbb{P}_1(K;\mathbb{R}^{3\times3}_{\text{sym}}))=24$. Since each $\sigma_h\in\Sigma_h$ must be continuous across faces shared by tetrahedra, σ_h is uniquely determined by its values at the vertices of the mesh. Because $\sigma_h(z)\in\mathbb{R}^{3\times3}_{\text{sym}}$ at each vertex z, and $\dim(\mathbb{R}^{3\times3}_{\text{sym}})=6$, $\dim(\Sigma_h)=6NV$, where NV is the number of vertices.

2.3 Linear System for Discrete Problem

We need to develop a linear system to solve for our stress tensor and displacement vector in equations (15)-(16). Let $\{\phi_j:1\le j\le3NT\}$ be a basis of V_h, and $\{\psi_j:1\le j\le6NV\}$ be a basis of Σ_h. Equations (15)-(16) hold for all $\tau_h\in\Sigma_h$ and $\mathbf{v}_h\in V_h$ if and only if,

$$\begin{aligned}a(\sigma_h,\psi_i)+b(\psi_i,\mathbf{u}_h)&=0,\quad 1\le i\le6NV\\-b(\sigma_h,\phi_k)+c(\mathbf{u}_h,\phi_k)&=\int_\Omega f\phi_k\,dx,\quad 1\le k\le3NT\end{aligned}\tag{21}$$

Using that

$$\mathbf{u}_h=\sum_{j=1}^{3NT}y_j\phi_j\quad\sigma_h=\sum_{j=1}^{6NV}x_j\psi_j\tag{22}$$

for some coefficient vectors $\mathbf{x} \in \mathbb{R}^{3NT}$, $\mathbf{y} \in \mathbb{R}^{6NV}$, (21) is equivalent to: Find $\begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} \in \mathbb{R}^{6NV+3NT}$ such that,

$$\begin{pmatrix} A & B^T \\ -B & C \end{pmatrix} \begin{pmatrix} \mathbf{x} \\ \mathbf{y} \end{pmatrix} = \begin{pmatrix} \mathbf{0} \\ \mathbf{f} \end{pmatrix} \tag{23}$$

where $A \in \mathbb{R}^{6NV \times 6NV}, B \in \mathbb{R}^{3NT \times 6NV}, C \in \mathbb{R}^{3NT \times 3NT}$ are given by

$$\begin{aligned} a_{ij} &= a(\psi_j, \psi_i), \quad 1 \le i, j \le 6NV \\ c_{ij} &= c(\phi_j, \phi_i), \quad 1 \le i, j \le 3NT \\ b_{ij} &= b(\psi_j, \phi_i), \quad 1 \le i \le 3NT \quad 1 \le j \le 6NV \end{aligned} \tag{24}$$

and $\mathbf{f} \in \mathbb{R}^{3NT}$ is given by $\mathbf{f}_k = \int_\Omega f \phi_k \, dx, \quad 1 \le k \le 3NT$

2.4 Discretization Error

The key theorem in [1] concerning discretization error is

Theorem 2.1. *Let* $(\sigma, \mathbf{u}) \in H(div, \Omega; \mathbb{R}^{3\times 3}_{sym}) \times L^2(\Omega; \mathbb{R}^3)$ *be the exact solution of problem* (13) *and* $(\sigma_h, \mathbf{u}_h) \in \Sigma_h \times V_h$ *the discrete solution of the stabilized mixed finite element method* (15)-(16). *If* $\sigma \in H^2(\Omega; \mathbb{R}^{3\times 3}_{sym})$ *and* $\mathbf{u} \in H^1(\Omega; \mathbb{R}^3)$, *then*

$$\|\sigma - \sigma_h\|_{H(div,\mathcal{A})} + \|\mathbf{u} - \mathbf{u}_h\|_{0,c} \le Ch(\|\sigma\|_2 + \|\mathbf{u}\|_1) \ , \tag{25}$$

where C *is a constant that is independent of* h, σ *and* $\mathbf{u}$.

The norms appearing in this theorem are defined by,

$$\begin{aligned} \|\tau\|^2_{H(div,\mathcal{A})} &:= a(\tau,\tau) + \|\text{div}\,\tau\|_0^2 \ , \\ \|\mathbf{v}_h\|^2_{0,c} &:= \|\mathbf{v}_h\|_0^2 + \|\mathbf{v}_h\|_c^2 = \int_\Omega \mathbf{v}_h \cdot \mathbf{v}_h \, dx + c(\mathbf{v}_h, \mathbf{v}_h) \\ \|\mathbf{u}\|_1^2 &:= \int_\Omega \mathbf{u} \cdot \mathbf{u} + \nabla \mathbf{u} : \nabla \mathbf{u} \, dx \ , \\ |\sigma|_2^2 &:= \int_\Omega \sum_{K,\ell=1}^{3} \sum_{i,j=1}^{4} \left(\frac{\partial^2 \sigma_{ij}}{\partial x_K \partial x_\ell} \right) dx \ , \\ \|\sigma\|_2^2 &:= \int_\Omega \sigma : \sigma + \sum_{k=1}^{3} \sum_{i,j=1}^{4} \left(\frac{\partial \sigma_{ij}}{\partial x_k} \right)^2 + \sum_{k,\ell=1}^{3} \sum_{i,j=1}^{4} \left(\frac{\partial^2 \sigma_{ij}}{\partial x_k \partial x_\ell} \right)^2 dx \ . \end{aligned}$$

