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Beam Propagation in Parabolically Tapered Graded-Index Waveguides

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Beam propagation in parabolically tapered graded-index waveguides

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To a good approximation, the electromagnetic-propagation characteristics of graded-index waveguides can be written in terms of polynomial-Gaussian modes. For uniform quadratic-index waveguides the behavior of these modes is well known. However, there are sometimes practical reasons for using tapered waveguides, but detailed propagation solutions are known for only a few specific taper functions. The parabolic taper is perhaps the most important special case, and the solution-generating techniques that we generalize are used to obtain analytic solutions for this case.

1. Introduction

Since the invention of the laser, there has been ever-increasing interest in the propagation of light in inhomogeneous dielectric media. Research interests in this field have increased even further with the development of relatively inexpensive low-loss glass fibers. By now it is well established that thin dielectric fiber waveguides provide an efficient and economical means for transmission of optical signals over large distances. A challenging problem in the development of fiber-optic communication systems involves the optimization of coupling techniques for the input and output of optical signals from other fibers or from thin-film waveguides. Among the many coupling techniques that have been developed, there are several that are based on waveguide tapering, and slight tapering and other distortions can also sometimes occur in the fabrication process. Tapered waveguides may also find applications as optical power concentrators, image-size reducers, and astigmatic image modifiers. Accordingly it is of some importance to understand in detail the propagation of electromagnetic modes in optical waveguides that have z-dependent characteristics.

An analysis by Burns *et al.* showed that the optimum shape for fiber couplers is a parabola,[1] and Campbell demonstrated the fabrication of parabolic tapers by the use of Ag-ion exchange in soda-lime glass.[2] Bures *et al.* proposed that a parabolic model should also be appropriate to describe the taper that results when a fiber is heated locally and stretched.[3] A schematic representation of such a stretched fiber is shown in Fig. 1. Experimental verification of the near-parabolic shape that resulted from fiber elongation was obtained by Burns *et al.*[4] and Rodrigues *et al.*[5] Transmission loss in parabolic tapers has been studied by Burns *et al.*,[6] and the wavelength dependence of the losses in such tapers has been explored by Cassidy *et al.*[7] The parabolic taper shape has been used by Bures *et al.*,[3] Rodrigues *et al.*,[5] Burns and Abebe,[8] and others in studies of coupling between parallel fibers.

Many of the fiber systems in modern communication networks are based on waveguides that have a continuous variation of the index of refraction. An advantage of such graded-index waveguides is their large bandwidth. In addition, graded-index waveguides that have a hyperbolic secant (or in the paraxial approximation, a quadratic) variation of the index of refraction transverse to the fiber axis have important image-transmitting characteristics[9,10] and are often known as lenslike media. The primary focus of this study is on such lenslike media that are also tapered along the axis of propagation. A new solution that corresponds to perhaps the most physically useful taper is discussed. This solution can be combined with other solutions by the use of the familiar *ABCD* methods of Gaussian beam theory to obtain a description of the behavior of more complex fiber tapers. Although the emphasis here is on tapered waveguides for fiber-optic applications, the results also apply to tapered gain profiles. Such

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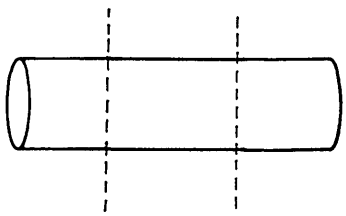
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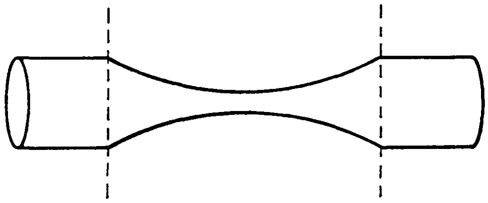
(b)

Fig. 1. Model of a fused fiber taper (a) before and (b) after the taper is formed.

profiles may be used, for example, for transverse-mode discrimination in lasers.

The basic wave-equation formalism for the propagation of Gaussian beams in z-dependent quadratic-index lenslike media is briefly reviewed in Section 2, and it is shown that for many purposes the propagation problem reduces to solution of a single second-order linear differential equation with nonconstant coefficients. Techniques for generation of solutions of this equation are reviewed and generalized in Section 3. These solution-generating techniques are applied to the specific and most important case of a parabolically tapered fiber in Section 4, and new analytical solutions are obtained.

