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A review on convolutions of Gamma random variables

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Abstract

Due to its wide range of applications, the topic of the distribution theory of convolutions of Gamma random variables has attracted the attention of many researchers. In this paper we review some of the latest developments on this problem.

Keywords Convolution; dispersive order; majorization; right spread order; skewness; star order.

1 Introduction

The convolution of independent random variables has attracted considerable attention in the literature due to its typical applications in many applied areas. For example, in reliability theory, it is used to study the lifetimes of redundant standby systems with independent components (cf. *Bon and Păltănea* [6]); in actuarial science, it is used to model the total claims on a number of policies in the individual risk model (cf. *Kaas, et al.* [10]); in nonparametric goodness-of-fit tests, the limiting distributions of U-statistics are the convolutions of independent random variables (cf. *Serfling* [25], Section 5.2). As another example, let X_i denote the random value of ith shock on a system, then if the convolutions of a number of X_i's exceed the threshold of the system, then the system fails (cf. *Marshall and Olkin* [23]). Therefore, study of lifetime of an standby system or a cumulative damage threshold model is based on stochastic properties of convolutions of random variables.

The gamma distribution is one of the most popular distributions in statistics, engineering and reliability applications. In particular, gamma distribution plays a prominent role in actuarial science since most total insurance claim distributions have roughly the same shape as gamma distributions: skewed to the right, non-negatively supported and unimodal (cf. Furman [9]). As is well known, the gamma distribution includes exponential and chi-square, two important distributions, as special cases. Due to the complicated nature of the distribution function of gamma random variable, most of the work in the literature discusses only the convolutions of exponential random variables. Some relevant references are *Khaledi* [12], *Boland, et al.* [5], *Kochar and Ma* [15], *Bon and Păltănea* [6], *Zhao and Balakrishnan* [29] and *Kochar and Xu* [17].

Let $X_1, \ldots, X_n$ be a random sample from a gamma distribution with shape parameter $a > 0$, scale parameter $\lambda > 0$ and with density function

$$f(x) = \frac{\lambda^a}{\Gamma(a)} x^{a-1} \exp\{-\lambda x\}, \quad x \geq 0.$$

We are interested in studying the stochastic properties of statistics of the form

$$W = \theta_1 X_1 + \theta_2 X_2 + \ldots + \theta_n X_n,$$

were $\theta_1, \ldots, \theta_n$ are positive weights (constants). *Bock, et al.* [4] showed that for $n = 2$, if

$$t \leq \frac{a(\theta_1 + \theta_2)}{\lambda},$$

then $P(W \leq t)$ is Schur-convex in (θ_1, θ_2); and if

$$t \geq \frac{(a + 1/2)(\theta_1 + \theta_2)}{\lambda},$$

then $P(W \geq t)$ is Schur-convex in (θ_1, θ_2). For general $n > 2$, $P(W \leq t)$ is Schur-convex in the region

$$\left\{ \boldsymbol{\theta} : \min_{1 \leq i \leq n} \theta_i \geq \frac{t\lambda}{na + 1} \right\},$$

where $\boldsymbol{\theta}$ is the vector of $(\theta_1, \ldots, \theta_n)$, and $P(W \geq t)$ is Schur-convex in $\boldsymbol{\theta}$ for

$$t \geq \frac{(na + 1)(\theta_1 + \theta_2 + \ldots + \theta_n)}{\lambda}.$$

Diaconis and Perlman [7] further studied the tail probabilities of convolution of gamma random variables. They pointed out that if

$$(\theta_1, \cdots, \theta_n) \overset{m}{\succeq} (\theta'_1, \cdots, \theta'_n) \tag{1.1}$$

then

$$\operatorname{Var}\left(\sum_{i=1}^{n} \theta_i X_i \right) \geq \operatorname{Var}\left(\sum_{i=1}^{n} \theta'_i X_i \right),$$

where $\overset{m}{\succeq}$ means the majorization order (see Definition 3.1).

This property states that if the weights are more dispersed in the sense of majorization, then the convolutions are more dispersed about their means as measured by their variances. *Diaconis and Perlman* [7] also wondered whether $\sum_{i=1}^{n} \theta_i X_i$ is more dispersed than $\sum_{i=1}^{n} \theta'_i X_i$ as measured by the stronger criterion of their tail probabilities. They tried to answer this question by proving that under the condition (1.1), the distribution functions of $\sum_{i=1}^{n} \theta_i X_i$ and $\sum_{i=1}^{n} \theta'_i X_i$ have only one crossing. However, they only proved this result for $n = 2$. For $n \geq 3$, they required further restrictions. Hence, this problem has been open for a long time, which is also known as *Unique Crossing Conjecture*.

The rest of paper is organized as follows. In Section 2, we first review some stochastic orders and majorization orders. In Section 3, we investigate the crossing properties of two convolutions of gamma random variables under various conditions on the parameters for $n = 2$. In Section 4, we establish the right spread ordering between two convolutions of independent gamma random variables for arbitrary n. We conclude our discussion with some remarks in the last Section.

2 Preliminaries

In this section, we will review some notions of stochastic orders and majorization orders.

Assume the positive random variables X and Y have distribution functions F and G, survival functions $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, density functions f and g, and failure rate functions $r_X = f/\bar{F}$ and $r_Y = g/\bar{G}$, respectively. The following orders are usually used to compare the magnitude of random variables.

