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Cockburn B, Gopalakrishnan J. (2004). A Characterization of Hybridized Mixed Methods for Second Order Elliptic Problems. SIAM Journal On Numerical Analysis, 42(1):283-301.

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SIAM J. NUMER. ANAL.
Vol. 42, No. 1, pp. 283–301
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A CHARACTERIZATION OF HYBRIDIZED MIXED METHODS FOR SECOND ORDER ELLIPTIC PROBLEMS*

BERNARDO COCKBURN† AND JAYADEEP GOPALAKRISHNAN‡

This paper is dedicated to Jim Douglas, Jr., on the occasion of his 75th birthday

Abstract. In this paper, we give a new characterization of the approximate solution given by hybridized mixed methods for second order self-adjoint elliptic problems. We apply this characterization to obtain an explicit formula for the entries of the matrix equation for the Lagrange multiplier unknowns resulting from hybridization. We also obtain necessary and sufficient conditions under which the multipliers of the Raviart–Thomas and the Brezzi–Douglas–Marini methods of similar order are identical.

Key words. mixed finite elements, hybrid methods, elliptic problems

AMS subject classifications. Primary, 54C40, 14E20; Secondary, 46E25, 20C20

DOI. 10.1137/S0036142902417893

1. Introduction.

In this paper, we give a new characterization of hybridized mixed methods. This characterization allows us to obtain an explicit formula for the entries of the matrix equation for the so-called Lagrange multipliers. It also allows comparison of hybridized versions of different mixed methods. For example, we give conditions under which the multipliers of the Raviart–Thomas (RT) method and those of the Brezzi–Douglas–Marini (BDM) method of comparable order coincide.

We consider the hybridized version [1] of the standard RT mixed method [13] for the elliptic boundary value problem

$$-\nabla \cdot (\, a\nabla u\,) + d\, u = f \qquad \text{in } \Omega \subset \mathbb{R}^2, \tag{1.1}$$

$$u = g \qquad \text{on } \partial\Omega, \tag{1.2}$$

where $a(\boldsymbol{x})$ is a symmetric positive definite matrix-valued function, $d(\boldsymbol{x})$ is a nonnegative function, and Ω is a polygonal domain in $\mathbb{R}^2$. We assume that $a(\boldsymbol{x})$ and $d(\boldsymbol{x})$ are bounded. We consider this a simple setting for transparent presentation of the main ideas. As will be clear later, our techniques can be applied to other hybridized methods and more general second order elliptic problems.

Before describing the results, recall that mixed finite element methods seek approximations $(\boldsymbol{q}_h, u_h)$ to $(-a\,\nabla u, u)$ in appropriate finite element spaces. They give rise to a matrix equation of the form

$$\begin{pmatrix} \mathbb{A} & -\mathbb{B}^t \\ \mathbb{B} & \mathbb{D} \end{pmatrix} \begin{pmatrix} \mathbb{Q} \\ \mathbb{U} \end{pmatrix} = \begin{pmatrix} \mathbb{G} \\ \mathbb{F} \end{pmatrix},$$

where $\mathbb{Q}$ and $\mathbb{U}$ are the vectors of coefficients of $\boldsymbol{q}_h$ and u_h with respect to their corresponding finite element basis, respectively. Since the system is not positive definite,

*Received by the editors November 13, 2002; accepted for publication (in revised form) July 28, 2003; published electronically January 28, 2004.
http://www.siam.org/journals/sinum/42-1/41789.html

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solving for $\mathbb{Q}$ and $\mathbb{U}$ is not always easy. Although one can arrive at a positive definite system by elimination of $\mathbb{Q}$ from the equations, this requires inverting $\mathbb{A}$ and maintaining $\mathbb{A}^{-1}$, which is typically a full matrix. Fortunately, by "hybridizing" the mixed method, this difficulty can be overcome. Let us now briefly recall the hybridization procedure.