2. Theory

The basic concepts of Gaussian beam propagation in quadratic-index media are well known, and it is sufficient here just to sketch some of the underlying principles. In particular we restrict our attention to the on-axis fundamental Gaussian beam mode. For more details and more general solutions, the reader may consult Refs. 11 and 12 and the further references cited therein.

As in most previous studies, it is assumed that the electric-field components are governed by the vector equation

$$\nabla^2\mathbf{E}'(x, y, z) + k^2(x, y, z)\mathbf{E}'(x, y, z) = -2\nabla\left[\frac{\nabla k(x, y, z)}{k(x, y, z)}\cdot\mathbf{E}'(x, y, z)\right], \quad (1)$$

where the prime is a reminder that Eq. (1) applies to the complex amplitude of the temporally harmonic electric field and the medium is assumed to be isotropic and nonmagnetic. When the right-hand side of Eq. (1) can be set approximately equal to 0, one obtains the standard scalar-wave equations for the field components. These equations are the basis for most treatments of light-ray and Gaussian beam propagation. This scalar approximation is valid for the dominant transverse-field components, provided that the propagation constant varies negligibly on a length scale of a wavelength. This restriction would be satisfied in the propagation direction for most practical fiber tapers, but it could sometimes be suspect in single-mode-fiber treatments that involve rapid radial variations of the index of refraction.

Because of the separability of the scalar approximated version of Eq. (1), a simplified y-independent model of the beam may usually be used without loss of generality. For the simplest case of a light beam traveling in the z direction in an x-dependent lenslike medium, the quadratically varying propagation constant can be written as

$$k^2(x, z) = k_0(z)[k_0(z) - k_2(z)x^2], \quad (2)$$

where all the coefficients are in general complex. For misaligned media one would need an additional term in Eq. (2) that is linear in x, and for elliptical beams in x- and y-dependent media a parallel development of the more general field variations is straightforward.[11,12] Equation (2) corresponds to a z-dependent medium in which the quadratic gain and index profiles may be independently specified. For an x-polarized wave propagating primarily in the z direction, a useful substitution is

$$\mathbf{E}'(x, z) = \mathbf{i}_x A(x, z)\exp\left[-i\int_0^z k_0(z')\mathrm{d}z'\right]. \quad (3)$$

With this substitution and Eq. (2), Eq. (1) reduces to

$$\frac{\partial^2}{\partial x^2}A(x, z) - 2ik_0(z)\frac{\partial}{\partial z}A(x, z) - i\frac{\mathrm{d}k_0(z)}{\mathrm{d}z}A(x, z) - k_0(z)k_2(z)x^2A(x, z) = 0, \quad (4)$$

where $A(x, z)$ is assumed to vary slowly enough with z that its second derivative can be neglected.

Equation (4) is a partial differential equation, but it may always be reduced exactly to a set of ordinary differential equations. For the simplest case of an on-axis fundamental Gaussian beam, the appropriate substitution is[13]

$$A(x, z) = A_0\exp\{-i[Q(z)x^2/2 + P(z)]\}. \quad (5)$$

With slightly more general substitutions, higher-order off-axis polynomial-Gaussian beams can also be described.[11,12] When Eq. (5) is substituted into Eq. (4) and the terms in equal powers of x are collected, one obtains the ordinary differential equations

$$Q^2(z) + k_0(z)\frac{\mathrm{d}Q(z)}{\mathrm{d}z} + k_0(z)k_2(z) = 0, \quad (6)$$

$$\frac{\mathrm{d}P(z)}{\mathrm{d}z} = -i\frac{Q(z)}{2k_0(z)}. \quad (7)$$

For the more general modes, additional terms and additional equations would be obtained as discussed in Ref. 11. It is important to note, however, that Eq. (6) would not change in form. Indeed it has been shown that once the beam-parameter equation (6) has been solved for a specific taper, all other facets of an off-axis polynomial-Gaussian beam are obtained analytically.[12]

