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Several techniques are described for studying the propagation of off-axis polynomial Gaussian beams in media having straight axes and periodic z variations of the quadratic refraction and loss coefficients. For some periodic variations, exact analytical solutions of the paraxial equations are possible, and for sufficiently slow variations, WKB solutions can always be obtained. All results are expressed in conventional beam matrix form.

I. Introduction

Dielectric waveguiding materials in which the index of refraction and gain vary at most quadratically in the transverse direction are generally referred to as lenslike materials, and their ray and beam transmission characteristics are closely related to those of ordinary lenses. In two recent studies, the beam propagation characteristics of tapered lenslike media have been considered.[1,2] Exact numerical and analytical solutions of the paraxial wave equation were obtained for a variety of potentially practical taper configurations. Such tapers have found many applications in coupling beams from one fiber (or other optical element) to another. In addition to tapers, other types of z dependence may occur accidentally or intentionally in the manufacture of graded-index fibers and other lenslike waveguiding media. Of particular interest in this study are the beam propagation characteristics of lenslike materials in which the index of refraction and loss profiles vary periodically in the propagation direction.

The idea that light beams could be guided by sequences of lenses has long been known.[3,4] The earliest studies of periodic lenslike media related to the development of gas lenses in which the waveguiding index profiles were obtained by introducing periodic temperature gradients in the gaseous propagation medium.[5-8] Gas lens waveguides are now largely obsolete because of less expensive graded-index fiber-optic waveguides, but interest in periodic lenslike structures has continued.[9-11] Periodically perturbed graded-index fibers are considered to be a possible consequence of defective fiber manufacturing techniques or distortions in multifiber cables. Intentionally introduced periodic perturbations may also be of value for beam couplers and mode filters.

For illustration, two specific types of periodic lenslike medium are shown schematically in Fig. 1, and the radius changes in the figure are meant to suggest changes in the loss and refraction profiles. In the first type, the parameters of the waveguide vary smoothly and continuously with propagation distance, while in the second type the waveguide is represented as a composite structure made up of segments that individually are easier to study. Techniques are described below for analyzing numerically the propagation of off-axis Gaussian beams in arbitrary z-dependent lenslike media. For a few types of continuously varying periodic medium, exact analytical solutions of the paraxial equations can be obtained. Since matrix representations are now available for many kinds of waveguide segment, very general periodic media can be analyzed exactly by a composite model of the type suggested in Fig. 1(b). For sufficiently gradual profile changes, WKB methods can also be applied. At the outset it may be noted that the assumption of an ideal quadratic lenslike profile can only be valid out to some finite radius in a realizable medium. However, for simplicity, it is assumed here as in previous studies that the propagating modes are confined entirely within the quadratic-profile region.

The fundamental beam equations and their interpretation are reviewed briefly in Sec. II. It is found that for periodic lenslike media, the most basic of the beam equations is of the Hill equation type. In the special case that the beam profile varies sinusoidally in the z direction, the general Hill equation reduces to a Mathieu equation, and solutions for beam propagation in sinusoidally periodic lenslike media are discussed in Sec. III. While the solutions of the Mathieu equation

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are most easily obtained numerically, very similar exact solutions can be obtained analytically for a new class of pseudosinusoidal variations discussed in Sec. IV. A method for composite periodic functions is presented in Sec. V, and an approximate WKB method for arbitrary weak periodic variations is described in Sec. VI. All the solutions are formulated in terms of conventional beam matrices, and hence it is straightforward to analyze or design complex optical systems which incorporate one or more segments of periodic lenslike medium.

II. Beam Equations

The reduction of Maxwell's equations to the ordinary differential equations of beam optics has been given in many places, so it is possible to be brief. A similar reduction may be found in Ref. 1 for the fundamental Gaussian mode and in Ref. 12 for higher-order beam modes. For conciseness, several simplifying restrictions are imposed at the outset. By using the results obtained here, together with the formulations just referenced, it is possible to investigate much more general beam and media configurations.

As in most previous studies, it is assumed that the dominant transverse electric field components are governed by the wave equation

$$\nabla^2 \bar{E}' + k^2 \bar{E}' = 0, \tag{1}$$

where the prime is a reminder that Eq. (1) applies to the complex amplitude of the electric field. For the simplified y-independent model of interest here, the quadratically varying propagation constant can be written

$$k^2(x,z) = k_0(z)[k_0(z) - k_2(z)x^2], \tag{2}$$

where all the coefficients are in general complex. For misaligned media one would need an additional term linear in x, and similar terms in y could also be included as appropriate.[1] Equation (2) corresponds to a z-dependent medium in which the quadratic gain and index profiles may be independently specified. For a wave propagating primarily in the z direction, a useful substitution is

$$E_x' = A(x,z) \exp\left[-i \int_0^z k_0(z')dz'\right] \cdot \tag{3}$$

Then Eq. (1) reduces to

$$\frac{\partial^2 A}{\partial x^2} - 2ik_0 \frac{\partial A}{\partial z} - i\frac{dk_0}{dz} A - k_0 k_2 x^2 A = 0, \tag{4}$$

where A is assumed to vary so slowly with z that its second derivative can be neglected. This last assumption is the familiar paraxial wave approximation, and from this point onward all solutions (except the WKB results of Sec. VI) will be exact.

