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Kernels of Directed Graph Laplacians

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Abstract. Let G denote a directed graph with adjacency matrix Q and in-degree matrix D. We consider the *Kirchhoff matrix* $L = D - Q$, sometimes referred to as the *directed Laplacian*. A classical result of Kirchhoff asserts that when G is undirected, the multiplicity of the eigenvalue 0 equals the number of connected components of G. This fact has a meaningful generalization to directed graphs, as was observed by Chebotarev and Agaev in 2005. Since this result has many important applications in the sciences, we offer an independent and self-contained proof of their theorem, showing in this paper that the algebraic and geometric multiplicities of 0 are equal, and that a graph-theoretic property determines the dimension of this eigenspace – namely, the number of reaches of the directed graph. We also extend their results by deriving a natural basis for the corresponding eigenspace. The results are proved in the general context of stochastic matrices, and apply equally well to directed graphs with non-negative edge weights.

Keywords: Kirchhoff matrix. Eigenvalues of Laplacians. Graphs. Stochastic matrix.

1 Definitions

Let G denote a directed graph with vertex set $V = \{1, 2, ..., N\}$ and edge set $E \subseteq V \times V$. To each edge $uv \in E$, we allow a positive weight ω_{uv} to be assigned. The *adjacency matrix* Q is the $N \times N$ matrix whose rows and columns are indexed by the vertices, and where the ij-entry is ω_{ji} if $ji \in E$ and zero otherwise. The *in-degree matrix* D is the $N \times N$ diagonal matrix whose ii-entry is the sum of the entries of the i^{th} row of Q. The matrix $L = D - Q$ is sometimes referred to as the *Kirchhoff matrix*, and sometimes as the *directed graph Laplacian* of G.

A variation on this matrix can be defined as follows. Let D^+ denote the pseudo-inverse of D. In other words, let D^+ be the diagonal matrix whose ii-entry is D_{ii}^{-1} if $D_{ii} \neq 0$

and whose ii-entry is zero if $D_{ii} = 0$. Then the matrix $\mathcal{L} = D^+(D - Q)$ has nonnegative diagonal entries, nonpositive off-diagonal entries, all entries between -1 and 1 (inclusive) and all row sums equal to zero. Furthermore, the matrix $S = I - \mathcal{L}$ is stochastic.

We shall see (in Section 4) that both L and $\mathcal{L}$ can be written in the form $D - DS$ where D is an appropriately chosen nonnegative diagonal matrix and S is stochastic. We therefore turn our attention to the properties of these matrices for the statement of our main results.

We show that for any such matrix $M = D - DS$, the geometric and algebraic multiplicities of the eigenvalue zero are equal, and we find a basis for this eigenspace (the *kernel* of M). Furthermore, the dimension of this kernel and the form of these eigenvectors can be described in graph theoretic terms as follows.

We associate with the matrix M a directed graph G, and write $j \rightsquigarrow i$ if there exists a directed path from vertex j to vertex i. For any vertex j, we define the *reachable set* $\mathcal{R}(j)$ to be the set containing j and all vertices i such that $j \rightsquigarrow i$. A maximal reachable set will be called a *reach*. We prove that the algebraic and geometric multiplicity of 0 as an eigenvalue for M equals the number of reaches of G.

We also describe a basis for the kernel of M as follows. Let $\mathcal{R}_1, \ldots \mathcal{R}_k$ denote the reaches of G. For each reach $\mathcal{R}_i$, we define the *exclusive part* of $\mathcal{R}_i$ to be the set $H_i = \mathcal{R}_i \setminus \cup_{j \neq i} \mathcal{R}_j$. Likewise, we define the *common part* of $\mathcal{R}_i$ to be the set $C_i = \mathcal{R}_i \setminus H_i$. Then for each reach $\mathcal{R}_i$ there exists a vector v_i in the kernel of M whose entries satisfy: (i) $(v_i)_j = 1$ for all $j \in H_i$; (ii) $0 < (v_i)_j < 1$ for all $j \in C_i$; (iii) $(v_i)_j = 0$ for all $j \notin \mathcal{R}_i$. Taken together, these vectors v_1, v_2,..., v_k form a basis for the kernel of M and sum to the all 1's vector $\mathbf{1}$.

