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Generalized Beam Matrices. IV. Optical System Design

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Generalized beam matrices. IV. Optical system design

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Systematic procedures are presented for determining the optical components needed to produce an arbitrary transformation of a Gaussian light beams's spot size, radius of curvature, displacement, and direction of propagation. As an example, an optical system is considered that spatially separates the two coincident Gaussian beams produced by a high-diffraction-loss resonator that uses a Gaussian variable-reflectivity output coupler. In addition, an *ABCDGH* reverse matrix theorem and an *ABCDGH* Sylvester theorem are also derived. These matrix theorems may be used to satisfy special constraints inherent in the design of multipass and periodic optical systems. © 1997 Optical Society of America [S0740-3232(97)00904-6]

1. INTRODUCTION

An oft-encountered problem in laser optics involves designing an optical system to convert a Gaussian beam with known characteristics into a Gaussian beam having some desired characteristics. Based on experience with many such systems, one is sometimes able to guess the type of system that is needed to produce a required beam transformation. The emphasis in this study, however, is on synthesis.[1–3] It ought not be necessary to rely on experience or good luck to design an optical system that will produce some required transformation of a Gaussian beam.

Though the design procedure for complex Gaussian-beam optical systems has been spelled out, the founding paper on the subject used 2×2 complex beam matrices.[1] However, many of today's optical systems incorporate optical elements, such as prisms, that do not fit within the framework of Kogelnik's 2×2 beam matrices.[4] Important effects such as misalignment, whether intentional or by accident, also require an alternative formalism. Recently, Kogelnik's 2×2 *ABCD* beam matrices were generalized to 3×3 *ABCDGH* matrices.[5] This generalization allows the inclusion of additional complex optical elements and such effects as misalignment, which changes a light beam's position and direction. Thus one purpose of this work is to generalize the synthesis process to include beam displacement and deflection.

The design process, as developed here, involves three more-or-less distinct steps: (1) converting the desired performance characteristics of an optical system into explicit values or constraints on the values of the transformation matrix elements; (2) factoring the matrix into certain primitive matrix forms; and (3) replacing each of these primitive matrices with realizable optical components. After some background about Gaussian-beam propagation in Section 2, various potential design constraints are considered in Section 3. With the information in those two sections, one can deduce the matrix needed for a desired transformation—this is step (1) of the design process. Almost any *ABCDGH* matrix that can be encountered in optics is factorable into primitive matrices of three basic types, and several possible factorizations are obtained in Section 4. These factorizations constitute step (2) of the design process. In Section 5 it is demonstrated how each of these primitive matrices can be realized by using actual laboratory components. This is step (3), and it concludes the design process. As an example, an optical system is designed in Section 6 that spatially separates two superimposed Gaussian beams.

2. PROPAGATION OF GAUSSIAN LASER BEAMS

A. Laser Beams in Optical Systems

For our purposes a Gaussian beam is characterized by four parameters: $1/e$ electric-field amplitude radius, or spot size (w); radius of curvature of the phase fronts (R); and displacement of the amplitude center, i.e., position (d_a) and slope (d_a'). Solutions of Maxwell's equations for beam propagation in many media of interest can be written most compactly in terms of combinations of these fundamental parameters. Thus we use a complex beam parameter (q) and a complex displacement parameter (S) defined at each place on the optical axis as

$$\frac{1}{q} = \frac{1}{R} - i\,\frac{\lambda_m}{\pi w^2}, \tag{1}$$

$$S = \frac{2\pi}{\lambda_m}\left(\frac{-1}{q}\,d_a + d_a'\right), \tag{2}$$

where λ_m is the wavelength of the laser beam in the optical medium (λ/n_0). With these definitions a wide range of optical elements and systems can be represented by 3 × 3 matrices, and Gaussian-beam propagation from reference plane 1 to reference plane 2 is determined from

$$\begin{pmatrix} u \\ u/q \\ Su \end{pmatrix}_2 = \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} \begin{pmatrix} u \\ u/q \\ Su \end{pmatrix}_1 . \qquad (3)$$

For the systems of interest here, the x and y variations can be treated independently. For brevity, we are omitting coordinate subscripts for most of the analysis, and the equations may be understood to govern independently the x and y field variations. For example, for the x variations each matrix element and field variable should be given an x subscript. For clarity, however, the x and y variations are indicated explicitly in Subsection 3.D, which deals with beam transformations.

As is common in transfer-matrix methods, the matrix representation for an optical system is the product of the matrices for the individual optical elements multiplied in the reverse of the order in which they are encountered by the laser beam. In general, the matrix representation for a given optical element is obtained by solving Maxwell's equations. These matrices have been derived for a wide variety of optical elements,[5] and a condensed table of these matrices is given in Fig. 1. An important element not in Ref. 5 is the grating whose *ABCDGH* matrix is derived in Appendix A. For simplicity, the matrices in Fig. 1 are specified in their unimodular form. The variable u in Eq. (3) is unimportant by itself, and only ratios of the components of the input and output vectors in Eq. (3) are observables. As Eq. (3) represents three equations, we divide the second by the first to obtain the Kogelnik transformation[4] and the third by the second to obtain the S-parameter transformation[5]:

$$\frac{1}{q_2} = \frac{C + D/q_1}{A + B/q_1}, \qquad (4)$$

$$S_2 = \frac{S_1}{A + B/q_1} + \frac{G + H/q_1}{A + B/q_1}. \qquad (5)$$

In the problem of synthesis, we assume that there is some desired relationship between the input values of spot size, radius of curvature, position, and slope and the output values of these same parameters. Thus q_1, S_1, q_2, and S_2 are given, and Eqs. (4) and (5) may be used in the determination of A, B, C, D, G, and H.

The determinant of the beam matrix is the ratio of the complex refractive indices of the media at the input and output planes.[1,5] In many cases these media are identical, and, if so, an additional constraint on the matrix elements would be

$$AD - BC = 1. \qquad (6)$$

Equations (4)–(6) represent six equations (since each is a complex equation) to be solved for the real and imaginary parts of the matrix elements, i.e., six equations with 12 unknowns. In general, there may be more than one appropriate beam matrix. However, several additional constraints may follow from other considerations, and some of these are discussed in Section 3.