Equation (4) is a partial differential equation, but it may be reduced to a set of ordinary differential equations by means of the substitution

$$A(x,z) = \exp[-i(Qx^2/2 + Sx + P)]. \tag{5}$$

This substitution leads to a description of the propagation of off-axis mismatched Gaussian beams, but with a slightly more complex substitution higher-order polynomial Gaussian beams can also be described.[12] The results in the present case are

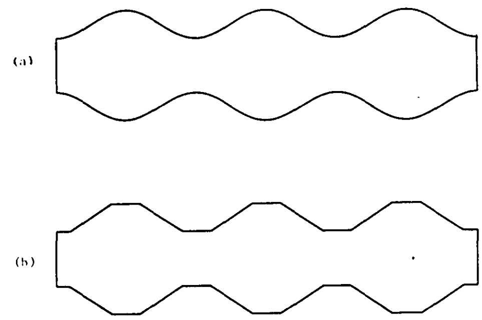

Fig. 1. Schematic representation of (a) a continuous periodic lenslike medium and (b) a composite periodic medium consisting of simpler uniform and tapered segments. The radius changes are meant to suggest changes in the loss and refraction profiles.

$$Q^2 + k_0 \frac{dQ}{dz} + k_0 k_2 = 0, \tag{6}$$

$$QS + k_0 \frac{dS}{dz} = 0, \tag{7}$$

$$\frac{dP}{dz} = -i\frac{Q}{2k_0} - \frac{S^2}{2k_0} \cdot \tag{8}$$

In the more general situations additional terms and additional equations would be obtained. It is important to note, however, that Eq. (6), which must be solved first, would not change its form.[12]

The complex beam parameter Q is related to the phase front curvature R and the $1/e$ amplitude spot size w by means of the familiar formula[13]

$$\frac{Q}{k_0} = \frac{1}{R} - \frac{2i}{k_0 w^2} \cdot \tag{9}$$

The complex displacement parameter S is related to the transverse displacement d_a of the center of the amplitude distribution and the displacement d_p of the center of the phase distribution according to the formulas

$$d_a = -S_i/Q_i, \tag{10}$$

$$d_p = -S_r/Q_r, \tag{11}$$

where the subscripts i and r refer respectively, to the imaginary and real parts. The imaginary part of the complex phase parameter P can be interpreted as a correction to the on-axis field amplitude resulting from the changing beam structure and position. Similarly, the real part of P can be interpreted as a phase correction.

Equations (6)–(8) may be solved in sequence and the main emphasis here is on exact solutions of Eq. (6). This is a Ricatti equation, and the first step in solving it is to introduce the well-known variable change[14]

$$Q = \frac{k_0}{r}\frac{dr}{dz} \cdot \tag{12}$$

With this substitution the Ricatti equation is transformed to a linear differential equation with nonconstant coefficients

$$\frac{d}{dz}\left[k_0(z)\frac{dr}{dz}\right] + k_2(z)r = 0, \tag{13}$$

where the z dependences of the coefficients are noted explicitly for emphasis. This equation has been obtained here from an exact reduction of the paraxial wave equation. The same equation may also be obtained from the paraxial ray equation, and in that case r would be interpreted as the z-dependent transverse displacement of a propagating light ray. While k_0 and k_2 may be allowed to be complex in wave optics studies, special care would be required to apply Eq. (13) in a ray optics analysis of media having a gain or loss profile.

Solutions of Eq. (13) are much easier to obtain if k_0 is a constant, and in that case one finds

$$\frac{d^2r}{dz^2} + \frac{k_2(z)}{k_0} r = 0. \tag{14}$$

It is important to note, however, that with an appropriate change of variables Eq. (13) can always be transformed into Eq. (14), and this transformation is discussed in Appendix A. Hence, there is no loss of generality in using Eq. (14) in place of Eq. (13).

When k_2 is a periodic function of z, Eq. (14) may be recognized as a Hill equation. A more common notation for such an equation is

$$\frac{d^2y}{dx^2} + f(x)y = 0, \tag{15}$$

where $f(x)$ is periodic. This type of equation was first investigated by Hill in connection with the theory of the moon's motion,[15] and such equations arise commonly in many branches of physics and engineering. Analytic solutions of the Hill equation are known for only a few special forms of $f(x)$, and the following sections apply those solutions to the problem of periodic graded-index waveguides.

III. Sinusoidal Variations

The simplest appearing Hill equation is one in which the periodic function $f(x)$ varies sinusoidally as shown schematically in Fig. 1(a). In this case the Hill equation reduces to the Mathieu equation

$$\frac{d^2y}{dx^2} + (p - 2q\cos 2x)y = 0. \tag{16}$$

This equation was introduced by Mathieu in his study of the vibrational modes of a stretched membrane having an elliptical boundary,[16] and it has been considered as a possible model for periodic graded-index media.[7,9,10] In spite of the simple form of the Mathieu equation, the solutions are not easy to evaluate. For most purposes they must still be obtained numerically or from the various published graphs and tables.[17] Nevertheless, this equation may be used to indicate schematically the solution procedure for rays and beams propagating in arbitrary continuous periodic media.

The Mathieu equation possesses a variety of periodic, aperiodic, and unstable solutions depending on the values of the parameters p and q. From a comparison of Eqs. (14) and (16), it may be seen that use of the Mathieu equation is equivalent to assuming that the quadratic term in the complex propagation parameter has the periodic z dependence

$$\frac{k_2(z)}{k_0} = p - 2q\cos 2z, \tag{17}$$

where p and q are in general complex constants. With an obvious renormalization, the Mathieu equation representation can, of course, apply for arbitrary spatial modulation frequencies.

The Mathieu equation has been introduced here because of its popularity and simplicity (apparent rather than real) and also to illustrate the solution procedures that for other periodic functions may be quite straightforward. Since Eq. (14) is an ordinary linear second-order differential equation, it follows that for any given values of the parameters p and q the general solution may be written in the form

$$r(z) = au(z) + bv(z), \tag{18}$$

where $u(z)$ and $v(z)$ are imagined to be linearly independent Mathieu functions and the coefficients a and b must be determined from initial conditions. Similarly, the rate of change of the parameter $r(z)$ is

$$r'(z) = au'(z) + bv'(z). \tag{19}$$

If the values of $r(z)$ and $r'(z)$ at the plane $z = z_1$ are r_1 and r_1', respectively, it follows from Eqs. (18) and (19) that the general equations for $r(z)$ and $r'(z)$ can be written in the usual matrix form:

$$\begin{bmatrix} r(z) \\ r'(z) \end{bmatrix} = \begin{bmatrix} A(z) & B(z) \\ C(z) & D(z) \end{bmatrix} \begin{bmatrix} r_1 \\ r_1' \end{bmatrix}, \tag{20}$$

where the matrix elements are

$$\mathrm{A}(z) = \frac{u'(z_1)v(z) - v'(z_1)u(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \tag{21}$$

$$\mathrm{B}(z) = \frac{v(z_1)u(z) - u(z_1)v(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \tag{22}$$

$$\mathrm{C}(z) = \frac{u'(z_1)v'(z) - v'(z_1u'(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}, \tag{23}$$

$$\mathrm{D}(z) = \frac{v(z_1)u'(z) - u(z_1)v'(z)}{v(z_1)u'(z_1) - u(z_1)v'(z_1)}. \tag{24}$$