Definition 2.1 X is said to be smaller than Y in the

(i) likelihood ratio order (denoted by $X \leq_{lr} Y$) if $g(x)/f(x)$ is increasing in x;

(ii) hazard rate order (denoted by $X \leq_{hr} Y$) if $\bar{G}(x)/\bar{F}(x)$ is increasing in x;

(iii) stochastic ordering (denoted by $X \leq_{st} Y$) if $\overline{F}(x) \leq \overline{G}(x)$ for every x.

(iv) mean residual life order, denoted by $X \leq_{mrl} Y$, if

$$\frac{\int_t^\infty \bar{F}(x)dx}{\bar{F}(t)} \leq \frac{\int_t^\infty \bar{G}(x)dx}{\bar{G}(t)}.$$

It is known that (cf. *Shaked and Shanthikumar* [26]),

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{mrl} Y \Rightarrow \mathrm{E}X \leq \mathrm{E}Y,$$

and

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow \mathrm{E}X \leq \mathrm{E}Y.$$

The following order, called the dispersive order, is used to compare the variabilities of two random variables.

Definition 2.2 X is said to be less dispersed than Y (denoted by $X \leq_{disp} Y$) if

$$F^{-1}(\beta) - F^{-1}(\alpha) \leq G^{-1}(\beta) - G^{-1}(\alpha)$$

for all $0 < \alpha \leq \beta < 1$.

A weaker order called the right spread order has also been proposed to compare the variabilities of two distributions (cf. *Fernández-Ponce, et al.* [8]).

Definition 2.3 X is said to be less right spread than Y (denoted by $X \leq_{RS} Y$) if

$$\int_{F^{-1}(p)}^\infty \bar{F}(x)\,dx \leq \int_{G^{-1}(p)}^\infty \bar{G}(x)dx, \qquad \text{for all } 0 \leq p \leq 1.$$

It is known that

$$X \leq_{disp} Y \Longrightarrow X \leq_{RS} Y \Longrightarrow \mathrm{Var}(X) \leq \mathrm{Var}(Y).$$

Bagai and Kochar (1986) proved the following result.

THEOREM **2.1** *If $X \leq_{disp} Y$ and F or G is IFR (increasing failure rate), then $X \leq_{hr} Y$.*

Definition 2.4 X is said to be smaller than Y in *the star order*, denoted by $X \leq_\star Y$ (or $F \leq_\star G$) if $G^{-1}F(x)/x$ is increasing in x on the support of X, where G^{-1} is the right continuous inverse of G.

It is known that if $X \leq_\star Y$, then $\overline{F}(x)$ crosses $\overline{G}(\theta x)$ at most once and from above as x increases from 0 to ∞, for each $\theta > 0$. If $X \leq_\star Y$, then Y is more skewed than X as explained in *Marshall and Olkin* [23]. The star order is also called more IFRA (increasing failure rate in average) order in reliability theory for reason explained below. The average failure of F at x is

$$\tilde{r}_X(x) = \frac{1}{x}\int_0^x r_X(u)du = \frac{-\ln \bar{F}(x)}{x}.$$

Thus $F \leq_\star G$ can be interpreted in terms of average failure rates as

$$\frac{\tilde{r}_X(F^{-1}(u))}{\tilde{r}_Y(G^{-1}(u))} = \frac{G^{-1}(u)}{F^{-1}(u)}$$

being increasing in $u \in (0,1)$. A random variable X is said to have an IFRA distribution if its average failure rate $\tilde{r}_X(x)$ is increasing. Note that X has an IFRA distribution if and only if F is star-ordered with respect to exponential distribution.

Definition 2.5 X is said to be more NBUE (new better than used in expectation) than Y or X is smaller than Y in the NBUE order (written as $X \leq_{\text{NBUE}} Y$) if

$$\frac{1}{\mu_F}\int_{F^{-1}(u)}^\infty \bar{F}(x)dx \leq \frac{1}{\mu_G}\int_{G^{-1}(u)}^\infty \bar{G}(x)dx, \quad \text{for all } u \in (0,1],$$

where $\mu_F(\mu_G)$ denotes the expectation of $X(Y)$.

It has been shown in *Kochar* [14] that

$$X \leq_\star Y \Longrightarrow X \leq_{\text{NBUE}} Y \Longrightarrow X \leq_{\text{Lorenz}} Y,$$

where $\leq_{\text{Lorenz}}$ means the Lorenz order, a well-known order in economics. It is also known that (*Marshall and Olkin* [23], p. 69),

$$X \leq_{\text{Lorenz}} Y \Longrightarrow \gamma_X \leq \gamma_Y,$$

where $\gamma_X = \sqrt{\text{Var}(X)}/\text{E}(X)$ denotes the coefficient of variation of X. A good discussion of those orders can be found in *Barlow and Proschan* [3], *Marshall and Olkin* [23] and *Shaked and Shanthikumar* [26].

When $\text{E}(X) = \text{E}(Y)$, the order $\leq_{RS}$ is equivalent to the order $\leq_{\text{NBUE}}$. However, they are distinct when $\text{E}(X) \neq \text{E}(Y)$. For more details, please refer to *Kochar et al.* [16].

