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Building a Knowledge Base: Understanding Prospective Elementary Teachers'

Mathematical Content Knowledge

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Abstract

This survey of the literature summarizes and reflects on research findings regarding elementary preservice teachers' (PSTs') mathematics conceptions and the development thereof. Despite the current focus on teacher education, peer-reviewed journals offer a surprisingly sparse insight in these areas. The limited research that exists chiefly presents views of PSTs' reasoning at singular points during a term, thus focusing on conceptions almost to the exclusion of their development. We summarize the current findings, which are a beginning of a collective understanding of PSTs' mathematical content knowledge. We believe much more work is needed to understand how PSTs can best develop their content knowledge. This is a call to the community to produce such peer-reviewed research.

Keywords: Mathematical content knowledge for teaching; preservice teacher education; elementary teacher education; research review

Introduction and Rationale

Over the last two decades, a consistent focus in major research publications and policy documents has been on the development of mathematical proficiency and conceptual understanding of learners of mathematics (Kilpatrick, Swafford, & Findell, 2001; Kirby, 2003; Lundin & Burton, 1998; National Council of Teachers of Mathematics, 2000; National Governors Association & Council of Chief State School Officers, 2010). This emphasis has in turn revitalized interest in mathematics teaching and teacher knowledge. In 2003, for example, the RAND Mathematics Study Panel (Kirby, 2003) called for a focus on research and development efforts in “developing teachers’ mathematical knowledge in ways that are directly useful for teaching” (p. 78). Called to action by this and other reports, mathematics educators are characterizing the knowledge needed to teach in a way that allows for sense making and the development of conceptual understanding (Adler & Ball, 2008; Hill, Ball, & Schilling, 2008; Ma, 1999; National Research Council, 2001; Shulman, 1986; Silverman & Thompson, 2008). Alder and Ball (2008), for example, define *mathematical knowledge for teaching* (MKT) as a construct that identifies the mathematical knowledge unique to the work of teachers of mathematics. A consensus seems to exist that mathematical content knowledge is an essential aspect of the mathematical knowledge needed to teach and that such knowledge must be deep (Ma, 1999) and multifaceted (Hill et al., 2008). In this paper we focus on *mathematical content knowledge* teachers need to teach. We include mathematical knowledge expected of elementary students as well as knowledge of representations, common elementary school students’ reasoning, and errors. In other words, we include *common content knowledge*, *specialized content knowledge*, and *knowledge of*

content and students as defined by Ball, Thames, and Phelps (2008) and Hill et al. (2008) as mathematical content knowledge.

To help PSTs develop such *mathematical content knowledge*, mathematics teacher educators need to understand two things: (a) the conceptions PSTs bring to teacher education (Bransford, Brown, & Cocking, 1999), since “the key to turning even poorly prepared prospective elementary teachers into mathematical thinkers is to work from what they *do* know” (Conference Board of the Mathematical Sciences, 2001, p. 17), and (b) how those conceptions can be further developed. We use the term *conceptions*, similar to Graeber and Tirosh (Graeber & Tirosh, 1988; Graeber, Tirosh, & Glover, 1989; Tirosh & Graeber, 1989, 1990a, 1990b), with a focus on mathematical content not including beliefs. Therefore, we modified Philipp’s (2007) definition of *conceptions* for our use and define *conceptions* as “general notions or mental structures encompassing ~~beliefs~~, [deleted by authors] meanings, concepts, propositions, rules, mental images, and preference” (p. 259). Yet, what does mathematics education research reveal about the mathematical content knowledge of PSTs and the development of PSTs’ conceptions?

Mewborn (2001) summarized the state of knowledge of mathematics content knowledge in the preparation and practice of K-8 teachers in 2001. She identified three weaknesses in her review: (a) the mathematical topics examined were narrow in focus (focusing on some topics and excluding others), (b) most research provided “snapshots” of teachers’ knowledge at a specific point in time with a lack of research focusing on the development or “videotaped version” of that knowledge, and (c) while the research suggested many teachers can perform algorithms but lacked the conceptual understanding to explain them, there was also a need for a rich description of strong concepts some teachers do possess. Mewborn called on the

mathematics education research community to provide a better understanding of mathematics content knowledge in various contexts, and the development thereof, emphasizing the need for longitudinal studies, and ultimately to develop "a theory of teachers' knowledge, teachers' practice, and student learning" (p. 34) through cross-case analyses. The current emphasis on teaching and teacher education encouraged us to examine whether the research community had responded to Mewborn's call, providing the field with a more recent update on the status of the mathematics content knowledge of preservice elementary teachers.

The authors of this report have taught specially designed content courses for elementary school teachers in the U.S. Their common pedagogy is to (a) identify PSTs' currently held conceptions to be in a position to build from them, (b) engage the PSTs in sense-making activities to develop conceptions, and (c) link mathematical content knowledge to other knowledge, including children's mathematical thinking (Thanheiser, Browning, Moss, Watanabe, & Garza-Kling, 2010). Limited insights for teaching were derived from the authors' individual explorations of the existing research literature regarding PSTs' conceptions and the development of conceptions within topics from mathematical content areas. In fact, there was little to no research conducted that connected to PSTs' mathematical knowledge in each specific topic area. Thus, to examine the landscape of research related to PSTs' content knowledge within and across content areas, the authors established a working group (Thanheiser et al., 2009; Thanheiser, Browning, Kastberg, et al., 2010) to summarize research addressing PSTs' content knowledge and its development more systematically, focusing on the years between 1998–2010 to examine what we have learned since Mewborn's (2001) study.

In our work, we summarized recent research findings about PSTs' mathematics conceptions and development of conceptions. Further, we identified needed areas of mathematics education research to ground and inform content and pedagogy of mathematics content courses for PSTs. Our guiding questions for this summary were:

1. *What research has been conducted on PSTs' content knowledge between 1998 and 2010?*
2. *What is known from this research about PSTs' content knowledge? In particular, what do we know about PSTs' conceptions and the development thereof?*

The first question yielded a database of articles, while findings from the second question are reflected in a more detailed summary developed from the articles. In the next section, we describe the research methods used to address these two questions.

