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# NÉDÉLEC SPACES IN AFFINE COORDINATES

JAYADEEP GOPALAKRISHNAN, LUIS E. GARCÍA-CASTILLO, AND LESZEK F. DEMKOWICZ

ABSTRACT. In this note we provide a conveniently implementable basis for simplicial Nédélec spaces of any order in any space dimension. The main feature of the basis is that it is expressed solely in terms of the barycentric coordinates of the simplex.

## 1. INTRODUCTION

Nédélec spaces are perhaps the most widely used finite element spaces in computational electromagnetics. They are very easy to describe: The simplicial Nédélec space of the first kind [7] of order  $k \geq 1$  is

$$(1.1) \quad \mathbf{R}_k = \mathbf{P}_{k-1} \oplus \mathbf{S}_k,$$

where  $\mathbf{P}_\ell$  denotes the set of all vector functions whose every component is a polynomial of degree at most  $\ell$  and  $\mathbf{S}_\ell = \{\mathbf{q} \in \mathbf{P}_\ell : \text{each component of } \mathbf{q}(\mathbf{x}) \text{ is a homogeneous polynomial of degree } \ell \text{ and } \mathbf{x} \cdot \mathbf{q}(\mathbf{x}) = 0 \text{ for all } \mathbf{x}\}$ .

Although many implementations using Nédélec spaces of the lowest order exist, there are very few codes that employ high order simplicial Nédélec spaces. One of the difficulties in programming methods using high order Nédélec spaces is the complexity of generating element basis functions. In this note we give a readily implementable basis in affine (or barycentric) coordinates of Nédélec spaces of any order. We present the basis for arbitrary orders as well as arbitrary space dimension  $N \geq 2$ . Although, only cases  $N = 2$  and  $N = 3$  are usually of interest, considering general  $N$  is not an over-generalization, as we shall see. For example, to get an affine basis for the  $N = 3$  case, we have to work with the highest degree part of a Nédélec space in  $\mathbb{R}^4$  and establish a correspondence with the Nédélec space in one less space dimension. Thus, for brevity, we consider general  $N$ .

The connection between the lowest order Nédélec elements and Whitney forms are well known. This leads to the following basis in affine coordinates for the lowest order Nédélec space on an  $N$ -simplex:

$$\{\lambda_i \nabla \lambda_j - \lambda_j \nabla \lambda_i\},$$

where  $\lambda_i, i = 1, \dots, N + 1$  are the affine coordinates of the simplex. We provide an elementary approach to generalize such forms to the higher order case.

Many papers devoted to the construction of specific shape functions for computational electromagnetics have appeared in engineering literature [1, 3, 2, 4, 5, 6, 8, 10, 11]. In particular, this note is motivated by an elegant presentation of Webb [9] devoted to the construction of hierarchical shape functions of arbitrary order in terms of barycentric

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coordinates. We provide an algebraic link between Nédélec's characterizations in [7] and results in [9]. It follows from our results that Webb's functions do not span the original Nédélec space. While [9] aims at constructing a hierarchical basis, our aim in this note is simply to rigorously establish a general form of the higher order Nédélec basis functions in affine coordinates. As we shall see, it is possible to achieve this aim by elementary arguments.

## 2. SOME CHARACTERIZATIONS OF THE NÉDÉLEC SPACE

In [7], Nédélec characterized  $\mathbf{R}_k$  in terms of the symmetric multilinear form

$$(2.1) \quad \varepsilon^k(\mathbf{q})(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{k+1}) = \frac{1}{(k+1)!} \sum_{\sigma} (\mathbf{d}^k \mathbf{q})(\mathbf{r}_{\sigma(1)}, \mathbf{r}_{\sigma(2)}, \dots, \mathbf{r}_{\sigma(k)}) \cdot \mathbf{r}_{\sigma(k+1)},$$

where the sum runs over all permutations  $\sigma$  of the set  $\{1, 2, \dots, k+1\}$ ,  $\mathbf{q}$  is any smooth vector function  $\mathbb{R}^N \mapsto \mathbb{R}^N$ ,  $\mathbf{r}_i$ 's are any set of  $k+1$  vectors in  $\mathbb{R}^N$ , and  $\mathbf{d}^k \mathbf{q}$  is the  $k$ -th order Fréchet derivative of  $\mathbf{q}$ , i.e., if  $\mathbf{q} = (q_1, \dots, q_N)^t$  and  $d^k q_i$  is the standard Fréchet derivative of  $q_i$  then  $\mathbf{d}^k \mathbf{q} = (d^k q_1, \dots, d^k q_N)^t$ . It is proved in [7] that a smooth function  $\mathbf{q}$  is in  $\mathbf{R}_k$  if and only if  $\varepsilon^k(\mathbf{q}) = 0$ . With this equivalence as a starting point, we will derive some other characterizations of the homogeneous part of the Nédélec space, namely  $\mathbf{S}_k$ . These characterizations help us give a basis for  $\mathbf{R}_k$  in affine coordinates.