We will also use the concept of majorization in the following discussion. Let $\{x_{(1)}, x_{(2)}, \cdots, x_{(n)}\}$ denote the increasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

Definition 2.6 The vector $\mathbf{x}$ in $\mathbb{R}^{+n}$ is said to majorize the vector $\mathbf{y}$ in $\mathbb{R}^{+n}$ (denoted by $\mathbf{x} \overset{m}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^j x_{(i)} \leq \sum_{i=1}^j y_{(i)}$$

for $j = 1, \cdots, n-1$ and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$.

Relaxing the equality condition gives the following weak majorization order.

Definition 2.7 The vector $\mathbf{x}$ in ${\mathbb{R}^+}^n$ is said to weakly submajorize the vector $\mathbf{y}$ in ${\mathbb{R}^+}^n$ (denoted by $\mathbf{x}\succeq_w\mathbf{y}$) if

$$\sum_{i=1}^{j} x_{[i]} \geq \sum_{i=1}^{j} y_{[i]}$$

for $j = 1, \cdots, n$, where $\{x_{[1]}, x_{[2]}, \cdots, x_{[n]}\}$ denotes the decreasing arrangement of the components of the vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$.

For extensive and comprehensive details on the theory of the majorization order and its applications, please refer to *Marshall and Olkin* [22].

Another interesting weaker order related to the majorization order introduced by *Bon and Păltănea* [6] is the p-larger order.

Definition 2.8 A vector $\mathbf{x}$ in ${\mathbb{R}^+}^n$ is said to be p-larger than another vector $\mathbf{y}$ in ${\mathbb{R}^+}^n$ (denoted by $\mathbf{x} \overset{p}{\succeq} \mathbf{y}$) if

$$\prod_{i=1}^{j} x_{(i)} \leq \prod_{i=1}^{j} y_{(i)}, \quad j = 1, \cdots, n.$$

Zhao and Balakrishnan [29] introduced the following order of reciprocal majorization.

Definition 2.9 A vector $\mathbf{x}$ in ${\mathbb{R}^+}^n$ is said to reciprocally majorize the another vector $\mathbf{y}$ in ${\mathbb{R}^+}^n$ (denoted by $\mathbf{x} \overset{rm}{\succeq} \mathbf{y}$) if

$$\sum_{i=1}^{j} \frac{1}{x_{(i)}} \geq \sum_{i=1}^{j} \frac{1}{y_{(i)}}, \quad j = 1, \cdots, n.$$

It has been pointed out in *Kochar and Xu* [17] that,

$$\mathbf{x} \overset{m}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{p}{\succeq} \mathbf{y} \Longrightarrow \mathbf{x} \overset{rm}{\succeq} \mathbf{y}.$$

3 Magnitude and dispersive orderings between convolutions of gamma random variables

Let $X_1, \ldots, X_n$ be independent exponential random variables with X_i having hazard rate λ_i, $i = 1, \ldots, n$, and $Y_1, \ldots, Y_n$ be another set of independent exponential random variables with Y_i having hazard rate λ_i', $i = 1, \ldots, n$. *Boland, El-Neweihi and Proschan* [5] showed that under the condition of the majorization order,

$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succeq} (\lambda_1', \cdots, \lambda_n') \Longrightarrow \sum_{i=1}^{n} X_i \geq_{lr} \sum_{i=1}^{n} Y_i.$$

Under the same condition, *Kochar and Ma* [15] proved that

$$(\lambda_1, \cdots, \lambda_n) \overset{m}{\succeq} (\lambda_1', \cdots, \lambda_n') \Longrightarrow \sum_{i=1}^{n} X_i \geq_{disp} \sum_{i=1}^{n} Y_i. \tag{3.1}$$

This topic has been extensively investigated by *Bon and Păltănea* [6]. They pointed out that, under the p-larger order, which is a weaker order than the majorization order,

$$(\lambda_1, \cdots, \lambda_n) \overset{p}{\succeq} (\lambda_1', \cdots, \lambda_n') \Longrightarrow \sum_{i=1}^{n} X_i \geq_{hr} \sum_{i=1}^{n} Y_i.$$

This result has been strengthened by *Khaledi* [12] as

$$(\lambda_1, \cdots, \lambda_n) \overset{p}{\succeq} (\lambda'_1, \cdots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{disp} \sum_{i=1}^{n} Y_i. \tag{3.2}$$

More recently, *Zhao and Balakrishnan* [29] proved that, under the condition of reciprocal order,

$$(\lambda_1, \cdots, \lambda_n) \overset{\mathrm{rm}}{\succeq} (\lambda'_1, \cdots, \lambda'_n) \Longrightarrow \sum_{i=1}^{n} X_i \geq_{mrl} \sum_{i=1}^{n} Y_i. \tag{3.3}$$

The result (3.1) of *Kochar and Ma* [15] can be immediately extended to convolutions of Erlang random variables as follows.

THEOREM **3.1** *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that for $i = 1, \ldots, n$, X_{λ_i} has gamma distribution with scale parameter λ_i and a common shape parameter a which is an integer such that $a \geq 1$. Then*

$$\boldsymbol{\lambda} \overset{m}{\succeq} \boldsymbol{\lambda}^* \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} X_{\lambda^*_i}. \tag{3.4}$$

Korwar [21] generalized Theorem (3.1) to the case of $a \geq 1$. Khaledi and Kochar [13] strengthened this result with majorization replaced by p-larger ordering in (3.1).