First, the so-called Lagrange multiplier λ_h is introduced. This gives rise to a matrix equation of the form

$$\begin{pmatrix} A & -B^t & -C^t \\ B & D & 0 \\ C & 0 & 0 \end{pmatrix} \begin{pmatrix} Q \\ U \\ \Lambda \end{pmatrix} = \begin{pmatrix} \mathcal{G} \\ \mathcal{F} \\ 0 \end{pmatrix}, \tag{1.3}$$

where Λ is the vector of degrees of freedom associated to the multiplier λ_h. We will precisely state the underlying finite element spaces later. As is now well known, the new vectors of degrees of freedom Q and U actually define the *same* approximation $(\boldsymbol{q}_h, u_h)$ as the original mixed method. Moreover, both Q and U can now be easily eliminated to obtain an equation for the multiplier only, namely,

$$\mathbb{E}\,\Lambda = \mathbb{H},$$

where $\mathbb{E}$ and $\mathbb{H}$ are given by

$$\begin{aligned} \mathbb{E} &= CA^{-1}\left(A - B^t(BA^{-1}B^t + D)^{-1}B\right)A^{-1}C^t, \\ \mathbb{H} &= \mathbb{H}_g + \mathbb{H}_f, \\ \mathbb{H}_g &= -CA^{-1}\left(A - B^t(BA^{-1}B^t + D)^{-1}B\right)A^{-1}\mathcal{G}, \\ \mathbb{H}_f &= -CA^{-1}B^t(BA^{-1}B^t + D)^{-1}\,\mathcal{F}. \end{aligned} \tag{1.4}$$

That the inverses taken above exist follows from the properties of the underlying finite element spaces. Considering this matrix equation instead of the previous one has several advantages: (i) the matrix $\mathbb{E}$ is symmetric and positive definite, so it can be numerically inverted by using methods like the conjugate gradient method; (ii) the number of degrees of freedom of the multiplier is remarkably smaller than the number of degrees of freedom of the original mixed method; (iii) once Λ has been obtained, both Q and U can be efficiently computed element by element; and (iv) the multiplier λ_h can actually be used to *improve* the approximation to u by means of a local postprocessing, as shown in [1]. This shows that the use of hybridized mixed methods is indeed very advantageous; however, the complicated relation between the matrices $\mathbb{E}$ and $\mathbb{H}$, and the matrices A, B, C, $\mathcal{F}$ and $\mathcal{G}$, can easily dissuade one from basing an implementation on $\mathbb{E}$ and $\mathbb{H}$.

In this paper, we show that the entries of the matrices $\mathbb{E}$ and $\mathbb{H}$ can be expressed as a weighted L^2-inner product of some *discontinuous* auxiliary functions, the weights being nothing but the matrix a^{-1} and the function d. These auxiliary functions are easily constructed in terms of the geometry of the mesh, the matrix a, the function d, and the spaces of the hybridized mixed finite element method. Their definition induces a natural decomposition of the approximate solution $(\boldsymbol{q}_h, u_h)$ of the form

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{q}_h, u_h)_{\lambda_h} + (\boldsymbol{q}_h, u_h)_g + (\boldsymbol{q}_h, u_h)_f,$$

where $(\boldsymbol{q}_h, u_h)_{\lambda_h}$ is a *lifting* of the Lagrange multiplier λ_h and $(\boldsymbol{q}_h, u_h)_g$ and $(\boldsymbol{q}_h, u_h)_f$ can be computed *locally* only in terms of the data. The introduction of other discrete

lifting operators has proved useful in another context earlier, namely, the analysis of discontinuous Galerkin methods for elliptic problems [2, 7, 8].

We then present two applications of this result. As a first application, we present a technique to assemble the matrix of the Lagrange multiplier equation using simple local element matrices. Next, we compare the matrices $\mathbb{E}$ and $\mathbb{H}$ of the RT method with those of the BDM method of similar order and give necessary and sufficient conditions for the multipliers to be *exactly* the same. This happens, for example, when $d = 0$ and $f = 0$, a case that occurs in many situations of practical interest.