Due to the recent appearance of Agaev and Chebotarev's notable paper [1], we would like to clarify the connections to their results. In that paper, the matrices studied have the form $M = \alpha(I - S)$ where α is positive and S stochastic. A simple check verifies that this is precisely the set of matrices of the form $D - DS$, where D is nonnegative diagonal. The number of reaches corresponds, in that paper, with the *in-forest dimension*. And where that paper concentrates on the location of the Laplacian eigenvalues in the complex plane, we instead have derived the form of the associated eigenvectors.

2 Stochastic matrices

A matrix is said to be *(row) stochastic* if the entries are nonnegative and the row sums all equal 1. Our first result is a special case of Geršgorin's theorem [3, p.344].

2.1 Lemma. *Suppose S is stochastic. Then each eigenvalue λ satisfies $|\lambda| \leq 1$.*

2.2 Definition. Given any real $N \times N$ matrix M, we denote by G_M the directed graph with vertices $1, \ldots, N$ and an edge $j \to i$ whenever $M_{ij} \neq 0$. For each vertex i, set $\mathcal{N}_i := \{j \mid j \to i\}$. We write $j \rightsquigarrow i$ if there exists a directed path in G_M from vertex j to vertex i. Furthermore, for any vertex j, we define $\mathcal{R}(j)$ to be the set containing j and all vertices i such that $j \rightsquigarrow i$. We refer to $\mathcal{R}(j)$ as the *reachable set* of vertex j. Finally, we say a matrix M is *rooted* if there exists a vertex r in G_M such that $\mathcal{R}(r)$ contains every vertex of G_M. We refer to such a vertex r as a *root*.

2.3 Lemma. *Suppose S is stochastic and rooted. Then the eigenspace $\mathcal{E}_1$ associated with the eigenvalue 1 is spanned by the all-ones vector* $\mathbf{1}$.

Proof. Conjugating S by an appropriate permutation matrix if necessary, we may assume that vertex 1 is a root. Since S is stochastic, $S\mathbf{1} = \mathbf{1}$ so $\mathbf{1} \in \mathcal{E}_1$. By way of contradiction, suppose $\dim(\mathcal{E}_1) > 1$ and choose linearly independent vectors $x, y \in \mathcal{E}_1$. Suppose $|x_i|$ is maximized at $i = n$. Comparing the n-entry on each side of the equation $x = Sx$, we see that

$$|x_n| \le \sum_{j \in \mathcal{N}_n} S_{nj}\,|x_j| \le |x_n| \sum_{j \in \mathcal{N}_n} S_{nj} = |x_n|.$$

Therefore, equality holds throughout, and $|x_j| = |x_n|$ for all $j \in \mathcal{N}_n$. In fact, since $\sum_{j \in \mathcal{N}_n} S_{nj}\, x_j = x_n$, it follows that $x_j = x_n$ for all $j \in \mathcal{N}_n$. Since S is rooted at vertex 1, a simple induction now shows that $x_1 = x_n$. So $|x_i|$ is maximized at $i = 1$. The same argument applies to any vector in $\mathcal{E}_1$ and so $|y_i|$ is maximized at $i = 1$.

Since $y_1 \neq 0$ we can define a vector z such that $z_i := x_i - \frac{x_1}{y_1} y_i$ for each i. This vector z, as a linear combination of x and y, must belong to $\mathcal{E}_1$. It follows that $|z_i|$ is also maximized at $i = 1$. But $z_1 = 0$ by definition, so $z_i = 0$ for all i. It follows that x and y are not linearly independent, a contradiction. □

2.4 Lemma. *Suppose S is stochastic $N \times N$ and vertex 1 is a root. Further assume $\mathcal{N}_1$ is empty. Let P denote the principal submatrix obtained by deleting the first row and column of S. Then the spectral radius of P is strictly less than 1.*

Proof. Since $\mathcal{N}_1$ is empty, S is block lower-triangular with P as a diagonal block. So the spectral radius of P cannot exceed that of S. Therefore, by Lemma 2.1, the spectral radius of P is at most 1. By way of contradiction, suppose the spectral radius of P is equal to 1. Then by the Perron-Frobenius theorem (see [3, p. 508]), we would have $Px = x$ for some nonzero vector x.