To reduce these matrix elements further, one would have to make use of any special properties of the functions $u(z)$ and $v(z)$. However, for the Mathieu equation, the results remain somewhat complicated, and it seems as easy to integrate the equation numerically.

In the case that the second-order paraxial equation is understood to represent the propagation of light rays, Eq. (20) describes completely the ray propagation process in periodic quadratic-index media. Similarly, however, it follows from Eqs. (12) and (20) that

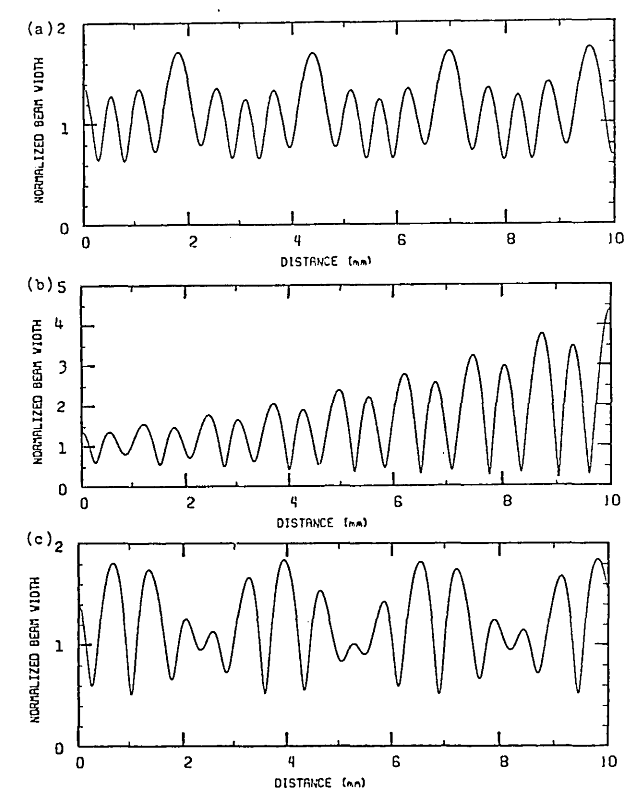

Fig. 2. Normalized beamwidth $w(z)/w_s$ as a function of distance along a periodically modulated quadratic-index waveguide with the modulation frequency (a) $\gamma = 0.5\gamma_0$, (b) $\gamma = 1.0\gamma_0$, and (c) $\gamma = 1.5\gamma_0$ (after Ref. 18).

the propagation of the complex beam parameter Q is governed by the Kogelnik transformation[13]

$$\frac{Q(z)}{k_0} = \frac{\mathrm{C} + \mathrm{D}Q(z_1)/k_0}{\mathrm{A} + \mathrm{B}Q(z_1)/k_0}. \tag{25}$$

A transformation of the same sort applies to the complex displacement parameter $S(z)$. To see this, we first write Eq. (7) in the form

$$\begin{aligned} \frac{d\ \ln S}{dz} &= -\frac{Q}{k_0} \\ &= -\frac{\mathrm{C} + \mathrm{D}Q(z_1)/k_0}{\mathrm{A} + \mathrm{B}Q(z_1)/k_0}, \end{aligned} \tag{26}$$

where Eq. (25) has also been used. But it follows from Eqs. (21) to (24) that C is the derivative of A, and D is the derivative of B. Hence, Eq. (26) can be written

$$\frac{d\ \ln S}{dz} = -\frac{d}{dz} \ln\left[\mathrm{A} + \frac{\mathrm{B}Q(z_1)}{k_0}\right]. \tag{27}$$

Using the facts $\mathrm{A}(z_1) = 1$ and $\mathrm{B}(z_1) = 0$, Eq. (27) may be integrated to obtain the displacement transformation

$$S(z) = S(z_1)[\mathrm{A} + \mathrm{B}Q(z_1)/k_0]^{-1}. \tag{28}$$

Similarly, if the determinant of the beam matrix is unity, Eq. (8) may be integrated to obtain the phase transformation

$$P(z) - P(z_1) = -\frac{i}{2} \ln\left[\mathrm{A} + \frac{\mathrm{B}Q(z_1)}{k_0}\right] - \frac{S^2(z_1)}{2k_0} \frac{\mathrm{B}}{\mathrm{A} + \mathrm{B}Q(z_1)/k_0}. \tag{29}$$

The conditions in which the determinant is unity are considered in Appendix B.

Equations (25), (28), and (29) provide a complete description of the amplitude distribution, phase front curvature, and displacement of Gaussian beams propagating in periodically varying lenslike materials. It was suggested at the beginning of this section that for the media of interest Eq. (14) might take the form of a Mathieu equation. However, this suggestion was only made for purposes of illustration, and the preceding results apply for arbitrary z-dependent graded-index media. When the paraxial equations actually take the form of the Mathieu equation, it is most straightforward to obtain the solutions by direct numerical integration.

