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## Citation Details

B.E. Sagan, J.S. Caughman IV. The multiplicities of a dual-thin Q -polynomial association scheme.  
Electron. J. Comb., 8 , Note 4 (2001), p. 5

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# The Multiplicities of a Dual-thin $Q$ -polynomial Association Scheme

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Submitted: June 23, 2000; Accepted: January 28, 2001.

MR Subject Classification: 05E30

## Abstract

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  denote a symmetric association scheme, and assume that  $Y$  is  $Q$ -polynomial with respect to an ordering  $E_0, \dots, E_D$  of the primitive idempotents. Bannai and Ito conjectured that the associated sequence of multiplicities  $m_i$  ( $0 \leq i \leq D$ ) of  $Y$  is unimodal. Talking to Terwilliger, Stanton made the related conjecture that  $m_i \leq m_{i+1}$  and  $m_i \leq m_{D-i}$  for  $i < D/2$ . We prove that if  $Y$  is dual-thin in the sense of Terwilliger, then the Stanton conjecture is true.

## 1 Introduction

For a general introduction to association schemes, we refer to [1], [2], [5], or [9]. Our notation follows that found in [3].

Throughout this article,  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  will denote a symmetric,  $D$ -class association scheme. Our point of departure is the following well-known result of Taylor and Levingston.

**1.1 Theorem.** [7] *If  $Y$  is  $P$ -polynomial with respect to an ordering  $R_0, \dots, R_D$  of the associate classes, then the corresponding sequence of valencies*

$$k_0, k_1, \dots, k_D$$

is unimodal. Furthermore,

$$k_i \leq k_{i+1} \quad \text{and} \quad k_i \leq k_{D-i} \quad \text{for } i < D/2. \quad \blacksquare$$

Indeed, the sequence is log-concave, as is easily derived from the inequalities  $b_{i-1} \geq b_i$  and  $c_i \leq c_{i+1}$  ( $0 < i < D$ ), which are satisfied by the intersection numbers of any  $P$ -polynomial scheme (cf. [5, p. 199]).

In their book on association schemes, Bannai and Ito made the dual conjecture.

**1.2 Conjecture.** [1, p. 205] *If  $Y$  is  $Q$ -polynomial with respect to an ordering  $E_0, \dots, E_D$  of the primitive idempotents, then the corresponding sequence of multiplicities*

$$m_0, m_1, \dots, m_D$$

*is unimodal.*

Bannai and Ito further remark that although unimodality of the multiplicities follows easily whenever the dual intersection numbers satisfy the inequalities  $b_{i-1}^* \geq b_i^*$  and  $c_i^* \leq c_{i+1}^*$  ( $0 < i < D$ ), unfortunately these inequalities do not always hold. For example, in the Johnson scheme  $J(k^2, k)$  we find that  $c_{k-1}^* > c_k^*$  whenever  $k > 3$ .

Talking to Terwilliger, Stanton made the following related conjecture.

**1.3 Conjecture.** [8] *If  $Y$  is  $Q$ -polynomial with respect to an ordering  $E_0, \dots, E_D$  of the primitive idempotents, then the corresponding multiplicities satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

Our main result shows that under a suitable restriction on  $Y$ , these last inequalities are satisfied.

To state our result more precisely, we first review a few definitions. Let  $\text{Mat}_X(\mathbb{C})$  denote the  $\mathbb{C}$ -algebra of matrices with entries in  $\mathbb{C}$ , where the rows and columns are indexed by  $X$ , and let  $A_0, \dots, A_D$  denote the associate matrices for  $Y$ . Now fix any  $x \in X$ , and for each integer  $i$  ( $0 \leq i \leq D$ ), let  $E_i^* = E_i^*(x)$  denote the diagonal matrix in  $\text{Mat}_X(\mathbb{C})$  with  $yy$  entry

$$(E_i^*)_{yy} = \begin{cases} 1 & \text{if } xy \in R_i, \\ 0 & \text{if } xy \notin R_i. \end{cases} \quad (y \in X). \quad (1)$$