Define a vector v with $v_1 = 0$ and $v_i = x_{i-1}$ for $i \in \{2, ..., N\}$. We find that

$$Sv = \left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline S_{21} & \\ \vdots & P \\ S_{N1} & \end{array}\right) \left(\begin{array}{c} 0 \\ \hline \\ x \\ \\ \end{array}\right) = \left(\begin{array}{c} 0 \\ \hline \\ x \\ \\ \end{array}\right) = v.$$

So $v \in \mathcal{E}_1$. But $v_1 = 0$, so Lemma 2.3 implies $x = 0$. This contradiction completes the proof. □

2.5 Corollary. *Suppose S is stochastic and $N \times N$. Assume the vertices of G_S can be partitioned into nonempty sets A, B such that for every $b \in B$, there exists $a \in A$ with $a \rightsquigarrow b$ in G_S. Then the spectral radius of the principal submatrix S_{BB} obtained by deleting from S the rows and columns of A is strictly less than 1.*

Proof. Define the matrix $\hat{S}$ by

$$\hat{S} = \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{u} & S_{BB} \end{pmatrix},$$

where $\mathbf{u}$ is chosen so that $\hat{S}$ is stochastic. We claim that $\hat{S}$ is rooted (at 1). To see this, pick any $b \in B$. We must show $1 \rightsquigarrow b$ in $G_{\hat{S}}$. By hypothesis there exists $a \in A$ with $a \rightsquigarrow b$ in G_S. Let

$$a = x_0 \to x_1 \to \cdots \to x_n = b$$

be a directed path in G_S from a to b. Let i be maximal such that $x_i \in A$. Then the x_{i+1}, x_i entry of S is nonzero, so the x_{i+1} row of S_{BB} has row sum strictly less than 1. Therefore, the x_{i+1} entry of the first column of $\hat{S}$ is nonzero. So $1 \to x_{i+1}$ in $G_{\hat{S}}$ and therefore $1 \rightsquigarrow b$ in $G_{\hat{S}}$ as desired. So $\hat{S}$ is rooted, and the previous lemma gives the result. □

2.6 Definition. A set $\mathcal{R}$ of vertices in a graph will be called a *reach* if it is a maximal reachable set; in other words, $\mathcal{R}$ is a *reach* if $\mathcal{R} = \mathcal{R}(i)$ for some i and there is no j such that $\mathcal{R}(i) \subset \mathcal{R}(j)$ (properly). Since our graphs all have finite vertex sets, such maximal sets exist and are uniquely determined by the graph. For each reach $\mathcal{R}_i$ of a graph, we define the *exclusive part* of $\mathcal{R}_i$ to be the set $H_i = \mathcal{R}_i \setminus \cup_{j \neq i} \mathcal{R}_j$. Likewise, we define the *common part* of $\mathcal{R}_i$ to be the set $C_i = \mathcal{R}_i \setminus H_i$.

2.7 Theorem. *Suppose S is stochastic $N \times N$ and let $\mathcal{R}$ denote a reach of G_S with exclusive part H and common part C. Then there exists an eigenvector $v \in \mathcal{E}_1$ whose entries satisfy*

(i) $v_i = 1$ *for all* $i \in H$,

(ii) $0 < v_i < 1$ *for all* $i \in C$,

(iii) 0 *for all* $i \notin \mathcal{R}$.

Proof. Let Y denote the set of vertices not in $\mathcal{R}$. Permuting rows and columns of S if necessary, we may write S as

$$S = \begin{pmatrix} S_{HH} & S_{HC} & S_{HY} \\ S_{CH} & S_{CC} & S_{CY} \\ S_{YH} & S_{YC} & S_{YY} \end{pmatrix} = \begin{pmatrix} S_{HH} & \mathbf{0} & \mathbf{0} \\ S_{CH} & S_{CC} & S_{CY} \\ \mathbf{0} & \mathbf{0} & S_{YY} \end{pmatrix}$$

Since S_{HH} is a rooted stochastic matrix, it has eigenvalue 1 with geometric multiplicity 1. The associated eigenvector is $\mathbf{1}_H$.