B. Laser Beams in Resonators

In the design of conventional laser resonators, one often finds that the eigenmode of the resonator has a Gaussian transverse distribution. By definition of a mode, the spot size and the radius of curvature repeat after the Gaussian beam propagates once around the resonator. If *ABCDGH* represents the round-trip matrix for a laser resonator starting at some particular reference plane, then

$$q_1 = q_2 = q_\infty , \qquad (7)$$

$$S_1 = S_2 = S_\infty , \qquad (8)$$

where q_∞ and S_∞ denote the values of q and S after a large number of round trips. These oscillation conditions, substituted into Eqs. (4) and (5), become

$$\frac{1}{q_\infty} = \frac{1}{R_\infty} - i\frac{\lambda_m}{\pi w_\infty^2} = \frac{D - A}{2B} \pm \frac{i}{B}\left[1 - \left(\frac{A + D}{2}\right)^2\right]^{1/2}, \qquad (9)$$

$$\frac{2\pi}{\lambda_m}\left(\frac{-d_{a\infty}}{q_\infty} + d'_{a\infty}\right) = \frac{G + H/q_\infty}{A + B/q_\infty - 1}, \qquad (10)$$

where Eqs. (1), (2), and (6) have also been used. Thus Eqs. (6), (9), and (10) are the initial constraints needed for designing laser resonators.

The ± sign in Eq. (9) indicates that there are two values for the steady-state beam parameter. These signs

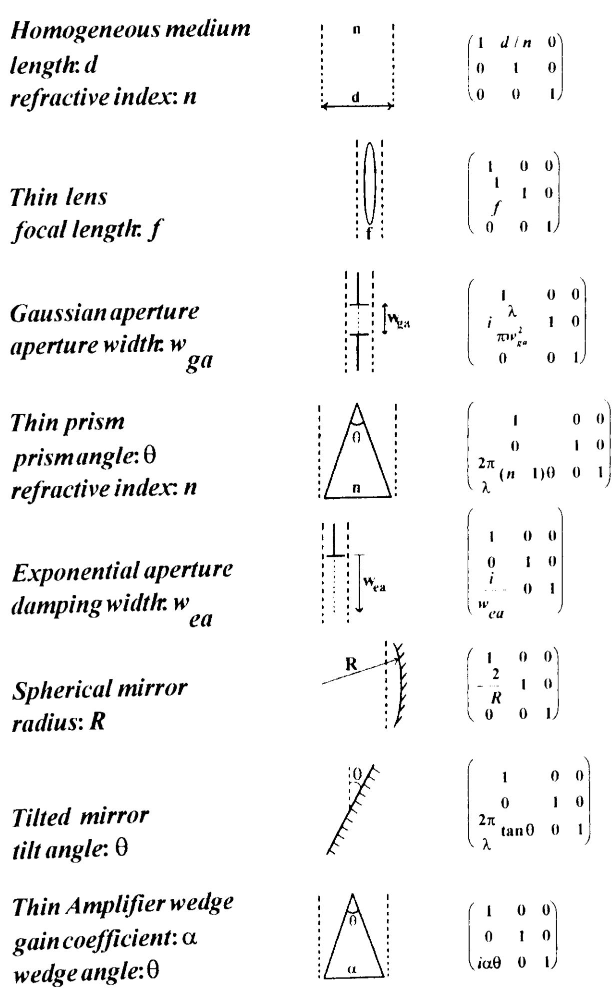


Fig. 1. *ABCDGH* matrix representation for several optical elements.

represent two different modes. Except when both are metastable, one of these modes will be stable, and the other will not be stable. Inevitable perturbations in real lasers do not allow the unstable mode to exist. The stable mode is the one in which[6]

$$F_s = \left|\frac{A + D}{2} \pm i\left[1 - \left(\frac{A + D}{2}\right)^2\right]^{1/2}\right| > 1. \tag{11}$$

The sign in Eq. (11) that yields a stability factor greater than one is the sign that should be used in Eq. (9). If one requires that the stability factor have a certain value, then that would be an additional design constraint.

Even if there is not a specifically designed spot size, it may be desired that

$$w_\infty^2 > 0. \tag{12}$$

This is known as the confinement condition, and if it is not satisfied, the laser beam mode will not be Gaussian.

3. DESIGN CONSTRAINTS

In the typical cascaded optical system design problem, the Gaussian beam's parameters at the input plane and the output plane are known. The *ABCDGH* beam matrix elements are determined from Eqs. (4)–(6). Similarly, the *ABCDGH* beam matrix elements for a laser resonator may be determined from Eqs. (6), (9), and (10). In either of these configurations, there may not be a unique *ABCDGH* matrix. However, it often occurs in practice that there are additional design constraints, and some of these are discussed below.

A. Length Constraint

Optical systems often have some type of length constraint. If there is no specific length constraint, there is still often a maximum or minimum length allowed for the system. Since length is not a property inherent in the *ABCDGH* matrix, length constraints must be met in the matrix factorization process.

B. Diffraction Angle Constraint

Constraints on the output beam of either a cascaded optical system or a laser resonator may be stated in terms of a diffraction angle. For an output beam with a given spot size and radius of curvature, the corresponding diffraction half-angle is

$$\theta_{\text{diff}} = \frac{\lambda}{\pi w}\left[1 + \left(\frac{z_0}{R}\right)^2\right]^{1/2}, \tag{13}$$

where $z_0 = \pi w^2/\lambda$ is the usual Rayleigh length. To minimize the diffraction angle, a lens may be placed at the output plane to collimate the beam. This is a common technique in resonator design.