The characterizations are stated in the next theorem. The statements of the theorem use multi-index notations which we now describe: The set of multi-indices is

$$\mathcal{I}(N, k) = \{\boldsymbol{\alpha} \equiv (\alpha_1, \alpha_2, \dots, \alpha_N) : \alpha_i \geq 0 \text{ are integers satisfying } \sum_{i=1}^N \alpha_i = k\}.$$

The notation  $\mathbf{x}^{\boldsymbol{\alpha}}$  for any  $\mathbf{x} \in \mathbb{R}^N$  and  $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_N) \in \mathcal{I}(N, p)$  denotes the product  $x_1^{\alpha_1} x_2^{\alpha_2} \dots x_N^{\alpha_N}$ . In a similar vein, the operator  $\partial^{\boldsymbol{\alpha}}$  for any  $\boldsymbol{\alpha} \in \mathcal{I}(N, p)$  is defined by

$$\partial^{\boldsymbol{\alpha}} = \frac{\partial^p}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \dots \partial x_N^{\alpha_N}}.$$

We denote by  $\mathbf{e}_\ell$  the multi-index whose indices are all zero except the  $\ell$ -th one, e.g.,  $\mathbf{e}_1 = (1, 0, \dots, 0)$ . We adopt the convention that any term involving a multi-index with negative components is zero. Let  $\mathbf{C}^k(\mathbb{R}^N)$  denote the set of  $k$  times continuously differentiable  $\mathbb{R}^N \mapsto \mathbb{R}^N$  maps. The following theorem is proved later.

### Theorem 2.1.

**A:** If  $\mathbf{q}$  is a function in  $\mathbf{C}^k(\mathbb{R}^N)$ , it is in  $\mathbf{R}_k$  if and only if

$$\sum_{\ell=1}^N \beta_\ell \partial^{\boldsymbol{\beta} - \mathbf{e}_\ell} q_\ell = 0, \quad \text{for all } \boldsymbol{\beta} \in \mathcal{I}(N, k+1).$$

**B:** If  $\mathbf{q}$  is the homogeneous polynomial given by

$$(2.2) \quad \mathbf{q} = \sum_{\ell=1}^N \left( \sum_{\boldsymbol{\alpha} \in \mathcal{I}(N, k)} c_{\boldsymbol{\alpha}, \ell} \mathbf{x}^{\boldsymbol{\alpha}} \right) \mathbf{e}_\ell,$$

it is in  $\mathbf{R}_k$  if and only if

$$(2.3) \quad \sum_{\ell=1}^N c_{\beta-\mathbf{e}_{\ell},\ell} = 0, \quad \text{for all } \beta \in \mathcal{I}(N, k+1),$$

where we understand that  $c_{\beta-\mathbf{e}_{\ell},\ell} = 0$  whenever any component of  $\beta - \mathbf{e}_{\ell}$  is negative.

*Remark 2.1.* Note that Statement B of the theorem characterizes  $\mathbf{S}_k$ . Since it is proved in [7] that  $\mathbf{q} \in \mathbf{S}_k$  if and only if  $\mathbf{x} \cdot \mathbf{q}(\mathbf{x}) = 0$ , we find that (2.3) is equivalent to  $\mathbf{x} \cdot \mathbf{q}(\mathbf{x}) = 0$ . This equivalence can also be directly seen easily.

*Remark 2.2.* One way to obtain *Nédélec-type spaces* is motivated by Theorem 2.1. By Nédélec-type spaces we mean spaces  $\tilde{\mathbf{R}}_k$  that satisfy the “exactness property”  $\nabla P_k = \{\mathbf{q} \in \tilde{\mathbf{R}}_k : \mathbf{curl} \mathbf{q} = 0\}$  (when  $N = 3$ ). Let  $\{a_{\beta,\ell}\}$  be any set of positive numbers. Define

$$\mathbf{S}_k^{(a)} = \left\{ \sum_{\ell=1}^N \sum_{\alpha \in \mathcal{I}(N,k)} c_{\alpha,\ell} \mathbf{x}^{\alpha} \mathbf{e}_{\ell} : \sum_{\ell=1}^N a_{\beta,\ell} c_{\beta-\mathbf{e}_{\ell},\ell} = 0, \text{ for all } \beta \in \mathcal{I}(N, k+1) \right\}.$$

Then it is easy to see that  $\overline{\mathbf{P}}_k = \mathbf{S}_k^{(a)} \oplus \nabla \overline{\mathbf{P}}_{k+1}$ , where  $\overline{\mathbf{P}}_{k+1}$  denote the space of scalar homogeneous polynomials of degree  $k+1$  and  $\overline{\mathbf{P}}_k$  denotes the space of vector polynomials whose components are in  $\overline{\mathbf{P}}_k$ . Hence  $\mathbf{R}_k^{(a)} \equiv \mathbf{P}_{k-1} \oplus \mathbf{S}_k^{(a)}$  defines a Nédélec-type space. When all  $a_{\beta,\ell} = 1$ , the space  $\mathbf{R}_k^{(a)}$  coincides with the Nédélec space. But  $\mathbf{R}_k^{(a)}$  and  $\mathbf{R}_k$  are not equal in general.

We can find a basis for the highest degree part of the Nédélec space, namely  $\mathbf{S}_k$ , by solving for the null space of the matrix implicit in (2.3). This will be done in the proof of the next theorem. To describe the basis, first partition the set of multi-indices  $\mathcal{I}(N, k)$  into

$$\mathcal{I}_j(N, k) = \{\beta \in \mathcal{I}(N, k) : \text{all except } j \text{ of the components of } \beta \text{ are zero}\}.$$

