10-1-1990

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Citation Details

Cresswell, A. A., & Zimmerman, R. L. (1990). Gravitational synchrotron radiation from cosmic strings. Physical Review D (Particles And Fields), 42(8), 2527-2532.

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(Received 6 June 1990)

This work studies the gravitational synchrotron radiation emitted from arbitrary cusps of cosmic strings. The results are expressed in terms of four parameters describing the motion of such a cusp. The power spectrum is derived for cusps moving at unit velocity. By using a phenomenological approach we also derive the power emitted when the radiation reaction on the cusps is taken into account. In both cases, the synchrotron nature of the radiation produces a power spectrum emitted in a narrow forward cone. If cosmic strings do exist, the radiation emitted by their cusps would seem to be a potential candidate for gravitational-wave detectors.

I. INTRODUCTION

Most grand unified theories predict the formation of cosmic strings[1] during phase transitions in the early universe. These cosmic strings are one-dimensional objects of the false vacuum of the more symmetric grand unified phase, in an otherwise homogeneous spacetime. Networks of strings intercommute, thus forming a distribution of loops of all sizes. These oscillating loops will lose their energy predominantly by gravitational radiation. A large class of string loops develop cusps during their cycle. The purpose of our work is to compute the beamed gravitational synchrotron radiation emitted by arbitrary cusps.

The strongest argument thus far in support of cosmic strings was that loops of string could provide a seed for galaxy formation and explain the observed galaxy correlation function.[2-4] Recent numerical work,[5] however, questions the plasusibilty of this scenario but does not rule out the importance of cosmic strings in galaxy formation. Whether or not cosmic strings prove to be the trigger for galaxy formation, their existence has great cosmological importance. The present work shows that cosmic strings could as well be reasonable candidates for gravitational-radiation detection.

Several authors have calculated the average gravitational radiation emitted by oscillating loops featuring cusps.[6,7] (Similar calculations also have been performed for loops with kinks;[8] we are not concerned with those here.) These calculations considered only a few simple classes of periodic string solutions. Such solutions are characterized by a single time scale that is of the order of the size of the loop. Therefore, the average radiation frequency is proportional to the oscillation frequency and thus lies, in general, outside the range of gravitational-wave detectors. On the other hand, more generic loops may have many time scales in which case the frequency of occurrence of the cusps need not be related to the frequency of oscillation of the loop. In fact most of the high-frequency power emitted by a loop will be radiated by the cusps. Moreover the synchrotron nature of this radiation will beam the power in a narrow forward cone. These considerations make arbitrary cusps good candidates for observation by gravitational-wave detectors.

In Sec. II, we calculate the spectral distribution of the synchrotron gravitational radiation coming from the most general cusp. As in the electromagnetic case, the result can be expressed in terms of parameters describing the instantaneous circular motion. In Sec. III we use the general expressions obtained in Sec. II to study the case of a cusp moving at the speed of light. In that section we also study the influence of radiation reaction on the result by phenomenologically constraining the cusp to move at a finite Lorentz factor. In both cases we derive the total power emitted per unit solid angle and compute the cutoff frequencies and the half-width of the beam.

II. SPECTRAL ENERGY RADIATED BY A CUSP

In this section we develop a formalism that allows us to compute the energy radiated by an arbitrary cusp. This formalism parallels the procedure followed in electromagnetism for the treatment of synchrotron radiation emitted by accelerated charges. We obtain the spectrum of radiation in terms of a few arbitrary parameters characterizing the observer's position and the motion of the cusp.

Letting $x^{\mu}=x^{\mu}(\sigma,\tau)$ represent the string's configuration where σ and τ parameters on the string, the string's dynamics are derived[1] from the constraints

$$\eta_{\mu\nu}\dot{x}^{\mu}x'^{\nu}=0\ ,\quad \eta_{\mu\nu}(\dot{x}^{\mu}\dot{x}^{\nu}+x'^{\mu}x'^{\nu})=0\ , \tag{1}$$

and the wave equations

$$\ddot{x}^{\mu}-x''^{\mu}=0\ , \tag{2}$$

which are obtained from varying the string's action functional which was first proposed by Nambu in the context of hadronic physics. In Eqs. (1) and (2), $\eta_{\mu\nu}$ represents the space-time metric $(-1,1,1,1)$. The dot and the prime indicate derivatives with respect to the timelike and spacelike parameters τ and σ of the string's world sheet.