C. Lossless Optical Systems

Most often one would also be inclined to require that the losses in a cascaded optical system be minimized. This is done by minimizing the number of components in the optical system and ensuring that each component has as appropriate optical coating. One would also often use optical elements that are represented by a strictly real *ABCDGH* matrix, since only these are lossless (and gainless). If the matrix elements are required to be real, six of our 12 unknowns are abruptly set equal to zero. In addition, the complex equations (4) and (5) reduce substantially to

$$w_2 = w_1\left[\left(A + \frac{B}{R_1}\right)^2 + \left(\frac{B}{z_0}\right)^2\right]^{1/2}, \tag{14}$$

$$R_2 = \frac{\left(A + \frac{B}{R_1}\right)^2 + \left(\frac{B}{z_0}\right)^2}{\left(A + \frac{B}{R_1}\right)\left(C + \frac{D}{R_1}\right) + \left(\frac{BD}{{z_0}^2}\right)}, \tag{15}$$

$$d_{a2} = Ad_{a1} + Bd'_{a1} + (BG - AH)/\beta_0, \tag{16}$$

$$d'_{a2} = Cd_{a1} + Dd'_{a1} + (DG - CH)/\beta_0, \tag{17}$$

where $\beta_0 = 2\pi/\lambda_m$.

D. Fourier-Transforming Optical Systems

In imaging and other applications, it is desired to design Fourier-transforming optical systems. An arbitrarily profiled beam propagating through an optical system represented by an *ABCDGH* matrix is transformed as follows[7]:

$$E'_{\text{out}}(x) = \exp(-i\hat{\phi})\int_{-\infty}^{\infty} K_{ABCDGH}(x_0, x)E'_{\text{in}}(x_0)\text{d}x_0, \tag{18}$$

where

$$K_{ABCDGH}(x_0, x) = K_{ABCD}(x_0, x)\exp\left[-i\left(\frac{H}{B}\right)x\right]\exp\left[-i\left(\frac{BG - AH}{B}\right)x_0\right], \tag{19}$$

$$K_{ABCD}(x_0, x) = \left(\frac{i}{\lambda_m B}\right)^{1/2}\exp\left[-i\frac{\pi}{\lambda_m}\left(\frac{Dx^2 - 2x_0x + A{x_0}^2}{B}\right)\right]. \tag{20}$$

The function $\hat{\phi}$ represents a space-independent phase shift. For astigmatic and other optical systems, there is a more general two-dimensional Fourier-transforming integral.[7] By examining the kernel, one can see that the optical system is a purely Fourier-transforming system when $A = 0, D = 0, H = 0$, and B, C, and G are strictly real. If B, C, and/or G are complex, then the optical system is Laplace transforming.

E. Periodic Optical Systems

There are a number of applications that make use of periodic optical systems. As a design problem, we suppose that the predetermined *ABCDGH* matrix representation for an optical system is to be factored into n identical submatrices. The initial design process then consists of obtaining the nth root of the system matrix. This may be obtained by using a Sylvester theorem[8] that specifies the sth power of a matrix, where s can be a fraction. However, the Sylvester theorem for a unimodular *ABCDGH* matrix has not been given previously. To derive it, we define

$$\begin{bmatrix} A_s & B_s & 0 \\ C_s & D_s & 0 \\ G_s & H_s & 1 \end{bmatrix} \equiv \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}^s . \tag{21}$$

It may be noted that A_s, B_s, C_s, and D_s are the same as the corresponding 2×2 matrix elements.[8] A simple way to obtain G_s and H_s is to use the commutativity requirement $M^s M = M M^s$. If this is done with the generalized beam matrix, then it immediately follows that

$$A_s G + C_s H + G_s = A G_s + C H_s + G, \tag{22}$$

$$B_s G + D_s H + H_s = B G_s + D H_s + H. \tag{23}$$

These two equations may be used to produce G_s and H_s, and the results are shown in Fig. 2.

F. Multipass Optical Systems

In the design of standing-wave lasers and reflective optical systems, where the light signal travels in both directions, it is useful to use a reverse matrix—a matrix that governs light propagation through an optical system backward. The derivation of the reverse theorem has not been given before for *ABCDGH* matrices, but it follows a similar derivations given in Ref. 9. The reverse matrix relates the input beam to the output beam for a beam traveling predominately in the negative z direction. The input beam properties in terms of the output beam properties may be obtained by multiplying both sides of Eq. (3) on the left by the matrix inverse. The result is

$$\begin{pmatrix} u \\ u/q \\ Su \end{pmatrix}_1 = M^{-1} \begin{pmatrix} u \\ u/q \\ Su \end{pmatrix}_2 . \tag{24}$$

Now the vectors must be written to reflect the fact that the beam is propagating predominately in the negative z direction. When the beam is traveling in the positive z direction, the governing differential equations are[5]

$$Q^2(z) + k_0(z) \frac{\partial Q(z)}{\partial z} + k_0(z) k_2(z) = 0, \tag{25}$$

$$Q(z) S(z) + k_0(z) \frac{\partial S(z)}{\partial z} + \frac{k_0(z) k_1(z)}{2}. \tag{26}$$

However, these may be rewritten for beams traveling in the $-z$ direction as

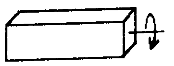
$$[-Q(z)]^2 + k_0(z) \frac{\partial[-Q(z)]}{\partial(-z)} + k_0(z) k_2(z) = 0, \tag{27}$$

$$[-Q(z)][-S(z)] + k_0(z) \frac{\partial[-S(z)]}{\partial(-z)} + \frac{k_0(z) k_1(z)}{2}, \tag{28}$$

and it follows that, for reverse propagation, $S \to -S$ and $Q \to -Q$ ($Q = k_0/q$). For this transformation Eq. (24) may be rewritten as

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{pmatrix} u \\ u/(-q) \\ (-S)u \end{pmatrix}_1 = M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{pmatrix} u \\ u/(-q) \\ (-S)u \end{pmatrix}_2 . \tag{29}$$

