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Citation Details

Kochar, S., & Xu, M. (2012). On the skewness of order statistics with applications. Appears in Annals of Operations Research. 198 (1).

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On the skewness of order statistics with applications

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July 25, 2012

Abstract

Order statistics from heterogenous samples have been extensively studied in the literature. However, most of the work focused on the effect of heterogeneity on the magnitude and dispersion of order statistics. In this paper, we study the skewness of order statistics from heterogeneous samples in the sense of star order. The main results extended the results in *Kochar and Xu* (2009, 2011). Examples and applications in statistical inference are highlighted.

Keywords exponential distribution; isotonic estimation; k-out-of-n system; skewness; star order

AMS 2000 subject classifications: Primary 60E15; 62N05; 62G30; 62D05

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1 Introduction

Skewness describes the departure of a distribution from symmetry, where one tail of the density is more "stretched out" than the other. Several well-known measures of skewness are available in the statistics literature, such as Pearson's coefficient of skewness and Edgeworth's coefficient. Interested readers may refer to *Arnold and Groeneveld* (1993) and *Marshall and Olkin* (2007, p.70) for more discussion and other measures of skewness. Skewed data is observed in many areas such as economics, engineering, medicine, insurance and psychology. It may be easy to recognize symmetric distributions but not so easy to determine whether one non-symmetric distribution is more skewed than another. Several partial orders have been introduced in the literature to compare the relative skewness of probability distributions. *Van Zwet* (1964) introduced the concept of *convex transform order* to compare two distributions according to skewness. Gamma distributions, which play a prominent role in actuarial science due to its skewness, are ordered according to the convex transform order in terms of their shape parameters. Another well-known partial order to compare the skewness of two probability distributions is *star order* (cf. *Barlow and Proschan*, 1981 and *Oja*, 1981). This ordering is weaker than the convex transform order. It is well known that the star order implies the Lorenz order, which is an important partial order in economics to compare income inequalities.

Order statistics have received a great amount of attention in the literature since they are widely used in reliability theory, data analysis, extreme value theory, goodness-of-fit tests, statistical inference and other applied probability and statistical areas. Most of these studies focused mainly on the case when order statistics are from independent and identically distributed (i.i.d.) random variables. Please refer to *David and Nagaraja* (2003) and *Balakrishnan and Rao* (1998a, b) for more details. Studies of order statistics from heterogeneous samples began in early 70s, motivated by robustness issues. After that, a lot of work has been done in single-outlier and multiple-outlier models. *Balakrishnan* (2007) synthesized recent developments on order statistics arising from independent but non-identically distributed random variables. One may also refer to *Kochar and Xu* (2007) and *Xu* (2010) for reviews on various recent developments.

In reliability engineering, an n component system that works if and only if at least k of the n components work is called a k-out-of-n system. Both parallel and series systems are special cases of the k-out-of-n system. Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denote the order statistics of random variables $X_1, X_2, \cdots, X_n$. The lifetime of a k-out-of-n system can be represented as $X_{n-k+1:n}$. In the literature, some work has been done on comparing the order statistics of a random sample according to Lorenz order (cf. *Arnold and Villasenor*, 1989 and *Kochar*, 2006). However, the problem of comparing the skewness of order statistics from two samples has not received much attention yet. *Kochar and Xu* (2009) began to fix this gap by studying the skewness of largest order statistics from heterogenous exponential distributions. They showed that the largest order statistics from heterogenous exponential samples are more skewed than the one from homogeneous exponential samples in the sense of convex transform order. This result has been partly extended in a subsequent paper by *Kochar and Xu* (2011), where they studied the general order statistics from multiple-outlier exponential models. They showed that under some suitable conditions, order statistics from two exponential samples can be ordered according to star ordering.

In this paper, we will further study the skewness of order statistics from two samples under the general framework. The main results in this paper are applicable to Weibull distribution, Pareto distribution and Lomax distribution, etc. Application in isotonic estimation is mentioned as well. Throughout the paper, all random variables are assumed to be nonnegative and continuous. The inverse functions defined in this paper are assumed to be right continuous.