Research Methods

In this survey of the literature we use the term *PST* to refer to a student who is enrolled in a university with the goal to become an elementary school teacher. In the U.S., elementary school teachers teach a variety of subjects. Therefore, PSTs are typically required to complete various content courses focused on the various areas they will be teaching (such as mathematics, English, etc.), as well as respective methods courses that focus on the pedagogy (such as how to teach mathematics, English, etc.). However, there is no common standard in the U.S. on how many or what type of mathematics classes a PST needs to complete before becoming a teacher. Some universities offer and require their PSTs to take content courses designed and developed for elementary school teachers. Other institutions may require no specially designed mathematics content courses but accept typical beginning collegiate mathematics courses, e.g., College Algebra, to provide mathematical content knowledge to

the PST. Some universities require a methods course focused specifically on how to teach mathematics; others require more general methods courses (focusing on teaching in general, or mathematics and science, etc.). Thus, while our requirement for the focus of a reviewed paper was elementary preservice teachers, the specific context in their education before teaching may vary widely across studies conducted in the U.S.

The literature review was conducted by a group of mathematics teacher educators who participated in a working group (Thanheiser, Browning, Kastberg, et al., 2010) focusing on PST content knowledge. Participants met at a conference (PME-NA) over a four-year period to establish and analyze what was at that time (2007, 2009, 2010) known about PSTs' content knowledge. The work was broken into two rounds. First, we identified literature within various content areas, and second, we identified themes across content areas. We describe these two rounds below.

Round 1

We began our summary effort by breaking into content groups based on the working group participants' own research interests, such as whole numbers, fractions, decimals, geometry and measurement, and algebra, ensuring to include several major topic areas of school mathematics in the U.S. Later, the results of whole numbers, fraction, and decimals were combined to form the number and operations content group. We limited the search to 1998–2010 to cover the 12-year range prior to the working group meetings. This time period coincided with the beginning of a renewed interest on teacher knowledge since the publication of Ma's work (1999) that compares elementary teachers' mathematical knowledge needed for teaching in the U.S. and China. Working under the assumption that current research would cite prior seminal and relevant work, we assumed articles published

in the time frame would build on findings from prior studies. We identified and used general search terms such as *preservice*, *prospective*, *elementary*, *teacher*, *education*, and *content knowledge*. Each content group then added specific content search terms such as *number*, *whole number*, *addition*, *subtraction*, *geometry*, and *algebra*. These search terms and combinations thereof were entered into the ERIC database. Since not all countries use the same grade-level classification system used in the U.S., we decided to look at findings from studies of PSTs preparing to teach children aged 3–14 to account for cases with combined middle and elementary certifications. We read the title and abstract of each report to determine whether the paper focused on elementary PSTs' mathematics content knowledge. If the title and abstract did not suffice to make a determination of fit, we read the whole paper. Thus, our database included peer-reviewed research papers focusing on the mathematical content knowledge of PSTs or the development thereof in any of the content areas described earlier.

Soon we realized that not all papers identified by the search provided insights into PSTs' content knowledge. For example, a paper by Newton (2009) was first identified through the ERIC keyword search using the keywords “preservice,” “elementary teachers,” and “fractions.” However, it was excluded from the database because its primary focus was on the change of PSTs' *beliefs* related to fractions rather than their content knowledge. Thus, discussion among group members led to the establishment of exclusion criteria. Excluded from our summary were studies that had (a) a general description of content knowledge that lacked specific attention to three primary content areas (thus our claims are restricted to these three content areas): numbers and operations (including whole numbers, fractions, decimals, and operations), geometry and measurement, and algebra; (b) a sole focus on perceptions

about mathematics not connected to content knowledge needed for teaching (we make no claims about PSTs' beliefs in this paper); (c) a focus on describing classroom practice or activities with a lack of attention to research design methods; and (d) a primary focus on high school PSTs, mathematics majors, or inservice elementary teachers (our claims are restricted to preservice elementary teachers). After applying these criteria, Round 1 resulted in 42 studies across all examined content areas.

Round 2

After this initial search, each content group summarized their findings and reported them at a working-group meeting (Thanheiser, Browning, Kastberg, et al., 2010). Each group shared initial themes from their respective content area, and we began to identify common themes across content areas by discussing whether the themes of the individual groups were echoed in other groups. The entire working group also discussed inclusion/exclusion criteria for the journals, focusing on whether the journals published empirical studies and were peer-reviewed. A final list of 23 journals from which articles were found for the summary was compiled (see Appendix A). Then each journal was carefully reviewed for additional articles focusing on PSTs' content knowledge within the given time frame to ensure all articles focusing on PSTs' content knowledge in those identified journals were found. At least two researchers met to discuss inclusion/exclusion of papers in each content area and to determine the entries into the database. All disagreements about inclusion/exclusion into the database were resolved through discussion. This review produced a few additional articles (a) published in 2010 (which had not been part of the original search as the working group met in 2007, 2009, and in 2010); (b) not indexed with any of the previously listed search

terms; (c) published in a year not included in ERIC database; or (d) not indexed with any of the key words.

The second round resulted in 13 additional studies. Thus, a total of 55 studies from 23 referred journals constituted the database for our analysis of research describing elementary PSTs' knowledge of numbers and operations, geometry and measurement, and algebra. The frequencies of the studies from each category follow: numbers and operations, 31¹; geometry and measurement, 8; and algebra, 16. (For a complete list of articles, see Appendix B). The studies in our database listed the reference, content area, research questions, study type, research design, lens or approach used, selection criteria, description of participants, conditions of and procedures for data collection, data analysis, findings, and conclusions and implications.

In this survey of the literature, we focus on these common themes, followed by highlights of the individual content areas and a reflection on the research that has been conducted thus far. For a detailed description of what is known in each content area, see the forthcoming 2014 special issue of *The Mathematics Enthusiast* focusing on PSTs' content knowledge.

Results

The most striking commonality among the content areas was the limited quantity of peer-reviewed papers exploring PSTs' conceptions and the development thereof. With the recent focus and interest on teacher knowledge, the authors had expected more papers focusing on PSTs' knowledge and its development. Collectively, the peer-reviewed research reports in journals provide a limited view of PSTs' content knowledge and how that knowledge is

¹ One paper is used in both the whole numbers and the fractions sections and listed in both reference lists, but only counted once for the summary.

learned. Without an understanding of the development of PSTs' conceptions, we believe it is challenging to design and develop courses that create meaningful opportunities for PSTs to learn mathematics. However, some common themes were found across the research studies and these are discussed in the next section.

Common Themes Across Content Areas

Three themes were found across the various content areas: (1) PSTs' conceptions were examined in narrow time frames; (2) PSTs use procedures, algorithms, and memorized rules to address problem situations; and (3) PSTs exhibit misconceptions found in explorations of children's conceptions. We discuss these three themes below.