Multiplying both sides on the left by the appropriate matrix yields

Operation	Description	Matrix $\cos\theta = (A + D)/2$
Sylvester's Theorem	$\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{pmatrix}^s$	$\frac{1}{\sin\theta}\begin{pmatrix} A\sin(s\theta) - \sin[(s-1)\theta] & B\sin(s\theta) & 0 \\ C\sin(s\theta) & D\sin(s\theta) - \sin[(s-1)\theta] & 0 \\ G_s' & H_s' & \sin\theta \end{pmatrix}$
Reverse Theorem		$\begin{pmatrix} D & B & 0 \\ C & A & 0 \\ CH - DG & AH - BG & 1 \end{pmatrix}$
Coordinate Reflection Theorem		$\begin{pmatrix} D & B & 0 \\ C & A & 0 \\ -G & -H & 1 \end{pmatrix}$
Matrix Inverse	$\begin{pmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{pmatrix}^{-1}$	$\begin{pmatrix} D & -B & 0 \\ -C & A & 0 \\ CH - DG & BG - AH & 1 \end{pmatrix}$

$$G_s' = \frac{(A-1)\sin(s\theta) + (D-1)\{\sin[(s-1)\theta] + \sin\theta\}}{A + D - 2} G + \frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{A + D - 2} CH$$

$$H_s' = \frac{\sin(s\theta) - \sin[(s-1)\theta] - \sin\theta}{A + D - 2} BG + \frac{(D-1)\sin(s\theta) + (A-1)\{\sin[(s-1)\theta] + \sin\theta\}}{A + D - 2} H$$

Fig. 2. *ABCDGH* matrix theorems.

$$\begin{pmatrix} u \\ u/(-q) \\ (-S)u \end{pmatrix}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \times \begin{pmatrix} u \\ u/(-q) \\ (-S)u \end{pmatrix}_2 . \tag{30}$$

Therefore the reverse matrix is

$$M_R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} M^{-1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} . \tag{31}$$

The resulting reverse matrix, obtained by multiplying out Eq. (31), is given in Fig. 2.

G. Ring Optical Systems

In a ring optical system, the x-transverse axis may be defined so that it points outward from the center of the ring. With this coordinate system, flat mirrors are represented by the matrix

$$M_{\text{flat mirror}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{32}$$

If one of the legs of the optical system has the x axis defined to be positive when it points in, then a mirror image transformation must be used on the elements in that leg:

$$M_{\text{coordinate reflection}} = \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ -G & -H & 1 \end{bmatrix} . \tag{33}$$

This result also appears in Fig. 2.

H. Other Constraints

Optical systems are sometimes astigmatic, and the results here are valid for each of the transverse axes. In dye lasers, for example, the pump angle and the emission angle are not collinear, and the resulting system may be astigmatic. A popular resonator design corrects for this astigmatism.[10] The frequency/wavelength dependence of an optical system may be accounted for by using a wavelength-dependent *ABCDGH* matrix. Requiring that an optical system have a certain dispersion response can put severe constraints on the optical design. Still other constraints might result from other practical considerations. For example, it might be specified that the optical system has a length of l between the reference planes or that the spot size and the beam displacement are everywhere less than some specified value. For the sections that follow, it is simply assumed that the matrix elements of the desired transformation have, by one means or another, been determined.

4. FACTORING THE MATRIX

Once the matrix elements are known, one is left with the task of finding actual optical components that, when placed in sequence, yield the required matrix. For many optical systems, the input and output planes are in the same type of optical medium, usually free space. For these types of optical system, the transfer matrix will be unimodular, and these types are given in Fig. 1. As a starting point, it may be observed that all of the optical elements from Fig. 1 can be readily expressed in one of the following three forms:

$$\hat{\alpha} = \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{34}$$

$$\hat{\beta} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{35}$$

$$\hat{\delta} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{bmatrix} , \tag{36}$$

where a caret denotes a matrix. It is thus reasonable to inquire how a broad class of *ABCDGH* matrices can be represented as a product of these matrix primitives (34)–(36). As an example of one interesting factorization, it can be shown by direct matrix multiplication that

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix} \begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG - AH)/B & 0 & 1 \end{bmatrix} . \tag{37}$$

The center matrix is essentially the usual *ABCD* matrix for an aligned optical system. It can be factored into matrices of the $\hat{\alpha}$ and $\hat{\beta}$ types [i.e., in terms of Eqs. (34) and (35)][1]:

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & (A-1)/C & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ C & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & (D-1)/C & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} , \tag{38}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ (A-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} . \tag{39}$$

Thus two different *ABCDGH* factorizations are obtained by substituting Eqs. (38) and (39) into Eq. (37). As expected, it requires five matrices to factor an *ABCDGH* matrix arbitrarily—the system matrix has six factors that are satisfied by five single-variable matrices and one unimodularity condition. Additional factorizations may be obtained by examining the commuting properties of $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\delta}$ matrices. Matrices of the $\hat{\alpha}$ and $\hat{\delta}$ types do not commute with each other. However, matrices of the $\hat{\delta}$ type and the $\hat{\beta}$ type do commute with each other, so that

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta & 0 & 1 \end{bmatrix} . \tag{40}$$

The validity of this expression can be confirmed by multiplying out each of the product pairs. However, this mathematical result makes sense physically, since both $\hat{\delta}$

and $\hat{\beta}$ matrices represent thin optical elements. Matrix commutativity involves matrix pairs. However, three-matrix combinations are also of interest. In particular, it may be shown that

$$\hat{\delta}(\delta_1)\hat{\alpha}(\alpha)\hat{\delta}(\delta_2) = \hat{\alpha}\left(\frac{\alpha\delta_2}{\delta_1+\delta_2}\right)\hat{\delta}(\delta_1+\delta_2)\hat{\alpha}\left(\frac{\alpha\delta_1}{\delta_1+\delta_2}\right), \tag{41}$$

which may be rewritten as

$$\hat{\alpha}(\alpha_1)\hat{\delta}(\delta)\hat{\alpha}(\alpha_2) = \hat{\delta}\left(\frac{\delta\alpha_2}{\alpha_1+\alpha_2}\right)\hat{\alpha}(\alpha_1+\alpha_2)\hat{\delta}\left(\frac{\delta\alpha_1}{\alpha_1+\alpha_2}\right). \tag{42}$$

These three-matrix conversions are analogous to Δ–Y (also known as Π–T) transformations used in electric circuit analysis.