2 Preliminaries

In this section, we recall some stochastic orders which will be used in the sequel.

Assume random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1-F$ and $\bar{G} = 1-G$, density functions f and g, and failure rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, respectively.

Definition 2.1 X is said to be smaller than Y in *the convex transform order*, denoted by $X \leq_c Y$ if and only if, $G^{-1}F(x)$ is convex in x on the support of X.

If $X \leq_c Y$, then Y is more skewed than X as explained in *van Zwet* (1964) and *Marshall and Olkin* (2007). The convex transform order is also called more IFR (increasing failure rate) order in reliability theory, since when f and g exist, the convexity of $G^{-1}F(x)$ means that

$$\frac{f\left(F^{-1}(u)\right)}{g\left(G^{-1}(u)\right)} = \frac{r_X\left(F^{-1}(u)\right)}{r_Y\left(G^{-1}(u)\right)},$$

is increasing in $u \in [0,1]$. Thus $X \leq_c Y$ can be interpreted to mean that X ages faster than Y in some sense.

Definition 2.2 X is said to be smaller than Y in *the star order*, denoted by $X \leq_\star Y$ (or $F \leq_\star G$) if the function $G^{-1}F(x)$ is star shaped in the sense that $G^{-1}F(x)/x$ is increasing in x on the support of X.

The star order is also called more IFRA (increasing failure rate in average) order in reliability theory, since the average failure of F at x is

$$\bar{r}_X(x) = \frac{1}{x}\int_0^x r_X(u)du = \frac{-\ln \bar{F}(x)}{x}.$$

Thus $F \leq_\star G$ can be interpreted in terms of average failure rates as

$$\frac{\bar{r}_X(F^{-1}(u))}{\bar{r}_Y(G^{-1}(u))}$$

is increasing in $u \in (0,1]$. Note that X has an increasing failure rate if and only if F is star-ordered with respect to exponential distribution.

The function

$$L_X(u) = \frac{1}{\mathrm{E}(X)}\int_{-\infty}^{F^{-1}(u)} x dF(x)$$

is known as Lorenz curve in the economics literature. It is often used to express inequality in incomes and has also been used to compare income inequalities.

Definition 2.3 X is said to be smaller than Y in the Lorenz order, denoted by $X \leq_{\text{Lorenz}} Y$, if

$$L_X(u) \geq L_Y(u), \quad \text{for all } u \in (0,1].$$

It is known in the literature (*Marshall and Olkin*, 2007, p. 69) that,

$$X \leq_c Y \Longrightarrow X \leq_\star Y \Longrightarrow X \leq_{\text{Lorenz}} Y \Longrightarrow cv(X) \leq cv(Y),$$

where $cv(X) = \sqrt{\mathrm{Var}(X)}/\mathrm{E}(X)$ denotes the coefficient of variation of X.

All the above partial orders are scale invariant. A good discussion of those orders can be found in *Barlow and Proschan* (1981) and *Marshall and Olkin* (2007).

Definition 2.4 X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{st} Y$), if $\bar{F}(x) \leq \bar{G}(x)$ for all x.

For more discussion on various stochastic orders, please refer to *Shaked and Shanthikumar* (2007) and references therein.

3 Main results

The following result due to *Kochar and Xu* (2009) will be used in the sequel.

Lemma 3.1 Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from an exponential distribution with common hazard rate λ. Then,

$$X_{n:n} \geq_c Y_{n:n}. \tag{3.1}$$

The following lemma, which is a modified version of Lemma 2.1 in *Kochar* (2006), plays a key role in the proof of man results.

