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The fluttering oscillations that have long been seen in falling sheets of water have recently been subjected to detailed theoretical analysis. In this study stability criteria for these oscillations are derived and discussed. These criteria provide insight into the physical mechanisms of the instability and may be employed in the design of fluttering waterfall systems.

I. INTRODUCTION

Fluttering oscillations are known to occur in the falling sheets of water associated with some dams and waterfall fountains.[1,2] While this effect has long been known, it usually goes unnoticed unless the observer has been appropriately sensitized to its possible occurrence. The many-hertz frequency of typical oscillations is a little high for obtaining a clear visual picture of the interesting dynamics; but, on the other hand, this frequency is a little low to be associated with an audible tone. In fact, most published investigations have been initiated because of the window-rattling vibrations caused by larger systems rather than because of any interest in the physics involved.

As noted previously, the fluttering fountain oscillations are often highly periodic and represent feedback oscillations of the air-water system. Small displacements of the water sheet are amplified by the Helmholtz mechanism as the sheet moves downward through the surrounding air.[3] The large-amplitude wave motions at the bottom tend to compress or expand the trapped air, which in turn pulls in our pushes out on the water surface at the top. This combination of gain and feedback can lead to sustained periodic and possibly chaotic oscillations.

In previous studies of fluttering fountain oscillations a theoretical model has been developed which provides at least qualitative agreement with available experimental data. However, the existence of a successful model does not mean that all of the important questions relating to this stability have been answered. Of particular interest are the stability criteria that indicate the conditions under which the fluttering oscillations will be observed. The model in this case is sufficiently complicated that meaningful stability criteria can only be obtained from numerical solutions. The purpose of this study is to derive and discuss the stability boundaries that correspond to the most likely ranges of experimental parameters. These results provide further insight into the underlying physics, and they also make possible the rational design of fountains which do or do not exhibit this effect.

The theoretical model for the fluttering fountain instability is briefly reviewed in Sec. II. Stability criteria are obtained in Sec. III showing the maximum flow rates for which the oscillations can be observed as a function of the fountain height. Other parameters explored in the stability plots include the depth of the air chamber behind the water sheet, the initial velocity of the water as it leaves the weir, and the velocity of any wind in the surrounding atmosphere. Quali-

tatively, it is found that a given system always becomes less stable as the height is increased, as the flow rate is decreased, as the depth of the air chamber is decreased, as the initial velocity of the water is increased, and as the velocity of any wind upward along the face of the fountain is increased.

II. MODEL

The fluttering fountains have been idealized to a nearly vertical water sheet flowing across a rectangular chamber which represents the air confined behind the water. The water becomes free of the weir and subject to air pressure fluctuations starting at the height y_0 with downward velocity v_0. The air chamber has a horizontal depth x_0. The equations that have been developed to describe this configuration are[1]

$$\frac{\partial u(y,t)}{\partial t} = \left(\frac{1}{\rho_w F} \left\{ \Delta p_f(t) + \rho_a \omega v(y) \left[\coth\left(\frac{\omega x_0}{v(y)}\right) \right. \right. \right.$$
$$\left. \left. \left. + \left(1 + \frac{v_a}{v(y)}\right)^2 \right] D(y,t) \right\} + \frac{\partial u(y,t)}{\partial y} \right) v(y) ,$$

$$\tag{1}$$

$$\frac{\partial D(y,t)}{\partial t} = u(y,t) + v(y)\frac{\partial D(y,t)}{\partial y} , \tag{2}$$

where $u(y,t)$ is the time and height dependent x component of the velocity, and $D(y,t)$ is the displacement in the x direction.

In Eq. (1) $\Delta p_f(t)$ is the time-dependent pressure difference across the sheet due to possible compression or expansion of the air in the chamber behind the water, and this pressure can be related approximately to the displacement by[2]

$$\Delta p_f(t) = -\frac{\gamma P_0}{A} \int_0^{y_0} D(y,t) dy . \tag{3}$$

The downward velocity of the water sheet is represented by $v(y)$, which can be written

$$v(y) = v_0 [1 - 2g(y - y_0)/v_0^2]^{1/2} . \tag{4}$$

The other parameters appearing in these equations include the air density ρ_a, the water density ρ_w, the flow rate per unit of weir length F, the radian frequency of the oscillations ω, the upward velocity of any wind against the waterfall v_a, the acceleration of gravity g, the specific heat ratio of the air
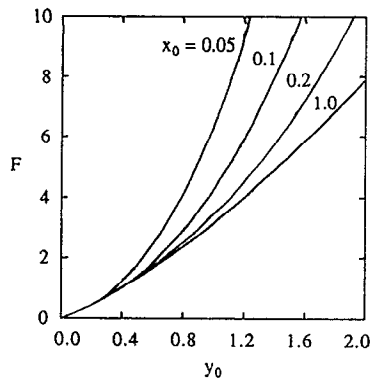
FIG. 1. Threshold water flow rate F in $/\,m^{-1}s^{-1}$ as a function of the fountain height y_0 in m for various values of the air chamber depth x_0 in m. The initial water velocity is assumed to be $v_0 = 1$ m s^{-1}, and the air velocity is $v_a = 0$ m s^{-1}.