The norm $\|\mathbf{v}_h\|_{0,c}$ is defined for all $\mathbf{v_h} \in \mathbf{V}_h + H^1(\Omega; \mathbb{R}^3)$. The mesh parameter h is the longest edge in the mesh. In other words, $h = \max_K h_K$, where h_K is the diameter of K. This theorem states the error in the stress tensor and displacement vector is approximately proportional to h.

3 Numerical Experiments

3.1 Numerical Results from [1]

In [1] the authors illustrate Theorem 2.1 by constructing an example where the exact solution is known. They let the Lamé parameters be $\lambda = 0.3$ and $\mu = 0.35$. The domain is a unit cube $\Omega = (0,1)^3$ in 3D. The displacement vector $\mathbf{u}$ is given by

$$\mathbf{u}(x) = \mathbf{u}(x_1, x_2, x_3) = \begin{pmatrix} 2^4 \\ 2^5 \\ 2^6 \end{pmatrix} x_1(1-x_1)x_2(1-x_2)x_3(1-x_3), \tag{26}$$

and the stress tensor σ and the load $\mathbf{f}$ can be calculated exactly from (1)-(3). To determine σ we will first use (3) to determine $\mathrm{tr}(\sigma)$. We can manipulate (3) to find $\mathrm{tr}(\sigma)$,

$$\mathrm{tr}(\sigma) = (3\lambda + 2\mu)\mathrm{tr}(\epsilon(\mathbf{u})) \ . \tag{27}$$

Once $\mathrm{tr}(\sigma)$ is determined we find that σ is

$$\sigma = \lambda(\mathrm{tr}(\epsilon(\mathbf{u})))I + 2\mu(\epsilon(\mathbf{u})). \tag{28}$$

From (1) the load $\mathbf{f}$ is equal to $(-\nabla \cdot \sigma)$. For this model problem, the authors obtained the errors in Table 1.

Table 1: Convergence of approximations of the stress tensor and displacement vector.

h	$\\|\sigma - \sigma_h\\|_{H(\mathrm{div}),\mathcal{A}}$	order	$\\|\mathbf{u}_h\\|_C$	order	$\\|\mathbf{u} - \mathbf{u}_h\\|_0$	order
2^{-1}	4.1723E+00	—	4.0747E-01	—	2.4720E-01	—
2^{-2}	2.3595E+00	0.82	3.5554E-01	0.20	1.7403E-01	0.51
2^{-3}	1.2849E+00	0.88	2.5527E-01	0.48	1.1168E-01	0.64
2^{-4}	6.8023E-01	0.92	1.5243E-01	0.74	6.3889E-02	0.81
2^{-5}	3.5167E-01	0.95	8.3310E-02	0.87	3.4309E-02	0.90

To estimate the order of convergence from convergence data one typically assumes the error model $E(h) = Ch^p$, where C and p are unknown, and p is the order of convergence. The order of convergence can be found by

$$p = \log_2\left(\frac{E(h)}{E(\frac{h}{2})}\right). \tag{29}$$

This is what the authors use to compute the orders in Table 1. As h gets smaller, the computed orders approach 1, in agreement with Theorem 2.1. Figure 1 is a sequence of four meshes. The coloring in Figure 1 is purely for visualizing the meshes—it has no physical significance. The first is our initial region, and each image after is a refinement of the previous and these are a similar region to Theorem 2.1. On each refinement the number of tetrahedra along an edge is doubled. Next, we take a look inside the third refinement (Figure 2) and view the norm of the displacement vector, $\mathbf{u}_h$ throughout the entire mesh. Figure 2 is visualizing the third row from Table 1 where $h = 2^{-3}$. The center of the mesh is where it experiences the majority of displacement. Maximum displacement is determined by the color red, and the minimum displacement is determined by the color blue.

3.2 A Non-Unit Cube, Stepahedron

In Figure 3 we took a new shape that contains the characteristics of a set of stairs. We shall call it a stepahedron. It contains the same load force $\mathbf{f}$ as the unit cube, but has an unknown displacement vector $\mathbf{u}_h$. The stepahedron was cut down the middle and at the steps to show $\mathbf{u}_h$ on the inside. At the top of each step the magnitude of $\mathbf{u}_h$ is not 0 like most of the boundary on the region. They do not protrude from the original region but it can be displaced along the surface or deform into the stepahedron. The outward normal to each step is 0.