The complex beam parameter $Q(z)$ is related to the phase-front curvature $R(z)$ and the $1/e$ amplitude spot size $w(z)$ by means of the formula

$$Q(z) = \frac{\mathrm{Re}[k_0(z)]}{R(z)} - \frac{2i}{w^2(z)}\cdot \tag{8}$$

The complex phase parameter $P(z)$ measures the relative on-axis complex phase of the propagating beam. This is the mode-dependent complex phase shift, excluding the plane-wave phase $-ik_0z$, reflection losses at dielectric boundaries, unknown constant phase shifts at thin lenses, etc. The real part of $P(z)$ represents the ordinary phase, and the imaginary part represents mode-dependent amplitude variations.

Equations (6) and (7) may be solved in sequence, and the main emphasis here is on exact solutions of Eq. (6). The solution of Eq. (7) simply involves an integration. Equation (6) is a Ricatti equation, and the first step in solving it is to introduce the well-known variable change[14]

$$Q(z) = \frac{k_0(z)}{r(z)}\frac{\mathrm{d}r(z)}{\mathrm{d}z}\cdot \tag{9}$$

With this substitution the Ricatti equation is transformed into a linear differential equation with nonconstant coefficients:

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[k_0(z)\frac{\mathrm{d}r(z)}{\mathrm{d}z}\right] + k_2(z)r(z) = 0. \tag{10}$$

Equation (10) has been obtained here from an exact reduction of the paraxial wave equation. Typically, the gain per wavelength of the optical medium is ignored, and $k_0(z)$ is approximated to be real.[15] When $k_2(z)$ is also real, this same equation is identical in form to the paraxial ray equation. In that case $r(z)$ could be interpreted as the z-dependent transverse displacement of a propagating light ray.

Solutions of Eq. (10) are much easier to obtain if k_0 is a constant, and in that case one finds that

$$\frac{\mathrm{d}^2r(z)}{\mathrm{d}z^2} + \frac{k_2(z)}{k_0}r(z) = 0. \tag{11}$$

It is important to note, however, that with an appropriate change of variables Eq. (10) can always be transformed into Eq. (11) without approximation, and this transformation has been discussed in Ref. 16. Hence there is no loss of generality in using Eq. (11) in place of Eq. (10). Also, in the fabrication of tapered media it would often be the case that the on-axis optical properties would be unaffected by the tapering process, so that k_0 would in fact be constant. A simpler notation for Eq. (11) is

$$\frac{\mathrm{d}^2r(z)}{\mathrm{d}z^2} + f(z)r(z) = 0, \tag{12}$$

and this form is used in some of the discussions below. In the special case that $f(z)$ is periodic, Eq. (12) is known as a Hill equation.[17]

3. Generating Solutions

Equation (12) cannot be solved analytically for an arbitrary function $f(z)$. It is the purpose of Section 3 to demonstrate a way of generating certain solvable equations of this type. The basic idea is to use the reverse procedure of assuming a solution and seeing what is obtained for the coefficient $f(z)$. In particular, Eq. (12) can be rewritten as

$$f(z) = -\frac{1}{r(z)}\frac{\mathrm{d}^2r(z)}{\mathrm{d}z^2}\cdot \tag{13}$$

Thus one may attempt to guess a form for $r(z)$ that leads to a useful form for $f(z)$.

It should also be noted that if one solution to Eq. (12) is known, the general solution can be found directly. It can be shown by the use of direct substitution that a second solution to Eq. (12) can be written as

$$r_2(z) = r_1(z)\int_0^z \frac{1}{r_1^2(z')}\,\mathrm{d}z'. \tag{14}$$

However, the Wronskian of $r_1(z)$ and $r_2(z)$ is unity. Therefore they are linearly independent functions, and the general solution may be written as

$$r(z) = r_1(z)\left(a + b\int_0^z \frac{1}{r_1^2(z')}\,\mathrm{d}z'\right), \tag{15}$$

where the constants a and b may be used to match input conditions. As an example one may consider the special case $f(z) = 1$ in Eq. (12). Of course, $\cos(z)$ is one of the solutions. From Eq. (14) the other linearly independent solution is

$$r_2(z) = \cos(z)\int_0^z \frac{1}{\cos^2(z')}\,\mathrm{d}z' = \sin(z), \tag{16}$$

as expected.