Clearly  $\mathcal{I}(N, k)$  is the disjoint union of the  $\mathcal{I}_1(N, k), \mathcal{I}_2(N, k), \dots, \mathcal{I}_N(N, k)$ . Note that sets  $\mathcal{I}_j(N, k+1)$  are empty for  $j > k+1$ . For  $N = 3$ , there is a natural correspondence between  $\mathcal{I}_j(N, k+1), j = 1, 2, 3$  and Nédélec’s edge, face and interior degrees of freedom. (This is made clearer after Theorem 3.1.) Consider  $\beta \in \mathcal{I}_j(N, k+1)$  and let  $\ell(1), \ell(2), \dots, \ell(j)$  be integers such that  $\beta_{\ell(m)} > 0$  and  $\beta_{\ell} = 0$  for all  $\ell \notin \{\ell(1), \dots, \ell(j)\}$ . Define the collection of  $j-1$  functions  $\mathcal{B}_k^{\beta}$  for  $j > 1$  by

$$\mathcal{B}_k^{\beta} = \{\mathbf{x}^{\beta-\mathbf{e}_{\ell(m)}} \mathbf{e}_{\ell(m)} - \mathbf{x}^{\beta-\mathbf{e}_{\ell(m+1)}} \mathbf{e}_{\ell(m+1)} : m = 1, 2, \dots, j-1\}.$$

Also define

$$\mathcal{B}_k^{(j)} = \bigcup_{\beta \in \mathcal{I}_j(N, k+1)} \mathcal{B}_k^{\beta}.$$

Note that in some cases  $\mathcal{B}_k^{(j)}$  is empty. For example, if  $N = 3$ , the set  $\mathcal{B}_1^{(3)}$  is empty.

**Theorem 2.2.** *The set  $\mathcal{B}_k \equiv \mathcal{B}_k^{(2)} \cup \mathcal{B}_k^{(3)} \cup \dots \cup \mathcal{B}_k^{(N)}$  is a basis for  $\mathbf{S}_k$ .*

We will now prove both the theorems.

*Proof of Theorem 2.1.* By Nédélec's characterization,  $\mathbf{q}$  is in  $\mathbf{R}_k$  if and only if  $\varepsilon^k(\mathbf{q}) = 0$ . We will show that  $\varepsilon^k(\mathbf{q}) = 0$  if and only if (2.3) holds. Let us first obviate some of the difficulties in dealing with the definition (2.1) by observing a simplification of  $\varepsilon^k$ : Since the Fréchet derivative  $\mathbf{d}^k \mathbf{q}(\mathbf{r}_1, \dots, \mathbf{r}_k)$  is symmetric with respect to interchanges of  $\mathbf{r}_i$ ,  $\varepsilon^k$  simplifies to

$$(2.4) \quad \varepsilon^k(\mathbf{q})(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_{k+1}) = \frac{1}{k+1} \sum_{j=1}^{k+1} \mathbf{r}_j \cdot \mathbf{d}^k \mathbf{q}(\mathbf{r}_1, \dots, \mathbf{r}_{j-1}, \mathbf{r}_{j+1}, \dots, \mathbf{r}_{k+1}).$$

Since  $\varepsilon^k(\mathbf{q})(\mathbf{r}_1, \dots, \mathbf{r}_{k+1})$  is linear in each  $\mathbf{r}_i$ , the function  $\mathbf{q}$  satisfies  $\varepsilon^k(\mathbf{q}) = 0$  if and only if

$$(2.5) \quad \varepsilon^k(\mathbf{q})(\mathbf{e}_{p(1)}, \mathbf{e}_{p(2)}, \dots, \mathbf{e}_{p(k+1)}) = 0$$

for all maps  $p : \{1, 2, \dots, k+1\} \mapsto \{1, 2, \dots, N\}$ . Note that

$$(2.6) \quad \begin{aligned} & \varepsilon^k(\mathbf{q})(\mathbf{e}_{p(1)}, \mathbf{e}_{p(2)}, \dots, \mathbf{e}_{p(k+1)}) \\ &= \frac{1}{k+1} \sum_{j=1}^{k+1} \mathbf{e}_{p(j)} \cdot \mathbf{d}^k \mathbf{q}(\mathbf{e}_{p(1)}, \dots, \mathbf{e}_{p(j-1)}, \mathbf{e}_{p(j+1)}, \dots, \mathbf{e}_{p(k+1)}) \\ &= \frac{1}{k+1} \sum_{j=1}^{k+1} d^k q_{p(j)}(\mathbf{e}_{p(1)}, \dots, \mathbf{e}_{p(j-1)}, \mathbf{e}_{p(j+1)}, \dots, \mathbf{e}_{p(k+1)}) \\ &= \frac{1}{k+1} \sum_{j=1}^{k+1} \frac{\partial^p q_{p(j)}}{\partial x_{p(1)} \partial x_{p(2)} \cdots \partial x_{p(j-1)} \partial x_{p(j+1)} \cdots \partial x_{p(k+1)}}. \end{aligned}$$

Now, construct multi-indices  $\boldsymbol{\beta}^{(p)}$  for each map  $p$  such that  $\boldsymbol{\beta}^{(p)} = (\beta_1, \dots, \beta_N)$  has its component  $\beta_i$  equal to the cardinality of the set  $\{j : p(j) = i\}$ . Then, by (2.6),

$$\varepsilon^k(\mathbf{q})(\mathbf{e}_{p(1)}, \mathbf{e}_{p(2)}, \dots, \mathbf{e}_{p(k+1)}) = \frac{1}{k+1} \sum_{j=1}^{k+1} \partial^{\boldsymbol{\beta}^{(p)} - \mathbf{e}_{p(j)}} q_{p(j)},$$

where we understand all terms involving  $\boldsymbol{\beta}^{(p)} - \mathbf{e}_{p(j)}$  to vanish whenever any component of  $\boldsymbol{\beta}^{(p)} - \mathbf{e}_{p(j)}$  is negative. In the above sum, the term  $\partial^{\boldsymbol{\beta}^{(p)} - \mathbf{e}_i} q_i$  appears exactly as many times as the value  $i$  is attained by the map  $p$ , i.e., exactly  $\beta_i$  times. Hence,

$$\varepsilon^k(\mathbf{q})(\mathbf{e}_{p(1)}, \mathbf{e}_{p(2)}, \dots, \mathbf{e}_{p(k+1)}) = \frac{1}{k+1} \sum_{\ell=1}^N \beta_\ell \partial^{\boldsymbol{\beta}^{(p)} - \mathbf{e}_\ell} q_\ell.$$

Thus, by (2.5),  $\mathbf{q}$  is in the Nédélec space  $\mathbf{R}_k$  if and only if

$$\sum_{\ell=1}^N \beta_\ell \partial^{\boldsymbol{\beta}^{(p)} - \mathbf{e}_\ell} q_\ell = 0$$

for all maps  $p : \{1, 2, \dots, k+1\} \mapsto \{1, 2, \dots, N\}$ .