As an example, several numerical solutions will be shown for the Mathieu equation

$$\frac{d^2r}{dz^2} + F\left(1 + \frac{1}{2}\cos\gamma z\right) r = 0 \tag{30}$$

in which the quadratic term in the index profile is strongly modulated. The coefficient F in real (nonabsorbing or amplifying) graded-index media corresponds to the unmodulated index ratio n_2/n_0, and for consistency with previous experimental and theoretical studies the value adopted for this ratio is 25 mm^{-2}. If the waveguide were not modulated, it would follow from Eqs. (6) and (9) that the steady-state spot size can be written

$$w_s = \left[\frac{\lambda}{\pi n_0}\left(\frac{n_0}{n_2}\right)^{1/2}\right]^{1/2}. \tag{31}$$

With the values $n_0 = 1.5$ and $\lambda = 1\ \mu$m, this steady-state spot size is $w_s = 0.0065$ mm, and this value will be the basis for our normalization of the spot size in periodically modulated waveguides.

A propagating off-axis light ray or polynomial Gaussian beam in an ordinary z-independent graded-index medium oscillates about the z axis with the spatial frequency

$$\gamma_0 = (n_2/n_0)^{1/2}, \tag{32}$$

and for these examples the oscillation frequency is $\gamma_0 = 5\ \mathrm{mm}^{-1}$. The actual oscillation in a modulated waveguide are typically much more complicated than the simple sinusoidal variations of frequency γ_0 that would be found in a z-independent waveguide. Nevertheless, γ_0 is a convenient reference frequency, and resonance effects are found to occur when the modulation frequency is close to γ_0.

Figure 2 shows the spot size oscillations that are obtained for modulation frequencies of $\gamma = 0.5\ \gamma_0$, $\gamma = 1.0\ \gamma_0$, and $\gamma = 1.5\ \gamma_0$. It is clear from this figure that there is a resonance effect when $\gamma = 1.0\ \gamma_0$, and this resonance causes the spot size fluctuations to increase without limit. This type of instability has been noted previously in a study of laser beam propagation in a

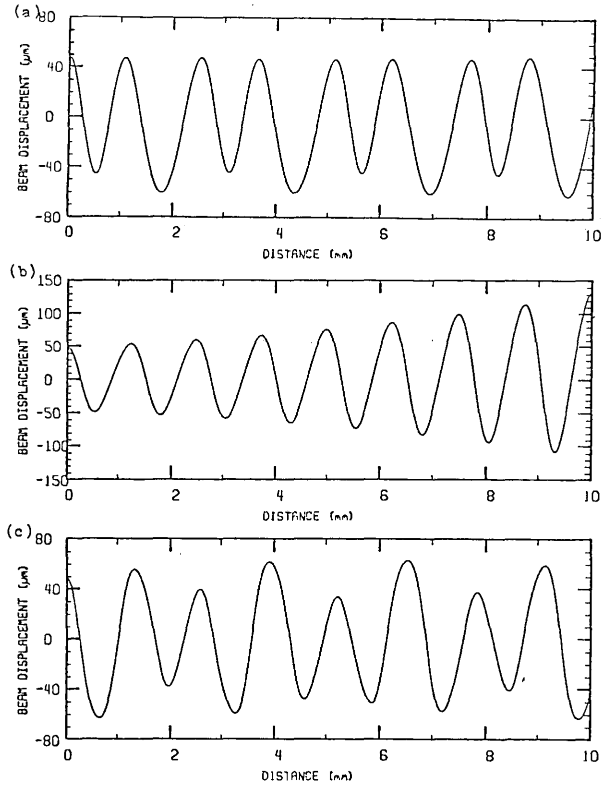

Fig. 3. Displacement of the amplitude center of the beam away from the waveguide axis in micrometers as a function of distance along a periodically modulated quadratic-index waveguide with the modulation frequency (a) $\gamma = 0.5\gamma_0$, (b) $\gamma = 1.0\gamma_0$, and (c) $\gamma = 1.5\gamma_0$ (after Ref. 18).

periodically perturbed plasma waveguide.[10] Similar results are obtained for the beam displacement, and plots of the displacement of the center of the amplitude distribution away from the z axis are shown in Fig. 3. Again there is a resonance effect when $\gamma = 1.0\,\gamma_0$.

IV. Pseudosinusoidal Variations

The previous section emphasized ray and beam propagation in periodic media governed by the simplest appearing paraxial equations. In particular the complex Hill equation was considered to be reducible to a Mathieu equation. However, except for some initial general considerations the solutions for beam propagation had to be obtained numerically. For this discussion it is assumed that the paraxial equation, Eq. (14), takes the specific form[19]

$$\frac{d^2r}{dz^2} + \left[\frac{F}{(1 + G\cos\gamma z)^4} + \frac{\gamma^2 G\cos\gamma z}{1 + G\cos\gamma z}\right] r = 0, \tag{33}$$

where in general F, G, and γ may be complex. While this new equation appears slightly more complicated than the similar Mathieu equation, it has the advantage that the general solution can be written explicitly in terms of elementary functions. In particular, the solution of Eq. (33) is

$$\begin{aligned} r(z) = {} & a\left(\frac{1 + G\cos\gamma z}{1 + G}\right)\cos\left(\frac{F^{1/2}}{\gamma(1 - G^2)}\left\{\frac{G\sin\gamma z}{1 + G\cos\gamma z} - \frac{2}{(1 - G^2)^{1/2}}\tan^{-1}\left[\frac{(1 - G^2)^{1/2}}{1 + G}\tan\left(\frac{\gamma z}{2}\right)\right]\right\}\right) \\ & + b\left(\frac{1 + G\cos\gamma z}{1 + G}\right)\sin\left(\frac{F^{1/2}}{\gamma(1 - G^2)}\left\{\frac{G\sin\gamma z}{1 + G\cos\gamma z} - \frac{2}{(1 - G^2)^{1/2}}\tan^{-1}\left[\frac{(1 - G^2)^{1/2}}{1 + G}\tan\left(\frac{\gamma z}{2}\right)\right]\right\}\right), \end{aligned} \tag{34}$$

where a and b are arbitrary constants. Various extensions and special cases of Eq. (34) are considered in Appendix C. The lenslike media governed by Eq. (33) are only of interest if the periodic z dependences can resemble the actual z dependences that one might expect to encounter in practical waveguides. This seems to be the case though, and the periodic term in Eq. (33) may be nearly sinusoidal or it may have narrow maxima or minima, depending on the values chosen for the parameters F and G. A more detailed discussion of the possible forms of the periodic term is included in Ref. 19.