There has been recent interest in solutions to the equation

$$\frac{\mathrm{d}^2r(z)}{\mathrm{d}z^2} + [f(z) + g(z)]r(z) = 0, \tag{17}$$

where the solution to the equation

$$\frac{d^2u(z)}{dz^2} + f(z)u(z) = 0 \tag{18}$$

is considered to be known. The interest in such solutions was fueled by a study of the specific Hill equation,

$$\frac{d^2r(z)}{dz^2} + \left\{\frac{F}{[1 + G\cos(\gamma z)]^4} + \frac{\gamma^2 G\cos(\gamma z)}{1 + G\cos(\gamma z)}\right\}r(z) = 0, \tag{19}$$

with $\gamma = 2$.[18] Equation (19) has a variety of applications, and $r(z)$ can be expressed solely in terms of elementary functions. A method of constructing the solution to Eq. (19) was shown by Wu and Shih.[19] They also introduced a few similar but more complicated solvable Hill equations. Renne pointed out that similar construction techniques have long been known.[20] Takayama noted that Eq. (19) and the Wu and Shih solutions contain singularities and demonstrated a method of constructing singularity-free solutions.[21] Nassar and Machado described a generalized theory for construction of solutions of equations of the form[22]

$$\frac{d^2r(z)}{dz^2} + [f_1(z) + f_2(z) + f_3(z) + \cdots]r(z) = 0. \tag{20}$$

These authors share the point of view that Eq. (19) is just one in a class of solvable Hill equations. Another more general construction technique is described below, and this technique is applied to the problem of the parabolically tapered waveguide.

As mentioned above, the basic idea behind construction techniques is to assume a solution and see what coefficient is obtained. With this in mind, we assume a modulated sinusoidal solution of the form

$$r(z) = u(z)\exp\left[i\int_0^z v(z')dz'\right]. \tag{21}$$

It follows that the differential equation is

$$\frac{d^2r(z)}{dz^2} - \left[\frac{1}{u(z)}\frac{d^2u(z)}{dz^2} + 2i\frac{1}{u(z)}\frac{du(z)}{dz}v(z) - v^2(z) + i\frac{dv(z)}{dz}\right]r(z) = 0, \tag{22}$$

which is of the form of Eq. (17). The coefficient in Eq. (22) has four terms. In the special case that the second and fourth terms cancel, $v(z)$ can be solved for, and the result is

$$v(z) = F^{1/2}/u^2(z). \tag{23}$$

The remaining equation is now

$$\frac{d^2r(z)}{dz^2} - \left[\frac{1}{u(z)}\frac{d^2u(z)}{dz^2} - \frac{F}{u^4(z)}\right]r(z) = 0. \tag{24}$$

This is the fundamental idea behind the Wu and Shih paper and the ones that followed it. In this case the other linearly independent solution is found easily. It is

$$r(z) = u(z)\exp\left[-i\int_0^z v(z')dz'\right], \tag{25}$$

where $v(z)$ is given by Eq. (23). In general the other linearly independent solution can be found from Eq. (14).

As an example of the applicability of Eq. (24) one may consider $u(z) = 1 + G\cos(\gamma z)$. It is not difficult to see that Eq. (24) becomes identical to the known result given in Eq. (19). Because $u(z)$ is known and $v(z)$ is defined in Eq. (23), $r(z)$ may be obtained as a linear combination of the solutions given in Eqs. (21) and (25). As an additional example, if $u(z) = [1 + G\cos(\gamma z)]^{1/2}$, then Eq. (24) leads to coefficient 7 of Table 1, which with $\gamma = 2$ is equivalent to the result given by Wu and Shih.[19]