Lemma 3.2 Let ϕ be a differentiable star-shaped function on $[0, \infty)$ such that $\phi(x) \geq x$ for all $x \geq 0$. Let ψ be an increasing differentiable function such that

$$x\frac{\psi'(x)}{\psi(x)} \quad \text{is incresing in } x.$$

Then the function

$$\psi\phi\psi^{-1}(x) \quad \text{is also star-shaped in } x.$$

Proof: Note that ϕ is star-shaped if and only if

$$\frac{\phi(x)}{x} \quad \text{is increasing in } x,$$

which can be represented as

$$\phi'(x) \geq \frac{\phi(x)}{x}. \tag{3.2}$$

Hence, for the required result, it is sufficient to show

$$\psi'\phi\psi^{-1}(x)\frac{\phi'\psi^{-1}(x)}{\psi'\psi^{-1}(x)} \geq \frac{\psi\phi\psi^{-1}(x)}{x}. \tag{3.3}$$

Using (3.2), the left side of (3.3) satisfies

$$\frac{\psi'\phi\psi^{-1}(x)}{\psi'\psi^{-1}(x)}\phi'\psi^{-1}(x) \geq \frac{\psi'\phi\psi^{-1}(x)}{\psi'\psi^{-1}(x)}\frac{\phi\psi^{-1}(x)}{\psi^{-1}(x)}.$$

So, it is enough to prove

$$\frac{\psi'\phi\psi^{-1}(x)}{\psi'\psi^{-1}(x)}\frac{\phi\psi^{-1}(x)}{\psi^{-1}(x)} \geq \frac{\psi\phi\psi^{-1}(x)}{x},$$

i.e.,

$$\frac{\psi'\phi\psi^{-1}(x)}{\psi\phi\psi^{-1}(x)}\phi\psi^{-1}(x)\frac{1}{\psi^{-1}(x)} \geq \frac{\psi'\psi^{-1}(x)}{x}$$

as ψ is increasing. Using the assumptions

$$x\frac{\psi'(x)}{\psi(x)} \quad \text{is incresing in } x \text{ and } \phi(x) \geq x,$$

the required result follows immediately. ■

Now, we are ready to present the following result.

Theorem 3.3 Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$, and let $Y_1, \ldots, Y_n$ be a random sample from a distribution with the common survival distribution $\bar{F}^{\lambda}$ where $\lambda \geq \tilde{\lambda} = \sqrt[n]{\prod_{i=1}^{n} \lambda_i}$, the geometric mean of λ_i's. If

$$\frac{R(x)}{xr(x)} \quad \text{is increasing in } x \geq 0,$$

then

$$X_{n:n} \geq_{\star} Y_{n:n},$$

where $R(x) = -\log \bar{F}(x)$ is the cumulative hazard rate function, and $r(x) = f(x)/\bar{F}(x)$ is the hazard rate function of F.

Proof: Since $R(x)$ is increasing and

$$R^{-1}(x) = \bar{F}^{-1}\left(e^{-x}\right),$$

it holds that, for $x \geq 0$, $i = 1, \ldots, n$,

$$P(R(X_i) > x) = P(X_i > R^{-1}(x)) = \bar{F}^{\lambda_i}(\bar{F}^{-1}(e^{-x})) = e^{-\lambda_i x}.$$

So, making the transform

$$X_i' = R(X_i), \quad i = 1, \cdots, n,$$

it follows that X_i' is exponential with hazard rate λ_i for $i = 1, \ldots, n$. Similarly, let $Y_i' = H(Y_i)$ be exponential with hazard rate λ for $i = 1, \ldots, n$.

Observing that

$$Y_{n:n}' \stackrel{st}{=} R(Y_{n:n}), \quad X_{n:n}' \stackrel{st}{=} R(X_{n:n}),$$

it holds that

$$P(Y_{n:n} \leq x) = P(R^{-1}(Y_{n:n}') \leq x) = P(Y_{n:n}' \leq R(x)) = G_{n:n}'(R(x)),$$

$$P(X_{n:n} \leq x) = P(R^{-1}(X_{n:n}') \leq x) = P(X_{n:n}' \leq R(x)) = F_{n:n}'(R(x)),$$

where $G_{n:n}'(\cdot)$, $F_{n:n}'(\cdot)$ are distribution functions of $Y_{n:n}'$ and $X_{n:n}'$. Now, we need to prove

$$R^{-1}F_{n:n}'^{-1}G_{n:n}'R(x) \quad \text{is star-shaped.}$$

From Theorem 3.1, $F_{n:n}'^{-1}G_{n:n}'(x)$ is star-shaped on $[0, \infty)$.