The matrix elements corresponding to Eq. (34) can be most easily obtained from a comparison with the general results of the previous section. Thus, from a comparison of Eqs. (18) and (34) one finds that the periodic functions $u(z)$ and $v(z)$ are

$$u(z) = \left(\frac{1 + G\cos\gamma z}{1 + G}\right)\cos\left(\frac{F^{1/2}}{\gamma(1 - G^2)}\left\{\frac{G\sin\gamma z}{1 + G\cos\gamma z} - \frac{2}{(1 - G^2)^{1/2}} \times\tan^{-1}\left[\frac{(1 - G^2)^{1/2}}{1 + G}\tan\left(\frac{\gamma z}{2}\right)\right]\right\}\right), \tag{35}$$

$$v(z) = \left(\frac{1 + G\cos\gamma z}{1 + G}\right)\sin\left(\frac{F^{1/2}}{\gamma(1 - G^2)}\left\{\frac{G\sin\gamma z}{1 + G\cos\gamma z} - \frac{2}{(1 - G^2)^{1/2}} \times\tan^{-1}\left[\frac{(1 - G^2)^{1/2}}{1 + G}\tan\left(\frac{\gamma z}{2}\right)\right]\right\}\right). \tag{36}$$

Using these formulas, the matrix elements are given by Eqs. (21)–(24). To actually evaluate the matrix elements, however, it is necessary to compute the derivatives of $u(z)$ and $v(z)$. For this purpose it is helpful to replace Eqs. (35) and (36) by the equivalent forms

$$u(z) = \left(\frac{1 + G\cos\gamma z}{1 + G}\right)\cos\left[F^{1/2}\int_0^z \frac{dz'}{(1 + G\cos\gamma z')^2}\right]. \tag{37}$$

$$v(z) = \left(\frac{1 + G\cos\gamma z}{1 + G}\right)\sin\left[F^{1/2}\int_0^z \frac{dz'}{(1 + G\cos\gamma z')^2}\right]. \tag{38}$$

Now the derivatives are found to be

$$u'(z) = -\frac{\gamma G\sin\gamma z}{1 + G}\cos[\ ] - \frac{F^{1/2}}{(1 + G)(1 + G\cos\gamma z)^2}\sin[\ ], \tag{39}$$

$$v'(z) = -\frac{\gamma G\sin\gamma z}{1 + G}\sin[\ ] + \frac{F^{1/2}}{(1 + G)(1 + G\cos\gamma z)^2}\cos[\ ], \tag{40}$$

where the empty brackets correspond to the bracketed quantities in Eqs. (37) and (38) or the equivalent quantities in Eqs. (35) and (36). Using Eqs. (37)–(40) the propagation matrix elements can now be obtained directly from Eqs. (21) to (24). With these elements the propagation characteristics of arbitrary off-axis rays and polynomial Gaussian beams are completely characterized by Eqs. (20), (25), (28), and (29). While these procedures may seem a bit cumbersome, the results are exact and explicit, and they are much faster to evaluate than the numerical solutions which would otherwise be required.

In concluding this section it may be well to emphasize that the solvable Hill equation with pseudosinusoidal coefficients is in many respects very similar to the relatively unsolvable Mathieu equation. Thus, for small values of G Eq. (33) reduces to

$$\frac{d^2r}{dz^2} + [F + (\gamma^2 - 4F)G \cos\gamma z]r = 0. \tag{41}$$

If γ is set equal to 2, this is

$$\frac{d^2r}{dz^2} + [F + 4(1 - F)G \cos 2z]r = 0, \tag{42}$$

which is of the same form as the Mathieu equation given above as Eq. (16).

V. Composite Periodic Profile Variations

In the previous sections it has been assumed that the gain and index profiles of the periodic waveguiding medium vary continuously along the waveguide axis. While this kind of behavior would be expected for most practical periodic media, it is important for completeness to also consider the possibility of periodic media in which the profiles alternate abruptly between analytically characterizable segments as suggested in Fig. 1(b). In this case it is also possible to obtain analytic solutions for the parameters of a propagating light ray or polynomial Gaussian beam at any position along the waveguide.

As the simplest possible illustration of these comments, one may consider a waveguide which includes segments of length d_a and quadratic propagation coefficient k_{2a} alternating with segments of length d_b and coefficient k_{2b}. The transformation matrix for one complete period of this waveguide is

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{bmatrix} \cos(k_{2b}/k_0)^{1/2}d_b & (k_0/k_{2b})^{1/2}\sin(k_{2b}/k_0)^{1/2}d_b \\ -(k_{2b}/k_0)^{1/2}\sin(k_{2b}/k_0)^{1/2}d_b & \cos(k_{2b}/k_0)^{1/2}d_b \end{bmatrix} \times \begin{bmatrix} \cos(k_{2a}/k_0)^{1/2}d_a & (k_0/k_{2a})^{1/2}\sin(k_{2a}/k_0)^{1/2}d_a \\ -(k_{2a}/k_0)^{1/2}\sin(k_{2a}/k_0)^{1/2}d_a & \cos(k_{2a}/k_0)^{1/2}d_a \end{bmatrix}, \tag{43}$$

If the fundamental propagation constant k_0 also changed between segments, it would be necessary to incorporate in this result the transformation matrices for the dielectric boundaries.

Once the transformation matrix for one stage of the waveguide is known, it becomes straightforward to obtain the transformation for an arbitrary number of stages by means of the theorem[20]

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^s = \frac{1}{\sin\vartheta}\begin{Bmatrix} A\sin(s\vartheta) - \sin[(s-1)\vartheta] & B\sin(s\vartheta) \\ C\sin(s\vartheta) & D\sin(s\vartheta) - \sin[(s-1)\vartheta] \end{Bmatrix}, \tag{44}$$

where $\vartheta = \cos^{-1}[(A + D)/2]$. From the known ray or beam parameters at the boundaries between stages it is straightforward to obtain these parameter values at any intermediate points along the waveguide.