Other classes of solutions can be obtained in a similar way. Suppose the second and third terms in Eq. (22) add to a constant multiplied by $v(z)$. This happens when $v(z)$ is given by

$$v(z) = \frac{2i}{u(z)}\frac{du(z)}{dz} - F. \tag{26}$$

Table 1. Taper Coefficients for Analytically Solvable Beam Equations

No.	Nonconstant Coefficient $f(z)$	Reference
1	$\frac{F}{(a + bz^2)^2}$	This paper
2	$\frac{F}{(1 - \gamma z)^2}$	23
3	$\frac{F}{(1 - \gamma z/2)^4}$	23
4	$F(1 + 2\gamma z)$	23
5	$a - 2q\cos(2z)$	24
6	$\frac{F}{[1 + G\cos(\gamma z)]^4} + \frac{\gamma^2 G\cos(\gamma z)}{1 + G\cos(\gamma z)}$	16
7	$1 + \frac{F + G^2 - 1}{[1 + G\cos(\gamma z)]^2}$	19
8	$\frac{\gamma G}{1 + G\cos(\gamma z)}\left[\gamma\cos(\gamma z) + \frac{\sin(\gamma z)}{F + z}\right]$	This paper
9	$\frac{V^2\,\mathrm{sech}^2(z/a) - B^2}{a^2}$	25
10	$\frac{g_0}{1 - (z/L)^2}$	26
11	$\frac{g_0}{1 - (z/L)}$	26

In this case Eq. (22) reduces to

$$\frac{d^2r(z)}{dz^2} + \left\{F^2 - 2iF\frac{1}{u(z)}\frac{du(z)}{dz} + \frac{1}{u(z)}\frac{d^2u(z)}{dz^2} - 2\left[\frac{1}{u(z)}\frac{du(z)}{dz}\right]^2\right\}r(z) = 0. \quad (27)$$

Note that the term $[1/u(z)][d^2u(z)/dz^2]$ has a different sign from that in Eq. (24).

Similarly, one can choose the third and fourth terms in Eq. (22) to cancel. This implies that

$$v(z) = \frac{-i}{F + z}\cdot \quad (28)$$

Therefore Eq. (22) reduces to

$$\frac{d^2r(z)}{dz^2} - \left[\frac{1}{u(z)}\frac{d^2u(z)}{dz^2} + \frac{2}{F + z}\frac{1}{u(z)}\frac{du(z)}{dz}\right]r(z) = 0. \quad (29)$$

As a new example suppose that $u(z) = 1 + G\cos(\gamma z)$, as above. It follows from Eq. (29) that we may obtain exact solutions to the interesting equation

$$\frac{d^2r(z)}{dz^2} + \frac{\gamma G}{1 + G\cos(\gamma z)}\left[\gamma\cos(\gamma z) + \frac{\sin(\gamma z)}{F + z}\right]r(z) = 0. \quad (30)$$

The taper function in Eq. (30) is also represented as coefficient 8 of Table 1.

4. Parabolic Taper

The focus of Section 4 is on practical optical fiber tapers. Such tapers are often manufactured by application of axial tension to a heated fiber, thus stretching it. As noted in Section 1, Burns *et al.* and others have demonstrated experimentally that the parabolic model is very accurate over most of the length of the taper.[4] The purpose here is to solve for propagation in such tapers in terms of the powerful *ABCD* matrix formalism. With this formalism, parabolically tapered media may be incorporated into multielement optical systems.

As mentioned above, the propagation of Gaussian beams in typical tapered media is governed by Eq. (11). For most fiber media the losses are negligible and so the propagation constant is real. Thus Eq. (11) becomes

$$\frac{d^2r(z)}{dz^2} + \frac{n_2(z)}{n_0}r(z) = 0. \quad (31)$$

We introduce the *ABCD* formalism by solving Eq. (31) and applying appropriate boundary conditions. If the index profile is weak, it follows from an expansion of Eq. (2) that the refractive-index profile can be written as

$$n(x, z) = n_0 - \tfrac{1}{2}n_2(z)x^2. \quad (32)$$

If n_{cl} denotes the index of the fiber at the boundary with the cladding and x_{cl} is the inner radius of the cladding, then it follows from Eq. (32) that these quantities are related by

$$n_{cl} = n_0 - \tfrac{1}{2}n_2(z)x_{cl}^2(z). \quad (33)$$

It is clear that along surfaces of constant index, $x_{cl}(z)$ varies in z if $n_2(z)$ does. This variation of the fiber radius is known as tapering. Hence $x_{cl}(z)$ will be referred to as the taper function. The notation used here describes only the tapering in the x direction, and there exist analogous definitions for tapering in the y direction when symmetric or astigmatic tapers are involved.