THEOREM **3.2** *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Rightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} X_{\lambda^*_i}. \tag{3.5}$$

The following result is an immediate consequence of Theorem 3.2, Theorem 2.1 and the fact that convolutions of IFR distributions are IFR.

COROLLARY 3.1 *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*

$$\boldsymbol{\lambda} \overset{p}{\succeq} \boldsymbol{\lambda}^* \Rightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{hr} \sum_{i=1}^{n} X_{\lambda^*_i}. \tag{3.6}$$

Since $(\lambda_1, \lambda_2, \ldots, \lambda_n) \overset{p}{\succeq} (\tilde{\lambda}, \tilde{\lambda}, \ldots, \tilde{\lambda})$, where $\tilde{\lambda}$ is the geometric mean of the λ_i's, the following lower bounds on various quantities of interest associated with convolutions of gamma random variables are given next.

COROLLARY 3.2 *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*
(a) $\sum_{i=1}^{n} X_{\lambda_i} \geq_{disp} \sum_{i=1}^{n} Y_i$
(b) $\sum_{i=1}^{n} X_{\lambda_i} \geq_{hr} \sum_{i=1}^{n} Y_i$ *which implies*
(c) $\sum_{i=1}^{n} X_{\lambda_i} \geq_{st} \sum_{i=1}^{n} Y_i$,
where $Y_1, \cdots, Y_n$ is a random sample from a Gamma distribution with shape parameter $a \geq 1$ and scale parameter $\tilde{\lambda}$, the geometric mean of λ_i's.

This result leads to better bounds for measures of variability for $\sum_{i=1}^{n} X_{\lambda_i}$ by replacing λ's by their geometric mean. On the other hand the bounds given by Korwar [21] uses arithmetic mean $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i$ instead of the geometric mean on the right hand sides of the above inequalities.

In Figures 2.2.1 and 2.2.2, we plot the distribution functions of convolutions of two independent gamma random variables along with the bounds given by Corollary 3.2 (c) and by Korwar (2002). In Figures 2.2.3 and 2.2.4, we plot the hazard functions of convolutions of two independent gamma random variables along with the bounds given by Corollary 3.2 (b) and by Korwar (2002). The vector of parameters in Figures 2.2.1 and 2.2.3 is $\boldsymbol{\lambda_1} = (1, 2)$ and that in Figures 2.2.2 and 2.2.4 is $\boldsymbol{\lambda_2} = (0.25, 2.75)$. Note that $\boldsymbol{\lambda_2} \overset{m}{\succeq} \boldsymbol{\lambda_1}$. It appears from these figures that the improvements on the bounds are relatively more if λ_i's are more dispersed in the sense of majorization. The fact that this is true follows because the geometric mean is Schur concave whereas the arithmetic mean is Schur constant and the distribution (hazard rate) of convolutions of i.i.d. gamma random variables with common parameter $\tilde{\lambda}$ is decreasing (increasing) in $\tilde{\lambda}$.