In the simplest possible illustration given above it was assumed that the waveguide consisted of alternating segments of uniform guiding media. It is apparent though that more complex waveguides can be modeled using the same techniques. For the example sketched in Fig. 1(b), one stage of the waveguide may be represented by a product of four matrices, and the matrices for several types of tapered medium have been reported.[2]

VI. Approximate Solutions

As noted above, ray and beam propagation in continuous media is governed by an equation of the form of Eq. (15). It is just for equations of this form that the WKB method provides approximate solutions, provided $f(x)$ does not change too rapidly and does not pass through zero. More specifically, it is essential that the function $f(x)$ must change negligibly in one wavelength of the oscillatory solution. If this condition is satisfied, the approximate solution of Eq. (15) can be written[2]

$$y(x) \approx [f(x)]^{-1/4}\left\{a \cos \int_0^x [f(x')]^{1/2}dx' + b \sin \int_0^x [f(x')]^{1/2}dx'\right\}, \tag{45}$$

where as usual the coefficients a and b are determined by initial conditions.

Using Eq. (45), it can be shown that the matrix form of the solutions to Eq. (14) is[2]

$$A = \left[\frac{k_2(z)}{k_2(z_1)}\right]^{-1/4} \cos \int_{z_1}^{z} \left[\frac{k_2(z')}{k_0}\right]^{1/2} dz', \tag{46}$$

$$B = \left[\frac{k_2(z)k_2(z_1)}{k_0^2}\right]^{-1/4} \sin \int_{z_1}^{z} \left[\frac{k_2(z')}{k_0}\right]^{1/2} dz', \tag{47}$$

$$C = -\left[\frac{k_2(z)k_2(z_1)}{k_0^2}\right]^{1/4} \sin \int_{z_1}^{z} \left[\frac{k_2(z')}{k_0}\right]^{1/2} dz', \tag{48}$$

$$D = \left[\frac{k_2(z)}{k_2(z_1)}\right]^{1/4} \cos \int_{z_1}^{z} \left[\frac{k_2(z')}{k_0}\right]^{1/2} dz'. \quad (49)$$

Equations (46)–(49) are applicable to any slowly varying quadratic-index fiber guide. For example, with z-dependent media governed by the Mathieu equation as represented in Eqs. (14) and (17), the matrix element A becomes

$$A = \left(\frac{p - 2q \cos 2z}{p - 2q \cos 2z_1}\right)^{-1/4} \cos \int_{z_1}^{z} (p - 2q \cos 2z')^{1/2} dz', \quad (50)$$

and similar results apply to the other elements. The integrals in these matrix elements can be evaluated by means of widely tabulated elliptic integrals.[21]

VII. Summary

Longitudinal variations of the refraction or loss profiles are often introduced intentionally in the manufacture of lenslike media, and sometimes they occur accidentally during the fabrication process. In this paper we have investigated several techniques for analyzing the propagation of polynomial Gaussian beams in periodic lenslike media. Among these techniques are (1) direct numerical solution of the beam equations, (2) identification with the widely studied Mathieu equation for media with sinusoidal perturbations, (3) identification with a new analytically solvable Hill equation, (4) decomposition of the periodic profile into discrete solvable segments, and (5) approximate solution using a WKB method. All these methods lead to ray or beam matrices of standard form, so that sections of periodic media, once analyzed, may be easily included as elements in more complex optical systems. Except for the WKB method all the techniques listed above yield exact solutions of the paraxial ray or beam equations. We have also included as appendices brief discussions on analytically eliminating any z dependence of one of the propagation constants, on the unimodularity of the resulting matrices, and on solution characteristics of the solvable Hill equation.

Appendix A: Equivalent Media

As noted in the text, it is most straightforward to analyze quadratic-index waveguides in which the k_0 term in the complex propagation constant is independent of distance z. However, this is not a fundamental restriction, because any lenslike medium having a z dependent k_0 can be transformed into another lenslike medium in which k_0 is a constant.[22] The most fundamental equation governing beam propagation in lenslike media is Eq. (14), which we write in more detailed form as

$$\frac{d}{dz}\left[k_0(z)\frac{dr(z)}{dz}\right] + k_2(z)r(z) = 0. \quad (A1)$$

It is postulated here that the z dependence of the complex variable r can be identical to the z' dependence of another variable r' which is a solution of the equation

$$\frac{d}{dz'}\left[k_0'(z')\frac{dr'(z')}{dz'}\right] + k_2'(z')r'(z') = 0, \quad (A2)$$

provided that the new coefficients $k_0'(z')$ and $k_2'(z')$ together with the new independent variable z' are suitably defined. In particular, it is always possible to transform the lenslike medium described by Eq. (A1) to another medium in which k_0' is independent of z'.

The transformation just described consists of the relationships

$$k_0'(z') = k_0[z(z')]\frac{dz'}{dz(z')}, \quad (A3)$$

$$k_2'(z') = k_2[z(z')]\frac{dz(z')}{dz'}, \quad (A4)$$

$$r'(z') = r[z(z')], \quad (A5)$$

where the function $z(z')$ is a completely arbitrary continuous monotonic function. The validity of this transformation can be confirmed by direct substitution into Eqs. (A1) and (A2) and use of the chain rule

$$\frac{d}{dz} = \frac{dz'}{dz}\frac{d}{dz'}. \quad (A6)$$

One thus concludes that there is an infinite set of lenslike media that are equivalent in the sense that the dependent variables transform in the same way between two reference planes.