From *Khaledi and Kochar* (2000), it is known that $\lambda \geq \tilde{\lambda}$ implies

$$F'^{-1}_{n:n} G'_{n:n}(x) \geq x.$$

By Lemma 3.2, it is enough to show

$$x \frac{\left(R^{-1}(x)\right)'}{R^{-1}(x)} \quad \text{is increasing in } x, \tag{3.4}$$

i.e.,

$$\frac{R(x)}{xr(x)} \quad \text{is increasing in } x,$$

which follows from the assumption. ■

As a direct consequence, we have the following result.

Corollary 3.4 Let $X_1, \ldots, X_n$ be independent random variables with X_i having survival function $\bar{F}^{\lambda_i}$, $i = 1, \ldots, n$, and let $Y_1, \ldots, Y_n$ be a random sample from a distribution with common survival distribution $\bar{F}^{\lambda}$ where $\lambda \geq \tilde{\lambda}$. If

$$\frac{R(x)}{xr(x)} \quad \text{is increasing in } x \geq 0,$$

then

$$X_{n:n} \geq_{\text{Lorenz}} Y_{n:n}.$$

One may wonder whether a similar result is true for other order statistics? The question can be partly answered by using the following two lemmas.

Lemma 3.5 (*Kochar and Xu*, 2011) Let $X_1, \ldots, X_p$ be i.i.d. exponential random variables with hazard rate λ_1, and let $X_{p+1}, \ldots, X_n$ be another set of i.i.d. exponential random variables with hazard rate λ_2. Let $Y_1, \ldots, Y_p$ be i.i.d. exponential random variables with hazard rate γ_1, and $Y_{p+1}, \ldots, Y_n$ be another set of i.i.d. exponential random variables with hazard rate γ_2. Then, for $k = 1, \ldots, n$,

$$\frac{\lambda_{(2)}}{\lambda_{(1)}} \geq \frac{\gamma_{(2)}}{\gamma_{(1)}} \Longrightarrow X_{k:n} \geq_{\star} Y_{k:n},$$

where $\lambda_{(2)} = \max\{\lambda_1, \lambda_2\}, \gamma_{(2)} = \max\{\gamma_1, \gamma_2\}$, and $\lambda_{(1)} = \min\{\lambda_1, \lambda_2\}, \gamma_{(1)} = \min\{\gamma_1, \gamma_2\}$.

Lemma 3.6 (*Bon and Păltănea*, 2006) Let $X_1, \ldots, X_p$ be independent exponential random variables with hazard rates λ_i, $i = 1, \ldots, n$. Let $Y_1, \ldots, Y_n$ be independent exponential random variables with a common hazard rate λ. Then,

$$\lambda \geq \hat{\lambda} \Longrightarrow X_{k:n} \geq_{st} Y_{k:n},$$

where

$$\hat{\lambda} = \left(\binom{n}{k}^{-1} \sum_{1 \leq i_1 < \ldots < i_k \leq n} \lambda_{i_1} \ldots \lambda_{i_k} \right)^{1/k}.$$

Using an argument similar to Theorem 3.3, one may prove the following result.