Observe now that  $\beta^{(p)} \in \mathcal{I}(N, k+1)$ , since the sum  $\beta_1 + \beta_2 + \dots + \beta_N$  equals the cardinality of the domain of  $p$ . Moreover, we can construct any multi-index  $\boldsymbol{\eta} \in \mathcal{I}(N, k+1)$  as  $\beta^{(p)}$  of some map  $p : \{1, 2, \dots, k+1\} \mapsto \{1, 2, \dots, N\}$ . Hence  $\mathbf{q} \in \mathbf{R}_k$  if and only if

$$(2.7) \quad \sum_{\ell=1}^N \eta_\ell \partial^{\boldsymbol{\eta} - \mathbf{e}_\ell} q_\ell = 0, \quad \text{for all } \boldsymbol{\eta} = (\eta_1, \dots, \eta_N) \in \mathcal{I}(N, k+1).$$

This proves Statement A.

To prove Statement B, we apply (2.7) with  $q_\ell$  equal to the  $\ell$ -th component of the expression in (2.2). Using the elementary identity

$$(2.8) \quad \partial^\alpha \mathbf{x}^\beta = \begin{cases} 0 & \text{if } \alpha \neq \beta, \\ \alpha! & \text{if } \alpha = \beta, \end{cases}$$

for all multi-indices  $\alpha, \beta \in \mathcal{I}(N, k+1)$  (where  $\alpha! = \alpha_1! \alpha_2! \dots \alpha_N!$ ), we find that (2.7) for each  $\boldsymbol{\eta} \in \mathcal{I}(N, k+1)$  is equivalent to

$$\sum_{\ell=1}^N \eta_\ell c_{\boldsymbol{\eta} - \mathbf{e}_\ell, \ell} (\boldsymbol{\eta} - \mathbf{e}_\ell)! = \boldsymbol{\eta}! \sum_{\ell=1}^N c_{\boldsymbol{\eta} - \mathbf{e}_\ell, \ell} = 0.$$

This proves the theorem.  $\square$

*Proof of Theorem 2.2.* It is obvious that the functions in  $\mathcal{B}_k$  satisfy (2.3). Hence by Theorem 2.1,  $\mathcal{B}_k \subseteq \mathcal{S}_k$ .

We now show that functions in  $\mathcal{B}_k$  are linearly independent. Any function in  $\mathcal{B}_k$  can be expressed in the form  $\mathbf{x}^{\alpha - \mathbf{e}_i} \mathbf{e}_i - \mathbf{x}^{\alpha - \mathbf{e}_j} \mathbf{e}_j$  for an appropriate multi-index  $\alpha$  and integers  $i, j$ . Obviously, without loss of generality, we can assume that  $i < j$  in all such expressions of basis functions in  $\mathcal{B}_k$ . Let  $\mathcal{C}_\ell = \{\mathbf{b} \in \mathcal{B}_k : \mathbf{b} = \mathbf{x}^{\alpha - \mathbf{e}_\ell} \mathbf{e}_\ell - \mathbf{x}^{\alpha - \mathbf{e}_m} \mathbf{e}_m \text{ with } \ell < m \text{ and some multi-index } \alpha\}$ . Then the sets  $\mathcal{C}_\ell$ ,  $\ell = 1, \dots, N-1$ , form a disjoint partitioning of  $\mathcal{B}_k$ . Hence we can write any linear combination of functions in  $\mathcal{B}_k$  as

$$\mathbf{p}(\mathbf{x}) = \sum_i d_i^{(1)} \mathbf{b}_i^{(1)} + \sum_i d_i^{(2)} \mathbf{b}_i^{(2)} + \dots + \sum_i d_i^{(N-1)} \mathbf{b}_i^{(N-1)},$$

where  $\mathbf{b}_i^{(\ell)} \equiv \mathbf{x}^{\alpha(i, \ell) - \mathbf{e}_\ell} \mathbf{e}_\ell - \mathbf{x}^{\alpha(i, \ell) - \mathbf{e}_{m_i}} \mathbf{e}_{m_i}$ ,  $i = 1, 2, \dots$ , is an enumeration of  $\mathcal{C}_\ell$  and the sums above run over these enumerations. We will now show that if  $\mathbf{p}(\mathbf{x}) = 0$  everywhere, then all the coefficients  $d_i^{(\ell)}$  are zero. Since the first component of the vector function  $\mathbf{p}(\mathbf{x})$  vanishes,