The *Terwilliger algebra* for  $Y$  with respect to  $x$  is the subalgebra  $T = T(x)$  of  $\text{Mat}_X(\mathbb{C})$  generated by  $A_0, \dots, A_D$  and  $E_0^*, \dots, E_D^*$ . The Terwilliger algebra was first introduced in [9] as an aid to the study of association schemes. For any  $x \in X$ ,  $T = T(x)$  is a finite dimensional, semisimple  $\mathbb{C}$ -algebra, and is noncommutative in general. We refer to [3] or [9] for more details.  $T$  acts faithfully on the vector space  $V := \mathbb{C}^X$  by matrix multiplication.  $V$  is endowed with the inner product  $\langle \cdot, \cdot \rangle$  defined by  $\langle u, v \rangle := u^t \bar{v}$  for all  $u, v \in V$ . Since  $T$  is semisimple,  $V$  decomposes into a direct sum of irreducible  $T$ -modules.

Let  $W$  denote an irreducible  $T$ -module. Observe that  $W = \sum E_i^* W$  (orthogonal direct sum), where the sum is taken over all the indices  $i$  ( $0 \leq i \leq D$ ) such that  $E_i^* W \neq 0$ . We set

$$d := |\{i : E_i^* W \neq 0\}| - 1,$$

and note that the dimension of  $W$  is at least  $d + 1$ . We refer to  $d$  as the *diameter* of  $W$ . The module  $W$  is said to be *thin* whenever  $\dim(E_i^*W) \leq 1$  ( $0 \leq i \leq D$ ). Note that  $W$  is thin if and only if the diameter of  $W$  equals  $\dim(W) - 1$ . We say  $Y$  is *thin* if every irreducible  $T(x)$ -module is thin for every  $x \in X$ .

Similarly, note that  $W = \sum E_i W$  (orthogonal direct sum), where the sum is over all  $i$  ( $0 \leq i \leq D$ ) such that  $E_i W \neq 0$ . We define the *dual diameter* of  $W$  to be

$$d^* := |\{i : E_i W \neq 0\}| - 1,$$

and note that  $\dim W \geq d^* + 1$ . A *dual thin* module  $W$  satisfies  $\dim(E_i W) \leq 1$  ( $0 \leq i \leq D$ ). So  $W$  is dual thin if and only if  $\dim(W) = d^* + 1$ . Finally,  $Y$  is *dual thin* if every irreducible  $T(x)$ -module is dual thin for every vertex  $x \in X$ .

Many of the known examples of  $Q$ -polynomial schemes are dual thin. (See [10] for a list.) Our main theorem is as follows.

**1.4 Theorem.** *Let  $Y$  denote a symmetric association scheme which is  $Q$ -polynomial with respect to an ordering  $E_0, \dots, E_D$  of the primitive idempotents. If  $Y$  is dual-thin, then the multiplicities satisfy*

$$m_i \leq m_{i+1} \quad \text{and} \quad m_i \leq m_{D-i} \quad \text{for } i < D/2.$$

The proof of Theorem 1.4 is contained in the next section.

We remark that if  $Y$  is bipartite  $P$ - and  $Q$ -polynomial, then it must be dual-thin and  $m_i = m_{D-i}$  for  $i < D/2$ . So Theorem 1.4 implies the following corollary. (cf. [4, Theorem 9.6]).

**1.5 Corollary.** *Let  $Y$  denote a symmetric association scheme which is bipartite  $P$ - and  $Q$ -polynomial with respect to an ordering  $E_0, \dots, E_D$  of the primitive idempotents. Then the corresponding sequence of multiplicities*

$$m_0, m_1, \dots, m_D$$

*is unimodal.*     ■

**1.6 Remark.** By recent work of Ito, Tanabe, and Terwilliger [6], the Stanton inequalities (Conjecture 1.3) have been shown to hold for any  $Q$ -polynomial scheme which is also  $P$ -polynomial. In other words, our Theorem 1.4 remains true if the words “dual-thin” are replaced by “ $P$ -polynomial”.