Theorem 3.7 Let $X_1, \ldots, X_p$ be i.i.d. random variables with the common survival distribution $\bar{F}^{\lambda_1}$, and let $X_{p+1}, \ldots, X_n$ be another set of i.i.d. random variables with with common survival distribution $\bar{F}^{\lambda_2}$, and let $Y_1, \ldots, Y_n$ be a random sample from a distribution with common survival distribution $\bar{F}^{\lambda}$ where $\lambda \geq \hat{\lambda}$. If

$$\frac{R(x)}{xr(x)} \quad \text{is increasing in } x \geq 0,$$

then

$$X_{k:n} \geq_{\star} Y_{k:n},$$

and hence,

$$X_{k:n} \geq_{\text{Lorenz}} Y_{k:n}, \quad k = 1, \ldots, n,$$

where

$$\hat{\lambda} = \left(\binom{n}{k}^{-1} \sum_{l \in \mathbb{L}} \binom{p}{l} \binom{n-p}{k-l} \lambda_1^l \lambda_2^{k-l} \right)^{1/k}$$

and $\mathbb{L} = \{l : \max\{k - n + p, 0\} \leq l \leq \min\{p, k\}\}$.

4 Examples and Applications

4.1 Examples

In this section, we will present some distributions for which Theorem 3.3 and Theorem 3.7 are applicable.

Weibull distribution

Weibull random variable $X_i \sim W(a, b_i)$ has survival function

$$\bar{F}_i(x) = \exp\left\{ -\left(\frac{x}{b_i} \right)^{\alpha} \right\}.$$

It is seen that $R(x) = x^{\alpha}$ and $\lambda_i = \frac{1}{b_i^{\alpha}}$, and

$$\frac{R(x)}{xr(x)} = \frac{1}{\alpha}.$$

As the star transform order is scale invariant, the following results hold.

Proposition 4.1 Let $X_1, \ldots, X_n$ be independent Weibull random variables $W(\alpha, b_i)$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a Weibull distribution $W(\alpha, b)$. Then,

$$X_{n:n} \geq_{\star} Y_{n:n}$$

Proposition 4.2 Let $X_1, \ldots, X_p$ be independent Weibull random variables $W(\alpha, b_1)$, and $X_{p+1}, \ldots, X_n$ be other independent Weibull random variables $W(\alpha, b_2)$. Let $Y_1, \ldots, Y_n$ be a random sample of size n from a Weibull distribution $W(\alpha, b)$. Then,

$$X_{k:n} \geq_{\star} Y_{k:n}, \quad k = 1, \ldots, n.$$

Pareto distribution

The survival function of a Pareto random variable X_i can be represented as

$$\bar{F}_i(x) = \left(\frac{b}{x}\right)^{\lambda_i}, \quad x \geq b.$$

Then, it follows that,

$$\frac{R(x)}{xr(x)} = \log(x/b)$$

is increasing in x. Hence, we have the following results.

Proposition 4.3 Let $X_1, \ldots, X_n$ be independent Pareto random variables with shape parameter λ_i for $i = 1, \cdots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with $\lambda \geq \tilde{\lambda}$. Then,

$$X_{n:n} \geq_\star Y_{n:n}.$$

Proposition 4.4 Let $X_1, \ldots, X_p$ be independent Pareto random variables with shape parameter λ_1, and $X_{p+1}, \ldots, X_n$ be independent Pareto random variables with shape parameter λ_2. Let $Y_1, \ldots, Y_n$ be a random sample from a Pareto distribution with parameter $\lambda \geq \hat{\lambda}$. Then,

$$X_{k:n} \geq_\star Y_{k:n}, \quad k = 1, \ldots, n.$$

<u>Lomax distribution</u>

The survival function of a Lomax random variable is

$$\bar{F}_i(x) = \left(1 + \frac{x}{b}\right)^{-\lambda_i}.$$

Then,

$$\frac{R(x)}{xr(x)} = \frac{b+x}{x} \log\left(1 + \frac{x}{b}\right).$$

Taking the derivative with respect to x and simplify, it holds that

$$\frac{d}{dx}\frac{R(x)}{xr(x)} = \frac{1}{x}\left[1 - \frac{b}{x}\log\left(1 + \frac{x}{b}\right)\right] \geq 0, \quad x \geq 0, b > 0.$$

Hence, we have the following Proposition.