Equations (37)–(41) may be combined to yield the following factorizations:

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & (A-1)/C & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ C & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & (D-1)/C & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG-AH)/B & 0 & 1 \end{bmatrix}, \tag{43}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ (A-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG-AH)/B & 0 & 1 \end{bmatrix}, \tag{44}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ (A-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG-AH)/B & 0 & 1 \end{bmatrix}, \tag{45}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG-AH)/B & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ (A-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{46}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ H/B & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & B & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (BG-AH)/B & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ (A-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{47}$$

$$\begin{bmatrix} A & B & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ (D-1)/B & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \dfrac{B(BG-AH)}{BG-AH+H} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{BG-AH+H}{B} & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & \dfrac{BH}{BG-AH+H} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ \dfrac{A-1}{B} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{48}$$

Equations (43) and (44) are obtained by substituting Eqs. (38) and (39) into Eq. (37), respectively. Equations (45)–

(47) are obtained by using the commutativity of $\hat{\beta}$ and $\hat{\delta}$ matrices [Eq. (40)] on Eq. (44). Equation (48) is obtained by using the matrix transformation (41) on Eq. (47).

Six different factorizations of *ABCDGH* matrices in terms of the specified matrix primitives have been obtained. In each of the above factorizations, it is assumed that B is nonzero. However, for system matrices that have $B = 0$, the system matrix may be rewritten as, for example, either one of the following:

$$\begin{bmatrix} A & 0 & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix} = \begin{bmatrix} A & -\alpha A & 0 \\ C & D - \alpha C & 0 \\ G & H - \alpha G & 1 \end{bmatrix} \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{49}$$

$$= \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} A - \alpha C & -\alpha D & 0 \\ C & D & 0 \\ G & H & 1 \end{bmatrix}. \tag{50}$$

Here α is an adjustable design parameter that may be chosen, for example, to match a length constraint. In each of these equations, the system matrix is rewritten in terms of a matrix primitive and a new system matrix that has a nonzero B element. In Eq. (49) the first matrix to the right of the equal sign may be factored using any of Eqs. (43)–(48). The rightmost matrix in Eq. (50) may be factored similarly.

The six factorizations represent all of the unimodular five-matrix factorizations by using only matrices of the $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\delta}$ types. If there exist design constraints that require the input and output beams to be in media with differing refracting properties, then the system matrix would be nonunimodular. In this case additional factorizations using the $\hat{\gamma}$ matrix of Ref. 1 are required. These would be obtained by using the nonunimodular factorizations of Ref. 1 in Eq. (37).

5. PHYSICAL REALIZATION OF THE MATRIX PRIMITIVES

Our discussions thus far have emphasized the factorization of 3×3 *ABCDGH* matrices into certain primitive matrix factors. It remains now to be shown that these factors can actually be represented by practical optical components. As a starting point, one finds that any $\hat{\beta}$-type matrix can be realized in practice. In particular, the product (in either order) of the matrix for a thin lens (or spherical mirror) with the matrix for a Gaussian transmission filter yields a $\hat{\beta}$ matrix having an arbitrary complex β element:

$$\begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{f} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -i\dfrac{\lambda}{\pi {w_{ga}}^2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -\dfrac{1}{f} - i\dfrac{\lambda}{\pi {w_{ga}}^2} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \beta_r + \beta_i & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{51}$$

It is, of course, true that the inverse Gaussian transmission characteristic (${w_{ga}}^2 < 0$) cannot be maintained to arbitrary radii, but it is necessary only that this profile be approximated to the largest radius of the beam. This same restriction applies to the radial phase-shift characteristics of finite-diameter lenses.

Just as an arbitrary $\hat{\beta}$ matrix can be realized, so can any $\hat{\delta}$ matrix. In this case a given $\hat{\delta}$ matrix can be represented by the combination of an exponential aperture (or exponential variable-reflectivity mirror) and a thin prism (or amplifier wedge):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{2\pi(n-1)}{\lambda}\theta_{\text{prism}} & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{i}{w_{ea}} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \dfrac{2\pi(n-1)}{\lambda}\theta_{\text{prism}} + \dfrac{i}{w_{ea}} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \delta_r + i\delta_i & 0 & 1 \end{bmatrix}. \tag{52}$$

The exponential aperture can be realized only out to some finite radius. However, the aperture profile need be realized only to the largest displacement of the beam. The width associated with the aperture may be either positive or negative, depending on the orientation of the aperture. Similarly, the prism angle may be either positive or negative, depending on the orientation of the prism. It should also be noted that a thin amplifier (or absorber) wedge has the same *ABCDGH* matrix as an exponential aperture, and, under certain conditions, it may be either desirable or inevitable to use an amplifier (or absorber) wedge.

The realization of arbitrary $\hat{\alpha}$ matrices is a bit more complicated. The only practical matrix that is automatically of the $\hat{\alpha}$ type is the matrix for a uniform medium of length l. But l is always a positive real number, so this matrix is totally inadequate for representing the negative or complex α elements that might result from the factorization of an arbitrary complex matrix. For this purpose a more general representation is needed, and one possibility consists of three complex lenses separated by two uniform media. For the system of interest this matrix product corresponds to the following matrix factorization[1]:

$$\begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2(\alpha^{-1} - l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 4(\alpha l^{-2} - l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 2(\alpha^{-1} - l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{53}$$

Thus an arbitrary complex $\hat{\alpha}$ matrix can be represented as a product of five realizable factors, and this result can be easily verified by multiplication. To illustrate the use of Eq. (53), let us imagine that we are trying to find a practical realization for a given unimodular *ABCDGH* matrix by using the factorization given in Eq. (46). With good luck the resulting five matrix factors can be realized by five optical elements (or perhaps three, depending on how one fabricates thin optical element combinations). With bad luck, however, the B element of the $\hat{\alpha}$ matrix may not be a positive real number. Then a more complicated representation of the $\hat{\alpha}$ matrix is required, and the system must be larger. The previous remarks have implied that this is unfortunate when one encounters an $\hat{\alpha}$ matrix in which the B element is not real and positive. However, a highly desirable feature of the expansion shown in Eq. (53) is that the distance l between the reference planes is totally arbitrary. In a practical situation one might like to specify the length of the optical system that is to produce a desired beam transformation. In the simpler five-matrix realizations, there is no length flexibility. In many situations, however, a length constraint can be met by multiplying the system matrix by the following representation for the identity matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 0 & 0 \\ -9/l & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & l/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right)^3 . \tag{54}$$