Of special interest is the transformation which takes a z dependent $k_0(z)$ and yields the arbitrarily specified constant $k_0'(z') = k_0'$. From Eq. (A3) this constraint yields

$$\frac{dz'}{dz} = \frac{k_0'}{k_0(z)}. \quad (A7)$$

The integral of this equation is the length transformation

$$z' = k_0' \int_0^z \frac{dz}{k_0(z)}. \quad (A8)$$

With Eq. (A4) the new quadratic term in the propagation constant is

$$k_2'(z') = k_2(z)k_0(z)/k_0'. \quad (A9)$$

Using these substitutions, Eq. (A2) can be written in the simpler form

$$\frac{d^2r'(z')}{dz'^2} + \frac{k_2'(z')}{k_0'}r'(z') = 0. \quad (A10)$$

This is the equation form included in the text as Eq. (14) with the restriction that k_0 be independent of z, and the same equation was also a basis of our recent studies of tapered waveguides. In this Appendix it has been shown that this form is actually general, since any z dependence of k_0 can always be transformed away.

Appendix B: Unimodularity of the Beam Matrices

It is well known that many of the ray matrices and beam matrices encountered in practical optical problems are unimodular. That is, they obey the relationship

$$\det M = \begin{vmatrix} A & B \\ C & D \end{vmatrix} = AD - BC = 1. \quad (B1)$$

This result is never essential for analyzing an optical system, but it may lead to simplifications of some of the resulting formulas. In the present study, for example, use of Eq. (B1) led to the simple phase transformation formula given in Eq. (29). Because of such reductions it is always helpful to know whether the matrices under study are unimodular. Thus it is reasonable to examine the conditions in which the matrices governing ray and beam propagation in z-dependent graded-index media are unimodular.

As noted in the text, it follows from Eqs. (21) to (24) that the matrix elements for an arbitrary z-dependent medium always obey the relationships

$$\frac{dA}{dz} = C, \tag{B2}$$

$$\frac{dB}{dz} = D. \tag{B3}$$

But when Eq. (14) is applied to Eq. (20) one also obtains

$$\frac{dC}{dz} = -\frac{k_2(z)}{k_0} A, \tag{B4}$$

$$\frac{dD}{dz} = -\frac{k_2(z)}{k_0} B. \tag{B5}$$

Using Eqs. (B2)–(B5), one finds

$$\begin{aligned} \frac{d}{dz}(AD - BC) &= D\frac{dA}{dz} + A\frac{dD}{dz} - B\frac{dC}{dz} - C\frac{dB}{dz} \\ &= DC - \frac{k_2(z)}{k_0}AB + \frac{k_2(z)}{k_0}AB - CD \\ &= 0. \end{aligned} \tag{B6}$$

Therefore, the determinant of the beam matrix is constant, even in a z-dependent medium. But from Eqs. (21) to (24) the initial value of the matrix elements at $z = z_1$ is A = 1, B = 0, C = 0, and D = 1. Since the determinant is constant and has an initial value of unity, it follows that the matrices for z-dependent graded-index media governed by Eq. (14) are always unimodular.

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Appendix C: Pseudosinusoidal Simplifications

The analysis in Sec. IV has been based on a certain solvable Hill equation for which the solutions can be obtained analytically. Although the general solution of that equation has been given analytically in Eq. (34), it may be worthwhile to consider explicitly some of the nonobvious special cases and extensions of that result for particular values of the parameters F, G, and γ. First of all, it may be noted that, if G is a real number greater than unity, Eq. (34) can be more conveniently written:

$$\begin{aligned} y(z) = a\left(\frac{1+G\cos\gamma z}{1+G}\right)\cos\Bigg(\frac{F^{1/2}}{\gamma(1-G^2)}\Bigg\{\frac{G\sin\gamma z}{1+G\cos\gamma z} - \frac{2}{(G^2-1)^{1/2}} \\ \times \tanh^{-1}\left[\frac{(G^2-1)^{1/2}}{1+G}\tan\left(\frac{\gamma z}{2}\right)\right]\Bigg\}\Bigg) + b\left(\frac{1+G\cos\gamma z}{1+G}\right) \\ \times \sin\Bigg(\frac{F^{1/2}}{\gamma(1-G^2)}\Bigg\{\frac{G\sin\gamma z}{1+G\cos\gamma z} - \frac{2}{(G^2-1)^{1/2}} \\ \times \tanh^{-1}\left[\frac{(G^2-1)^{1/2}}{1+G}\tan\left(\frac{\gamma z}{2}\right)\right]\Bigg\}\Bigg). \end{aligned} \tag{C1}$$

If G is equal to unity, Eq. (34) reduces to the form

$$\begin{aligned} y(z) = a\left(\frac{1+\cos\gamma z}{2}\right)\cos\left\{\frac{F^{1/2}}{2\gamma}\left[\tan\left(\frac{\gamma z}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\gamma z}{2}\right)\right]\right\} \\ + b\left(\frac{1+\cos\gamma z}{2}\right)\sin\left\{\frac{F^{1/2}}{2\gamma}\left[\tan\left(\frac{\gamma z}{2}\right) + \frac{1}{3}\tan^3\left(\frac{\gamma z}{2}\right)\right]\right\}. \end{aligned} \tag{C2}$$

Similar modifications are helpful for special values of the parameter F. For example, if F is a negative real number, Eq. (34) can be replaced by

$$\begin{aligned} y(z) = a\left(\frac{1+G\cos\gamma z}{1+G}\right) \\ \times \cosh\Bigg(\frac{(-F)^{1/2}}{\gamma(1-G^2)}\Bigg\{\frac{G\sin\gamma z}{1+G\cos\gamma z} - \frac{2}{(1-G^2)^{1/2}} \\ \times \tan^{-1}\left[\frac{(1+G^2)^{1/2}}{1+G}\tan\left(\frac{\gamma z}{2}\right)\right]\Bigg\}\Bigg) + b\left(\frac{1+G\cos\gamma z}{1+G}\right) \\ \times \sinh\Bigg(\frac{(-F)^{1/2}}{\gamma(1-G^2)}\Bigg\{\frac{G\sin\gamma z}{1+G\cos\gamma z} - \frac{2}{(1-G^2)^{1/2}} \\ \times \tan^{-1}\left[\frac{(1-G^2)^{1/2}}{1+G}\tan\left(\frac{\gamma z}{2}\right)\right]\Bigg\}\Bigg). \end{aligned} \tag{C3}$$