The parameter τ can be chosen as the time coordinate x^0, in which case Eqs. (1) and (2) take the form

$$\dot{\mathbf{x}}\cdot\mathbf{x}'=0\ , \quad \mathbf{x}^2+\mathbf{x}'^2=1\ , \tag{3}$$

where boldface indicates the space part of the string's position four-vector.

A general solution to Eq. (3) is written as

$$\vec{x}(\sigma,t)=\tfrac{1}{2}[\mathbf{a}(\zeta)+\mathbf{b}(\eta)]\ , \tag{4}$$

where we have defined the null parameters

$$\zeta=\sigma-t \quad \text{and} \quad \eta=\sigma+t\ . \tag{5}$$

The conditions (3) thus read

$$a'^2=b'^2=1\ , \tag{6}$$

where the prime now indicates a derivative with respect to the argument of the function. Thus the three-vectors $\mathbf{a}'$ and $\mathbf{b}'$ can be thought of as describing trajectories on a unit sphere, the so-called Kibble-Turok sphere. Whenever the two trajectories intersect, a cusp, moving at unit velocity, develops at that instant.

Equations (6) do not include the effect of the gravitational-radiation reaction back on the string. This effect could be significant at the location of the cusp but is very difficult to compute. For our purpose here, we introduce radiation damping phenomenologically by constraining the cusp to move at a finite Lorentz factor.[4,9] Taking v (where $v<1$) to be the speed of the cusp, we thus impose on the cusp the constraints

$$a'^2=b'^2=v^2\ , \tag{7}$$

which are to replace Eq. (6). Since our attention is focused on the cusp, this constraint is only needed in the neighborhood of the cusp and thus does not impose any unphysical requirement to the whole extent of the loop. Equations (7) reduce the size of the Kibble-Turok sphere around the cusp and result in a smoothing of the cusp.

We assume that the cusp develops at the origin of coordinates and moves with a positive velocity v along the z axis. Thus

$$\begin{aligned}&\mathbf{a}(0)=\mathbf{b}(0)=0\ ,\\&\mathbf{a}'(0)=-\mathbf{b}'(0)=-v\mathbf{e}_z\ .\end{aligned} \tag{8}$$

The most general solutions $\mathbf{a}'$ and $\mathbf{b}'$ to Eqs. (7), can be written as

$$\begin{aligned}\mathbf{a}'(\zeta)&=-v\begin{bmatrix}\cos f_a(\zeta)\sin g_a(\zeta)\\ \sin f_a(\zeta)\sin g_a(\zeta)\\ \cos g_a(\xi)\end{bmatrix}\ ,\\ \mathbf{b}'(\eta)&=v\begin{bmatrix}\cos f_\beta(\eta)\sin g_b(\eta)\\ \sin f_b(\eta)\sin g_b(\eta)\\ \cos g_b(\eta)\end{bmatrix}\ ,\end{aligned} \tag{9}$$

where f_a, g_a, f_b, and g_b are smooth but otherwise arbitrary functions (See Fig. 1). The functions $\mathbf{a}$ and $\mathbf{b}$ an be expanded around the cusp ($\zeta=\eta=0$). We have

$$\begin{aligned}&\mathbf{a}(\zeta)=-\zeta v\mathbf{e}_z+\tfrac{1}{2}\zeta^2\mathbf{a}''(0)+\tfrac{1}{6}\zeta^3\mathbf{a}'''(0)+\cdots\ ,\\&\mathbf{b}(\eta)=\eta v\mathbf{e}_z+\tfrac{1}{2}\eta^2\mathbf{b}''(0)+\tfrac{1}{6}\eta^3\mathbf{b}'''(0)+\cdots\ ,\end{aligned} \tag{10}$$

where the derivatives of a and b, evaluated at the cusp, are given by the equations

$$\begin{aligned}\mathbf{a}''(0)&=-v\omega_a\begin{bmatrix}\cos f_a\\ \sin f_a\\ 0\end{bmatrix}\ ,\\ \mathbf{a}'''(0)&=-v\begin{bmatrix}\Omega_a\\ g_a''\sin f_a+2\omega_a f_a'\cos f_a\\ -\omega_a^2\end{bmatrix}\end{aligned} \tag{11}$$

and

$$\begin{aligned}\mathbf{b}''(0)&=v\omega_b\begin{bmatrix}\cos f_b\\ \sin f_b\\ 0\end{bmatrix}\ ,\\ \mathbf{b}'''(0)&=v\begin{bmatrix}\Omega_b\\ g_b''\sin f_b+2\omega_b f_b'\cos f_b\\ -\omega_b^2\end{bmatrix}\ .\end{aligned} \tag{12}$$