6. EXAMPLE DESIGN: LASER BEAM SEPARATION

One of the goals of this section is to demonstrate the design procedure detailed above by synthesizing a useful optical system. This example also demonstrates the ability of complex optical systems to discriminate spatially between Gaussian beams based on their spot size and phase curvature. In particular, an optical system is designed that spatially separates two coincident Gaussian beams with different spot sizes.

There are a variety of conditions in which two coincident Gaussian beams with different spot sizes may be encountered. One example involves the laser output of a high-diffraction-loss resonator (also known as an unstable resonator) that employs a Gaussian variable-reflectivity mirror. These resonators have important beam quality advantages over hard-aperture resonators. In the Gaussian mirror lasers the mode inside the laser is Gaussian, and thus the output beam that is transmitted through the Gaussian mirror has an intensity

$$I'_{\text{out}} = I'_{\text{in}}(1 - R) \tag{55}$$

$$= I'_0 \exp(-2x^2 w^{-2}) \times [1 - \exp(-2x^2 w_{ga}{}^{-2})] \tag{56}$$

$$= I'_0 \exp(-2x^2 w^{-2}) - I'_0 \exp[-2x^2(w^{-2} + w_{ga}{}^{-2})], \tag{57}$$

which is the sum of two Gaussians with different spot sizes.

Before beginning the design, it is useful to combine Eqs. (1), (2), (4), and (5) to produce explicit formulas for output radius of curvature, spot size, position, and slope of a beam given these properties at the input plane:

$$\frac{1}{R_2} = \frac{\left(C_r + \frac{D_r}{R_1} + \frac{2D_i}{\beta_{01}w_1{}^2}\right)\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1{}^2}\right) + \left(C_i + \frac{D_i}{R_1} - \frac{2D_r}{\beta_{01}w_1{}^2}\right)\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right)}{\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1{}^2}\right)^2 + \left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right)^2}, \tag{58}$$

$$\frac{2}{\beta_{02}w_2{}^2} = \frac{\left(C_r + \frac{D_r}{R_1} + \frac{2D_i}{\beta_{01}w_1{}^2}\right)\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right) - \left(C_i + \frac{D_i}{R_1} - \frac{2D_r}{\beta_{01}w_1{}^2}\right)\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1{}^2}\right)}{\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1{}^2}\right)^2 + \left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right)^2}, \tag{59}$$

$$d_{a2} = \frac{1}{\beta_{02}} \frac{\left[\frac{2}{\beta_{01}w_1{}^2}(\beta_{01}d_{a1} - H_r) + G_i + \frac{H_i}{R_1}\right]\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1{}^2}\right) - \left[-\frac{1}{R_1}(\beta_{01}d_{a1} - H_r) + \beta_{01}d'_{a1} + G_r + \frac{2H_i}{\beta_{01}w_1{}^2}\right]\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right)}{\left(C_r + \frac{D_r}{R_1} + \frac{2D_i}{\beta_{01}w_1{}^2}\right)\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1{}^2}\right) - \left(C_i + \frac{D_i}{R_1} - \frac{2D_r}{\beta_{01}w_1{}^2}\right)\left(A_r + \frac{B_r}{R_1} - \frac{2B_i}{\beta_{01}w_1{}^2}\right)}, \tag{60}$$

$$
\begin{aligned}
d'_{a2} ={}& \frac{1}{\beta_{02}} \frac{\left[-\frac{1}{R_1}(\beta_{01}d_{a1} - H_r) + \beta_{01}d'_{a1} + G_r + \frac{2H_i}{\beta_{01}w_1^2}\right]\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right) + \left[\frac{2}{\beta_{01}w_1^2}(\beta_{01}d_{a1} - H_r) + G_i + \frac{H_i}{R_1}\right]\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right)}{\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right)^2 + \left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right)^2} \\
&+ \frac{1}{\beta_{02}} \frac{\left(C_r + \frac{D_r}{R_1} + \frac{2D_i}{\beta_{01}w_1^2}\right)\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right) + \left(C_i + \frac{D_i}{R_1} - \frac{2D_r}{\beta_{01}w_1^2}\right)\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right)}{\left(C_r + \frac{D_r}{R_1} + \frac{2D_i}{\beta_{01}w_1^2}\right)\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right) - \left(C_i + \frac{D_i}{R_1} - \frac{2D_r}{\beta_{01}w_1^2}\right)\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right)} \\
&\times \frac{\left[\frac{2}{\beta_{01}w_1^2}(\beta_{01}d_{a1} - H_r) + G_i + \frac{H_i}{R_1}\right]\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right) - \left[-\frac{1}{R_1}(\beta_{01}d_{a1} - H_r) + \beta_{01}d'_{a1} + G_r + \frac{2H_i}{\beta_{01}w_1^2}\right]\left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right)}{\left(A_r + \frac{B_r}{R_1} + \frac{2B_i}{\beta_{01}w_1^2}\right)^2 + \left(A_i + \frac{B_i}{R_1} - \frac{2B_r}{\beta_{01}w_1^2}\right)^2}. \qquad (61)
\end{aligned}
$$

A. Determination of the *ABCDGH* Beam Matrix

Our goal is to design an optical system that has differing effects on two different Gaussian beams. Both beams are initially on the system axis ($d_{a1} = 0$), are traveling along it ($d'_{a1} = 0$), and are initially collimated ($R_1 = \infty$). Both beams are to be propagating parallel to each other and the optical axis at the output plane ($d'_{a2} = 0$). This input plane and the output plane are to be in free space; thus $\beta_{01} = \beta_{02} \equiv \beta_0$, and $AD - BC = 1$.