The paper is organized as follows. In section 2, we introduce the hybridized version of the mixed method of Raviart and Thomas and then state, discuss, and prove the characterization result, Theorem 2.1. In section 3, we show how to assemble the matrices for the multipliers, and in section 4, we compare the matrices of the RT method with those of the BDM method of similar order. Finally, in section 5, we end with some concluding remarks.

2. The main result. We begin this section by introducing the classical mixed method of Raviart and Thomas [13]; then, following [1], we hybridize the method. Finally, we state, discuss, and prove the main result, Theorem 2.1.

2.1. The hybridized mixed method. Given a triangulation of Ω, $\mathcal{T}_h$, made of triangles, the mixed method seeks an approximation $(\boldsymbol{q}_h, u_h)$ to the solution $(\boldsymbol{q}, u)$ of the model problem

$$c\,\boldsymbol{q} = -\nabla u \qquad \text{in } \Omega, \tag{2.1}$$

$$\nabla \cdot \boldsymbol{q} + d\,u = f \qquad \text{in } \Omega, \tag{2.2}$$

$$u = g \qquad \text{on } \partial\Omega, \tag{2.3}$$

where $c = a^{-1}$. The approximation $(\boldsymbol{q}_h, u_h)$ is sought in the finite element space $\mathbb{V}_h \times \mathbb{W}_h$ given by

$$\mathbb{V}_h = \{\boldsymbol{v} \in H(\mathrm{div}, \Omega) : \quad \boldsymbol{v}|_K \in P^k(K) \times P^k(K) + \boldsymbol{x}\, P^k(K) \quad \text{for all } \ K \in \mathcal{T}_h\},$$
$$\mathbb{W}_h = \{w \in L^2(\Omega) : \quad w|_K \in P^k(K) \quad \text{for all } \ K \in \mathcal{T}_h\},$$

where $P^k(K)$ denotes the space of polynomials on K of degree at most k, $k \geq 0$, and is defined by requiring that, for all $(\boldsymbol{v}, w) \in V_h \times W_h$,

$$\int_\Omega c\,\boldsymbol{q}_h \cdot \boldsymbol{v}\,dx - \int_\Omega u_h \nabla \cdot \boldsymbol{v}\,dx = -\int_{\partial\Omega} g\,\boldsymbol{v} \cdot \boldsymbol{n}\,ds, \tag{2.4}$$

$$\int_\Omega w \nabla \cdot \boldsymbol{q}_h\,dx + \int_\Omega d\,u_h\,w\,dx = \int_\Omega f\,w\,dx. \tag{2.5}$$

It is easy to see that the above weak formulation gives rise to a system of equations of the form

$$\begin{pmatrix} \mathbb{A} & -\mathbb{B}^t \\ \mathbb{B} & \mathbb{D} \end{pmatrix} \begin{pmatrix} \mathbb{Q} \\ \mathbb{U} \end{pmatrix} = \begin{pmatrix} \mathbb{G} \\ \mathbb{F} \end{pmatrix}.$$

We can try to solve this equation by first eliminating $\mathbb{Q}$ from the equations and then solving the resulting equation for $\mathbb{U}$, namely,

$$\left(\mathbb{B}\,\mathbb{A}^{-1}\mathbb{B}^t + \mathbb{D}\right)\,\mathbb{U} = \mathbb{F} + \mathbb{B}\,\mathbb{A}^{-1}\mathbb{G}.$$

Unfortunately, the matrix $\mathbb{A}$ is not easy to invert since the elements of V_h, being functions in $H(\mathrm{div},\Omega)$, have their normal components continuous across element interfaces. If $\boldsymbol{q}_h$ were totally discontinuous, $\mathbb{A}$ would be block-diagonal and hence easily invertible. The idea of the hybridized mixed methods is to relax this continuity constraint to render $\mathbb{A}$ block-diagonal.