We have defined the constants

$$\begin{aligned}&\Omega_a=g_a''\cos f_a-2\omega_a f_a'\sin f_a,\quad \omega_a=g_a'(0),\quad g_a''=g_a''(0)\equiv 0,\\&\Omega_b=g_b''\cos f_b-2\omega_b f_b'\sin f_b,\quad \omega_b=g_b'(0),\quad g_b''=g_b''(0)\equiv 0,\\&g_a=g_a(0)\equiv 0,\quad f_a=f_a(0),\quad \phi_a'=f_a'(0)\ ,\\&g_b=g_b(0)\equiv 0,\quad f_b=f_b(0),\quad f_b'=f_b'(0)\ .\end{aligned} \tag{13}$$

As in the electromagnetic case, the emitted gravitational synchrotron radiation will be determined completely by the parameters describing the instantaneous circular motion of the emitter, namely, the radius of curvature, the velocity of that motion and the position of the observer with respect to that motion. For arbitrary cusps, the radius of curvature of the motion will be given by the square of the velocity divided by the acceleration at $t=0$: namely,

$$R=2v^2/|\mathbf{a}''(0)+\mathbf{b}''(0)|\ . \tag{14}$$

Written in terms of the angular frequencies of $\mathbf{a}$ and $\mathbf{b}$, given by ω_a and ω_b in Eq. (13), the expression for the radius of curvature takes the form

$$R=\frac{2v}{[\omega_a^2+\omega_b^2-2\omega_a\omega_b\cos(f_a-f_b)]^{1/2}}\ , \tag{15}$$

where f_a-f_b is the angle between the planes tangent to the motions of $\mathbf{a}$ and $\mathbf{b}$, respectively (See Fig. 1). We notice that when $\omega_a=\omega_b$ and $f_a=f_b$, the motion of the cusp is linear along the z axis.

The power per solid angle, emitted by the cusp in the direction $\mathbf{N}=(\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)$ of an observer, is given in the local inertial frame by

$$\frac{dP}{d\Omega}=r^2N^i\langle t^{i0}\rangle\ , \tag{16}$$

where $\langle t^{\mu\nu}\rangle$ is the average, over several wavelengths, of the stress-energy tensor of the gravitational waves. We choose a common form[10] of $t_{\mu\nu}$ given by

$$t_{\mu\nu}=\frac{1}{8\pi G}(R^{(2)}_{\mu\nu}-\tfrac{1}{2}\eta_{\mu\nu}\eta^{\lambda\rho}R^{(2)}_{\lambda\rho}) , \tag{17}$$

where $R^{(2)}_{\mu\nu}$ is the second-order part of the Ricci tensor. Following a development that parallels Ref. 9 one obtains the energy radiated over all times in a solid angle $d\Omega$ by performing a time integration of Eq. (16). The result is

$$\frac{dE}{d\Omega}=\frac{G}{2\pi^2}\int_0^{+\infty}\omega^2 d\omega[T^{\mu\nu}(\omega,\mathbf{k})T^*_{\mu\nu}(\omega,\mathbf{k}) -\tfrac{1}{2}T^2(\omega,\mathbf{k})] , \tag{18}$$

where

$$\mathbf{k}=\omega\mathbf{N} . \tag{19}$$

Clearly, the result is valid only because gravity is weak everywhere.