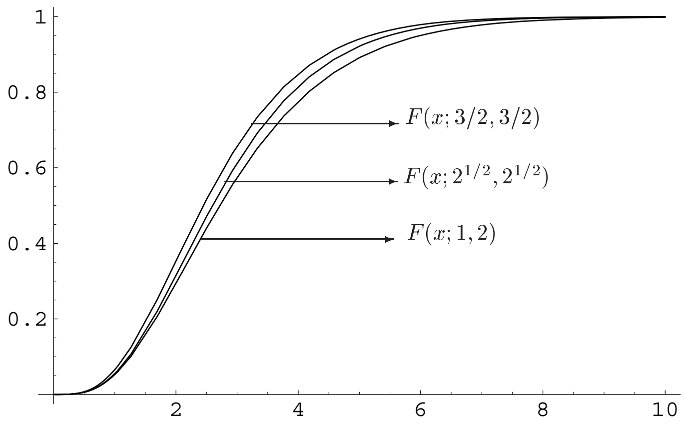

Figure 2.2.1. Graphs of distribution functions of $S(\lambda_1, \lambda_2)$

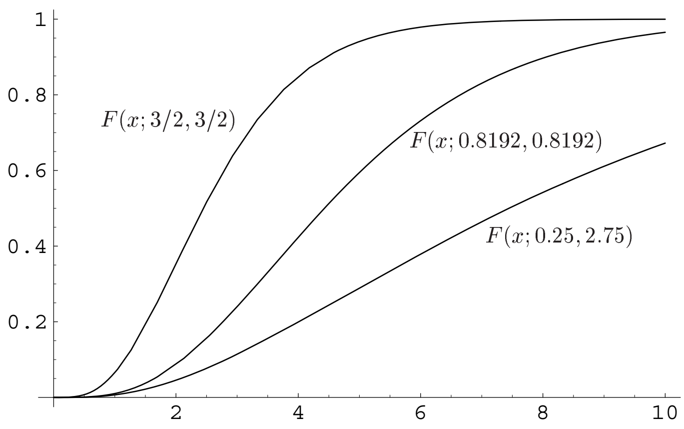

Figure 2.2.2. Graphs of distribution functions of $S(\lambda_1, \lambda_2)$

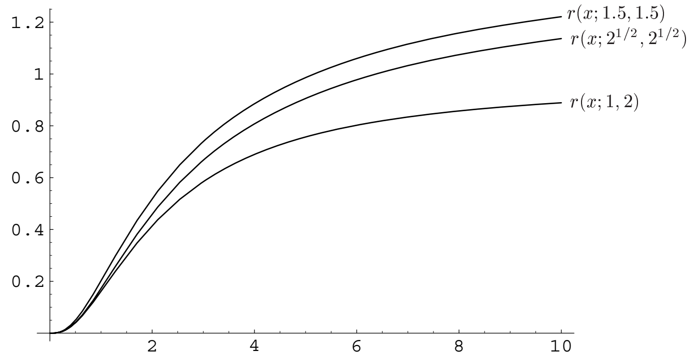

Figure 2.2.3. Graphs of hazard rates of $S(\lambda_1,\lambda_2)$

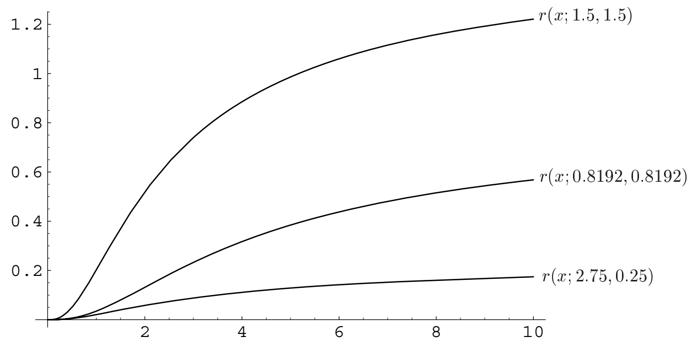

Figure 2.2.4. Graphs of hazard rates of $S(\lambda_1,\lambda_2)$

The following result due to Amiri, Khaledi and Samaniego [1], is a generalization of Theorem 4.1 of Zhao and Balakrishnan [29] and Corollary 3.8 in Kochar and Xu [17] from convolutions of independent exponential distributions to convolutions of gamma distributions with common shape parameters $a \geq 1$.

THEOREM **3.3** *Let $X_{\lambda_1},\dots,X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i=1,\dots,n$. Then,*

$$(\lambda_1,\dots,\lambda_n) \overset{rm}{\succeq} (\lambda_1^*,\dots,\lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{mrl} \sum_{i=1}^{n} X_{\lambda_i^*}.$$

COROLLARY 3.3 *Let $X_{\lambda_1},\dots,X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_i, for $i=1,\dots,n$ and $Y_1,\dots,Y_n$ be a random sample from a gamma distribution with shape parameter $a \geq 1$ and scale parameter λ_H, where λ_H is harmonic mean of λ_i's. Then,*

$$(\lambda_1,\dots,\lambda_n) \overset{rm}{\succeq} (\lambda_1^*,\dots,\lambda_n^*) \Longrightarrow \sum_{i=1}^{n} X_{\lambda_i} \geq_{mrl} \sum_{i=1}^{n} Y_i.$$

This corollary provide a computable lower bound on mrl function of convolutions of gamma random variables which is sharper than those that can be obtained from Theorem 3.4 of Korwar [21] in terms of arithmetic mean and from Corollary 2.2 of Khaledi and Kochar [13] in terms of geometric mean of λ_i's. To justify these observations, in Figures 2.3.1 and 2.3.2 we plot the mean residual life functions of convolutions of two independent gamma random variables with bound given in terms of arithmetic mean, geometric mean and harmonic mean of λ_i's. In Figure 2.3.1, we plot the mean residual functions for $\lambda_1 = 3.6$ and $\lambda_2 = 0.4$.

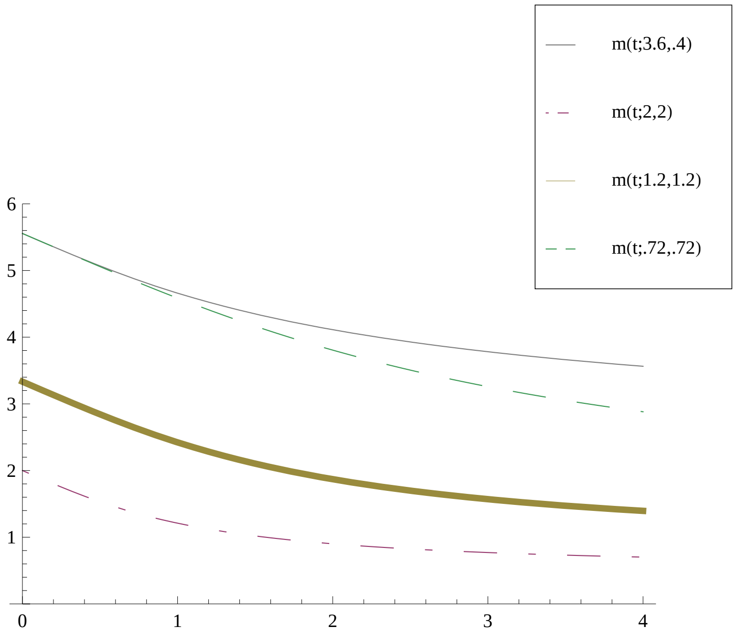

Figure 2.3.1. Mean residual function of convolutions of gamma random variables

We also plot the mean residual life functions of convolutions independent gamma random variables for different sets of λ_i's

$$(2,6) \overset{rm}{\succeq} (5.2,2.4) \overset{rm}{\succeq} (3,6) \overset{rm}{\succeq} (4,4)$$

that shows that how rm ordering between λ_i's will affect the mean residual life function of convolutions of gamma random variables.