$$\sum_i d_i^{(1)} \mathbf{x}^{\alpha(i, 1) - \mathbf{e}_1} = 0.$$

But for two distinct basis functions  $\mathbf{b}_i^{(\ell)}$  and  $\mathbf{b}_j^{(\ell)}$ , the multi-indices  $\alpha(i, 1)$  and  $\alpha(j, 1)$  are not equal, so  $\mathbf{x}^{\alpha(i, 1) - \mathbf{e}_1} \neq \mathbf{x}^{\alpha(j, 1) - \mathbf{e}_1}$  for  $i \neq j$ . Hence, by the linear independence of distinct monomials, all coefficients  $d_i^{(1)}$  are zero. Next, consider  $\{d_i^{(2)}\}$ . Since the second component of  $\mathbf{p}(\mathbf{x})$  vanishes,

$$\sum_i d_i^{(2)} \mathbf{x}^{\alpha(i, 2) - \mathbf{e}_2} = 0.$$

Note that in this sum there are no terms of the form  $-d_i^{(1)} \mathbf{x}^{\alpha(i,1)-\mathbf{e}_1}$  because we have already shown that  $d_i^{(1)} = 0$ . By an argument similar to the above one, we prove that  $d_i^{(2)} = 0$ . Continuing, we find that all  $d_i^{(\ell)} = 0$ . Hence  $\mathcal{B}_k$  is a linearly independent set.

It now only remains to prove that the cardinality of  $\mathcal{B}_k$  equals the dimension of  $\mathcal{S}_k$ . It is easy to count the dimension of  $\mathcal{S}_k$ : For  $\alpha, \beta \in \mathcal{I}(N, k)$ ,  $\alpha \neq \beta$ , the conditions

$$\sum_{\ell=1}^N c_{\alpha-\mathbf{e}_{\ell}, \ell} = 0 \quad \text{and} \quad \sum_{\ell=1}^N c_{\beta-\mathbf{e}_{\ell}, \ell} = 0$$

are independent. Indeed, none of the terms in the first sum occurs in the second sum. Therefore,

$$(2.9) \quad \dim(\mathcal{S}_k) = N \operatorname{card}(\mathcal{I}(N, k)) - \operatorname{card}(\mathcal{I}(N, k+1)).$$

On the other hand,

$$(2.10) \quad \begin{aligned} \operatorname{card} \mathcal{B}_k &= \sum_{j=2}^N \operatorname{card} \mathcal{B}_k^{(j)} = \sum_{j=2}^N \sum_{\beta \in \mathcal{I}_j(N, k+1)} \operatorname{card} \mathcal{B}_k^{\beta} \\ &= \sum_{j=2}^N (j-1) \operatorname{card} \mathcal{I}_j(N, k+1). \end{aligned}$$

We can compute  $\operatorname{card} \mathcal{I}_j(N, k+1)$  using the recursion

$$\operatorname{card} \mathcal{I}_j(N, k+1) = \binom{N}{j} \left( \operatorname{card} \mathcal{I}(j, k+1) - \sum_{m=1}^{j-1} \operatorname{card} \mathcal{I}_m(j, k+1) \right),$$

and observing that  $\operatorname{card} \mathcal{I}_2(N, k+1) = \binom{N}{2} k$ . By induction,

$$\operatorname{card} \mathcal{I}_j(N, k+1) = \binom{N}{j} \binom{k}{j-1}.$$

Hence, by (2.10), we find that  $\operatorname{card} \mathcal{B}_k$  equals the sum

$$s_{N,k} \equiv \sum_{j=2}^N (j-1) \binom{N}{j} \binom{k}{j-1},$$

where we use the convention that  $\binom{i}{j}$  is zero if  $i < j$ . Now using the identity

$$k \binom{N}{j} \binom{k-1}{j-1} - N \binom{k}{j-1} \binom{N-1}{j} = (k-N+1) \binom{N}{j} \binom{k}{j-1},$$

we find that  $s_{N,k} = (k s_{N,k-1} - N s_{N-1,k}) / (k - N + 1)$ . With the help of this identity it is easy to simplify the sum  $s_{N,k}$  to closed form:

$$\operatorname{card} \mathcal{B}_k = s_{N,k} = \frac{(N+k-1)! k}{(N-2)! (k+1)!}.$$

The right hand side expression can now be easily seen to equal  $\dim \mathcal{S}_k$  given in (2.9).  $\square$

Degree	Basis functions		
	Interior	Face( $l, m, n$ )	Edge( $l, m$ )
$k = 1$	None	None	$\lambda_l \nabla \lambda_m - \lambda_m \nabla \lambda_l$
$k = 2$	None	$\lambda_l \lambda_m \nabla \lambda_n - \lambda_m \lambda_n \nabla \lambda_l$ $\lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l \nabla \lambda_m$	$\lambda_l \lambda_m \nabla \lambda_l - \lambda_l^2 \nabla \lambda_m$ $\lambda_m^2 \nabla \lambda_l - \lambda_l \lambda_m \nabla \lambda_m$
$k = 3$	$\lambda_2 \lambda_3 \lambda_4 \nabla \lambda_1 - \lambda_1 \lambda_2 \lambda_3 \nabla \lambda_4$ $\lambda_3 \lambda_4 \lambda_1 \nabla \lambda_2 - \lambda_2 \lambda_3 \lambda_4 \nabla \lambda_1$ $\lambda_4 \lambda_1 \lambda_2 \nabla \lambda_3 - \lambda_3 \lambda_4 \lambda_1 \nabla \lambda_2$	$\lambda_l^2 \lambda_m \nabla \lambda_n - \lambda_l \lambda_m \lambda_n \nabla \lambda_l$ $\lambda_l \lambda_m^2 \nabla \lambda_n - \lambda_m^2 \lambda_n \nabla \lambda_l$ $\lambda_n \lambda_l \lambda_m \nabla \lambda_n - \lambda_m \lambda_n^2 \nabla \lambda_l$ $\lambda_l \lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l^2 \nabla \lambda_m$ $\lambda_m^2 \lambda_n \nabla \lambda_l - \lambda_m \lambda_n \lambda_l \nabla \lambda_m$ $\lambda_m \lambda_n^2 \nabla \lambda_l - \lambda_n^2 \lambda_l \nabla \lambda_m$	$\lambda_l^2 \lambda_m \nabla \lambda_l - \lambda_l^3 \nabla \lambda_m$ $\lambda_m^2 \lambda_l \nabla \lambda_m - \lambda_m^3 \nabla \lambda_l$ $\lambda_l \lambda_m^2 \nabla \lambda_l - \lambda_l^2 \lambda_m \nabla \lambda_m$