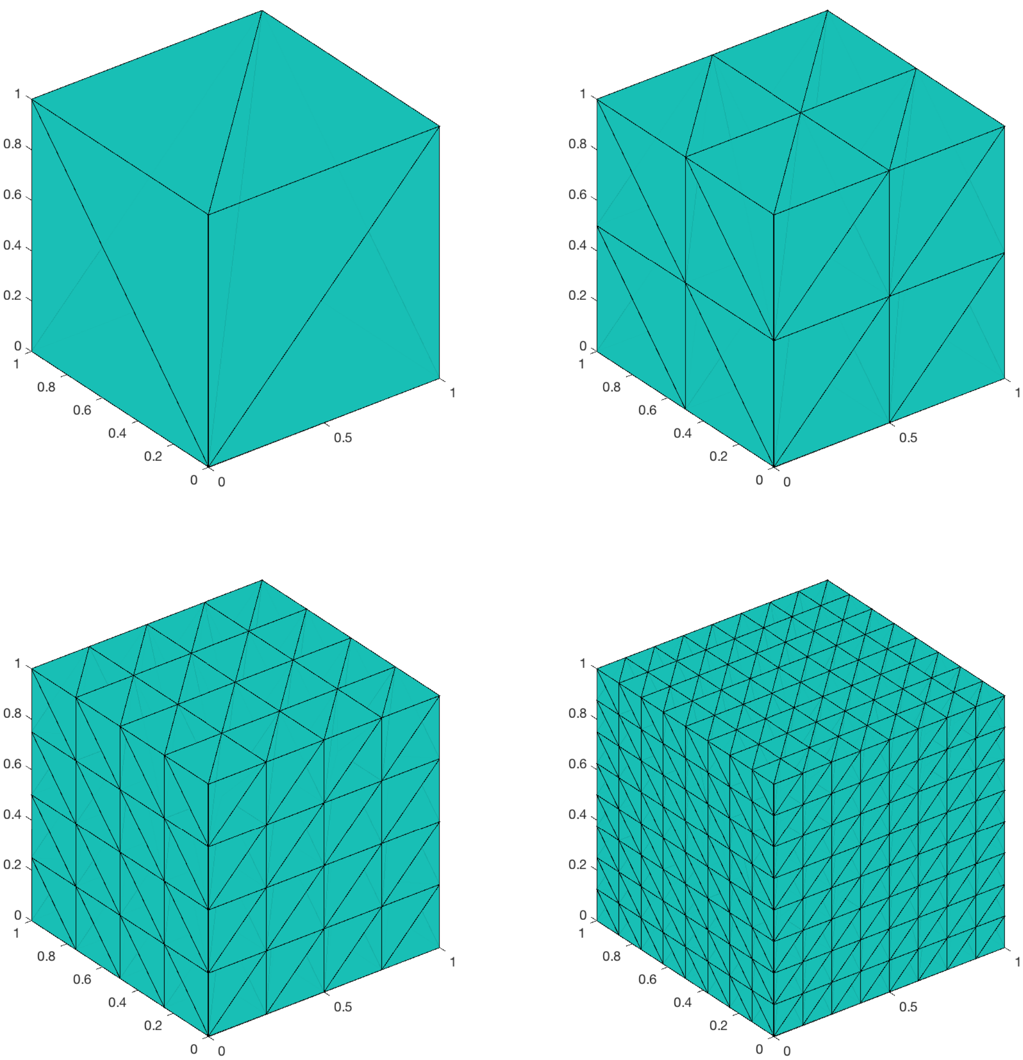

Figure 1: An initial mesh on the cube $(0,1)^3$, and three refinements.

