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General formula for the inelastic scattering of relativistic spin-$\frac{1}{2}$ particles by heavy atoms

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A general differential-cross-section formula for the inelastic scattering of relativistic spin-$\frac{1}{2}$ particles by heavy atoms is derived in the Born approximation and the structure and polarization of the incident particle is incorporated. The results reduce to those of Fano and of Turner *et al.* in well-known limiting cases.

It is known that a detailed experimental and theoretical investigation of the relativistic effects in the inelastic scattering of high-energy projectiles by medium-heavy and heavy atoms is essential to calculate their contribution to the stopping power. The results obtained also find applications in the absorption of the penetrating component of the cosmic rays by materials in space environment as well as in studies related to the K vacancy produced in target atoms. The problem considered here is the scattering of a relativistic spin-$\frac{1}{2}$ particle with internal structure and arbitrary polarization by an electron, bound to an atom, via the one-photon-exchange process. Specifically, we shall consider a proton with its finite distribution of charge and magnetic moment as the projectile; the applicability of the result to other incident particles is obvious.

Following Bjorken and Drell's notation,[1] the proton current can be written as ($\hbar = c = 1$):

$$\bar{u}'\Gamma_\mu u = e\bar{u}'\left[\gamma_\mu F_1 + \frac{i\sigma_{\mu\nu}q^\nu}{2M}\kappa F_2\right]u \quad , \tag{1}$$

where $u'(E',\vec{P}')$ and $u(E,\vec{P})$ are, respectively, the Dirac spinors for the outgoing and incoming proton; γ_μ are the Dirac matrices; $q = P' - P$ represents the four-momentum transfer; κ is the anomalous magnetic moment; and F_1, F_2 are the Dirac and Pauli form factors, respectively. $\sigma_{\mu\nu}$ is defined as

$$\sigma_{\mu\nu} = \frac{i}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu) \quad . \tag{2}$$

For the incident proton of mass M and spin S with arbitrary direction of polarization, the overall projection operator may be written in the form

$$\Lambda_+\Sigma_S = \left[\frac{\not{P}+M}{2M}\right]\left[\frac{1+\gamma_5\not{S}}{2}\right] \quad , \tag{3}$$

where $\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3$. The differential cross section can be expressed as

$$\frac{d\sigma}{d\Omega} = \frac{4M^2e^4}{q^4}\frac{|\vec{P}'|}{|\vec{P}|}\Sigma_{S,S'}|\bar{u}'\Gamma_\mu u J^\mu|^2 \quad , \tag{4}$$

where

$$J^\mu = \langle n|e^{i\vec{q}\cdot\vec{r}}\gamma^0\gamma^\mu|0\rangle \tag{5}$$

is the electronic form factor that constitutes the atomic current between the ground state $|0\rangle$ and the excited state $|n\rangle$. Though in this paper we have assumed a one-electron atom, the generalization for many electrons is straightforward. Using Eqs. (1), (3), and (4), the cross section can finally be written as

$$\frac{d\sigma}{d\Omega} = \frac{4M^2e^4}{q^4}\frac{|\vec{P}'|}{|\vec{P}|}M_{\mu\nu}J^\mu J^{\nu *} \quad . \tag{6}$$

Here $M_{\mu\nu}J^\mu J^\nu$ can be expressed as the sum of three terms here

$$M_{\mu\nu}J^\mu J^\nu = M_{\rm I} + M_{\rm II} + M_{\rm III} \quad ,$$

$$M_{\rm I} = \mathrm{Tr}\left[\gamma^\mu\left[\frac{1+\gamma_5\not{S}}{2}\right]\left[\frac{\not{P}+M}{2M}\right]\gamma^\nu\left[\frac{\not{P}'+M}{2M}\right]\right]F_1^2 J_\mu J_\nu^* \quad ,$$

$$M_{\rm II} = \frac{\kappa F_1 F_2}{M}\,\mathrm{Re}\left\{\mathrm{Tr}\left[\left[\frac{1+\gamma_5\not{S})}{2}\right]\left[\frac{\not{P}+M}{2M}\right]\times(-i\sigma^{\nu\lambda})\left[\frac{\not{P}'+M}{2M}\right]\right]q_\lambda J_\mu J_\nu^*\right\} \quad ,$$

$$M_{\rm III} = \left[\frac{\kappa}{2M}\right]^2 \mathrm{Tr}\left[\sigma^{\mu\lambda}\left[\frac{1+\gamma_5\not{S}}{2}\right]\left[\frac{\not{P}+M}{2M}\right]\times\sigma^{\nu\sigma}\left[\frac{\not{P}'+M}{2M}\right]\right]q_\lambda q_\sigma F_2^2 J_\mu J_\nu^* \quad . \tag{7}$$