TABLE 3.1. Basis for Nédélec spaces on a tetrahedron in barycentric coordinates for some degrees

### 3. A BASIS IN AFFINE COORDINATES

In this section we use the previous results to construct a basis for  $\mathbf{R}_k$  on an  $N$ -simplex in terms of barycentric coordinates. Let  $\lambda_\ell$ ,  $\ell = 1, 2, \dots, N + 1$  denote the barycentric coordinate functions of the  $N$ -simplex and set

$$\boldsymbol{\lambda} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{N+1} \end{bmatrix}.$$

We will need to use our basis for  $\mathbf{S}_k$ , as given by Theorem 2.2, but in one dimension higher. To emphasize the dimension, we shall occasionally use an additional subscript in our notations: E.g.,  $\mathbf{S}_{k,N}$ ,  $\mathcal{B}_{k,N}$ , and  $\mathcal{B}_{k,N}^{(j)}$  are the same as  $\mathbf{S}_k$ ,  $\mathcal{B}_k$ , and  $\mathcal{B}_k^{(j)}$  defined in  $N$  space dimensions, respectively. Now, pick any function in the basis for  $\mathbf{S}_{k,N+1}$  given by Theorem 2.2, and write it as

$$(3.1) \quad b_\beta(\mathbf{x}, \mathbf{e}_l, \mathbf{e}_m) \equiv \mathbf{x}^{\beta - \mathbf{e}_l} \mathbf{e}_l - \mathbf{x}^{\beta - \mathbf{e}_m} \mathbf{e}_m,$$

for some  $\beta \in \mathcal{I}(N + 1, k + 1)$  and some integers  $l$  and  $m$ . Replace  $\mathbf{x}$  by  $\boldsymbol{\lambda}$  in (3.1). (We are now viewing both  $\mathbf{x}$  and  $\boldsymbol{\lambda}$  as vectors in  $\mathbb{R}^{N+1}$ .) Also replace  $\mathbf{e}_j$  by  $\nabla \lambda_j$  for  $j = l$  and  $m$  in (3.1) to get functions

$$(3.2) \quad b_\beta(\boldsymbol{\lambda}, \nabla \lambda_l, \nabla \lambda_m) = \boldsymbol{\lambda}^{\beta - \mathbf{e}_l} \nabla \lambda_l - \boldsymbol{\lambda}^{\beta - \mathbf{e}_m} \nabla \lambda_m,$$

for same multi-indices  $\beta$  allowed in (3.1). At this point, this is a formal replacement (because  $\mathbf{e}_j$  and  $\nabla \lambda_j$  are vectors of different dimensions), but we will justify it later. Applying the transformation (3.1)  $\mapsto$  (3.2) to every element of  $\mathcal{B}_{k,N+1}^{(j)}$  we get a new set of functions:

$$\Lambda_k^{(j)} \equiv \{b_\beta(\boldsymbol{\lambda}, \nabla \lambda_l, \nabla \lambda_m) : b_\beta(\mathbf{x}, \mathbf{e}_l, \mathbf{e}_m) \in \mathcal{B}_k^{(j)}\}.$$

Note that although  $\mathcal{B}_{k,N+1}$  consisted of functions defined on  $\mathbb{R}^{N+1}$ , the new set  $\Lambda_k^{(j)}$  consists of functions defined on  $\mathbb{R}^N$ . (Now we view  $\boldsymbol{\lambda}$  as a function on  $\mathbb{R}^N$ .)



**Theorem 3.1.** *The set  $\Lambda_k^{(2)} \cup \Lambda_k^{(k)} \cup \dots \cup \Lambda_k^{(N+1)}$  is a basis in affine coordinates for the Nédélec space in  $N$  dimensions. (The basis functions are of the form (3.2) for some multi-indices  $\beta \in \mathcal{I}_j(N+1, k+1)$ ,  $j \geq 2$ .)*

Let us now discuss in some detail the application of our results to the particular case of a tetrahedron with vertices  $\mathbf{v}_1, \dots, \mathbf{v}_4$ . Let the barycentric coordinate functions be enumerated such that  $\lambda_i(\mathbf{v}_j) = \delta_{ij}$ . The face of the tetrahedron formed by vertices  $\mathbf{v}_l, \mathbf{v}_m, \mathbf{v}_n$  is called  $\text{Face}(l, m, n)$  and its edge formed by vertices  $\mathbf{v}_l, \mathbf{v}_m$  is called  $\text{Edge}(l, m)$ . Table 3 lists basis functions in affine coordinates for  $\mathbf{R}_k$  for some  $k$ , as given by (3.2).