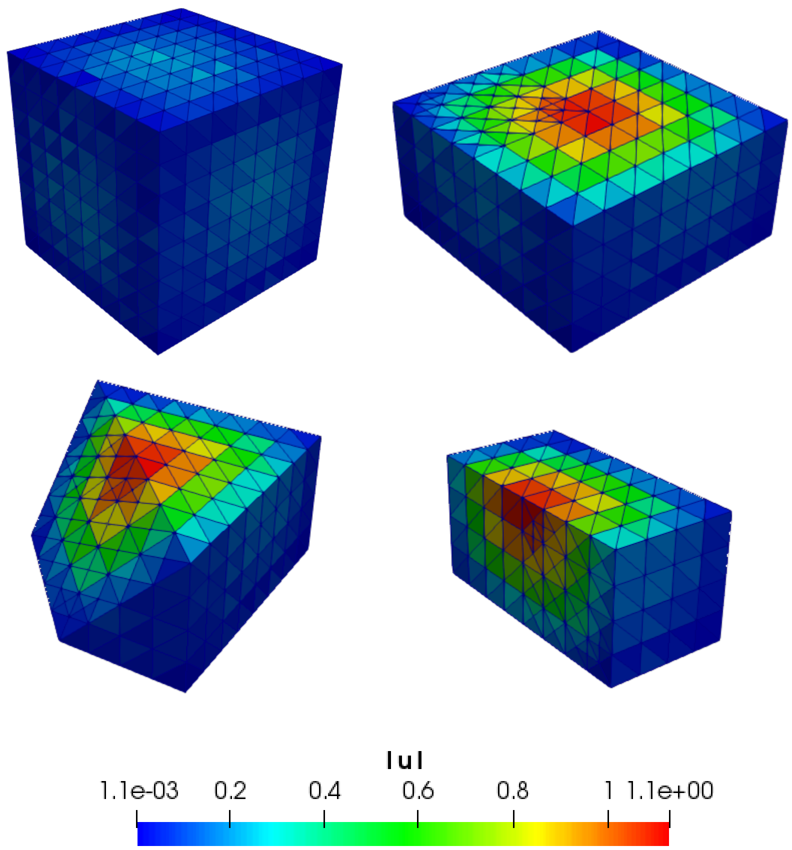

Figure 2: The third refined sequence cube $(0,1)^3$ cut to view different magnitudes of $\mathbf{u}_h$.

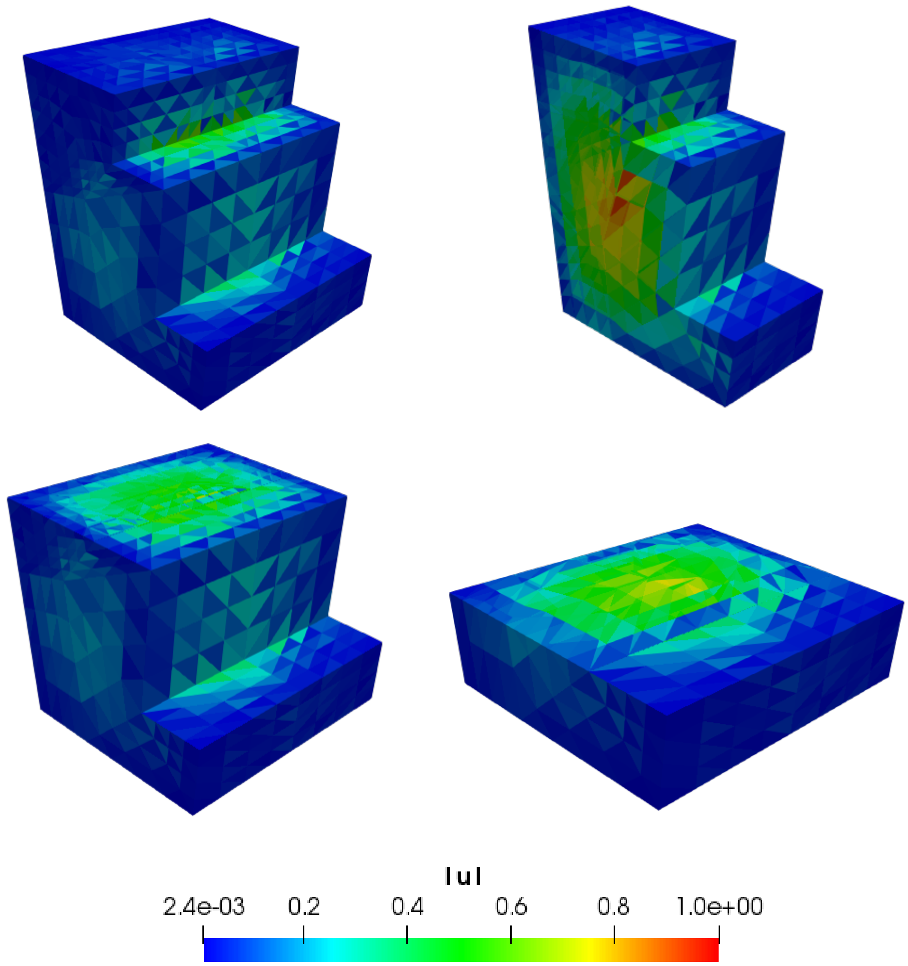

Figure 3: A stepahedron to cut to view different magnitudes of $\mathbf{u}_h$.

References

[1] L. Chen, J. Hu, and X. Huang. Stabilized Mixed Finite Element Methods for Linear Elasticity on Simplicial Grids in $\mathbb{R}^n$. *Computational Methods in Applied Mathematics*, 17(1):17–31, January 2017.