Tr signifies taking trace here. We shall evaluate the cross section in two steps; in the first step terms without γ_5 will be kept and in the second step only the γ_5 terms will be included. Making use of the well-known trace theorems, we obtain the following:

(i) *For terms without* γ_5:

$$M_{\mathrm{I}} \sim \frac{1}{2M^2}[P^\mu P'^\nu + P^\nu P'^\mu - g^{\mu\nu}(P\cdot P' - M^2)] \ . \tag{8}$$

To evaluate terms containing $\sigma^{\mu\nu}$, we apply the Gordon decomposition[1]

$$\bar{u}'\gamma^\mu u = \bar{u}'\left(\frac{(P+P')^\mu}{2M} + \frac{i\sigma^{\mu\nu}q_\nu}{2M}\right)u \ . \tag{9}$$

By replacing the term $i\sigma^{\mu\nu}q_\nu$ by $2M\gamma^\mu - (P+P')^\mu$ in (1), we obtain for M_{II} in Eq. (7)

$$M_{\mathrm{II}} \sim \frac{1}{8M^2}\mathrm{Re}(\mathrm{Tr}\{\gamma^\mu(\not{P}+M)[2M\gamma^\nu - (P+P')^\nu](\not{P}'+M)\}J_\mu J_\nu^*) = \frac{1}{2M}\mathrm{Re}[(g^{\mu\nu}q^2 - q^\mu q^\nu)J_\mu J_\nu^*] \ , \tag{10}$$

where we have made use of the relation

$$M^2 - P\cdot P' = q^2/2 \ . \tag{11}$$

Similarly,

$$M_{\mathrm{III}} \sim \frac{1}{8M^2}\mathrm{Tr}\{[2M\gamma^\mu - (P+P')^\mu](P+M)[2M\gamma^\nu - (P+P')^\nu](P'+M)\}$$
$$= \frac{1}{2M^2}[4M^2(P^\mu P'^\nu + P^\nu P'^\mu) + 4M^2(M^2 - P\cdot P')g^{\mu\nu} + (P+P')^\mu(P+P')^\nu(P\cdot P' - 3M^2)] \ . \tag{12}$$

By making use of (11) and the condition for the conservation of current

$$q\cdot J = 0 \ , \tag{13}$$

i.e., $P\cdot J = P'\cdot J$, we finally obtain from Eqs. (8), (9), and (10) the following:

$$M_{\mathrm{I}} + M_{\mathrm{II}} + M_{\mathrm{III}} \sim \frac{1}{2M^2}\left[2|P\cdot J|^2\left(F_1{}^2 - \frac{q^2k^2}{4M^2}F_2{}^2\right) + \frac{q^2J^2}{2}(F_1 + kF_2)^2\right] \ . \tag{14}$$

(ii) *For terms with* γ_5:

Starting with Eq. (7), we have similarly

$$M_{\mathrm{I}} \sim \frac{1}{8M^2}\mathrm{Tr}[\gamma^\mu\gamma_5\not{S}(\not{P}+M)\gamma^\nu(\not{P}'+M)] = \frac{i}{2M}\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta \ , \tag{15}$$

$$M_{\mathrm{II}} \sim \frac{1}{8M^2}\mathrm{Re}(\mathrm{Tr}\{\gamma^\mu\gamma_5\not{S}(\not{P}+M)[2M\gamma^\nu - (P+P')^\nu](\not{P}'+M)\}J_\mu J_\nu^*)$$
$$= \mathrm{Re}\left[i\left(\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta + \frac{1}{2M^2}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu\right)J^\mu J^{\nu *}\right] \ , \tag{16}$$