If Eqs. (31) and (33) are combined, one obtains the governing equation

$$\frac{d^2r(z)}{dz^2} + 2\frac{1 - n_{cl}/n_0}{x_{cl}^2(z)}r(z) = 0. \quad (34)$$

As mentioned above, the interest here lies in parabolically tapered media. Hence it is postulated that the taper is described by the known taper function

$$x_{cl}(z) = a + bz^2, \quad (35)$$

where the origin of the z coordinate is at the minimum of the taper. An index of refraction contour for a typical symmetric taper is shown in Fig. 2(a). With this substitution, Eq. (34) becomes

$$\frac{d^2r(z)}{dz^2} + 2\frac{1 - n_{cl}/n_0}{(a + bz^2)^2}r(z) = 0. \quad (36)$$

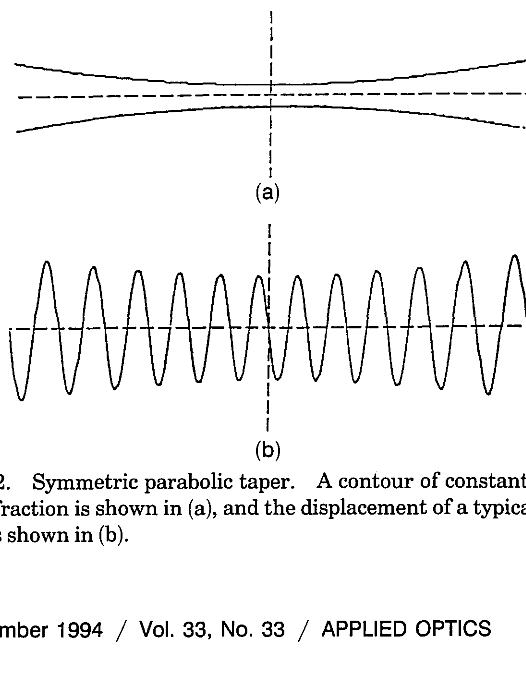

Fig. 2. Symmetric parabolic taper. A contour of constant index of refraction is shown in (a), and the displacement of a typical light ray is shown in (b).

One may show by direct substitution that the solution to Eq. (36) is

$$r(z) = (a + bz^2)^{1/2}\left(c_1 \cos\left\{\left[\frac{ab + 2(1 - n_{cl}/n_0)}{ab}\right]^{1/2} \times \tan^{-1}\left(\frac{b^{1/2}z}{a^{1/2}}\right)\right\} + c_2 \sin\left\{\left[\frac{ab + 2(1 - n_{cl}/n_0)}{ab}\right]^{1/2} \times \tan^{-1}\left(\frac{b^{1/2}z}{a^{1/2}}\right)\right\}\right), \quad (37)$$

where the coefficients c_1 and c_2 are to be determined from the initial conditions. This result can be generated by letting $u = (a + bz^2)^{1/2}$ in Eq. (24), redefining the constant F, and applying Eqs. (21), (23), and (25). It should be noted that in the limit of small b, the solution becomes the expected undamped sinusoid.

The propagation formulas for a tapered medium can also be written in terms of the familiar *ABCD* matrix formalism. First it may be noted that Eq. (37) can be written in the more abbreviated form

$$r(z) = c_1 u(z) + c_2 v(z), \quad (38)$$

where the definitions of the functions $u(z)$ and $v(z)$ are obvious but different from the definitions in Section 3. Similarly, the rate of change of the parameter $r(z)$ is

$$r'(z) = c_1 u'(z) + c_2 v'(z). \quad (39)$$

It follows from Eqs. (38) and (39) that the general equations for $r(z)$ and $r'(z)$ can be written in the usual matrix form

$$\begin{pmatrix} r(z) \\ r'(z) \end{pmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{pmatrix} r(z_1) \\ r'(z_1) \end{pmatrix}, \quad (40)$$

where the matrix elements are[16]

$$A(z) = \frac{u'(z_1)v(z) - v'(z_1)u(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \quad (41)$$

$$B(z) = \frac{v(z_1)u(z) - u(z_1)v(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \quad (42)$$

$$C(z) = \frac{u'(z_1)v'(z) - v'(z_1)u'(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \quad (43)$$

$$D(z) = \frac{v(z_1)u'(z) - u(z_1)v'(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}. \quad (44)$$