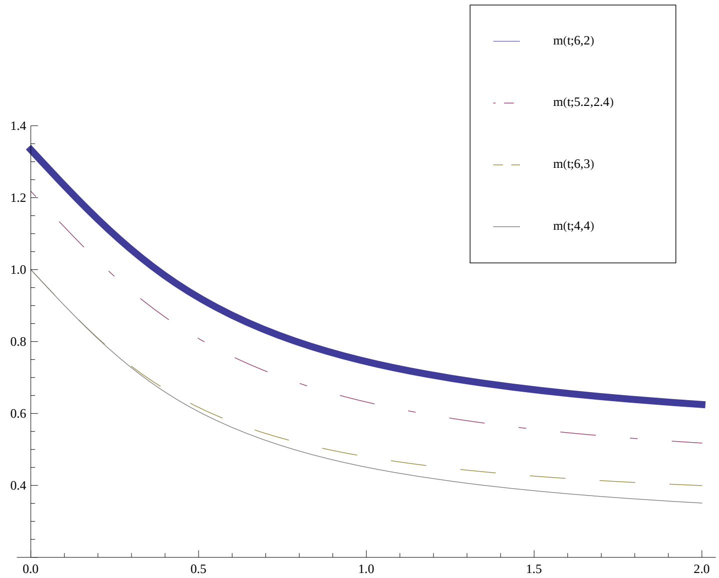

Figure 2.3.2. Mean residual function of convolutions of gamma random variables

Mi, Shi and Zhou [24] studied linear combinations of independent gamma random variables with different integer shape parameters (i.e., Erlang random variables). They established likelihood ratio ordering between two linear combinations of Erlang random variables under some restrictions on the coefficients and shape parameters. It is interesting to note that *Yu* [27] proved that

$$\sum_{i=1}^{n} \beta_i X_i \geq_{st} \sum_{i=1}^{n} \beta X_i \Longleftrightarrow \prod_{i=1}^{n} \beta_i^{a_i} \geq \prod_{i=1}^{n} \beta^{a_i}, \tag{3.7}$$

where $\beta, \beta_i \in \mathbb{R}_+$, and X_i's are gamma random variables $\Gamma(a_i, \lambda)$ for $i = 1, \ldots, n$, respectively.

Kochar and Xu [20] gave the following equivalent characterization of stochastic ordering between two linear combinations of independent gamma random variables.

Lemma 3.1 Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $\beta_{(2)}/\beta_{(1)} \geq \beta'_{(2)}/\beta'_{(1)}$, then the following statements are equivalent:

(a) $\beta_{(1)}^{a_1} \beta_{(2)}^{a_2} \geq {\beta'}_{(1)}^{a_1} {\beta'}_{(2)}^{a_2}$;

(b) $\beta_{(1)} X_1 + \beta_{(2)} X_2 \geq_{st} \beta'_{(1)} X_1 + \beta'_{(2)} X_2$.

They also proved following result, which recovers Theorem 3.3 in *Zhao* [28]

Theorem **3.4** Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. If $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$, then

$$(\log(\beta_1), \cdots, \log(\beta_n)) \succeq_{\mathrm{w}} (\log(\beta'_1), \cdots, \log(\beta'_n)) \Longrightarrow \sum_{i=1}^{n} \beta_{(i)} X_i \geq_{disp} \sum_{i=1}^{n} \beta'_{(i)} X_i.$$

4 Star ordering between convolutions of gamma random variables

Kochar and Xu [17] proved the following result on convolutions of exponential random variables.

$$(\frac{1}{\lambda_1},\ldots,\frac{1}{\lambda_n}) \stackrel{m}{\succeq} (\frac{1}{\lambda_1^*},\ldots,\frac{1}{\lambda_n^*}) \Rightarrow \sum_{i=1}^{n} E_{\lambda_i} \geq_{\triangle} \sum_{i=1}^{n} E_{\lambda_i^*}, \tag{4.1}$$

where E_{λ_i}, $i = 1,\ldots,n$ is exponential random variable with hazard rate λ_i and $\triangle$ order stands for NBUE and Lorenz order. For more details of Lorenz order the reader is referred to section 3.A. in Shaked and Shanthikumar (2007).

Let X_{θ_1}, X_{θ_2}, $X_{\theta'_1}$ and $X_{\theta'_2}$ be independent gamma random variables with a common shape parameter a and scale parameters $\theta_1 = 1/\lambda_1$, $\theta_2 = 1/\lambda_2$, $\theta'_1 = 1/\lambda'_1$ and $\theta'_2 = 1/\lambda'_2$, respectively. Proposition 2.1 of *Diaconis and Perlman* [7] shows that if

$$(\lambda_1,\lambda_2) \stackrel{m}{\succeq} (\lambda'_1,\lambda'_2),$$

then the distribution function of $X_{\theta_1} + X_{\theta_2}$ crosses that of of $X_{\theta'_1} + X_{\theta'_2}$ exactly once.

In this section, it will be shown that under various conditions on the scale parameters, one can establish star ordering between convolutions of gamma random variables .