If we also require that the optical system not change the values of the spot sizes ($w_2 = w_1$) and the radii of curvature ($R_2 = R_1 = \infty$), then it is clear that one solution that satisfies Eqs. (58) and (59) is $A = 1$, $B = 0$, $C = 0$, and $D = 1$ (the optical system has no effect on the spot size and the radius of curvature of the beam; therefore the *ABCD* submatrix would sensibly be the identity matrix). These values of *ABCD* satisfy Eqs. (58) and (59) under all input conditions, and Eqs. (60) and (61) reduce to

$$\beta_0 d_{a2} = \frac{G_i - \dfrac{2H_r}{\beta_0 w_1^2}}{\dfrac{2}{\beta_{01} w_1^2}}, \qquad (62)$$

$$0 = G_r + \frac{2H_i}{\beta_0 w_1^2}. \qquad (63)$$

The optical system is to separate the two beams, beam a and beam b. If beam a is undisplaced by the optical system, then when

$$w_1 = w_a, \qquad (64a)$$

it follows that

$$d_{a2} = 0. \qquad (64b)$$

We suppose that beam b is displaced some distance off the axis. For specificity, we choose that when

$$w_1 = w_b, \qquad (65a)$$

then

$$d_{a2} = \Delta. \qquad (65b)$$

Substituting Eqs. (64) and (65) each into Eqs. (62) and (63) yields four equations with four unknowns:

$$0 = \frac{G_i - \dfrac{2H_r}{\beta_0 w_a^2}}{\dfrac{2}{\beta_0 w_a^2}}, \qquad (66)$$

$$0 = G_r + \frac{2H_i}{\beta_0 w_a^2}, \qquad (67)$$

$$\Delta\beta_0 = \frac{G_i - \dfrac{2H_r}{\beta_0 w_b^2}}{\dfrac{2}{\beta_0 w_b^2}}, \qquad (68)$$

$$0 = G_r + \frac{2H_i}{\beta_0 w_b^2}. \qquad (69)$$

It can readily be found that the solutions to these equations are

$$G_r = 0, \qquad (70)$$

$$G_i = \frac{2\Delta}{w_b^2 - w_a^2}, \qquad (71)$$

$$H_r = \frac{\beta_0 \Delta}{\dfrac{w_b^2}{w_a^2} - 1}, \qquad (72)$$

$$H_i = 0. \qquad (73)$$

Now that the system matrix has been obtained, it must be factored into matrix primitives.

B. Factoring the Matrix

The matrix may be initially factoring by using Eq. (49):

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ iG_i & H_r & 1 \end{bmatrix} = \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ iG_i & H_r + idG_i & 1 \end{bmatrix} \begin{bmatrix} 1 & -d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad (74)$$

Neither of the two matrices on the right-hand side of Eq. (74) is a matrix primitive, and therefore they must be factored further. The left matrix on the right-hand side of Eq. (74) may be factored by using Eq. (47):

$$\begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ iG_i & H_r + idG_i & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (H_r + idG_i)/d & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -H_r/d & 0 & 1 \end{bmatrix}. \quad (75)$$

The rightmost matrix in Eq. (74) may be rewritten by using Eq. (53):

$$\begin{bmatrix} 1 & -d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2(d^{-1} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -4(dl^{-2} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -2(d^{-1} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (76)$$

Equations (74)–(76) may be combined to yield the designed system matrix in terms of matrix primitives:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ iG_i & H_r & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ (H_r + idG_i)/d & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & d & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -H_r/d & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -2(d^{-1} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -4(dl^{-2} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & l/2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -2(d^{-1} + l^{-1}) & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (77)$$

C. Physical Realization of the Matrix Primitives

The above matrix factorization is valid for any d and l. If we choose $l = 2d = 2L/3$, then Eq. (77) becomes

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ iG_i & H_r & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ iG_i & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3H_r/L & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & L/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3H_r/L & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -9/L & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -9/L & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & L/3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 0 & 0 \\ -9/L & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (78)$$

and the length of the optical system is L. By comparing the rightmost matrix in Eq. (78) with the matrix representation for a thin lens from Fig. 1, it follows that the first optical element in the system is a lens whose focal length is

$$f = L/9. \quad (79)$$

All of the other lenses in the system have the same focal length. It can similarly be seen that the second optical element [second from the last matrix in Eq. (78)] represents free space whose length is

$$d = L/3. \quad (80)$$

There is also a prism in the optical system, whose properties obey

$$(n_{\text{prism}} - 1)\theta_{\text{prism}} = \frac{3\Delta/L}{\dfrac{w_b^2}{w_a^2} - 1}. \quad (81)$$

Finally, there is an exponential aperture whose transmission profile width is

$$w_{ea} = \frac{w_b^2 - w_a^2}{2\Delta}. \quad (82)$$

A schematic of the design is given in Fig. 3. This example could not be done with previously developed matrix design techniques.

7. SUMMARY

The design of optical systems and laser resonators is usually done by either extensive numerical simulations or by analyzing simple two-lens optical systems and working backward to produce design information. An alternative approach is to consider the design problem directly, and optical transfer matrices are well suited to this problem. In this study this latter approach has been used to produce an arbitrary transformation of a Gaussian light beam's spot size, radius of curvature, displacement, and direction.