Theme 1: PSTs' conceptions were examined in narrow time frames. Mewborn's (2001) suggestion of more research examining PSTs' development of conceptions longitudinally was not met with most studies found based on snapshots of PSTs' reasoning drawn from brief interviews and/or paper-and-pencil items at single points or several points in time (e.g., Thanheiser, 2009). The focus of those studies was to establish a description of PSTs' conceptions at certain times. Findings from snapshots serve a valuable role in describing knowledge and gaps in knowledge, in building tasks to be used as a baseline for studies of development, in suggesting impacts of particular treatments to experimental groups, and to stimulate consideration of possible origins of the misconceptions identified. Although isolated snapshot studies cannot provide evidence of how PSTs' mathematics conceptions develop, we do see value in repeated-snapshot studies that could monitor change or development of concepts over time.

Theme 2: PSTs use procedures, algorithms, and memorized rules to address problem situations. A second theme that arose across the studies was that PSTs tended to use procedures, algorithms, and memorized rules to address problem situations. For areas of fractions, whole numbers, decimals, algebra, and measurement, PSTs struggled when asked to explain why the algorithms work. For geometry, PSTs struggled to understand, construct, and use definitions with research showing that “memorizing the concept definition does not guarantee success” (Cunningham & Roberts, 2010, p. 10).

Theme 3: PSTs exhibit misconceptions found in explorations of children's conceptions. A third theme across the content areas was that PSTs exhibit misconceptions identified in the school mathematics research literature and associated with emergent understandings. These emergent understandings are often based on overgeneralizations drawn from familiar domains of numbers and geometry. For example, consistent with the results from studies conducted by Tirosh and her colleagues in the 1990s (e.g., Tirosh & Graeber, 1990b), the application of whole number understanding was overgeneralized to apply to fraction situations such as “the quotient must be less than the dividend” (Rizvi, 2004; Rizvi & Lawson, 2007), the application of base-ten algorithms to non-base-ten situations (Thanheiser, 2009), or all altitudes of a given triangle must exist within the triangle (Gutierrez & Jaime, 1999). These overgeneralizations have been identified in descriptions of children’s reasoning (Moloney & Stacey, 1997; Resnick et al., 1989), suggesting that adults may retain the misconceptions they develop in elementary grades.

Themes 1 and 2 are consistent with Mewborn’s (2001) findings and suggest there is still work to be done with respect to those themes. Themes 2 and 3 are consistent with the findings from the Teacher Education and Development Study in Mathematics (TEDS-M)

(Tatto & Senk, 2011). TEDS-M reported preservice elementary teacher education content knowledge using a system of “anchor points” established by international experts in mathematics education. The PSTs who reached Anchor Point 1 are able to “apply whole number arithmetic in simple problem-solving situations, [however] they tended to overgeneralize and had difficulty solving abstract problems and those requiring multiple steps” (p. 128). In addition, PSTs who reached Anchor Point 2 were more successful in solving fraction story problems and “generally did well on items testing ‘knowing’ and on standard problems about number, geometry, and algebra, classified as ‘applying,’ but they had more difficulty answering problems that required more complex reasoning in applied or nonroutine situations” (p. 129). Only about 50% of the PSTs in the U.S. reached Anchor Point 2. While the research studies we reviewed and the TEDS-M study give us insight into what PSTs can and cannot do, they give us less insight into why PSTs struggle and how their knowledge may be developed and with what provocations. We discuss one possible approach in the next section.

Promising Directions

We discuss two promising directions found in the literature thus far: (a) the use of artifacts of children’s mathematical thinking to help PSTs develop MKT, and (b) the beginning development of studies moving beyond snapshots.

The use of artifacts of children’s mathematical thinking to help PSTs develop MKT.

Of considerable power in supporting PSTs’ recognition of the limitations of their existing conceptions and moving the development of their mathematical conceptions toward more robust understanding is the introduction and analysis of children’s work. While analysis of children’s work is often used in methods courses to address issues of teaching, research has

shown that analysis of children's mathematical thinking can help PSTs increase their content knowledge (Philipp et al., 2007). Artifacts of children's mathematical thinking can be selected to address specific misconceptions or address specific mathematical topics (Thanheiser, Strand, & Mills, 2011). Mathematical content knowledge is required to be able to analyze student work and identify student errors; we consider this a component of the *specialized content knowledge*.

Moving beyond snapshot studies.

In some of the content areas (e.g., whole numbers), researchers (e.g., Thanheiser, under review) are examining the development of PSTs' conceptions, thus focusing on a videotaped version of PSTs' conceptions and the development thereof resulting in hypothetical learning trajectories (Simon, 1995). However, this work is sparse to date and additional work across content areas is needed to provide a complete picture that can lead to the theory of teacher knowledge, as called for by Mewborn (2001). To give some insight into the current picture of PSTs' conceptions and the development of their conceptions, we describe highlights from the three content areas.

Highlights from the Three Content Areas

Highlights of PSTs' content knowledge of number and operation. The PSTs' computational proficiency with numbers and operations is generally good except on rational number multiplication and division (Li & Kulm, 2008; Menon, 2009). However, PSTs' decisions about whether an algorithm is appropriate in a given context are less reliable. For example, when presented with a child's incorrect application of the standard (base-ten) subtraction algorithm in a time (non base-ten) context, more than half of the PSTs did not

recognize the problem (i.e., that this algorithm could not be applied without modification) and accepted the child's incorrect solution as correct (Thanheiser, 2009).

Consistent with Ma's (1999) findings regarding inservice teachers, PSTs struggle to explain why the algorithms work (Li & Kulm, 2008; Luo, 2009; Thanheiser, 2009, 2010). The majority of the PSTs do not possess adequate conceptual understanding of numbers and operations that they would need for their teaching (Li & Kulm, 2008; Newton, 2008; Thanheiser, 2009, 2010). They have difficulty in carrying out teacher-like tasks, such as modeling operations with multiple representations (Luo, 2009; Rizvi & Lawson, 2007), interpreting students' alternative algorithms (Li & Kulm, 2008; Son & Crespo, 2009), and identifying the roots of student errors (Tirosh, 2000).