The partitioning of the basis into  $\Lambda_k^{(2)}$ ,  $\Lambda_k^{(3)}$ , and  $\Lambda_k^{(4)}$  now has a geometrical interpretation considering the geometry of the tetrahedron. The set  $\Lambda_k^{(4)}$  gives “interior” basis functions in the sense that the face and edge Nédélec degrees of freedom (see [7]) are zero for these functions. Indeed, functions in  $\Lambda_k^{(4)}$  can be expressed as a difference of two terms each of which has a factor of the form  $\lambda_p \lambda_l \lambda_m \nabla \lambda_n$ . This factor is zero on faces containing the vertex  $\mathbf{v}_n$ , while on  $\text{Face}(p, l, m)$  its tangential component is zero. Hence functions in  $\Lambda_k^{(4)}$  have zero face and edge Nédélec moments.

The set  $\Lambda_k^{(3)}$  gives “face” basis functions in the sense that these functions have edge Nédélec degrees of freedom equal to zero. Indeed, these functions (see (3.2)) are of the form

$$\lambda_l^r \lambda_m^s \lambda_n^t (\lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l \nabla \lambda_m),$$

for some nonnegative powers  $r, s$ , and  $t$ . It is easily verified that the tangential components of the function  $\lambda_m \lambda_n \nabla \lambda_l - \lambda_n \lambda_l \nabla \lambda_m$  is zero on all faces except  $\text{Face}(l, m, n)$ . Its tangential component is zero on all edges of the tetrahedron.

The remaining functions (those in  $\Lambda_k^{(2)}$ ) are “edge” basis functions. These functions (again by (3.2)) take the form

$$\lambda_l^r \lambda_m^s (\lambda_l \nabla \lambda_m - \lambda_m \nabla \lambda_l).$$

Their tangential component is nonzero on  $\text{Edge}(l, m)$ , but vanishes on every other edge of the tetrahedron.

Note that our edge basis functions may not have zero Nédélec face degrees of freedom and our face basis functions may not have zero interior moments. In the unlikely event that nodal basis functions dual to the original Nédélec degrees of freedom [7] are needed, they can be easily written out in terms of the affine expressions we introduced. For example, to construct a face basis function that has all its interior Nédélec degrees of freedom equal to zero, we only need to pick up one of our face basis functions exhibited above and subtract from it a linear combination of our interior basis functions. This will suffice, because by construction, our interior basis functions have no face moments. Similarly, to get an edge basis function that has zero face and interior moments, we only need to take one of our edge basis functions and subtract from it a linear combination of face and interior basis functions.

In the remainder of this section, we prove Theorem 3.1. The proof and the construction of the basis is motivated by well known correspondences between homogeneous polynomials and the full set of polynomials. As before, let us use an additional subscript to emphasize dimension and write  $P_{k,N}$  for the set of polynomials of degree  $k$  in the  $N$ -dimensional variable  $\mathbf{x}$ . Let  $\overline{P}_{k,N+1}$  denote the space of homogeneous polynomials

of degree  $k$  in the variable  $\widehat{\mathbf{x}} \in \mathbb{R}^{N+1}$ . Writing any  $p \in \overline{P}_{k,N+1}$  as  $p = \sum_{\alpha \in \mathcal{I}(N+1,k)} c_\alpha \widehat{\mathbf{x}}^\alpha$ , we define the transformation  $A_0 : \overline{P}_{k,N+1} \mapsto P_{k,N}$  by  $A_0 p = \sum_{\alpha \in \mathcal{I}(N+1,k)} c_\alpha \boldsymbol{\lambda}^\alpha$ . Since  $\boldsymbol{\lambda}(\mathbf{x})$  represents barycentric coordinates of some  $N$ -simplex in the  $N$ -dimensional  $\mathbf{x}$ -space,  $A_0 p$  is a polynomial in  $P_{k,N}$ . The procedure that maps basis functions of  $\mathbf{S}_{k,N+1}$  to a basis of  $\mathbf{R}_{k,N}$  defines a transformation, which although not  $A_0$ , is related to  $A_0$  as follows: It is a map  $\mathbf{A}$  that makes the following diagram commute:

$$\begin{array}{ccc} \overline{P}_{k+1,N+1} & \xrightarrow{\nabla_{\widehat{\mathbf{x}}}} & \overline{P}_{k,N+1} \\ \downarrow A_0 & & \downarrow \mathbf{A} \\ P_{k+1,N} & \xrightarrow{\nabla_{\mathbf{x}}} & P_{k,N}, \end{array}$$

where the subscripts on  $\nabla$  indicate the differentiation variable. We define  $\mathbf{A}$  precisely in the following proof.

*Proof of Theorem 3.1.* This proof proceeds by showing that the transformation of (3.1) into (3.2) can be extended to a homeomorphism  $\mathbf{A}$  from  $\mathbf{S}_{k,N+1}$  to the Nédélec space  $\mathbf{R}_{k,N}$ . To define  $\mathbf{A}$ , let  $\mathbf{x} \in \mathbb{R}^N$  denote the physical coordinate, so that the the first  $N$  barycentric coordinates are related to  $\mathbf{x}$  via an invertible matrix  $B \in \mathbb{R}^{N \times N}$  and a vector  $\mathbf{b} \in \mathbb{R}^N$  such that

$$\begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} = B\mathbf{x} + \mathbf{b}$$

Let  $n = [-1, -1, \dots, -1]$  and  $M$  be the  $(N+1) \times N$  matrix given by

$$M = \begin{bmatrix} B \\ nB \end{bmatrix}$$

Then, since  $\boldsymbol{\lambda} = M\mathbf{x} + \mathbf{d}$  for some  $\mathbf{d} \in \mathbb{R}^{N+1}$ ,

$$(3.3) \quad \nabla \lambda_\ell = M^t \mathbf{e}_\ell, \quad \text{for all } \ell = 1, 2, \dots, N+1.$$

Define  $\mathbf{A} : \mathbf{S}_{k,N+1} \mapsto \mathbf{P}_{k,N}$  by

$$(\mathbf{A}\widehat{\mathbf{q}})(\mathbf{x}) = M^t \widehat{\mathbf{q}}(\boldsymbol{\lambda}(\mathbf{x}))$$

for any  $\widehat{\mathbf{q}} \in \mathbf{S}_{k,N+1}$ .