Indeed, it was Fraejis de Veubeque [10], back in 1965, who realized that this can be achieved by introducing additional unknowns λ_h, associated to element interfaces, called Lagrange multipliers. Let $\mathcal{E}_{i,h}$ denote the set of edges of the mesh $\mathcal{T}_h$ that are in the interior of the domain Ω. The multipliers are then nothing but approximations to the trace of u on each $e \in \mathcal{E}_{i,h}$. As we show next, their introduction allows elimination of *both* $\boldsymbol{q}_h$ and u_h and reduction of the system to a single matrix equation for the multipliers.

In our particular case, the hybridized mixed method seeks an approximation $(\boldsymbol{q}_h, u_h, \lambda_h)$ to $(\boldsymbol{q}, u, u|_{\mathcal{E}_{i,h}})$ in the finite element space $V_h \times W_h \times M_h$ given by

$$\begin{aligned}
V_h &= \{\boldsymbol{v} \in L^2(\Omega)\times L^2(\Omega): \quad \boldsymbol{v}|_K \in P^k(K)\times P^k(K) + \boldsymbol{x}P^k(K) \quad \text{for all } K \in \mathcal{T}_h\},\\
W_h &= \{w \in L^2(\Omega): \quad w|_K \in P^k(K) \quad \text{for all } K \in \mathcal{T}_h\},\\
M_h &= \{\mu \in L^2(\mathcal{E}_{i,h}): \quad \mu|_e \in P^k(e) \quad \text{for all } e \in \mathcal{E}_{i,h}\}.
\end{aligned}$$

It is defined by requiring that, for all $(\boldsymbol{v}, u, \mu) \in V_h \times W_h \times M_h$,

$$\int_\Omega c\,\boldsymbol{q}_h\cdot\boldsymbol{v}\,dx - \sum_{K\in\mathcal{T}_h}\int_K u_h\nabla\cdot\boldsymbol{v}\,dx + \sum_{e\in\mathcal{E}_{i,h}}\int_e \lambda_h\,[\![\boldsymbol{v}]\!]\,ds = -\int_{\partial\Omega} g\,[\![\boldsymbol{v}]\!]\,ds, \tag{2.6}$$

$$\sum_{K\in\mathcal{T}_h}\int_K w\nabla\cdot\boldsymbol{q}_h\,dx + \int_\Omega d\,u_h\,w\,dx = \int_\Omega f\,w\,dx, \tag{2.7}$$

$$\sum_{e\in\mathcal{E}_{i,h}}\int_e \mu\,[\![\boldsymbol{q}_h]\!]\,ds = 0, \tag{2.8}$$

where $[\![\boldsymbol{v}]\!] = \boldsymbol{v}\cdot\boldsymbol{n}$ on $\partial\Omega$ and $[\![\boldsymbol{v}]\!] = \boldsymbol{v}_e^+\cdot\boldsymbol{n}_e^+ + \boldsymbol{v}_e^-\cdot\boldsymbol{n}_e^-$ on $e \in \mathcal{E}_h$. Here, $\boldsymbol{n}$ denotes the outward unit normal to Ω, $\boldsymbol{n}_e^+ = -\boldsymbol{n}_e^-$ is an arbitrary unit vector normal to the $e \in \mathcal{E}_{i,h}$, and $\boldsymbol{v}_e^\pm(\boldsymbol{x}) = \lim_{\epsilon\downarrow 0}\boldsymbol{v}(\boldsymbol{x} - \epsilon\,\boldsymbol{n}_e^\pm)$.

2.2. Two local mappings. Next, we introduce two mappings in terms of which the characterization result will be expressed. They are defined using (2.6) and (2.7).