Observe that S_{CC} has spectral radius < 1 by Corollary 2.5. Further, notice that $S(\mathbf{1}_H, \mathbf{0}_C, \mathbf{0}_Y)^T = (\mathbf{1}_H, S_{CH}\mathbf{1}_H, \mathbf{0}_Y).^T$ Using this, we find that solving the equation

$$S(\mathbf{1}_H, \mathbf{x}, \mathbf{0}_C)^T = (\mathbf{1}_H, \mathbf{x}, \mathbf{0}_C)^T$$

for $\mathbf{x}$ amounts to solving

$$\begin{pmatrix} \mathbf{1}_H \\ S_{CH}\mathbf{1}_H + S_{CC}\mathbf{x} \\ \mathbf{0}_Y \end{pmatrix} = \begin{pmatrix} \mathbf{1}_H \\ \mathbf{x} \\ \mathbf{0}_Y \end{pmatrix}.$$

Solving the above, however, is equivalent to solving $(I - S_{CC})\mathbf{x} = S_{CH}\mathbf{1}_H$. Since the spectral radius of S_{CC} is strictly less than 1, the eigenvalues of $I - S_{CC}$ cannot be 0. So $I - S_{CC}$ is invertible. It follows that $\mathbf{x} = (I - S_{CC})^{-1} S_{CH}\mathbf{1}_H$ is the desired solution.

Conditions (i) and (iii) are clearly satisfied by $(\mathbf{1}_H, \mathbf{x}, \mathbf{0}_Y),^T$ so it remains only to verify (ii). To see that the entries of $\mathbf{x}$ are positive, note that $(I - S_{CC})^{-1} = \sum_{i=0}^{\infty} S_{CC}^i$, so the entries of $\mathbf{x}$ are nonnegative and strictly less than 1. But every vertex in C has a path from the root, where the eigenvector has value 1. So since each entry in the eigenvector for S must equal the average of the entries corresponding to its neighbors in G_S, all entries in C must be positive. □

3 Matrices of the form $D - DS$

We now consider matrices of the form $D - DS$ where D is a nonnegative diagonal matrix and S is stochastic. We will determine the algebraic multiplicity of the zero eigenvalue. We begin with the rooted case.

3.1 Lemma. *Suppose $M = D - DS$, where D is a nonnegative diagonal matrix and S is stochastic. Suppose M is rooted. Then the eigenvalue 0 has algebraic multiplicity 1.*

Proof. Let $M = D - DS$ be given as stated. First we claim that, without loss of generality, $S_{ii} = 1$ whenever $D_{ii} = 0$. To see this, suppose $D_{ii} = 0$ for some i. If $S_{ii} \neq 1$, let S' be the stochastic matrix obtained by replacing the i^{th} row of S by the i^{th} row of the identity matrix I, and let $M' = D - DS'$. Observe that $M = M'$, and this proves our claim. So we henceforth assume that

$$S_{ii} = 1 \qquad \text{whenever} \qquad D_{ii} = 0. \tag{1}$$

Next we claim that, given (1), $\ker(M)$ must be identical with $\ker(I - S)$. To see this, note that if $(I - S)v = 0$ then clearly $Mv = D(I - S)v = 0$. Conversely, suppose $Mv = 0$. Then $D(I - S)v = 0$ so the vector $w = (I - S)v$ is in the kernel of D. If w has a nonzero entry w_i then $D_{ii} = 0$. Recall this implies $S_{ii} = 1$ and the i^{th} row of $I - S$ is zero. But $w = (I - S)v$, so w_i must be zero. This contradiction implies w must have no nonzero entries, and therefore $(I - S)v = 0$. So M and $I - S$ have identical nullspaces as desired.

By Lemma 2.3, $S\mathbf{1} = \mathbf{1}$, so $M\mathbf{1} = 0$. Therefore the geometric multiplicity, and hence the algebraic multiplicity, of the eigenvalue 0 must be at least 1. By way of contradiction, suppose the algebraic multiplicity is greater than 1. Then there must be a nonzero vector x and an integer $d \geq 2$ such that

$$M^{d-1}x \neq 0 \qquad \text{and} \qquad M^d x = 0.$$

Now, since $\ker M = \ker(I - S)$, Lemma 2.3 and the above equation imply that $M^{d-1}x$ must be a multiple of the vector $\mathbf{1}$. Scaling $M^{d-1}x$ appropriately, we find there exists a vector v such that