Recently, *Kochar and Xu* [18] studied the problem of comparing the skewness of linear combinations of independent gamma random variables. Let X_1 and X_2 be independent and identically distributed gamma random variables $\Gamma(a,\lambda)$. They proved that for $(\beta_i,\beta'_i) \in \mathbb{R}^2_+$, $i = 1,2$, if either

$$(\beta_1,\beta_2) \stackrel{\mathrm{m}}{\succeq} (\beta'_1,\beta'_2) \tag{4.2}$$

or

$$\left(\frac{1}{\beta_1},\frac{1}{\beta_2}\right) \stackrel{\mathrm{m}}{\succeq} \left(\frac{1}{\beta'_1},\frac{1}{\beta'_2}\right) \tag{4.3}$$

then

$$\beta_1 X_1 + \beta_2 X_2 \geq_\star \beta'_1 X_1 + \beta'_2 X_2, \tag{4.4}$$

where $\geq_\star$ denotes the star order, and $\stackrel{\mathrm{m}}{\succeq}$ denotes the majorization order. *Amiri, et al.* [1] also independently proved the above results when $a \geq 1$. These results are closely related to a result of *Yu* [27], who proved that, for $\beta_i \in \mathbb{R}_+$,

$$\sum_{i=1}^{n} \beta_i X_i \geq_\star \sum_{i=1}^{n} X_i, \tag{4.5}$$

where X_i's are gamma random variables $\Gamma(a_i,\lambda)$ for $i = 1,\ldots,n$, respectively. These results reveal that if the coefficients are more dispersed, then the linear combinations are more skewed as compared by star ordering.

This topic is further pursued by *Zhao* [28] who extended the results of Eqs (4.2) - (4.4) to the case of independent gamma random variables with different shape parameters. More precisely, let X_1 and X_2 be independent gamma random variables $\Gamma(a_1,\lambda)$ and $\Gamma(a_2,\lambda)$. *Zhao* (2011) proved that, for $\beta_1 \leq \beta_2$ and $\beta'_1 \leq \beta'_2$,

$$(\beta_1,\beta_2) \stackrel{\mathrm{m}}{\succeq} (\beta'_1,\beta'_2) \Longrightarrow \beta_1 X_1 + \beta_2 X_2 \geq_\star \beta'_1 X_1 + \beta'_2 X_2, \tag{4.6}$$

and if $\beta_1 \le \beta_2$, $\beta_1' \le \beta_2'$, and $a_1 \le a_2$, then,

$$\left(\frac{1}{\beta_1}, \frac{1}{\beta_2}\right) \overset{\text{m}}{\succeq} \left(\frac{1}{\beta_1'}, \frac{1}{\beta_2'}\right) \Longrightarrow \beta_1 X_1 + \beta_2 X_2 \ge_\star \beta_1' X_1 + \beta_2' X_2. \tag{4.7}$$

Kochar and Xu [19] gives a different sufficient condition on the scale parameters of the convoluting gamma random variables for star ordering to hold.

THEOREM **4.1** Let X_{θ_1}, X_{θ_2}, $X_{\theta_1'}$, $X_{\theta_2'}$ be independent gamma random variables with a common shape parameter a and scale parameters $\theta_1 = 1/\lambda_1$, $\theta_2 = 1/\lambda_2$, $\theta_1' = 1/\lambda_1'$ and $\theta_2' = 1/\lambda_2'$. Then,

$$(\lambda_1, \lambda_2) \overset{m}{\succeq} (\lambda_1', \lambda_2') \Longrightarrow X_{\theta_1} + X_{\theta_2} \ge_\star X_{\theta_1'} + X_{\theta_2'}.$$

Remark: Theorem 4.1 implies that the distribution function of $X_{\theta_1} + X_{\theta_2}$ crosses that of $X_{\theta_1'} + X_{\theta_2'}$ at most once, no matter how $X_{\theta_1} + X_{\theta_2}$ is scaled. As a special case, they have exactly one crossing when both sides have the same mean which strengthens Proposition 2.1 in *Diaconis and Perlman* [7].

Recently *Kochar and Xu* [20] has given a new sufficient condition for ordering the skewness of linear combinations of two independent gamma random variables with arbitrary shape parameters and this result unifies the previous results on this topic as given above.

THEOREM **4.2** Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. Then,

$$\frac{\beta_{(2)}}{\beta_{(1)}} \ge \frac{\beta_{(2)}'}{\beta_{(1)}'} \Longrightarrow \beta_{(1)} X_1 + \beta_{(2)} X_2 \ge_\star \beta_{(1)}' X_1 + \beta_{(2)}' X_2,$$

where $\{\beta_{(1)}, \beta_{(2)}\}$ denotes the increasing arrangement of the components of the vector $(\beta_1, \beta_2) \in \mathbb{R}_+^2$.

Remark 1: The condition given in the Theorem 4.2 is very general. It is weaker than any of the following conditions, which are commonly used in the literature:

(a) $(\beta_1, \beta_2) \overset{\text{m}}{\succeq} (\beta_1', \beta_2')$;

(b) $(\log(\beta_1), \log(\beta_2)) \overset{\text{m}}{\succeq} (\log(\beta_1'), \log(\beta_2'))$;

(c) $(1/\beta_1, 1/\beta_2) \overset{\text{m}}{\succeq} (1/\beta_1', 1/\beta_2')$.

Remark 2: Conditions (a) and (c) have been used to prove Theorems 4.2 and 4.3 in *Zhao* [28] (see also Eqs. (4.6) and (4.7)). The proof of Theorem 4.2 of *Zhao* [28] is quite involved. However, it follows immediately from Remark 1.