The first mapping lifts functions on edges of the triangulation $\mathcal{T}_h$ to functions on Ω. Let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$. Notwithstanding a slight abuse of notation, we shall denote the set of all square integrable functions on the union of all edges of $\mathcal{E}_h$ by $L^2(\mathcal{E}_h)$. The lifting associates to each $\mathsf{m} \in L^2(\mathcal{E}_h)$ the pair of functions $(\boldsymbol{q}_h, u_h)_\mathsf{m} \equiv (\boldsymbol{q}_{h,\mathsf{m}}, u_{h,\mathsf{m}}) \in V_h \times W_h$ defined by requiring that

$$\int_\Omega c\,\boldsymbol{q}_{h,\mathsf{m}}\cdot\boldsymbol{v}\,dx - \sum_{K\in\mathcal{T}_h}\int_K u_{h,\mathsf{m}}\nabla\cdot\boldsymbol{v}\,dx = -\sum_{e\in\mathcal{E}_h}\int_e \mathsf{m}\,[\![\boldsymbol{v}]\!]\,ds, \tag{2.9}$$

$$\sum_{K\in\mathcal{T}_h}\int_K w\nabla\cdot\boldsymbol{q}_{h,\mathsf{m}}\,dx + \int_\Omega d\,u_{h,\mathsf{m}}\,w\,dx = 0 \tag{2.10}$$

hold for all $(\boldsymbol{v}, w) \in V_h \times W_h$.

The second mapping associates to the function $f \in L^2(\Omega)$ the element $(\boldsymbol{q}_h, u_h)_f \equiv (\boldsymbol{q}_{h,f}, u_{h,f}) \in V_h \times W_h$ and is defined by requiring that

$$\int_\Omega c\,\boldsymbol{q}_{h,f} \cdot \boldsymbol{v}\,dx - \sum_{K\in\mathcal{T}_h} \int_K u_{h,f} \nabla \cdot \boldsymbol{v}\,dx = 0, \tag{2.11}$$

$$\sum_{K\in\mathcal{T}_h} \int_K w \nabla \cdot \boldsymbol{q}_{h,f}\,dx + \int_\Omega d\,u_{h,f}\,w\,dx = \int_\Omega f\,w\,dx \tag{2.12}$$

hold for all $(\boldsymbol{v}, w) \in V_h \times W_h$.

Note that these mappings can be computed in an element-by-element fashion. Indeed, they are uniquely defined on each element because of the surjectivity of the map $(\nabla\cdot) : V_h \mapsto W_h$ restricted to an element. Moreover, on each element $K \in \mathcal{T}_h$, the lifting $(\boldsymbol{q}_h, u_h)_{\mathsf{m}}$ can be thought of as a result of a one element discretization of the boundary value problem

$$\begin{aligned} c\,\boldsymbol{q}_{\mathsf{m}} &= -\nabla u_{\mathsf{m}} && \text{in } K,\\ \nabla\cdot\boldsymbol{q}_{\mathsf{m}} + d\,u_{\mathsf{m}} &= 0 && \text{in } K,\\ u_{\mathsf{m}} &= \mathsf{m} && \text{on } \partial K, \end{aligned}$$

and that the mapping $(\boldsymbol{q}_h, u_h)_f$ is an approximation to the solution of

$$\begin{aligned} c\,\boldsymbol{q}_f &= -\nabla u_f && \text{in } K,\\ \nabla\cdot\boldsymbol{q}_f + d\,u_f &= f && \text{in } K,\\ u_f &= 0 && \text{on } \partial K. \end{aligned}$$

2.3. Characterization of the approximate solution. Before stating the result, let us introduce the following convention: The extension by zero of the function $\eta \in L^2(\mathcal{F}_h)$, where $\mathcal{F}_h$ is a subset of $\mathcal{E}_h$, to $\mathcal{E}_h$ is also denoted by η. In this way, if $\mathsf{m} = \lambda_h$ on $\mathcal{E}_{i,h}$ and $\mathsf{m} = g$ on $\partial\Omega$, we simply write $\mathsf{m} = \lambda_h + g$; as a consequence we also write

$$(\boldsymbol{q}_h, u_h)_{\mathsf{m}} = (\boldsymbol{q}_h, u_h)_{\lambda_h} + (\boldsymbol{q}_h, u_h)_g.$$

We now have all that is needed to state the main result.