γ, the background atmospheric pressure P_0, and the cross-sectional area of the air chamber in the absence of oscillations $A = x_0 y_0$.

The equations given as Eqs. (1)–(4) form a complete set. Numerical solutions have been shown to provide good agreement with the waveforms and frequencies of the fluttering oscillations exhibited by the water sheets of two fountains in Dunedin, New Zealand. Other features of the experimental results such as hysteresis and possibly chaotic behavior would require a more complicated nonlinear model.

The purpose of this study is to obtain detailed stability criteria for the fluttering fountain oscillations. These criteria show the conditions under which a fountain can produce this oscillatory behavior. The easiest conditions to vary experimentally include the fountain height and the water flow rate. Accordingly, the stability criteria that we have obtained are displayed as plots of the flow rate at the oscillation threshold versus the fountain height for various values of the other parameters of interest. In this procedure numerical solutions of Eqs. (1)–(4) corresponding to an initially perturbed fountain are obtained as a function of time for a period of time longer than the oscillation period. If the oscillations are found to be growing in time, the flow rate is increased and the computations are continued. If the oscillations are diminishing in time, the flow rate is decreased. In this way one can determine the value of the flow rate at the oscillation threshold. This procedure is the basis for the stability criteria described in the following section.

III. RESULTS

For our first set of stability boundaries, we explore the dependence of the threshold flow rate F on fountain height y_0 for various values of the depth of the air chamber behind the fountain x_0. The results are plotted in Fig. 1. For a given value of the height and depth, oscillations occur for flow rates below the corresponding plotted contours, and the behavior is stable for flow rates above the contours. These plots extend over a range of heights from 0 up to 2 m, and this range includes most fountains that one is likely to encounter. The depth parameter values in this plot extend from 0.05 to 1

m, and again these values include most practical cases. There are, of course, many other parameters in the model given above in Eqs. (1)–(4), but it is not possible in a single plot to represent all of the possible values for all of these parameters. Thus, for Fig. 1 it has been assumed that the problem of interest involves water ($\rho_w = 10^3$ kg m^{-3}) flowing through air ($\rho_a = 1.225$ kg m^{-3}, $\gamma = 1.4$, $P_0 = 1.013 \times 10^5$ N m^{-2}) in a gravitational field ($g = 9.8$ m s^{-2}). Also, the initial velocity of the water as it leaves the weir is 1 m s^{-1}, and the air velocity in front of the fountain is zero.

It is clear from Fig. 1 that the threshold value of the flow rate increases as the height of the fountain increases. This is mainly a result of the tendency for the overall Helmholtz gain to increase with height and decrease with increasing flow rate. It may also be seen from the figure that for a given height the threshold flow rate increases as the depth of the air chamber decreases. This effect is due to two separate physical phenomena.

First, as discussed by Rayleigh, the Helmholtz gain is always enhanced by the presence of the back wall on the air chamber; and this effect is represented by the coth function in Eq. (1).[4] The smaller the value of x_0 the larger the value of this function. Second, it follows from Eq. (3) and the definition of the area A that smaller values of x_0 will also lead to larger values of the pressure feedback. It turns out from our numerical solutions that these two effects are not very similar in magnitude or consequences.

As suggested by the figure, for decreasing values of x_0 all of the curves in Fig. 1 approach the same limit. This limit is very close to the $x_0 = 1$ curve in the figure. We find that the departure of the other curves from this curve is due almost entirely to the x_0 dependence in the coth function. One can work out from the figure that as long as the fountain height is not more than about five times larger than the fountain depth the stability boundary is not raised by the presence of the back wall of the air chamber. This simply means that as long as the depth is not too small compared to the oscillation wavelength (fountain height) the back wall is almost irrelevant.

It is true, of course, that the reduction in feedback with increasing chamber depth must also have some effect on the threshold boundaries. However, our numerical experiments show that this effect is imperceptible for any reasonable values of the depth. Thus, for purposes of this instability effect the air behind the water sheet acts much like an incompressible fluid, and the depth of the air chamber has little effect on the feedback. This also means that the stability boundaries are almost independent of the specific heat ratio γ and the pressure P_0, which also enter into the pressure feedback.