The Fourier transform of the stress-energy tensor of the string is given by

$$T_{\mu\nu}(k)=\int d^4x\, T_{\mu\nu}(x)e^{-ik_\mu x^\mu} \tag{20}$$

For a string loop of length L and mass per unit length μ the stress energy is given by

$$T^{\mu\nu}(t,x)=\mu\int_0^L d\sigma\, \delta^{(3)}(\mathbf{x}-\mathbf{f})\{\dot f^\mu \dot f^\nu - f'^\mu f'^\nu\} , \tag{21}$$

where $f^\mu=x^\mu(\sigma,t)$ represents the string configuration. Performing the change to the null string parameters of Eq. (5) and then computing the Fourier transform of Eq. (21) we obtain

$$T^{\mu\nu}(k)=\frac{\mu}{2}\int_{-\infty}^{+\infty}d\zeta\, e^{-i\omega(\zeta+\mathbf{N}\cdot\mathbf{a})/2} \times\int_{-\infty}^{+\infty}d\eta\, e^{i\omega(\eta-\mathbf{N}\cdot\mathbf{b})/2}f^{\mu\nu}(\zeta,\eta) , \tag{22}$$

where $F^{\mu\nu}$ is given by

$$F^{00}=1,\quad \{F^{0i}\}\equiv\mathbf{F}=\tfrac{1}{2}(\mathbf{b}'-\mathbf{a}') ,\quad \{F^{ij}\}\equiv F=-\tfrac{1}{2}\{\mathbf{a}'\mathbf{x}\mathbf{b}'+\mathbf{b}'\mathbf{x}\mathbf{a}'\} . \tag{23}$$

Since our purpose is to focus on the radiation emitted at the cusp we expand the arguments of the exponentials around the origin to obtain

$$T^{\mu\nu}(k)=\frac{\mu}{2}\int_{-\infty}^{+\infty}d\zeta\, e^{i[a_1\zeta+a_2\zeta^2+a_2\zeta^3]} \times\int_{-\infty}^{+\infty}d\eta\, e^{i[b_1\eta+b_2\eta^2+b_3\eta^3]}F^{\mu\nu}_{\rm lin} , \tag{24}$$

where the coefficients a_i and b_i are given in the Appendix. Terms of fourth and higher orders have been neglected in the exponentials. $F_{\rm lin}$ is the first-order expansion of F around the origin and is given by

$$\begin{gathered}
F^{00}_{\rm lin}=1,\quad \mathbf{F}_{\rm lin}=v\begin{pmatrix}0\\0\\1\end{pmatrix}+\tfrac{1}{2}v\zeta\omega_a\begin{pmatrix}\cos f_a\\ \sin f_a\\ 0\end{pmatrix}+\tfrac{1}{2}v\eta\omega_b\begin{pmatrix}\cos f_b\\ \sin f_b\\ 0\end{pmatrix} ,\\
F^{xx}_{\rm lin}=v^2\omega_a\omega_b\cos f_a\cos f_b\eta\zeta,\quad F^{xy}_{\rm lin}=\tfrac{1}{2}v^2\omega_a\omega_b\sin(f_a+f_b)\eta\zeta ,\\
F^{yy}_{\rm lin}=v^2\omega_a\omega_b\sin f_a\sin f_b\eta\zeta,\quad F^{xz}_{\rm lin}=\tfrac{1}{2}v^2\{\omega_b\cos f_b\eta+\omega_a\cos f_a\zeta\} ,\\
F^{zz}_{\rm lin}=v^2,\quad F^{yz}_{\rm lin}=\tfrac{1}{2}v^2\{\omega_b\sin f_b\eta+\omega_a\sin f_a\xi\} .
\end{gathered} \tag{25}$$

In order to express Eqs. (24) in terms of modified Bessel functions we perform the following change of integration variables:

$$\zeta'=\zeta+b_2/3b_3,\quad \eta'=\eta+a_2/3a_3 . \tag{26}$$

This change of variables is consistent with the ζ and η expansions appearing in (24) if the new variables defined in (26) remain small. This condition is verified in the case we shall consider, namely, the forward direction, where N lies in a narrow cone along the z axis thus having $\theta\approx 0$. The majority of the radiation, as in the analog electromagnetic case, is emitted in that direction.