$$M_{\mathrm{III}} \sim \frac{1}{8M^2}\mathrm{Tr}\{[2M\gamma_\mu - (P+P')_\mu]\gamma_5\not{S}(\not{P}+M)[2M\gamma_\nu - (P+P')_\nu](\not{P}'+M)\}J^\mu J^{\nu *}$$
$$= 2Mi(\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta J^\mu J^{\nu *}) + \frac{i}{M}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu[J^\mu, J^{\nu *}] \ , \tag{17}$$

where

$$[J^\mu, J^{\nu *}] \equiv J^\mu J^{\nu *} - J^\nu J^{\mu *} \ . \tag{18}$$

Putting together Eqs. (15)–(17), we obtain

$$M_{\mathrm{I}} + M_{\mathrm{II}} + M_{\mathrm{III}} \sim \frac{1}{2M^2}\left[iM(F_1+\kappa F_2)^2\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta J^\mu J^{\nu *} + 2M\kappa F_1F_2\,\mathrm{Re}\left(\frac{i}{2M^2}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu J^\mu J^{\nu *}\right)\right.$$
$$\left. + \frac{i\kappa^2F_2{}^2}{2M}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu[J^\mu, J^{\nu *}]\right] \ . \tag{19}$$

Adding the results in (14) and (19) and substituting in (6) we obtain finally the covariant form of the differential cross section:

$$\frac{d\sigma}{d\Omega} = \frac{2e^4}{q^4}\frac{|\vec{P}'|}{|\vec{P}|}\left[2|P\cdot J|^2\left(F_1{}^2 - \frac{K^2q^2}{4M^2}F_2{}^2\right) + \frac{q^2J^2}{2}(F_1+\kappa F_2)^2 + iM(F_1+\kappa F_2)^2\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta J^\mu J^{\nu *}\right.$$
$$\left. + 2M\kappa F_1F_2\,\mathrm{Re}\left(\frac{i}{2M^2}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu J^\mu J^{\nu *}\right) + \frac{i\kappa^2F_2^2}{2M}\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu[J^\mu, J^{\nu *}]\right] \ . \tag{20}$$

This is the desired generalization of the proton-atom interaction cross section from which various well-known limiting cases can be reduced. For example, by setting $F_1=1$, $F_2=0$, we obtain

$$\frac{d\sigma}{d\Omega}=\frac{2e^4}{q^4}\frac{|\vec{P}'|}{|\vec{P}|}\left(2|P\cdot J|^2\frac{q^2J^2}{2}+iM\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta J^\mu J^{\nu *}\right) , \quad (21)$$

which is the case when the incident proton is treated as a point particle. If we further set $S^\alpha=0$ in (21), we get back the case for the unpolarized point proton as projectile.[1,2] On the other hand, if we set $S^\alpha=0$ in (20), we reproduce a certain generalization which has been obtained previously.[3,4]

Next we give the three-dimensional form of Eq. (20). By defining

$$J^\mu=(J^0,\vec{J})=(F_n,\vec{G}_n) , \quad (22)$$

one can show that[3]

$$|P\cdot J|^2=E^2\left|\frac{q^2}{|\vec{q}|^2}F_n+\vec{\beta}_t\cdot\vec{G}_n\right|^2 , \quad (23)$$

where $\vec{\beta}_t$ is the component of the incident proton's velocity perpendicular to $\vec{q}$. Furthermore, for transitions between atomic states of definite parity, $0\neq\mu\neq\nu\neq 0$ in all the J^μ, J^ν terms in Eq. (20). Hence we have

$$\epsilon_{\alpha\beta\mu\nu}S^\alpha q^\beta J^\mu J^{\nu *}=[(E-E')\vec{S}-S_0(\vec{P}-\vec{P}')]\cdot(\vec{G}_n\times\vec{G}_n^*) , \quad (24)$$

$$\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu J^\mu J^{\nu *}=(S_0\vec{P}-E\vec{S})\cdot(\vec{G}_n\times\vec{P}')(\vec{P}+\vec{P}')\cdot\vec{G}_n^* , \quad (25)$$

and

$$\epsilon_{\alpha\beta\mu\delta}S^\alpha P^\beta P'^\delta(P+P')_\nu[J^\mu,J^{\nu *}]=(S_0\vec{P}-E\vec{S})\cdot(\vec{G}_n\times\vec{P}')(\vec{P}+\vec{P}')\cdot\vec{G}_n^*-(S_0\vec{P}-E\vec{S})\cdot(\vec{G}_n^*\times\vec{P}')(\vec{P}+\vec{P}')\cdot\vec{G}_n . \quad (26)$$