THEOREM 2.1 (characterization of $(\boldsymbol{q}_h, u_h, \lambda_h)$). *Let $(\boldsymbol{q}_h, u_h, \lambda_h)$ be the solution of the hybridized RT method* (2.6), (2.7), *and* (2.8). *Then*

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{q}_h, u_h)_{\lambda_h} + (\boldsymbol{q}_h, u_h)_g + (\boldsymbol{q}_h, u_h)_f.$$

The Lagrange multiplier $\lambda_h \in M_h$ is the unique solution of

$$a_h(\lambda_h, \mu) = b_h(\mu) \quad \textit{for all } \mu \in M_h, \tag{2.13}$$

where

$$a_h(\lambda_h, \mu) = \int_\Omega c\,\boldsymbol{q}_{h,\lambda_h} \cdot \boldsymbol{q}_{h,\mu}\,dx + \int_\Omega d\,u_{h,\lambda_h}\,u_{h,\mu}\,dx$$

and

$$b_h(\mu) = \int_{\partial\Omega} g\,[\![\boldsymbol{q}_{h,\mu}]\!]\,ds + \int_\Omega f\,u_{h,\mu}\,dx.$$

Remark 2.1. Although the normal components of the functions $\boldsymbol{q}_{h,g}$, $\boldsymbol{q}_{h,f}$ (which can be computed locally only in terms of the data) and $\boldsymbol{q}_{h,\lambda_h}$ are not necessarily continuous across interelement boundaries, the normal components of their sum, namely, $\boldsymbol{q}_h$, are. Marini [12] pointed out this fact for the lowest order RT method when $d = 0$ and a is a piecewise-constant scalar function.

Remark 2.2. Just as for classical finite element methods, the variational formulation (2.13) gives rise to a matrix equation for the degrees of freedom of the multiplier Λ of the form

$$\mathbb{E}\,\Lambda = \mathbb{H}.$$

Thus the entries of the matrices $\mathbb{E}$ and $\mathbb{H}$ are obtained as a weighted L^2-inner product of discontinuous functions, as claimed in the introduction.

We end this section with a proof of Theorem 2.1.

2.4. Proof of Theorem 2.1. We prove this result in three steps. In the first, we observe some identities that result from the equations of the method. In the next step, we show that the continuity condition on the jumps of the fluxes results in a variational equation for the Lagrange multiplier unknowns. In the last step, we collect these two results and conclude.

2.4.1. Step 1.

LEMMA 2.2 (elementary identities). *We have, for any* $\mathsf{m}, \mu \in L^2(\mathcal{E}_h)$, *and* $f \in L^2(\Omega)$,

$$\text{(i)} -\sum_{e\in\mathcal{E}_h}\int_e \mu\,[\![\boldsymbol{q}_{h,\mathsf{m}}]\!]\,ds = \int_\Omega c\,\boldsymbol{q}_{h,\mathsf{m}}\cdot\boldsymbol{q}_{h,\mu}\,dx + \int_\Omega d\,u_{h,\mathsf{m}}\,u_{h,\mu}\,dx,$$

$$\text{(ii)} \int_\Omega c\,\boldsymbol{q}_{h,f}\cdot\boldsymbol{q}_{h,\mathsf{m}}\,dx + \int_\Omega d\,u_{h,f}\,u_{h,\mathsf{m}}\,dx = 0,$$

$$\text{(iii)} -\sum_{e\in\mathcal{E}_h}\int_e \mathsf{m}\,[\![\boldsymbol{q}_{h,f}]\!]\,ds = -\int_\Omega f\,u_{h,\mathsf{m}}\,dx.$$

Proof. Let us begin by proving the identity (i). First, take $\boldsymbol{v} = \boldsymbol{q}_{h,\mu}$ in (2.9). Then, replace m by μ in (2.10) and take $w = u_{h,\mathsf{m}}$. The identity (i) follows by simply adding these two equations.