After we perform the integration of Eq. (21) we obtain

$$\begin{gathered}
T^{00}=\tfrac{1}{2}me^{i\Phi}I_aI_b,\quad T^{0z}=vT^{00},\quad T^{zz}=v^2T^{00} ,\\
T^{xx}=\tfrac{1}{2}me^{i\Phi}v^2\omega_a\omega_b\cos f_a\cos f_bH_aH_b ,\\
T^{yy}=\tfrac{1}{2}me^{i\Phi}v^2\omega_a\omega_b\sin f_a\sin f_bH_aH_b ,\\
T^{xy}=\tfrac{1}{4}me^{i\Phi}v^2\omega_a\omega_b\sin(f_a+f_b)H_aH_b ,\\
T^{0x}=\tfrac{1}{4}me^{i\Phi}v(\omega_b\cos f_bI_aH_b+\omega_a\cos f_aI_bH_a) ,\\
T^{0y}=\tfrac{1}{4}me^{i\Phi}v(\omega_b\sin f_bI_aH_b+\omega_a\sin f_aI_bH_a) .
\end{gathered} \tag{27}$$

Φ is a phase factor and we have defined the following integrals

$$
\begin{aligned}
I_a &= \int dx\, e^{i(A_1 x + A_2 x^3)} , \\
H_a &= \int dx\, x e^{i(A_1 x + A_2 x^3)} , \\
I_b &= \int dx\, e^{i(B_1 x + B_2 x^3)} , \\
H_b &= \int dx\, x e^{i(B_1 x + B_2 x^3)} ,
\end{aligned} \tag{28}
$$

where

$$
\begin{aligned}
A_1 &= -\tfrac{1}{2}\omega(1-v\cos\theta) - \tfrac{1}{4}\omega v \left[\frac{\omega_a^2 \cos^2\phi_a \sin^2\theta}{\Omega_a \sin\theta - \omega_a^2 \cos\theta} \right] , \\
B_1 &= \tfrac{1}{2}\omega(1-v\cos\theta) + \tfrac{1}{4}\omega v \left[\frac{\omega_b^2 \cos^2\phi_b \sin^2\theta}{\Omega_b \sin\theta - \omega_b^2 \cos\theta} \right] , \\
A_2 &= \frac{\omega v}{12}(\Omega_a \sin\theta - \omega_a^2 \cos\theta) , \\
B_2 &= -\frac{\omega v}{12}(\Omega_b \sin\theta - \omega_b^2 \cos\theta) .
\end{aligned} \tag{29}
$$

The ω's and Ω's are defined in Eq. (13). $\phi_a = f_a - \phi$ and $\phi_b = f_b - \phi$ are the angles that the observer makes with, respectively, the planes of motion of **a** and **b** (see Fig. 1). The integrals in Eqs. (28) are expressed in terms of modified Bessel functions:

$$
\begin{aligned}
I_a &= \frac{2}{3}\left[\frac{A_1}{A_2}\right]^{1/2} K_{1/3}(\xi_a) , \\
H_a &= \frac{2iA_1}{3\sqrt{3}A_2} K_{2/3}(\xi_a) , \\
I_b &= \frac{2}{3}\left[\frac{B_1}{B_2}\right]^{1/2} K_{1/3}(\xi_b) , \\
H_b &= \frac{2iB_1}{3\sqrt{3}B_2} K_{2/3}(\xi_b) ,
\end{aligned} \tag{30}
$$

where

$$
\xi_a = \frac{2A_1^{3/2}}{3\sqrt{3}A_2^{1/2}}, \quad \xi_b = \frac{2B_1^{3/2}}{3\sqrt{3}B_2^{1/2}} . \tag{31}
$$

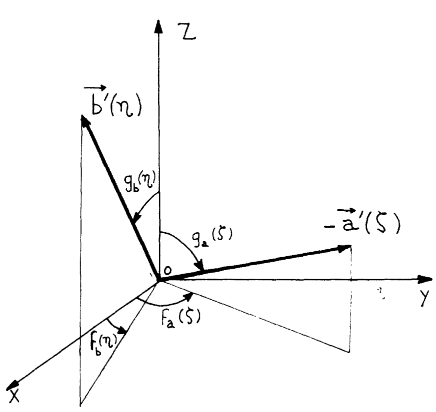

FIG. 1. String velocity $(\mathbf{b}',\mathbf{a}')$ in terms of g_a, g_b, F_a, and F_b.