$$Mv = -\mathbf{1}.$$

Suppose $\text{Re}(v_i)$ is maximized at $i = n$. Comparing the n-entries above, we find

$$D_{nn}\text{Re}(v_n) + 1 = D_{nn} \sum_{j \in \mathcal{N}_n} S_{nj}\text{Re}(v_j) \le D_{nn}\text{Re}(v_n) \sum_{j \in \mathcal{N}_n} S_{nj} = D_{nn}\text{Re}(v_n),$$

which is clearly impossible. $\square$

3.2 Theorem. *Suppose $M = D - DS$, where D is a nonnegative diagonal matrix and S is stochastic. Then the number of reaches of G_M equals the algebraic and geometric multiplicity of 0 as an eigenvalue of M.*

Proof. Let $\mathcal{R}_1, ..., \mathcal{R}_k$ denote the reaches of G_M and let H_i denote the exclusive part of $\mathcal{R}_i$ for each $1 \le i \le k$, and let $C = \cup_{i=1}^k C_i$ denote the union of the common parts of all the reaches. Simultaneously permuting the rows and columns of M, D, and S if necessary, we may write $M = D - DS$ as

$$M = \left(\begin{array}{cccc|c} D_{H_1H_1}(I - S_{H_1H_1}) & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \ddots & \cdots & \mathbf{0} & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & D_{H_kH_k}(I - S_{H_kH_k}) & \mathbf{0} \\ \hline -D_{CC}S_{CH_1} & \cdots & \cdots & -D_{CC}S_{CH_k} & D_{CC}(I - S_{CC}) \end{array}\right)$$

The characteristic polynomial $\det(M - \lambda I)$ is therefore given by

$$\det(D_{H_1H_1}(I - S_{H_1H_1}) - \lambda I) \cdots \det(D_{H_kH_k}(I - S_{H_kH_k}) - \lambda I) \cdot \det(D_{CC}(I - S_{CC}) - \lambda I).$$

By Lemma 3.1, each submatrix $D_{H_1H_1}(I - S_{H_1H_1})$ has eigenvalue 0 with algebraic and geometric multiplicity 1. But observe that D_{CC} has nonzero diagonal entries since C is the union of the common parts C_i, so $D_{CC}(I - S_{CC})$ is invertible by Corollary 2.5. The theorem now follows. $\square$

We now offer the following characterization of the nullspace.

3.3 Theorem. *Suppose $M = D - DS$, where D is a nonnegative $N \times N$ diagonal matrix and S is stochastic. Suppose G_M has k reaches, denoted $\mathcal{R}_1, ..., \mathcal{R}_k$, where we denote the exclusive and common parts of each $\mathcal{R}_i$ by H_i, C_i respectively. Then the nullspace of M has a basis $\gamma_1, \gamma_2, ..., \gamma_k$ in $\mathbb{R}^N$ whose elements satisfy:*

(i) $\gamma_i(v) = 0$ *for* $v \notin \mathcal{R}_i$;

(ii) $\gamma_i(v) = 1$ *for* $v \in H_i$;

(iii) $\gamma_i(v) \in (0, 1)$ *for* $v \in C_i$;

(iv) $\sum_i \gamma_i = \mathbf{1}_N$.

Proof. Let $M = D - DS$ be given as stated. As in the proof of Theorem 3.2 above, we may assume without loss of generality that

$$S_{ii} = 1 \qquad \text{whenever} \qquad D_{ii} = 0. \tag{2}$$

We further observe, as in the proof of Theorem 3.2, that M and $I - S$ have identical nullspaces, given (2).

Notice that the diagonal entries of a matrix do not affect the reachable sets in the associated graph, so the reaches of G_{I-S} are identical with the reaches of G_S. Furthermore, scaling rows by nonzero constants also leaves the corresponding graph unchanged, so $G_M = G_{D(I-S)} = G_{I-S}$. Therefore the reaches of G_M are identical with the reaches of G_S.