If F is set equal to zero the general solution becomes

$$\begin{aligned} y(z) = a\left(\frac{1+G\cos\gamma z}{1+G}\right) \\ + b\left(\frac{1+G\cos\gamma z}{1+G}\right)\left[\frac{1}{\gamma(1-G^2)}\right]\Bigg\{\frac{G\sin\gamma z}{1+G\cos\gamma z} - \frac{2}{(1-G^2)^{1/2}} \\ \times \tan^{-1}\left[\frac{(1-G^2)^{1/2}}{1+G}\tan\left(\frac{\gamma z}{2}\right)\right]\Bigg\}, \end{aligned} \tag{C4}$$

where the constant b has been replaced by $bF^{-1/2}$.

In the limit that γ goes to zero Eq. (34) reduces to

$$y(z) = a\cos\left[\frac{F^{1/2}z}{(1+G)^2}\right] + b\sin\left[\frac{F^{1/2}z}{(1+G)^2}\right], \tag{C5}$$

where b has been replaced by minus b. In this limit there is no reason not to also set G to zero since it adds no generality to Eq. (33). Thus Eq. (C5) simplifies further to

$$y(z) = a\cos(F^{1/2}z) + b\sin(F^{1/2}z), \tag{C6}$$

and this is the standard result for a ray or beam propagating in a z-independent lenslike medium.

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Long-gain-length solar-pumped box laser

A new laser cavity configuration efficiently couples solar radiation to the laser mode volume. C_3F_7I gas is optically pumped by two xenon-arc solar simulators, creating a population inversion and subsequent lasing from atomic iodine at 1.3 μm. The laser cavity is a stainless-steel box with internal high-reflectivity mirrors that guide the laser mode volume through the optically excited C_3F_7I gas. Lasing output powers of ~300 mW have been achieved for durations of 150 msec.

Previous solar-pumped gas laser systems have been limited to laser gain lengths of less than 10 cm, requiring very high solar concentrations to achieve lasing. This new system allows lasing at substantially lower solar simulator intensities (150 suns) and much longer laser gain lengths (60 cm).

The laser was constructed using O-ring grooves to allow a vacuum seal with the quartz glass plates on each side of the stainless-steel laser frame. Brewster windows in the upper left-hand corner allow external laser cavity mirrors to be mounted and easily aligned. The back cavity mirror has maximum reflectivity at 1.3 μm, and the output mirror has a reflectivity of either 97 or 85%. In each of the three internal corners of the laser cavity high-reflectivity dielectric mirrors are placed. These mirrors allow the laser optical path (mode volume) to be aligned with the incoming solar radiation pattern. The laser beam is detected by a germanium linear-array detector, which is used to resolve the laser beam profile. Two xenon solar simulators produce a maximum of 5 kW each of light. A mechanical shutter is used to allow rapid excitation of the laser cavity. The input simulator light is concentrated in the form of a doughnut.

The system allows long gain lengths at much lower solar concentrations, substantially increasing the practicality of solar pumping. The system was originally developed for studies of power transmission over long distances through space.

This work was done by Russell J. De Young of Langley Research Center. No further documentation is available. This invention is owned by NASA, and a patent application has been filed. Inquiries concerning license for its commercial development should be addressed to the Patent Counsel, Langley Research Center, Mail Code 279, Hampton, Va. 23665. Refer to LAR-13256.

Optical scanner for linear array

An optical scanner instantaneously reads contiguous lines forming a scene or target in the object plane. The reading may be active or passive and the scans continuous or discrete. The scans are essentially linear with scan angle and are symmetric about the axial ray. A nominal focal error, resulting from a curvature of the scan, is well within the Rayleigh limit. The scanner was specifically designed to be fully compatible with the general requirements of linear arays.

The essential elements of the optical system are shown in Fig. 7. The corner mirrors M_1, M_2, M_3, and M_4 are perpendicularly oriented so that any ray incident on the front surface of the scan mirror will be directed by the corner mirrors along a parallelogrammic path arriving at the back surface of the scan mirror where it will be reflected in a direction parallel to the incident ray. Except for the imaging lens, all the optical elements are plane mirrors. The rotation of the scan mirror is the only motion required.

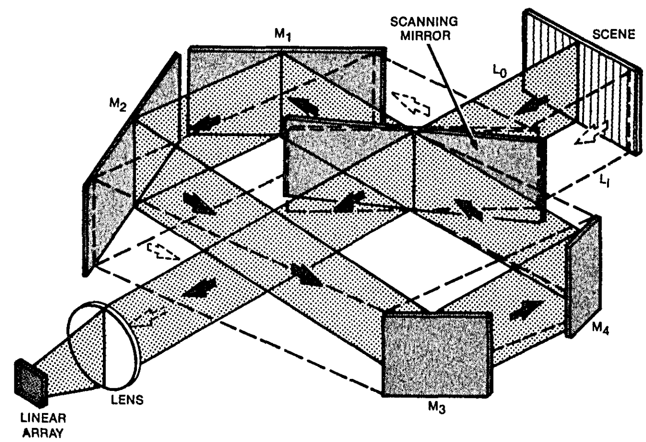

Fig. 7. Optical system and path followed by a fan of axial rays, L_o, and some arbitrary fan of rays, L_i, are shown.

Since the angle of incidence on the front and back surfaces of the scan mirror is equal, it follows that the incident and emergent rays must be parallel. Moreover, since each ray must pass through the center of the scan, there is a lateral shifting of frames so that each ray leaves the system collinear with the axial ray. Effectively then, the front surface of the scan mirror scans the object plane frame by frame while the back surface of the scan mirror descans each frame onto the image plane. All this indicates that each scan line could be identified with a particular scan angle.

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