Furthermore, studies are just beginning to explore ways that mathematics teacher educators can support the development of better conceptual understanding and model good teaching (Green, Piel, & Flowers, 2008; Toluk-Ucar, 2009). For example, using the teaching-experiment methodology, McClain (2003) examined the development of PSTs' conceptions of place value through activities that were grounded in the research on elementary students' ways of reasoning with place value and multi-digit addition and subtraction. She found that PSTs recorded and developed pictorial and numerical notational schemes so that they communicate their thinking about the multiplicative structure of place value. Using composition and decomposition of quantities, similar to those used by elementary school students as a basis, they developed a stronger understanding of algorithms for addition and subtraction. Finally, cognitive conflict (placing PSTs in a situation that conflicts with their currently held conceptions), identified by Tirosh and Graeber (1990a) as effective in producing changes in PSTs' conception in division, has currently been reported to effect

change in PSTs' conceptions in number theory (Zazkis & Chernoff, 2008). Zazkis and Chernoff created situations for PSTs in which their conceptions did not hold. For example, one PST claimed that 477 is a prime number because it is the result of multiplying two prime numbers (19 and 23). Through successive contexts (i.e., Is 15 a prime number? Is 77 a prime number?), Zazkis and Chernoff were able to guide the PSTs toward recognizing that 477 is not a prime number but the composition of two prime numbers.

Highlights of PSTs' content knowledge of geometry and measurement. The sparse amount of peer-reviewed research related to PSTs' geometry and measurement content knowledge shows PSTs to have a limited understanding across several concepts (Gutierrez & Jaime, 1999; Halat, 2008; Menon, 2009; Pickreign, 2007; Zevenbergen, 2005). All of these research studies took snapshots of the PSTs' understanding, either at a single point or multiple points in time, with no detail on *how* the PSTs' understanding developed.

In taking these snapshots of PSTs' understanding, characteristics of lower levels of geometric thinking (Halat, 2008), procedural knowledge (Gerretson, 2004; Menon, 2009; Zevenbergen, 2005), and limited concept images (Cunningham & Roberts, 2010; Gutierrez & Jaime, 1999) were found across almost all studies. Halat (2008) found more than half of the 125 Turkish elementary PSTs in his study tested at van Hiele levels 0, 1, or 2 (out of 5 levels) after they had completed a one-semester university geometry course. Menon's (1998) work suggests a potential future impact of PSTs' limited understanding with respect to the concept of perimeter. When PSTs were asked to create a question that would assess if a child "really understood the *meaning* of perimeter" (p. 362), he found PSTs' questions chiefly focused on procedural and well-practiced skills, mirroring the PSTs' own instrumental understandings of the concept. And, when creating altitudes of triangles external to the shape, Gutierrez and

Jaime (1999) found PSTs had poor concept images (Vinner, 1991) and relied chiefly on visual cues, even when explicit definitions of triangle altitudes were provided with the task.

Several studies found PSTs exhibited the same misconceptions as children, such as believing a definition of rectangle must include that two sides are shorter than the remaining two (Pickreign, 2007); thinking that diagonals of concave polygons must lie within the shape (Cunningham & Roberts, 2010); reasoning that since there are 100 cm in a meter, there would be 100 cm^3 in a cubic meter (Zevenbergen, 2005); and thinking that triangle altitudes must be different from any of the three given sides and exist within the shape (Gutierrez & Jaime, 1999).

Highlights of PSTs' content knowledge of algebra. The small body of research on PSTs' understandings of algebra suggests that PSTs are challenged across many topics within this content strand. For example, research suggests that PSTs are typically able to produce mathematically sound generalizations of linear patterns and arithmetic sequences of multiples, yet they struggle to justify their own generalizations by connecting them back to the pattern (Richardson, Berenson, & Staley, 2009; Rivera & Becker, 2007), or by recognizing the connections between their symbolic generalizations and their own algebraic thinking (Zazkis & Liljedahl, 2002).

Within some areas of algebra, such as strategies for solving algebraic word problems, using and interpreting variables, and using and interpreting the equals sign, research suggests that change in PSTs' understandings is possible with instruction that is focused on having PSTs explore multiple solution strategies, analyze children's work, and—in the case of variables specifically—write simple computer program commands. With respect to strategies for solving word problems, the findings of a cross-sectional study of first- and third-year

PSTs suggest that the latter group of PSTs solve word problems correctly more often after experiencing classroom instruction that focuses on multiple solution strategies (Van Dooren, Verschaffel, & Onghena, 2003). However, the third-year PSTs in the study demonstrated the same inflexibility in choosing a type of solution strategy as the first-year PSTs. With respect to change in PSTs' conceptions of variables, research suggests that their understandings can become more sophisticated through writing simple computer commands in a drawing program (Mohr, 2008). Finally, with respect to the equals sign, research suggests that PSTs' understandings of the equals sign can develop toward a relational understanding of equivalence by analyzing and discussing children's use of the equals sign (Prediger, 2010).

Studies examining the nature of PSTs' concept development in algebra are limited in number, especially in light of the robust knowledge base for children's development in this content area. However, one exception to this is a three-week teaching experiment focused on helping PSTs develop their abilities to justify generalizations of linear patterns by having students work in small groups on a sequence of interrelated linear patterns. The results of this study suggested that PSTs' generalizations of linear patterns and their justifications can be characterized by a five-stage developmental framework (see Figure 1).

0	Generalizes a recursive rule with no justification of the coefficient and y-intercept
1	Generalizes an explicit rule with no justification of the coefficient and y-intercept
2	Generalizes an explicit rule with partial or faulty justifications of the coefficient and y-intercept
3	Justifies the coefficient and y-intercept and generalizes an explicit rule inconsistently or inefficiently
4	Generalizes an explicit rule and justifies the coefficient and y-intercept

Figure 1. Richardson et al.'s (2009) five-stage framework for PSTs' generalizations and justifications of linear figural patterns.

Limitations

There are several limitations to our summary. First, we reviewed peer-reviewed research in the time frame from 1998 through 2010. Thus, while covering a little more than a decade, this work is not comprehensive. In particular, there have been reports of PSTs' understanding of mathematics conducted prior to 1998 that contributed significantly to the work of the field. For example, the efforts of Graeber, Tirosh, and their colleagues in 1990s set the stage for later studies exploring cognitive conflict and exploring number and operation. While research in the time frame built on findings from reports published prior to 1998, scholars studying particular topics, such as decimals, may want to explore earlier literature to gain further insights into PSTs' conceptions. Second, we were able to identify and review only journals published in English. Future work should include expanding our review to include research published in other languages. Third, some research papers were excluded from our summary because they focused on a general description of content knowledge that lacked specific attention to number and operations (including whole numbers, fractions, decimals, and operations), geometry and measurement, and algebra. Exploration of findings in such excluded papers may contribute to understandings of conceptual linkages between content areas. In addition, our restriction of the key word choice within particular content areas may have excluded articles in our search that were still connected in some form to PSTs' content knowledge. For example, a paper focusing on justification and argumentation in the context of geometry lessons may have yielded rich information about PSTs' conceptions of important geometric ideas, yet it was not captured by our summary. And finally, we used only the ERIC database. While a comprehensive database, it still does not account for all possible research journals that may accept research in mathematics education.