Applying Theorems 2.7 and 3.2, we find that the nullity of the matrix M equals k and the nullspace of M has a basis satisfying (i)–(iii). To see (iv), observe that the all 1's vector $\mathbf{1}$ is a null vector for M, and notice that the only linear combination of these basis vectors that assumes the value 1 on each of the H_i is their sum. □

4 Graph Laplacians

In this section, we simply apply our results to the Laplacians L and $\mathcal{L}$ of a (weighted, directed) graph, as discussed in Section 1.

4.1 Corollary. *Let G denote a weighted, directed graph and let $\mathcal{L}$ denote the (directed) Laplacian matrix $\mathcal{L} = D^+(D - Q)$. Suppose G has N vertices and k reaches. Then the algebraic and geometric multiplicity of the eigenvalue 0 equals k. Furthermore, the associated eigenspace has a basis $\gamma_1, \gamma_2, ..., \gamma_k$ in $\mathbb{R}^N$ whose elements satisfy: (i) $\gamma_i(v) = 0$ for $v \in G - \mathcal{R}_i$; (ii) $\gamma_i(v) = 1$ for $v \in H_i$; (iii) $\gamma_i(v) \in (0, 1)$ for $v \in C_i$; (iv) $\sum_i \gamma_i = \mathbf{1}_N$.*

Proof. The matrix $S = I - \mathcal{L}$ is stochastic and the graphs G and G_S have identical reaches. The result follows by applying Theorem 3.3. □

We next observe that the same results hold for the Kirchhoff matrix $L = D - Q$.

4.2 Corollary. *Let G denote a directed graph and let L denote the Kirchhoff matrix $L = D - Q$. Suppose G has N vertices and k reaches. Then the algebraic and geometric multiplicity of the eigenvalue 0 equals k. Furthermore, the associated eigenspace has a basis $\gamma_1, \gamma_2, ..., \gamma_k$ in $\mathbb{R}^N$ whose elements satisfy: (i) $\gamma_i(v) = 0$ for $v \in G - \mathcal{R}_i$; (ii) $\gamma_i(v) = 1$ for $v \in H_i$; (iii) $\gamma_i(v) \in (0, 1)$ for $v \in C_i$; (iv) $\sum_i \gamma_i = \mathbf{1}_N$.*

Proof. One simply checks that the matrix L has the form $D - DS$ where S is the stochastic matrix $I - \mathcal{L}$ from above, and D is the in-degree matrix of G. The result follows by applying Theorem 3.3. □

In numerous applications, in particular those related to difference - or differential equations (see [6]), it is a crucial fact that any nonzero eigenvalue of the Laplacian has a strictly positive real part. Using some of the stratagems already exhibited, the proof of this fact is easy, and we include the result for completeness.

4.3 Theorem. *Any nonzero eigenvalue of a Laplacian matrix of the form $D-DS$, where D is nonnegative diagonal and S is stochastic, has (strictly) positive real part.*

Proof. Let $\lambda \neq 0$ be an eigenvalue of $D - DS$ and v a corresponding eigenvector, so $(D - DS)v = \lambda v$. Thus for all i,

$$D_{ii} v_i = \lambda v_i + D_{ii} \sum_j S_{ij} v_j. \tag{3}$$

Suppose D_{ii} is zero. Then $\lambda v_i = 0$. Since $\lambda \neq 0$ it follows that $v_i = 0$. Since $\lambda \neq 0$, the vector v is not a multiple of $\mathbf{1}$. Let n be such that $|v_i|$ is maximized at $i = n$. Multiply v by a nonzero complex number so that v_n is real. Since v_n is nonzero, the above argument shows that $D_{nn} \neq 0$. Dividing (3) for $i = n$ by D_{nn} and taking the real and imaginary parts separately, we obtain

$$\sum_j S_{nj} \text{ Re } (v_j) = (1 - \frac{\text{Re } (\lambda)}{D_{nn}}) v_n, \qquad \sum_j S_{nj} \text{ Im } (v_j) = -\frac{\text{Im } (\lambda)}{D_{nn}} v_n.$$

The first of these equations implies that $\text{Re}(\lambda) \geq 0$. Now if $\text{Re}(\lambda) = 0$ then for all $j \in \mathcal{N}_n$ we have $v_j = v_n$ and thus $\text{Im}(v_j) = 0$. Notice that in this case, the imaginary part of λ must be nonzero. So in the second equation above, the left hand side is zero but the right hand side is not. The conclusion is now immediate. □

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