The spectral energy radiated is given by the integrand of Eq. (18); namely,

$$
\frac{d^2E}{d\omega\, d\Omega} = \frac{G\omega^2}{2\pi^2}[T^{\mu\nu}(\omega,\mathbf{k})T^*_{\mu\nu}(\omega,\mathbf{k}) - \tfrac{1}{2}T^2(\omega,\mathbf{k})] . \tag{32}
$$

At high frequency, the dominant contribution to (32) comes from the product of H integrals that represent $K_{2/3}$ functions. Thus, when neglecting the I integrals contributions to Eq. (32), the expression for the spectral energy takes the simple form

$$
\frac{d^2E}{d\omega\, d\Omega} = \frac{G\omega^2\mu^2}{16\pi^2} v^4 \omega_a^2 \omega_b^2 H_a^2 H_b^2 . \tag{33}
$$

After writing out the expressions for H in Eq. (33), we obtain

$$
\frac{d^2E}{d\omega\, d\Omega} = \frac{G\omega^2\mu^2}{729\pi^2} v^4 \omega_a^2 \omega_b^2 \left[\frac{A_1 B_1}{A_2 B_2}\right]^2 \times K_{2/3}^2(\xi_a) K_{2/3}^2(\xi_b) , \tag{34}
$$

where the coefficients A_1, A_2, B_1, and B_2 are given in (29).

Expression (34) represents the spectral energy radiated by a cusp moving at velocity $v \le 1$. The answer depends on cusp parameters but not on the overall string configuration. Thus it allows us to study gravitational radiation from the most general cusp of string. These results are reminiscent of the electromagnetic case.[11]

Since we are interested in the forward radiation, we can expand the coefficients of Eqs. (29) for small θ. After setting $v=1$ whenever possible, we obtain

$$
\begin{aligned}
A_1 &= -\frac{\omega}{4}\left[\frac{1}{\gamma^2} + \theta^2 \sin^2\phi_a - \alpha \cos^2\phi_a \theta^3 + O(\theta^4)\right] , \\
B_1 &= \frac{\omega}{4}\left[\frac{1}{\gamma^2} + \theta^2 \sin^2\phi_b - \beta \cos^2\phi_b \theta^3 + O(\theta^4)\right] , \\
A_2 &= -\frac{\omega\omega_a^2}{12}[1 - \alpha\theta - \tfrac{1}{2}\theta^2 + O(\theta^3)] , \\
B_2 &= \frac{\omega\omega_b^2}{12}[1 - \beta\theta - \tfrac{1}{2}\theta^2 + O(\theta^3)] ,
\end{aligned} \tag{35}
$$

where

$$
\alpha \equiv \Omega_a/\omega_a^2 \quad \text{and} \quad \beta \equiv \Omega_b/\omega_b^2 .
$$

Thus Eq. (34) becomes

$$
\frac{d^2E}{d\omega\, d\Omega} = \frac{16G\mu^2}{9\pi^2} \frac{\omega^2}{\omega_a^2 \omega_b^2} g(\theta, v) K_{2/3}^2(\xi_a) K_{2/3}^2(\xi_b) , \tag{36}
$$

where

$$
g(\theta,v) = \{(1-v)^2[(1-v)^2 + O(\theta)] + \tfrac{1}{16}\theta^8 \sin^4\phi_a \sin^4\phi_b [1 + O(\theta)]\} \tag{37}
$$

and

$$
\xi_a = \omega/\omega_{ca}, \quad \xi_b = \omega/\omega_{cb} \tag{38}
$$

with

$$\omega_{ca}=6\omega_a\left\{\frac{1}{\gamma^2}\pm\frac{\alpha\theta}{3\gamma^2}+\theta^2\left[\sin^2\phi_a+\frac{1}{\gamma^2}\left[\frac{2\alpha^2}{9}+\frac{1}{6}\right]\right]+O(\theta^3)\right\}^{-3/2},$$

$$\omega_{cb}=6\omega_b\left\{\frac{1}{\gamma^2}\pm\frac{\beta\theta}{3\gamma^2}+\theta^2\left[\sin^2\phi_b+\frac{1}{\gamma^2}\left[\frac{2\beta^2}{9}+\frac{1}{6}\right]\right]+O(\theta^3)\right\}^{-3/2}. \quad (39)$$

In Eq. (37), $g(v,\theta)$ has been written in order to study the cases $v=1$ and $v\neq 1$ separately. γ represents the usual Lorentz factor $(1-v^2)^{1/2}$ and we assumed $v=1$ wherever possible. Since the modified Bessel functions drop exponentially when their argement becomes greater than one, ω_{ca} and ω_{cb} represent cutoff frequencies for the radiation.