Discussion

Increased focus on teacher content knowledge by policy makers and researchers, the emphasis on the kind of knowledge needed to teach mathematics, and Mewborn's (2001) call for more research on PSTs' content knowledge may have led mathematics educators to expect active research programs in PSTs' conceptions and their development of these conceptions. Thus, in this past decade or so, mathematics educators may have anticipated a flurry of peer-reviewed research articles in these areas. Yet the number of peer-reviewed articles reported in this study is quite small. Collectively, the peer-reviewed research provides a limited view of PSTs' conceptions and how PSTs develop their mathematics conceptions. However, as discussed above, we did identify three themes across content areas.

1. *PSTs' conceptions were examined in narrow time frames.*
2. *PSTs use procedures, algorithms, and memorized rules to address problem situations.*
3. *PSTs exhibit misconceptions found in explorations of children's conceptions.*

Reflecting on the challenge to clearly articulate a relationship between teachers' content knowledge and pedagogical practices, Neubrand et al. (2009) state, "(a) there is still a lack of comprehensive and categorical descriptions that frame teachers' knowledge, particularly for content-oriented viewpoints, and (b) there is apparently no broad consensus about the status of that knowledge..." (p. 211); our findings confirm these statements and those made by Mewborn (2001). We do not yet have a clear enough picture of what conceptions PSTs bring to teacher education and how those conceptions develop. This paper (and others coming out of this work) represents a collection that is (at best) the beginning of a summary on PSTs' content knowledge and puts forth a call to the field to examine PSTs' content knowledge and

the development thereof. This knowledge base is needed to enable mathematics teacher educators to create learning opportunities to support PSTs to develop the knowledge they need to teach.

To describe development, one would need to move beyond snapshots of conceptions, as suggested by Mewborn (2001). Building on this metaphor, researchers would need to create *videotapes*, or longitudinal studies that reveal the development of concepts. Such research studies would follow PSTs closely as they construct conceptions, for example, in a teaching experiment (Steffe & Thompson, 2000) or using design-study methodology (Cobb, Confrey, diSessa, Lehrer, & Schauble, 2003) to establish hypothetical learning trajectories (Simon, 1995). While the overall picture on the development of PSTs' content knowledge is not yet sufficient to provide clear guidance to the design of mathematics courses for PSTs, findings from some well-studied areas, such as fraction multiplication/division or whole number addition/subtraction, could serve as the basis for designing units of instruction, and thereby could be used by researchers to explore questions of how PSTs' conceptions develop. The answers to such questions will have wider implications across content areas.

One interesting observation made by the group that explored the research related to fraction work is that while the majority of the research literature on children's fraction knowledge focused on basic concepts, such as part-whole, unitizing, or comparison (e.g., Pothier & Sawada, 1983; Sáenz-Ludiow, 1994; Steffe, 2001), the majority of the studies on PSTs' fraction knowledge were on fraction multiplication and division. PSTs' difficulties representing fraction operations or making sense of children's common errors might be rooted in their insufficient understanding of some foundational fraction concepts. More studies on PSTs using tasks and ideas similar to those used by Steffe (2001, 2003) and Olive

(2003) in their studies on children's fraction concepts will highlight the similarity and difference between children's and PSTs' content knowledge.

Finally, the applicability of the mathematical knowledge for teaching to the research on PSTs' content knowledge needs to be further examined. Given the fact that the majority of the mathematics content courses for PSTs in the U.S. are housed in mathematics departments while the mathematics method courses typically reside in colleges of education, coordinations are needed in order to examine the role of mathematical content knowledge in the development of mathematics teaching.

References

- Adler, J., & Ball, D. (2008). Mathematical knowledge for teaching. Retrieved October 9, 2011, from <http://tsg.icme11.org/tsg/show/30>
- Ball, D., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5), 389-407.
- Bransford, J. D., Brown, A. L., & Cocking, R. R. (Eds.). (1999). *How people learn*. Washington, DC: National Academy Press.
- Cobb, P., Confrey, J., diSessa, A., Lehrer, R., & Schauble, L. (2003). Design experiments in educational research. *Educational Researcher*, 32, 9-13.
- Conference Board of the Mathematical Sciences. (2001). *The Mathematical Education of Teachers*. Providence RI and Washington DC: American Mathematical Society and Mathematical Association of America.
- Cunningham, R. F., & Roberts, A. (2010). Reducing the mismatch of geometry concept definitions and concept images held by pre-service teachers. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1(Content Knowledge), 1-17.
- Gerretson, H. (2004). Pre-service elementary teachers' understanding of geometric similarity: The effect of dynamic geometry software. *Focus on Learning Problems in Mathematics*, 26(3), 12-23.
- Graeber, A. O., & Tirosh, D. (1988). Multiplication and division involving decimals: Preservice elementary teachers' performance and beliefs. *The Journal of Mathematical Behavior*, 7(3), 263-280.
- Graeber, A. O., Tirosh, D., & Glover, R. (1989). Preservice teachers' misconceptions in solving verbal problems in multiplication and division. *Journal for Research in Mathematics Education*, 20(1), 95-102.
- Green, M., Piel, J. A., & Flowers, C. (2008). Reversing education majors' arithmetic misconceptions with short-term instruction using manipulative. *The Journal of Educational Research*, 101(4), 234-242.
- Gutierrez, A., & Jaime, A. (1999). Pre-service primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher Education*, 2(3), 253-275.
- Halat, E. (2008). Pre-service elementary school and secondary mathematics teachers' Van Hiele levels and gender differences. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1.
- Hill, H. C., Ball, D., & Schilling, S. G. (2008). Unpacking pedagogical content knowledge: Conceptualizing and measuring teachers' topic-specific knowledge of students. *Journal for Research in Mathematics Education*, 39(4), 372-400.
- Kilpatrick, J., Swafford, J., & Findell, B. (Eds.). (2001). *Adding it up. Helping children learn mathematics*. Washington, DC: National Academy Press.
- Kirby, S. N. (2003). Developing a strategic research and development program in mathematics education. Retrieved from http://www.rand.org/pubs/research_briefs/RB8023.html
- Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM Mathematics Education*, 40, 833-843.