There are three basic steps in the design process: (1) converting the desired performance characteristics of the

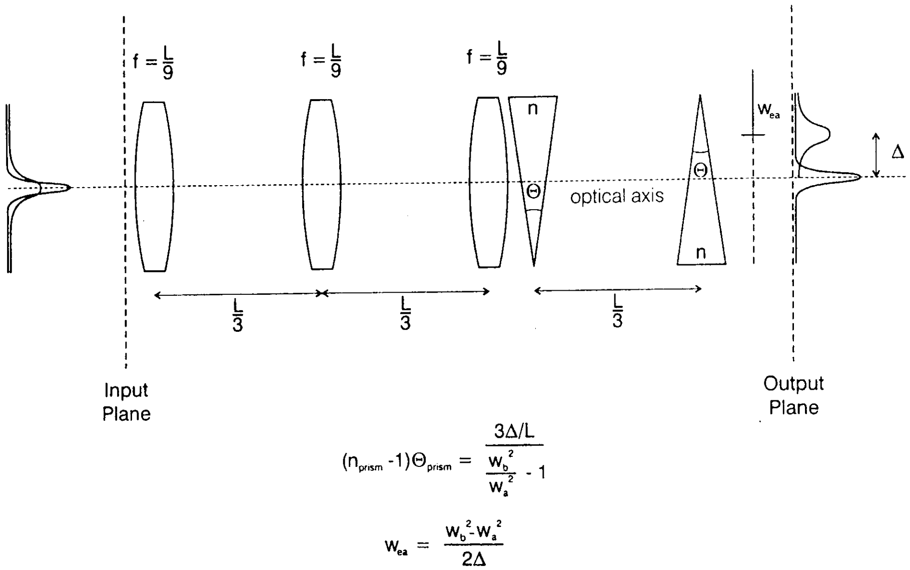


Fig. 3. Designed optical system that spatially separates two input Gaussian beams. The spot sizes of the two input beams (w_a and w_b), the displacement between the two output beams (Δ), and the length of the optical system (L) may be chosen arbitrarily.

optical systems into explicit values or constraints on the values of the transformation matrix elements; (2) factoring the matrix into certain primitive forms; and (3) replacing each of these primitive matrices with realizable optical components. The first step can be achieved for optical systems by using Eqs. (1) and (2) with Eqs. (4)–(6). For laser resonators Eqs. (6), (9), and (10) may be used to determine the *ABCDGH* matrix. There are often other constraints, and several of these have been discussed. Once the *ABCDGH* matrix is known, the next step of the design procedure is to factor the matrix into matrix primitives that are realizable as optical components. The six different five-matrix factorizations are given in Eqs. (43)–(48). The design process is completed by replacing the matrix primitives with optical components. This is done by using Fig. 1. An example of the complete design process is given in Section 6.

The design process as outlined here takes the formal analytical tools developed in the first three papers in this series and transforms them into a powerful explicit procedure for the design of lasers and complex optical systems.

APPENDIX A: DIFFRACTION GRATINGS

In this appendix the *ABCDGH* matrix for a diffraction grating is obtained, and it is shown to have the same form as a thin prism. The matrix may be determined from the grating equation, which is

$$\sin \theta_{\text{incident}} = \sin \theta_{\text{reflected}} + \frac{m\lambda}{\Lambda}, \tag{A1}$$

where m is the grating order and Λ is the grating period. If we define the axis for the center wavelength λ_0 so that

$$\sin \theta_I = \sin \theta_R + \frac{m\lambda_0}{\Lambda}, \tag{A2}$$

then wavelengths near λ_0 are paraxial to that axis, so that

$$\sin(\theta_I + \theta_1) = \sin(\theta_R + \theta_2) + \frac{m\lambda}{\Lambda}, \tag{A3}$$

where $\sin\theta_1 \approx d'_{a1}$, $\sin\theta_2 \approx d'_{a2}$, $\cos\theta_1 \approx 1$, and $\cos\theta_2 \approx 1$. Under these conditions Eq. (A3) becomes

$$d'_{a2} = \left(\frac{\cos\theta_I}{\cos\theta_R}\right) d'_{a1} - \frac{m(\lambda - \lambda_0)}{\Lambda \cos\theta_R}. \tag{A4}$$

From Eqs. (A8) and (A9) of Ref. 5 and the unimodularity condition, it follows that

$$M_{\text{grating}} = \begin{bmatrix} \dfrac{\cos\theta_R}{\cos\theta_I} & 0 & 0 \\ 0 & \dfrac{\cos\theta_I}{\cos\theta_R} & 0 \\ -\dfrac{2\pi m}{\Lambda\cos\theta_I}\left(1 - \dfrac{\lambda_0}{\lambda}\right) & 0 & 1 \end{bmatrix}. \tag{A5}$$

This has a form similar to a prism matrix. Thus a grating may be used as an alternative to prisms in the final design.

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REFERENCES

1. L. W. Casperson, "Synthesis of Gaussian beam optical systems," Appl. Opt. **20**, 2243–2249 (1981).
2. F. Keijzer, R. J. W. E. Lahaye, and S. Stolte, "Midfocus matching: a method for calculating lens position for a Gaussian beam in a two-lens system," Appl. Opt. **31**, 7537–7540 (1992).
3. J.-P. Weber, "Device design using Gaussian beams and ray matrices in planar optics," IEEE J. Quantum Electron. **30**, 2407–2416 (1994).
4. H. W. Kogelnik, "Imaging of optical modes—resonators with internal lenses," Bell Syst. Tech. J. **44**, 455–494 (1965).

5. A. A. Tovar and L. W. Casperson, "Generalized beam matrices: Gaussian beam propagation in misaligned complex optical systems," J. Opt. Soc. Am. A **12**, 1522–1533 (1995).
6. L. W. Casperson, "Mode stability of lasers and periodic optical systems," IEEE J. Quantum Electron. **QE-10**, 629–634 (1974).
7. A. A. Tovar and L. W. Casperson, "Generalized beam matrices. III. Application to diffraction analysis," J. Opt. Soc. Am. A **13**, to be published (1996).
8. A. A. Tovar and L. W. Casperson, "Generalized Sylvester theorems for periodic applications in matrix optics," J. Opt. Soc. Am. A **12**, 578–590 (1994).
9. A. A. Tovar and L. W. Casperson, "Generalized reverse theorems for multipass applications in matrix optics," J. Opt. Soc. Am. A **11**, 2633–2642 (1994).
10. H. W. Kogelnik, E. P. Ippen, A. Dienes, and C. V. Shank, "Astigmatically compensated cavities for cw dye lasers," IEEE J. Quantum Electron. **QE-8**, 373–379 (1972).