When the observer lies in the plane of motion of **a** or **b** then ϕ_a or ϕ_b vanish and the cutoff frequencies take a different form: namely,

$$\omega_{ca}=\frac{6\omega_a}{\gamma^{-3}[1+O(\theta)]-\alpha^{3/2}\theta^{9/2}[1+O(\theta)]},$$

$$\omega_{cb}=\frac{6\omega_b}{\gamma^{-3}[1+O(\theta)]-\beta^{3/2}\theta^{9/2}[1+O(\theta)]}. \quad (40)$$

III. DISCUSSION

In this section we study the preceding results in the case where the radiation reaction effect on the string can be neglected ($v=1$) as well as the more realistic case where this effect is taken into account. We find that in all cases a strong beaming of the radiation in the forward direction can be expected.

A. Case $v=1$

Neglecting radiation reaction we find that Eq. (37) becomes

$$g(\theta,1)=\tfrac{1}{16}\theta^8\sin^4\phi_a\sin^4\phi_b[1+O(\theta)]. \quad (41)$$

The cutoff frequencies [Eqs. (39)] take the form

$$\omega_{ca}=\frac{6\omega_a}{\theta^3\sin^3\phi_a}\quad\text{and}\quad\omega_{cb}=\frac{6\omega_b}{\theta^3\sin^3\phi_b}. \quad (42)$$

In the limit of vanishing θ (i.e., when the observer is in line with the motion of the cusp), the cutoff frequencies are infinite and thus all frequencies participate to the spectrum of radiation. For cusps having ϕ_a or ϕ_b vanishing, the corresponding cutoff frequencies are derived from Eqs. (40).

We can write an approximate expression for the spectral energy radiated by using the asymptotic expressions for the modified Bessel functions in (36). The asymptotic form of $K_{2/3}(\xi)$ for $\xi \ll 1$ is given by

$$K_{2/3}(\xi)\sim\Gamma(\tfrac{2}{3})/(2^{1/3}\xi^{2/3}). \quad (43)$$

The energy then takes the form

$$\frac{d^2E}{d\omega\, d\Omega}=G'\frac{\mu^2}{(\omega_a\omega_b)^{2/3}}\frac{1}{\omega^{2/3}}[1+O(\theta)],$$

$$G'=(1/\pi^2)\Gamma(\tfrac{2}{3})^4 12^{2/3}G. \quad (44)$$

Next obtain the power in the beam by multiplying the above expression by v/R where we can assume that $v=1$; R is given by Eq. (15). The total power radiated, per solid angle, up to the cutoff frequency ω_c is derived by integrating the resulting expression. The result is

$$\frac{dP}{d\Omega}=G'\frac{3\mu^2}{R(\omega_a\omega_b)^{2/3}}\omega_c^{1/3}[1+O(\theta)]. \quad (45)$$

Thus in the limiting case of cusps moving at unit velocities the power radiated per unit solid angle becomes arbitrarily large as θ goes to zero since the cutoff frequencies [Eqs. (42)] in the forward direction are infinite. However this singularity is integrable since $\omega_c^{1/3}\propto\theta^{-1}$ and thus the total power is finite.

We can estimate the angular half-width of the beam at a given frequency by setting the arguments of the modified Bessel functions equal to unity. Namely, imposing $\xi_a=1$ and $\xi_b=1$ in Eqs. (38) and (42), we obtain

$$\theta_c=\operatorname{Inf}\left[\left[\frac{6\omega_a}{\omega}\right]^{1/3}\frac{1}{\sin\phi_a};\left[\frac{6\omega_b}{\omega}\right]^{1/3}\frac{1}{\sin\phi_b}\right], \quad (46)$$

and thus we find that the beaming increases with frequency as expected. In Eq. (45) it is assumed that $\sin\phi_a$ and $\sin\phi_b$ remain strictly positive. If either vanish, Eq. (40) is used instead in order to compute the half width of the beam and we obtain qualitatively the same result.