- Lundin, J., & Burton, S. (Eds.). (1998). *Mathematics framework for California public schools. Kindergarten through grade twelve*. Sacramento, CA: CDE Press.
- Luo, F. (2009). Evaluating the effectiveness and insights of pre-service elementary teachers' abilities to construct word problems for fraction. *Journal of Mathematics Education*, 2, 83-98.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Erlbaum.
- McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning*, 5(4), 281-306.
- Menon, R. (1998). Preservice teachers' understanding of perimeter and area. *School Science and Mathematics*, 98(7), 361-368.
- Menon, R. (2009). Preservice teachers' subject matter knowledge of mathematics. *The International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimplymouth.ac.uk/jjournal/menon.pdf>
- Mewborn, D. (2001). Teachers content knowledge, teacher education, and their effects on the preparation of elementary teachers in the United States. *Mathematics Teacher Education & Development*, 3, 28-36.
- Mohr, D. J. (2008). Pre-service elementary teachers make connections between geometry and algebra through the use of technology. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 3.
- Moloney, K., & Stacey, K. (1997). Changes with age in students' conceptions of decimal notation. *Mathematics Education Research Journal*, 9(1), 25-38.
- National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.
- National Governors Association & Council of Chief State School Officers. (2010). *Common Core State Standards Initiative*. Retrieved February, 10, 2012, from <http://www.corestandards.org/>
- National Research Council. (2001). *Knowing and learning mathematics for teaching: Proceedings of a workshop*. Washington DC: National Academy Press.
- Neubrand, M., Seago, N., Agudelo-Valderrama, C., DeBlois, L., Leikin, R., & Wood, T. (2009). The balance of teacher knowledge: Mathematics and pedagogy. In R. Even & D. L. Ball (Eds.), *The professional education and development of teachers of mathematics: The 15th ICMI study*. New York: NY: Springer.
- Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, 45(4), 1080-1110.
- Newton, K. J. (2009). Instructional practices related to preservice elementary teachers' motivation for fractions. *Journal of Mathematics Teacher Education*, 12, 89-109.
- Olive, J. (1999). From fractions to rational numbers of arithmetic: A reorganization hypothesis. *Mathematical Thinking and Learning*, 1, 279-314.
- Philipp, R. (2007). Mathematics teachers' beliefs and affect. In F. K. Lester, Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 257-315). Reston, VA: National Council of Teachers of Mathematics.
- Philipp, R., Ambrose, R., Lamb, L. L. C., Sowder, J. T., Schappelle, B. P., Sowder, L., et al. (2007). Effects of early field experiences on the mathematical content knowledge and

- beliefs of prospective elementary school teachers: An experimental study. *Journal for Research in Mathematics Education*, 38(5), 438-476.
- Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them? *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1(Content Knowledge). Retrieved from <http://www.k-12prep.math.ttu.edu>
- Pothier, Y., & Sawada, D. (1983). Partitioning: The emergence of rational number ideas in young children. *Journal for Research in Mathematics Education*, 14, 307-317.
- Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education*, 13(1), 73-93.
- Resnick, L., Nesher, P., Leonard, F., Magone, M., Omanson, S., & Peled, I. (1989). Conceptual bases of arithmetic errors: The case of decimal fractions. *Journal for Research in Mathematics Education*, 20, 8-27.
- Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers' use of representation to reason algebraically. *Journal of Mathematical Behavior*, 28(2-3), 188-199.
- Rivera, F. D., & Becker, J. R. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior*, 26(2), 140-155.
- Rizvi, N. F. (2004). Prospective teachers' ability to pose word problems. *The International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimt.plymouth.ac.uk/journal/rizvi.pdf>
- Rizvi, N. F., & Lawson, M. J. (2007). Prospective teachers' knowledge: Concept of division. *International Education Journal*, 8(2), 377-392.
- Sáenz-Ludiow, A. (1994). Michale's fraction schemes. *Journal for Research in Mathematics Education*, 25, 50-85.
- Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher*, 15(2), 4-14.
- Silverman, J., & Thompson, P. (2008). Toward a framework for the development of mathematical knowledge for teaching. *Journal of Mathematics Teacher Education*, 11(6), 499-511. doi:10.1007/s10857-008-9089-5
- Simon, M. (1995). Reconstructing mathematics pedagogy from a constructivist perspective. *Journal for Research in Mathematics Education*, 26, 114-145.
- Son, J.-W., & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's non-traditional strategy when dividing fractions. *Journal of Mathematics Teacher Education*, 12(4), 235-261.
- Steffe, L. (2001). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*, 20, 267-307.
- Steffe, L. (2003). Fractional commensurate, composition, and adding schemes: Learning trajectories of Jason and Laura: Grade 3-5. *Journal of Mathematical Behavior*, 22, 237-295.
- Steffe, L., & Thompson, P. W. (2000). Teaching experiment methodology: Underlying principles and essential elements. In A. E. Kelly & R. A. Lesh (Eds.), *Handbook of research design in mathematics and science education* (pp. 267-306). Mahwah, NJ: Erlbaum.

- Tatto, M. T., & Senk, S. (2011). The mathematics education of future primary and secondary teachers: Methods and findings from the teacher education and development study in mathematics. *Journal of Teacher Education*, 62, 121-137.
- Thanheiser, E. (2009). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education*, 40(3), 251-281.
- Thanheiser, E. (2010). Investigating further preservice teachers' conceptions of multidigit whole numbers: Refining a framework. *Educational Studies in Mathematics*, 75(3), 241-251.
- Thanheiser, E. (under review). Developing preservice elementary teachers conceptions with well-designed tasks: Explaining successes and analyzing conceptual difficulties. *Journal for Mathematics Teacher Education*.
- Thanheiser, E., Browning, C., Jacobbe, T., Kastberg, S., Lo, J., Moss, M., . . . Watanabe, T. (2009). *Preservice elementary school teachers' content knowledge in mathematics - working group*. Paper presented at the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), Atlanta, GA.
- Thanheiser, E., Browning, C., Kastberg, S., Lioutikova, E., Lo, J., Moss, M., . . . Roy, G. J. (2010). *Preservice elementary school teachers' content knowledge in mathematics - working group*. Paper presented at the North American Chapter of the International Group for the Psychology of Mathematics Education (PME-NA), Columbus, OH.
- Thanheiser, E., Browning, C. A., Moss, M., Watanabe, T., & Garza-Kling, G. (2010). Developing mathematical content knowledge for teaching elementary school mathematics. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 1.
- Thanheiser, E., Strand, K., & Mills, B. (2011). *Using interactions with children and/or artifacts of children's mathematical reasoning to promote the development of preservice teachers' subject matter knowledge*. Paper presented at the Research on Undergraduate Mathematics Education, Portland, OR.
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, 31(1), 5-25.
- Tirosh, D., & Graeber, A. O. (1989). Elementary teachers' explicit beliefs about multiplication and division. *Educational Studies in Mathematics*, 20(1), 79-96.
- Tirosh, D., & Graeber, A. O. (1990a). Evoking cognitive conflict to explore preservice teachers' thinking about division. *Journal for Research in Mathematics Education*, 21(2), 98-108.
- Tirosh, D., & Graeber, A. O. (1990b). Inconsistencies in preservice elementary teachers' beliefs about multiplication and division. *Focus on Learning Problems in Mathematics*, 12(3-4), 65-74.
- Toluk-Ucar, Z. (2009). Developing pre-service teachers understanding of fractions through problem-posing. *Teaching and Teacher Education: An International Journal of Research and Studies*, 25(1), 166-175.
- Van Dooren, W., Verschaffel, L., & Onghena, P. (2003). Pre-service teachers' preferred strategies for solving arithmetic and algebra word problems. *Journal of Mathematics Teacher Education*, 6(1), 27-52.