## 2 Proof of the Theorem

Let  $Y = (X, \{R_i\}_{0 \leq i \leq D})$  denote a symmetric association scheme which is  $Q$ -polynomial with respect to the ordering  $E_0, \dots, E_D$  of the primitive idempotents. Fix any  $x \in X$  and let  $T = T(x)$  denote the Terwilliger algebra for  $Y$  with respect to  $x$ . Let  $W$  denote any irreducible  $T$ -module. We define the *dual endpoint* of  $W$  to be the integer  $t$  given by

$$t := \min\{i : 0 \leq i \leq D, E_i W \neq 0\}. \tag{2}$$

We observe that  $0 \leq t \leq D - d^*$ , where  $d^*$  denotes the dual diameter of  $W$ .

**2.1 Lemma.** [9, p.385] *Let  $Y$  be a symmetric association scheme which is  $Q$ -polynomial with respect to the ordering  $E_0, \dots, E_D$  of the primitive idempotents. Fix any  $x \in X$ , and write  $E_i^* = E_i^*(x)$  ( $0 \leq i \leq D$ ),  $T = T(x)$ . Let  $W$  denote an irreducible  $T$ -module with dual endpoint  $t$ . Then*

$$(i) \ E_i W \neq 0 \quad \text{iff} \quad t \leq i \leq t + d^* \quad (0 \leq i \leq D).$$

(ii) *Suppose  $W$  is dual-thin. Then  $W$  is thin, and  $d = d^*$ . ■*

**2.2 Lemma.** [3, Lemma 4.1] *Under the assumptions of the previous lemma, the dual endpoint  $t$  and diameter  $d$  of any irreducible  $T$ -module satisfy*

$$2t + d \geq D. \quad \blacksquare$$

*Proof of Theorem 1.4.* Fix any  $x \in X$ , and let  $T = T(x)$  denote the Terwilliger algebra for  $Y$  with respect to  $x$ . Since  $T$  is semisimple, there exists a positive integer  $s$  and irreducible  $T$ -modules  $W_1, W_2, \dots, W_s$  such that

$$V = W_1 + W_2 + \dots + W_s \quad (\text{orthogonal direct sum}). \quad (3)$$

For each integer  $j$ ,  $1 \leq j \leq s$ , let  $t_j$  (respectively,  $d_j^*$ ) denote the dual endpoint (respectively, dual diameter) of  $W_j$ . Now fix any nonnegative integer  $i < D/2$ . Then for any  $j$ ,  $1 \leq j \leq s$ ,

$$\begin{aligned} E_i W_j \neq 0 &\Rightarrow t_j \leq i && \text{(by Lemma 2.1(i))} \\ &\Rightarrow t_j < i + 1 \leq D - i \leq D - t_j && \text{(since } i < D/2) \\ &\Rightarrow t_j < i + 1 \leq D - i \leq t_j + d_j^* && \text{(by Lemmas 2.1(ii), 2.2)} \\ &\Rightarrow E_{i+1} W_j \neq 0 \text{ and } E_{D-i} W_j \neq 0 && \text{(by Lemma 2.1(i)).} \end{aligned}$$

So we can now argue that, since  $Y$  is dual thin,

$$\begin{aligned} \dim(E_i V) &= |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}| \\ &\leq |\{j : 0 \leq j \leq s, E_{i+1} W_j \neq 0\}| \\ &= \dim(E_{i+1} V). \end{aligned}$$

In other words,  $m_i \leq m_{i+1}$ . Similarly,

$$\begin{aligned} \dim(E_i V) &= |\{j : 0 \leq j \leq s, E_i W_j \neq 0\}| \\ &\leq |\{j : 0 \leq j \leq s, E_{D-i} W_j \neq 0\}| \\ &= \dim(E_{D-i} V) \end{aligned}$$

This yields  $m_i \leq m_{D-i}$ . ■

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