Putting Eqs. (22)–(26) into Eq. (20), we obtain

$$\begin{aligned}\frac{d\sigma}{d\Omega}=2e^4\frac{|\vec{P}'|}{|\vec{P}|}\Bigg[&2E^2\left(F_1^2-\frac{\kappa^2q^2}{4M^2}F_2^2\right)\left(\frac{1}{|\vec{q}|^4}|F_n|^2+\frac{1}{q^4}|\vec{\beta}_t\cdot\vec{G}_n|^2\right)+\frac{1}{2q^2}(F_1+\kappa F_2)^2(|F_n|^2-|\vec{G}_n|^2)\\&+\frac{iM}{q^4}(F_1+\kappa F_2)^2[(E-E')\vec{S}-S_0(\vec{P}-\vec{P}')]\cdot(\vec{G}_n\times\vec{G}_n^*)+\frac{2M\kappa F_1F_2}{q^4}\mathrm{Re}[(S_0\vec{P}-E\vec{S})\cdot(\vec{G}_n\times\vec{P}')(\vec{P}+\vec{P}')\cdot\vec{G}_n^*]\\&+\frac{\kappa^2F_2^2}{Mq^4}\mathrm{Im}[(S_0\vec{P}-E\vec{S})\cdot(\vec{G}_n^*\times\vec{P}')(\vec{P}+\vec{P}')]\cdot\vec{G}_n)\Bigg] ,\end{aligned}$$

which is a generalization of the results obtained by Fano[5] and by Turner *et al.*[3]

Finally, for the unpolarized free-electron target, one can show that $\langle J^\mu J^{\nu *}\rangle$ is real and symmetric in μ and ν. Hence all the terms involving the polarization effects in (20) vanish, yielding the well-known result that a pointed incident beam does not give any different scattering cross sections in the first Born approximation.[1]

It might be desirable to know the limitation of an approximation of a proton as a point particle. Let us define the structure factor for the proton S as

$$S=\frac{G_E^2+\tau G_M^2}{1+\tau} , \quad (27)$$

where G_E and G_M are the electric and magnetic form factors of the proton[6] and are related to F_1 and F_2 defined earlier by the following equations:

$$G_E=F_1+\kappa q^2F_2/(4M^2) , \quad (28)$$

$$G_M=F_1+\kappa F_2 . \quad (29)$$

In (27), $\tau=q^2/4M$ with q being the timelike four-momentum transfer.

If one uses the phenomenological relation,[7] $G_M=\kappa G_E$ ($\kappa=2.793$ for proton), that holds very well for a large range of energy transfer, the structure factor may be written as

$$S=\frac{(1+\tau\kappa^2)}{1+\tau}G_E^2 , \quad (30)$$

where in the "exponential model"

$$G_E(q^2)=\frac{1}{[1+(q^2/18.5)]^2} .$$

If we neglect the binding energy of the electron and denote by m and $\bar{Q}$ the rest mass and energy transferred to the electron, then we have approximately $q^2=2m\bar{Q}/(0.197)^2$, where use has been made of the fact that $1F^{-1}=197$ MeV/c. In terms of $\bar{Q}$, (30) may be written as

$$S=\frac{1+2.26\times10^{-3}\bar{Q}}{1+0.29\times10^{-3}\bar{Q}}\frac{1}{(1+1.43\times10^{-3}\bar{Q})^4} .$$

Since $|\vec{q}|^2=\bar{Q}^2+2m\bar{Q}$, therefore in atomic units, $Q=|q|^2/Z^2$ and $\bar{Q}=m[(1+QZ^2)^{1/2}-1]$, S may be expressed as a function of Q.

It is found that for medium mass (such as Ni) and heavy (such as Pb or U) atoms, the structure effects become noticeable for proton energies ~ 1000 GeV, and are negligible for lower energies.

The additional effects in Eq. (27) applied to a particular atom will be studied in a future paper.

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