5 Right spread order between linear combinations of gamma random variables

Amiri et al [1] proved the following result on RS ordering between convolutions of gamma random variables with a common shape parameter.

THEOREM **5.1** *Let $X_{\lambda_1}, \ldots, X_{\lambda_n}$ be independent random variables such that X_{λ_i} has gamma distribution with shape parameter $a \ge 1$ and scale parameter λ_i, for $i = 1, \ldots, n$. Then,*

$$(\lambda_1, \ldots, \lambda_n) \overset{rm}{\succeq} (\lambda_1^*, \ldots, \lambda_n^*) \Longrightarrow \sum_{i=1}^n X_{\lambda_i} \ge_{RS} \sum_{i=1}^n X_{\lambda_i^*}.$$

Theorem 5.1 generalizes Corollary 3.9 of *Kochar and Xu* [17] from convolutions of independent Erlang distributions to convolutions of gamma distributions with common shape parameters $a \geq 1$.

Now we consider the case when the shape parameters of the Gamma random variables are not necessarily equal. The first result (cf. Kochar and Xu [20]) gives the following characterization of right spread order for linear combinations of two gamma random variables.

Lemma 5.1 Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. If $\beta_{(2)}/\beta_{(1)} \geq \beta'_{(2)}/\beta'_{(1)}$, then the following statements are equivalent:

(a) $\beta_{(1)}a_1 + \beta_{(2)}a_2 \geq \beta'_{(1)}a_1 + \beta'_{(2)}a_2$;

(b) $\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{RS} \beta'_{(1)}X_1 + \beta'_{(2)}X_2$.

Proof: It follows from Theorem 4.3 in *Fernández-Ponce, et al.* [8] that for two nonnegative random variables X and Y, if $X \leq_\star Y$, then

$$\mathrm{E}X \leq \mathrm{E}Y \Longleftrightarrow X \leq_{RS} Y.$$

It follows from Theorem 4.2 that under the given assumption

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_\star \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Hence,

$$\beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{RS} \beta'_{(1)}X_1 + \beta'_{(2)}X_2$$

is equivalent to

$$\mathrm{E}\left(\beta_{(1)}X_1 + \beta_{(2)}X_2\right) \geq \mathrm{E}\left(\beta'_{(1)}X_1 + \beta'_{(2)}X_2\right).$$

So, the required result follows. ■

Remark: Theorem 4.5 in *Zhao* [18] states that if $1 \leq a_1 \leq a_2$, then

$$(\beta_1, \beta_2) \succeq_{\mathrm{w}} (\beta'_1, \beta'_2) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{RS} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

Lemma 5.1 removes the restriction on the shape parameters.

As a direct consequence, we have the following result.

Corollary 5.2 Let X_1 and X_2 be independent gamma random variables $\Gamma(a_1, \lambda)$ and $\Gamma(a_2, \lambda)$, respectively. Then,

$$(\beta_1, \beta_2) \overset{\mathrm{m}}{\succeq} (\beta'_1, \beta'_2) \Longrightarrow \beta_{(1)}X_1 + \beta_{(2)}X_2 \geq_{RS} \beta'_{(1)}X_1 + \beta'_{(2)}X_2.$$

The following result of *Zhao* [18] immediately follows from Corollary 5.2, Theorem 3.C.7 of *Shaked and Shanthikumar* [26] and similar argument to Theorem 3.4.

Corollary 5.3 Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. If $1 \leq a_1 \leq a_2 \leq \ldots \leq a_n$, then

$$(\beta_1, \cdots, \beta_n) \succeq_{\mathrm{w}} (\beta'_1, \cdots, \beta'_n) \Longrightarrow \sum_{i=1}^n \beta_{(i)}X_i \geq_{RS} \sum_{i=1}^n \beta'_{(i)}X_i.$$

Yu [27] gave necessary and sufficient conditions for stochastically comparing linear combinations of heterogeneous and homogeneous gamma random variables. The following result gives necessary and sufficient conditions for comparing linear combinations of gamma random variables according to right spread order.

Proposition 5.4 Let $X_1, \ldots, X_n$ be independent gamma random variables $\Gamma(a_1, \lambda), \ldots, \Gamma(a_n, \lambda)$, respectively. Then,

$$\sum_{i=1}^{n} \beta_i X_i \geq_{RS} \beta \sum_{i=1}^{n} X_i \Longleftrightarrow \beta \leq \frac{\sum_{i=1}^{n} \beta_i a_i}{\sum_{i=1}^{n} a_i}.$$

Proof: It follows from *Yu* [27] (see also (4.5)) that

$$\sum_{i=1}^{n} \beta_i X_i \geq_{\star} \beta \sum_{i=1}^{n} X_i.$$

Using Theorem 4.3 in *Fernández-Ponce, et al.* [8] again, we have

$$\sum_{i=1}^{n} \beta_i X_i \geq_{RS} \beta \sum_{i=1}^{n} X_i \Longleftrightarrow \mathrm{E}\left(\sum_{i=1}^{n} \beta_i X_i\right) \leq \mathrm{E}\left(\sum_{i=1}^{n} \beta X_i\right).$$

Hence, the required result follows.

Remark: Compared to Corollary 5.3, there is no restriction on the shape parameters.

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