In the case of the simple periodic loops considered by Vachaspati and Vilenkin,[6] for instance, the cusp frequencies are given by $\omega_a=\omega_0$, and $\omega_b=\omega_0$ where ω_0 is the fundamental frequency of the loop, namely $\omega_0=4\pi/L$. For such loops, we recover their result and obtain for the total power radiated in the nth Fourier harmonic directly from Eq. (45):

$$\frac{dP}{d\Omega}=3G'\mu^2 n_c^{1/3}, \quad (47)$$

where n_c is the critical harmonic above which there is little radiation.

B. $v<1$ case

Assuming that the radiation reaction has a significant effect on the characteristics of the cusp as observed in the forward direction, we find from Eq. (37) that

$$g(\theta,v)=(1-v)^4[1+O(\theta)]. \quad (48)$$

Equation (48) reflects the assumption that the angle θ at which the cusp is observed remains small enough in order for the second term in the expression of $g(\theta,v)$, given by (37), to remain small. From the approximate expressions for the Bessel functions in expression (37) and the value of $g(\theta,v)$ given by (48), we express the spectral energy as

$$\frac{d^2E}{d\omega\, d\Omega} = G' \frac{\mu^2}{(\omega_a \omega_b)^{2/3}} \frac{1}{\omega^{2/3}} [1 + O(\theta)] \,. \tag{49}$$

Notice that the expression for the energy per unit frequency is, to lowest order, independent of the effects of the radiation reaction on the cusp [compare to Eq (44)]. Thus the expression for the power in the beam given by Eq. (45) remains valid for the $v < 1$ case. However the cutoff frequencies for the energy radiated do depend on the radiation effects. We have

$$\omega_{ca} = 6\omega_a \gamma^3, \quad \omega_{cb} = 6\omega_b \gamma^3 \,, \tag{50}$$

which is obviously bounded at $\theta = 0$ for all cusps, unlike the $v = 1$ case. However since γ can be expected to be large although not infinite, the cutoff frequencies will remain very high.

Lastly we derive the beam width. We must realize that when $v \approx 1$, we are dealing with two expansions (in $1-v$ and in θ) whose terms cannot always be compared (e.g., a θ^2 and a $(1-v)\theta$ term). It is thus important to keep the physics in mind when we neglect terms. Since γ can be assumed to be very large although finite, we make the assumption, in Eq. (39), that for most strings the half-width θ_c of the beam will satisfy the following inequalities: $\alpha/\gamma^2 \ll \sin^2\phi_a \theta_c$ and $\beta/\gamma^2 \ll \sin^2\phi_b \theta_c$. In that case, defining the half-width to be the angle for which $\xi(\theta_c) = \xi(0) + 1$, we obtain from Eq. (39)

$$\theta_c = \inf \left[(2\omega_{ca}/3\omega)^{1/2} \frac{1}{\gamma \sin\phi_a} ; (2\omega_{bc}/\omega)^{1/2} \frac{1}{\gamma \sin\phi_b} \right] . \tag{51}$$

Once again we see that for frequencies of the order of the critical frequencies the radiation is very strongly beamed forward if ϕ_a and ϕ_b are nonvanishing [when ϕ_a and/or ϕ_b vanish a result similar to Eq. (51) is derived from Eq. (40)].

APPENDIX

The expansion coefficients in Eq. (21) are given by

$$a_1 = -\frac{\omega}{2}(1 - v\cos\theta) \,,$$

$$a_2 = \tfrac{1}{4}\omega v \omega_a \sin\theta \cos\phi_a \,,$$

$$a_3 = \tfrac{1}{12}\omega v [\sin\theta (g_a'' \cos\phi_a - 2\omega_a f_a' \sin\phi_a) - \omega_a^2 \cos\theta] \,,$$

$$b_1 = \frac{\omega}{2}(1 - v\cos\theta) \,,$$

$$b_2 = -\tfrac{1}{4}\omega v \omega_b \sin\theta \cos\phi_b \,,$$

$$b_3 = -\tfrac{1}{12}\omega v [\sin\theta (g_b'' \cos\phi_b - 2\omega_b f_b' \sin\theta_b) - \omega_b^2 \cos\theta] \,,$$

where

$$N = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta) \,,$$

$$\phi_a = f_a - \phi, \quad \phi_b = f_b - \phi \,.$$

[1]For a review see, e.g., A. Vilenkin, in *300 Years of Gravitation*, edited by S. Hawking and W. Israel (Cambridge University Press, Cambridge, England, 1987), p. 499.

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