Note that the transformation (3.1) $\mapsto$ (3.2) is the restriction of  $\mathbf{A}$  to  $\mathcal{B}_{k,N+1}$ . Indeed, expressing  $\widehat{\mathbf{q}} \in \mathbf{S}_{k,N+1}$  as the function  $\widehat{\mathbf{q}}(\widehat{\mathbf{x}})$  of an  $N+1$  dimensional coordinate variable  $\widehat{\mathbf{x}}$ , the first step in the transformation (3.1) $\mapsto$ (3.2) is the replacement of  $\widehat{\mathbf{x}}$  by  $\boldsymbol{\lambda}$ . The next step is the replacement of  $\mathbf{e}_\ell$  by  $\nabla \lambda_\ell$ . By (3.3), the end result of these replacements is the function  $\mathbf{A}\widehat{\mathbf{q}}$  (of the  $N$ -dimensional variable  $\mathbf{x}$ ).

We will now show that the range of the map  $\mathbf{A}$ , namely

$$\widetilde{\mathbf{R}}_{k,N} = \{\mathbf{r}(\mathbf{x}) : \mathbf{r} = \mathbf{A}\widehat{\mathbf{q}} \text{ for some } \widehat{\mathbf{q}} \in \mathbf{S}_{k,N+1}\}$$

equals the Nédélec space  $\mathbf{R}_{k,N}$  and that  $\mathbf{A}$  is injective. To show that  $\widetilde{\mathbf{R}}_{k,N} \subseteq \mathbf{R}_{k,N}$ , we use the following identity which can easily be verified using (2.4): For all  $\mathbf{v}_i \in \mathbb{R}^N$ ,

$$(3.4) \quad \varepsilon_{N+1}^k(\widehat{\mathbf{q}})(M\mathbf{v}_1, \dots, M\mathbf{v}_{k+1}) = \varepsilon_N^k(\mathbf{A}\widehat{\mathbf{q}})(\mathbf{v}_1, \dots, \mathbf{v}_{k+1}),$$

where the subscripts of  $\varepsilon^k$  distinguishes the domain of the symmetrization operator to be of  $N$  or  $N + 1$  dimensional functions. Since  $\widehat{\mathbf{q}} \in \mathbf{S}_{k,N+1}$ , the left hand side is zero, so  $\mathbf{A}\widehat{\mathbf{q}} \in \mathbf{R}_{k,N}$ . Thus  $\widetilde{\mathbf{R}}_{k,N} \subseteq \mathbf{R}_{k,N}$ .

Now, to show that  $\widetilde{\mathbf{R}}_{k,N} = \mathbf{R}_{k,N}$ , it suffices to prove that the map  $\mathbf{A}$  is injective. Suppose there is a  $\widehat{\mathbf{q}}(\widehat{\mathbf{x}}) \in \mathbf{S}_{k,N+1}$  such that  $(\mathbf{A}\widehat{\mathbf{q}})(\mathbf{x}) = 0$  for all  $\mathbf{x}$ . Then,  $M^t\widehat{\mathbf{q}}(\boldsymbol{\lambda}(\mathbf{x})) = 0$ . Since the principal  $N \times N$  submatrix of  $M$  is invertible, this implies that the first  $N$  components of  $\widehat{\mathbf{q}}(\boldsymbol{\lambda}(\mathbf{x}))$  equal its last component. Thus

$$\widehat{\mathbf{q}}(\boldsymbol{\lambda}(\mathbf{x})) = \phi_k(\mathbf{x})\mathbf{t}, \quad \text{where } \mathbf{t} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

for some scalar polynomial  $\phi_k$  of degree  $k$ . Since  $\widehat{\mathbf{x}} \cdot \widehat{\mathbf{q}}(\widehat{\mathbf{x}})$  vanishes, we find that

$$\boldsymbol{\lambda}(\mathbf{x}) \cdot \widehat{\mathbf{q}}(\boldsymbol{\lambda}(\mathbf{x})) = \boldsymbol{\lambda}(\mathbf{x}) \cdot (\phi_k(\mathbf{x})\mathbf{t}) = \phi_k(\mathbf{x}) = 0$$

for all  $\mathbf{x}$ . Thus,  $\widehat{\mathbf{q}}(\widehat{\mathbf{x}})$  vanishes on the hyperplane  $\{\widehat{\mathbf{x}} : \widehat{\mathbf{x}} \cdot \mathbf{t} = 1\}$  in  $\mathbb{R}^{N+1}$ . Since homogeneous polynomials that vanish on a plane not containing the origin vanish everywhere, we conclude that  $\widehat{\mathbf{q}}$  vanishes everywhere and  $\mathbf{A}$  is injective.

The injectivity of  $\mathbf{A}$  also proves that a basis of  $\mathbf{S}_{k,N+1}$  is mapped to a basis of  $\mathbf{R}_{k,N}$ . Since Theorem 2.2 asserts that  $\mathbf{B}_{k,N+1}$  is a basis for  $\mathbf{S}_{k,N+1}$ , we conclude that  $\mathbf{A}\mathbf{B}_{k,N+1}$  gives a basis for  $\mathbf{R}_{k,N}$ .  $\square$

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