The practical consequence of the above discussion is that one could sometimes assume that the appropriate instability boundary would be independent of x_0 and in this case would correspond approximately to the $x_0 = 1$ curve in Fig. 1. This is so because one would not want the chamber depth to be very small compared to the height. For too small a depth the water sheet tends to hit the back wall of the chamber and sometimes be drawn against it by the air-pumping action of the flowing water. In modeling most practical fountains one might also be able to simplify the theoretical analysis by
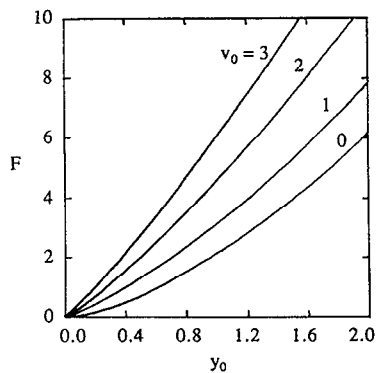
FIG. 2. Threshold water flow rate F in $/\,m^{-1}s^{-1}$ as a function of the fountain height y_0 in m for various values of the initial water velocity v_0 in $m\,s^{-1}$. The coth function in Eq. (1) has been approximated by unity, and the air velocity is $v_a = 0\,m\,s^{-1}$.

FIG. 3. Threshold water flow rate F in $/\,m^{-1}s^{-1}$ as a function of the fountain height y_0 in m for various values of the air velocity v_a in $m\,s^{-1}$. The coth function in Eq. (1) has been approximated by unity, and the initial water velocity is $v_0 = 1\,m\,s^{-1}$.

replacing the coth function with unity. This approximation was made in Ref. 2 in extracting approximate solutions for the fluttering fountain waveforms, and it is also made for the remainder of the results described here.

With the above approximation it becomes possible to display the stability boundaries with some other parameter as a variable. In Fig. 2 is a set of plots of the threshold flow rate as a function of height for various values of the initial velocity v_0. The curve labeled $v_0 = 1$ is the same as the curve labeled $x_0 = 1$ in Fig. 1. For lower values of the initial flow rate the fountain is more stable, and lower values of the flow rate are necessary to reach the instability. For higher values of the flow rate the fountain is less stable. This effect is due to the velocity dependence of the basic Helmholtz amplification mechanism.

Another parameter that can affect the instability boundaries is the air velocity on the outer surface of the water sheet. The Helmholtz gain results from the relative motion of the water and the adjacent air. If the air in front of the fountain is moving upward there is an increase in the gain, and if the air is moving downward the gain is decreased. The effects of such air motion are represented approximately by the term v_a in Eq. (1). In Fig. 3 is a plot of the threshold flow rate as a function of the fountain height for various values of the upward air velocity v_a in front of the fountain. The initial downward water velocity is again $v_0 = 1\,m\,s^{-1}$ for all of these plots. It is clear from this figure that, as one should expect, the presence of a modest wind can have a substantial effect toward the encouragement or discouragement of the fluttering oscillations. Such wind effects were conspicuous in our studies of the Dunedin fountains,[1] and Schwartz has also reported that the draft from a fan could induce oscillations in an otherwise stable nappe.[5]

IV. CONCLUSION

Many of the features of the fluttering oscillations sometimes seen in waterfall fountains can be interpreted by means of a mathematical model consisting of coupled differential
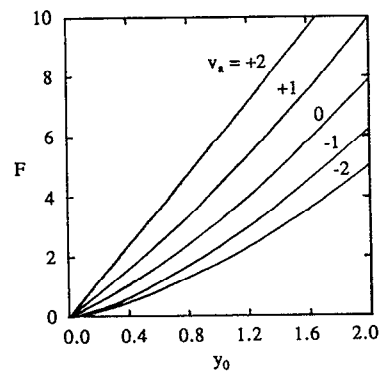
equations. One of the most important questions relating to these oscillations concerns the stability criteria for their occurrence, and these criteria can be extracted by means of numerical solutions. We have carried out such solutions, and the results have been presented here in the form of stability contours. These contours are in qualitative agreement with available experimental data and should permit the rational design of systems which do or do not display this effect.

Qualitatively, one finds that a given system always becomes less stable as the height is increased, as the flow rate is decreased, as the depth of the air chamber is decreased, as the initial velocity of the water is increased, and as the velocity of any wind upward along the face of the fountain is increased. The depth of the air chamber has been shown to be most important for relatively shallow chambers, which also are at risk of having the sheet hit or attach to the back wall of the chamber. For many practical systems this depth would be too great to be a significant factor. It has also been shown that for practical fountain conditions the air in the chamber is compressed very little and acts as if it were incompressible in providing feedback for the oscillations. While the results described here are strictly applicable to fountains involving nearly plane water sheets moving vertically through air, the underlying principles should also provide guidance for the design of oscillators based on different fluids or having different geometries.

ACKNOWLEDGMENT

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