Substituting the matrix formalism of Eq. (40) into Eq. (9) yields the well-known Kogelnik transformation[15]:

$$\frac{Q(z)}{k_0(z)} = \frac{C(z) + D(z)Q(z_1)/k_0(z_1)}{A(z) + B(z)Q(z_1)/k_0(z_1)}. \quad (45)$$

Thus, with the aid of Eq. (8), one may obtain the z dependence of the spot-size and phase-front curvature in such a taper or in optical systems that contain these tapered media. In a similar manner, Eq. (7) may be integrated and the result is [27]

$$P(z) - P(z_1) = -\frac{i}{2}\ln[A(z) + B(z)Q(z_1)/k_0(z_1)]. \quad (46)$$

Because the real part of this phase parameter represents the Gaussian beam's axial phase and the imaginary part represents the axial amplitude, the z dependence of these beam properties may be found as well.

For a medium or an optical system that is represented by a beam matrix that consists of only real elements, such as a lossless tapered waveguide, the center of a Gaussian beam propagates with the same trajectory as a paraxial light ray. In this case the ray matrix is identical to the beam matrix[28] $r(z)$ and $r'(z)$ may also be interpreted as the position and slope of the amplitude center of the Gaussian beam. Higher-order modes and off-axis beams in complex media can be treated by the use of well-known extensions of these methods. In any such generalization the *ABCD* matrix elements are unchanged from those calculated here. As a representative example, a plot of the amplitude center of an off-axis Gaussian beam is shown in Fig. 2(b), and this result was obtained with the equations of Section 4. It is clear from Fig. 2(b) that near the region of the minimum fiber diameter the amplitude and period may be significantly less than in the regions away from the diameter minimum, as one might expect. Indeed, it can be seen from Eq. (37) that the frequency of the beam's oscillation is inversely proportional to the square of the amplitude of beam oscillation. Interestingly, this property is not unique to the parabolic taper. Any solution to Eq. (11) that has the form of Eq. (24) has this property.

The propagation formulas for the parabolic taper that have been derived here now take their place with the several other analytic taper solutions that have been reported previously. Table 1 includes a listing of several of the more elementary taper functions for which solutions have been obtained. The form of these solutions and other details may be obtained from the references indicated in Table 1.

The purpose of the discussion thus far has been to construct mathematical solutions to specific fiber tapers of interest. Needless to say, there might be many taper profiles that one would wish to investigate but that would not be susceptible to analytical solutions. In this case another possible approach is to represent the taper by a sequence of shorter fiber segments for which the transformation characteristics are known. In Ref. 16, for example, it was suggested that the propagation through a periodically tapered fiber could be described in terms of a sequence of alternating uniform and linearly tapered segments. More generally, any taper configuration can be described in terms of such composite elements. However, the effectiveness of such a representation is limited by the fact that at the junction of two finite segments one cannot obtain continuity of the first derivative of the index of refraction. On the other hand, it has been demonstrated above that the parabolic taper function can be characterized analytically. Therefore sequences of parabolic elements may also be used to represent an arbitrarily tapered fiber. An advantage of such parabolic splines is that one can use a small number of such elements to represent an arbitrary taper while always maintaining continuity of the first derivative. This method could be very effective for modeling of periodic tapers, and a sinusoidally modulated index function, for example, can be well represented by just a pair of alternating parabolic tapers.

5. Conclusion

In this paper a physically interesting and important taper function has been identified, and analytic solutions for the propagating fields and ray trajectories have been obtained. In this taper function a quadratic graded-index medium is tapered in such a way that the cladding radius is a parabolic function of position along the fiber. This tapering has been shown to occur naturally when a fiber is heated and drawn, and hence it is a very natural configuration for use in practical fiber systems. It is also useful analytically because more complex smooth tapers can be well represented in terms of a small number of such parabolic taper segments.

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