- Vinner, S. (1991). The role of definitions in the teaching and learning of mathematics. In D. Tall (Ed.), *Advanced mathematical thinking* (pp. 65–81). Dordrecht, The Netherlands: Kluwer.
- Zazkis, R., & Chernoff, E. (2008). What makes a counterexample exemplary? *Educational Studies in Mathematics*, *68*(3), 195-208.
- Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, *49*, 379-402.
- Zevenbergen, R. (2005). Primary preservice teachers' understandings of volume: The impact of course and practicum experiences. *Mathematics Education Research Journal*, *17*(1), 3-23.

Appendix 1: List of Journals

- *American Educational Research Journal*
- *Asia Pacific Education Review*
- *Canadian Journal of Science, Mathematics and Technology Education*
- *College Student Journal*
- *Educational Studies in Mathematics*
- *Focus on Learning Problems in Mathematics* (renamed *Investigations in Mathematics Learning* in 2008)
- *International Education Journal* (renamed *International Education Journal: Comparative Perspectives* in 2007)
- *International Journal for Mathematics Teaching and Learning*
- *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal* (research article)
- *Journal for Research in Mathematics Education*
- *Journal of Educational Research*
- *Journal of Mathematical Behavior*
- *Journal of Mathematics Education*
- *Journal of Mathematics Teacher Education*
- *Journal of Science and Mathematics Education in Southeast Asia*
- *Mathematics Education Research Journal*
- *Mathematical Thinking and Learning*
- *Research in Mathematics Education*
- *School Science and Mathematics* (research article)
- *Teaching and Teacher Education: An International Journal of Research and Studies*
- *The International Journal for Technology in Mathematics Education*
- *The International Journal of Computer Algebra in Mathematics* (renamed *The Journal of Computers in Mathematics and Science Teaching* in 2005)
- *ZDM: The International Journal on Mathematics Education*

Appendix 2: List of All Reviewed Articles

Number and Operation

Whole Numbers:

- Chapman, O. (2007). Facilitating preservice teachers' development of mathematics knowledge for teaching arithmetic operations. *Journal of Mathematics Teacher Education, 10*, 341-349.
- Crespo, S., & Nicol, C. (2006). Challenging preservice teachers' mathematical understanding: The case of division by zero. *School Science and Mathematics, 106*(2), 84-97.
- Glidden, P. L. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics, 108*(4), 130-136.
- Green, M., Piel, J. A., & Flowers, C. (2008). Reversing education majors' arithmetic misconceptions with short-term instruction using manipulatives. *Journal of Educational Research, 101*(4), 234-242.
- Harkness, S. S., & Thomas, J. (2008). Reflections on "Multiplication as Original Sin": The implications of using a case to help preservice teachers understand invented algorithms. *Journal of Mathematical Behavior, 27*(2), 128-137.
- Kaasila, R., Pehkonen, E., & Hellinen, A. (2010). Finnish pre-service teachers' and upper secondary students' understanding of division and reasoning strategies used. *Educational Studies in Mathematics, 73*(3), 247-261.
- Liliedahl, P., Sinclair, N., & Zazkis, R. (2006). Number concepts with number worlds: Thickening understandings. *International Journal of Mathematical Education in Science and Technology, 37*(3), 253-275.
- Lo, J.-J., Grant, T. J., & Flowers, J. (2008). Challenges in deepening prospective teachers' understanding of multiplication through justification. *Journal of Mathematics Teacher Education, 11*(1), 5-22.
- McClain, K. (2003). Supporting preservice teachers' understanding of place value and multidigit arithmetic. *Mathematical Thinking and Learning, 5*(4), 281-306.
- Menon, R. (2003). Exploring preservice teachers understanding of two-digit multiplication. *The International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimt.plymouth.ac.uk/journal/ramakrishnanmenon.pdf>
- Menon, R. (2004). Preservice teachers' number sense. *Focus on Learning Problems in Mathematics, 26*(2), 49-61.
- Thanheiser, E. (2009). Preservice elementary school teachers' conceptions of multidigit whole numbers. *Journal for Research in Mathematics Education, 40*(3), 251-281.
- Thanheiser, E. (2010). Investigating further preservice teachers' conceptions of multidigit whole numbers: Refining a framework. *Educational Studies in Mathematics, 75*(3), 241-251.

- Tsao, Y.-L. (2005). The number sense of preservice elementary school teachers. *College Student Journal*, 39(4), 647-679.
- Yackel, E., Underwood, D., & Elias, N. (2007). Mathematical tasks designed to foster a reconceptualized view of early arithmetic. *Journal of Mathematics Teacher Education*, 10(4-6), 351-367.
- Yang, D.-C. (2007). Investigating the strategies used by pre-service teachers in Taiwan when responding to number sense questions. *School Science and Mathematics*, 107(7), 293.
- Zazkis, R. (2005). Representing numbers: Primes and irrational. *International Journal of Mathematical Education in Science and Technology*, 36, 207-218.

Decimals:

- Stacey, K., Helme, S., Steinle, V., Baturo, A., Irwin, K., & Bana, J. (2001). Preservice teachers' knowledge of difficulties in decimal numeration. *Journal of Mathematics Teacher Education*, 4(3), 205-225.
- Widjaja, W., Stacey, K., & Steinle, V. (2008). Misconceptions about density of decimals: Insights from Indonesian pre-service teachers. *Journal of Science and Mathematics Education in Southeast Asia*, 31(2), 117-131.