Proposition 4.5 Let $X_1, \ldots, X_n$ be independent Lomax random variables with parameter λ_i for $i = 1, \cdots, n$. Let $Y_1, \ldots, Y_n$ be a random sample from a Lomax distribution with parameter $\lambda \geq \hat{\lambda}$. Then,

$$X_{n:n} \geq_\star Y_{n:n}.$$

Proposition 4.6 Let $X_1, \ldots, X_p$ be independent Lomax random variables with parameter λ_1, and $X_{p+1}, \ldots, X_n$ be independent Lomax random variables with parameter λ_2. Let $Y_1, \ldots, Y_n$ be a random sample from a Lomax distribution with parameter $\lambda \geq \hat{\lambda}$. Then,

$$X_{k:n} \geq_\star Y_{k:n}.$$

4.2 Estimation under order restriction

In this section, we will discuss the estimation under star order restriction. Let $X_1, \dots, X_p$ be independent exponential random variables with hazard rate λ_1, and $X_{p+1}, \dots, X_{p+q}$ be other independent exponential random variables with hazard rate λ_2, where $p+q=n$. According to Theorem 3.7, it holds that

$$X_{k:n} \geq_{\star} Y_{k:n},$$

where $Y_{k:n}$ is the kth order statistics from the standard exponential distribution. Mimicking the procedure in *Barlow et al.* (1972), one may derive the estimator $\hat{F}_n$ for the distribution function F of $X_{k:n}$ under the restriction

$$\hat{F}_n \geq_{\star} G,$$

where G is the distribution of $Y_{k:n}$, which could be easily derived, and

$$\hat{F}_n(t) = \begin{cases} 0, & t < X_{1:n}, \\ G\left[\lambda_n(X_{n-i+1:n})t\right], & X_{n-i+1} \leq t < X_{n-i+2:n} \quad 2 \leq i \leq n, \\ 1, & t \geq X_{n:n}. \end{cases}$$

where

$$\lambda_n(X_{n-i+1:n}) = \frac{G^{-1}F_n(X_{n-i+1:n})}{X_{n-i+1:n}}$$

is nondecreasing in i. The isotonic regression with respect to weights $w_i > 0$ could be derived as

$$\lambda_n^*(X_{n-i+1:n}) = \min_{t \geq i} \max_{s \leq i} \frac{\sum_{j=s}^{t} \lambda_n(X_{n-j+1:n}) w_j}{\sum_{j=s}^{t} w_j},$$

and

$$\hat{\lambda}_n^*(t) = \begin{cases} 0, & t < X_{1:n}, \\ \lambda_n^*(X_{n-i+1:n}), & X_{n-i+1:n} \leq t < X_{n-i+2:n}, \quad 2 \leq i \leq n, \\ 1, & t \geq X_{n:n}. \end{cases}$$

For illustration purpose, we simulate the distributions of largest order statistics with $n=5$, $p=3$, $q=2$ and $\lambda_1 = 2$, $\lambda_2 = 5$. For G, we simply use standard exponential distribution. We plot the empirical distribution (Emp), isotonic distribution (Iso) and the simulated real distribution (Real) of largest order statistics for different sample sizes $(25, 50, 100, 200)$ in Figure 1-Figure 4.

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Figure 1: sample size $n = 25$

Figure 2: sample size $n = 50$

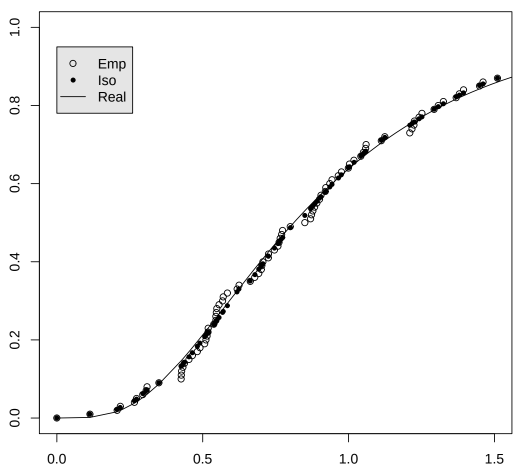

Figure 3: sample size $n = 100$

Figure 4: sample size $n = 200$

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