Fractions:

- Chinnappan, M. (2000). Preservice teachers' understanding and representation of fractions in a javabars environment. *Mathematics Education Research Journal*, 12(3), 234-253.
- Domoney, B. (2002). Student teachers' understanding of rational number: Part-whole and numerical constructs. *Research in Mathematics Education*, 4(1), 53-67.
- Green, M., Piel, J. A., & Flowers, C. (2008). Reversing education majors' arithmetic misconceptions With short-term instruction using manipulatives. *Journal of Educational Research*, 101(4), 234-242. [double listed in whole numbers and fractions]
- Li, Y., & Kulm, G. (2008). Knowledge and confidence of pre-service mathematics teachers: The case of fraction division. *ZDM: The International Journal on Mathematics Education*, 40, 833-843.
- Lin, C. (2010). Web-based instruction on preservice teachers' knowledge of fraction operations. *School Science and Mathematics*, 110(2), 59-70.
- Luo, F. (2009). Evaluating the effectiveness and insights of pre-service elementary teachers' abilities to construct word problems for fraction. *Journal of Mathematics Education*, 2, 83-98.
- Menon, R. (2009). Preservice teachers' subject matter knowledge of mathematics. *International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimt.plymouth.ac.uk/journal/menon.pdf>

- Newton, K. J. (2008). An extensive analysis of preservice elementary teachers' knowledge of fractions. *American Educational Research Journal*, 45(4), 1080-1110.
- Rizvi, N. F. (2004). Prospective teachers' ability to pose word problems. *International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimt.plymouth.ac.uk/journal/rizvi.pdf>
- Rizvi, N. F., & Lawson, M. J. (2007). Prospective teachers' knowledge: Concept of division. *International Education Journal*, 8(2), 377-392.
- Son, J. W., & Crespo, S. (2009). Prospective teachers' reasoning and response to a student's non-traditional strategy when dividing fraction. *Journal of Mathematics Teacher Education*, 12(4), 235-261.
- Tirosh, D. (2000). Enhancing prospective teachers' knowledge of children's conceptions: The case of division of fractions. *Journal for Research in Mathematics Education*, 31(1), 5-25.
- Toluk-Ucar, Z. (2009). Developing pre-service teachers understanding of fractions through problem-posing. *Teaching and Teacher Education: An International Journal of Research and Studies*, 25(1), 166-175.

Geometry and Measurement

- Cunningham, R. F., & Roberts, A. (2012). Reducing the mismatch of geometry concept definitions and concept images held by pre-service teachers. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1(Content Knowledge). Retrieved from <http://www.k-12prep.math.ttu.edu>
- Fujita, T., & Jones, K. (2007). Learners' understanding of the definitions and hierarchical classification of quadrilaterals: Towards a theoretical framing. *Research in Mathematics Education*, 9(1-2), 3-20.
- Gerretson, H. (2004). Pre-service elementary teachers' understanding of geometric similarity: The effect of dynamic geometry software. *Focus on Learning Problems in Mathematics*, 26(3), 12-23.
- Gutierrez, A., & Jaime, A. (1999). Preservice primary teachers' understanding of the concept of altitude of a triangle. *Journal of Mathematics Teacher Education*, 2(3), 253.
- Halat, E. (2008). Pre-service elementary school and secondary mathematics teachers' Van Hiele levels and gender differences. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1. Retrieved from <http://www.k-12prep.math.ttu.edu>
- Menon, R. (1998). Preservice teachers' understanding of perimeter and area. *School Science and Mathematics*, 98(7), 361.
- Pickreign, J. (2007). Rectangles and rhombi: How well do preservice teachers know them. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1. Retrieved from <http://www.k-12prep.math.ttu.edu>.

Zevenbergen, R. (2005). Primary preservice teachers' understandings of volume: The impact of course and practicum experiences. *Mathematics Education Research Journal*, 17(1), 3-23.

Algebra

Billings, E. M. H., & Klanderma, D. (2000). Graphical representations of speed: Obstacles preservice K-8 teachers experience. *School Science and Mathematics*, 100(8), 440-450.

Briscoe, C., & Stout, D. (2001). Prospective elementary teachers' use of mathematical reasoning in solving a lever mechanics problem. *School Science and Mathematics*, 101(5), 228-235.

Meel, D. E. (1999). Prospective teachers' understandings: Function and composite function. *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1, 1-12.

Mohr, D. (2008). Pre-service elementary teachers make connections between geometry and algebra through the use of technology. *Issues in the Undergraduate Mathematics Preparation of School Teachers*, 3.

Nillas, L. A. (2010). Characterizing preservice teachers' mathematical understanding of algebraic relationships. *International Journal for Mathematics Teaching and Learning*. Retrieved from <http://www.cimt.plymouth.ac.uk/journal/nillas.pdf>.

Otto, C. A., Everett, S. A., & Luera, G. R. (2008). Using a functional model to develop a mathematical formula. *School Science and Mathematics*, 108(6), 228-237.

Pomerantsev, L., & Korostelva, O. (2003). Do prospective elementary and middle school teachers understand the structure of algebraic expressions? *Issues in the Undergraduate Mathematics Preparation of School Teachers: The Journal*, 1.

Prediger, S. (2010). How to develop mathematics-for-teaching and for understanding: The case of meanings of the equal sign. *Journal of Mathematics Teacher Education*, 13(1), 73-93.

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers' use of representation to reason algebraically. *Journal of Mathematical Behavior*, 28(2-3), 188-199.

Rivera, F. D., & Becker, J. R. (2007). Abduction-induction (generalization) processes of elementary majors on figural patterns in algebra. *Journal of Mathematical Behavior*, 26(2), 140-155.

Schmidt, S., & Bednarz, N. (2002). Arithmetical and algebraic types of reasoning used by pre-service teachers in a problem-solving context. *Canadian Journal of Mathematics, Science and Technology Education*, 2(1), 67-90.

- Stylianou, D. A., Smith, B., & Kaput, J. J. (2005). Math in motion: Using CBRs to enact functions. *The Journal of Computers in Mathematics and Science Teaching*, 24(3), 299-324.
- Van Dooren, W., Verschaffel, L., & Onghena, P. (2003). Pre-service teachers' preferred strategies for solving arithmetic and algebra word problems. *Journal of Mathematics Teacher Education*, 6(1), 27-52.
- You, Z., & Quinn, R. J. (2010). Prospective elementary and middle school teachers' knowledge of linear functions: A quantitative approach. *Journal of Mathematics Education*, 3, 66-76.
- Zazkis, R., & Liljedahl, P. (2002a). Arithmetic sequence as a bridge between conceptual fields. *Canadian Journal of Mathematics, Science and Technology Education*, 2(1), 93-120.
- Zazkis, R., & Liljedahl, P. (2002b). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 4, 379-402.