To prove (ii), simply take $w = u_{h,f}$ in (2.10) and $\boldsymbol{v} = \boldsymbol{q}_{h,\mathsf{m}}$ in (2.11) and add the equations.

Finally, to prove (iii), take $\boldsymbol{v} = \boldsymbol{q}_{h,f}$ in (2.9) and $w = u_{h,\mathsf{m}}$ in (2.12) and add the two equations to obtain

$$-\sum_{e\in\mathcal{E}_h}\int_e \mathsf{m}\,[\![\boldsymbol{q}_{h,f}]\!]\,ds = -\int_\Omega f\,u_{h,\mathsf{m}}\,dx + \Theta,$$

where

$$\Theta = \int_\Omega c\,\boldsymbol{q}_{h,f}\cdot\boldsymbol{q}_{h,\mathsf{m}}\,dx + \int_\Omega d\,u_{h,f}\,u_{h,\mathsf{m}}\,dx.$$

Thus (iii) follows from (ii). This completes the proof. ∎

2.4.2. Step 2. In the following lemma, we explore equivalent characterizations of the continuity requirement on fluxes imposed by the method.

LEMMA 2.3 (jump condition). *Let* $(\boldsymbol{q}_h, u_h, \lambda_h)$ *be the solution of the hybridized RT method* (2.6), (2.7), *and* (2.8), *and let* m *be an arbitrary member of* M_h. *Then the following statements are equivalent:*

1. $\sum_{e\in\mathcal{E}_{i,h}} \int_e \mu \, [\![\boldsymbol{q}_{h,\mathsf{m}} + \boldsymbol{q}_{h,g} + \boldsymbol{q}_{h,f}]\!] \, ds = 0$ *for all* $\mu \in M_h$.
2. $(\boldsymbol{q}_h, u_h) = (\boldsymbol{q}_h, u_h)_{\mathsf{m}} + (\boldsymbol{q}_h, u_h)_g + (\boldsymbol{q}_h, u_h)_f$.
3. $\mathsf{m} = \lambda_h$.
4. $a_h(\mathsf{m}, \mu) = b_h(\mu)$ *for all* $\mu \in M_h$.

Proof. (1) $\Longrightarrow$ (2): Set

$$(\widetilde{\boldsymbol{q}}_h, \widetilde{u}_h) = (\boldsymbol{q}_h, u_h)_{\mathsf{m}} + (\boldsymbol{q}_h, u_h)_g + (\boldsymbol{q}_h, u_h)_f.$$

Then, combining the equations defining the local mappings, we get that

$$\int_\Omega c\, \widetilde{\boldsymbol{q}}_h \cdot \boldsymbol{v}\, dx - \sum_{K\in\mathcal{T}_h} \int_K \widetilde{u}_h \nabla\cdot \boldsymbol{v}\, dx + \sum_{e\in\mathcal{E}_{i,h}} \int_e \mathsf{m}\, [\![\boldsymbol{v}]\!]\, ds = -\int_{\partial\Omega} g\, [\![\boldsymbol{v}]\!]\, ds,$$
$$\sum_{K\in\mathcal{T}_h} \int_K w \nabla\cdot \widetilde{\boldsymbol{q}}_h\, dx + \int_\Omega d\, \widetilde{u}_h\, w\, dx = \int_\Omega f\, w\, dx$$

for all $(\boldsymbol{v}, w) \in V_k \times W_h$. Therefore, whenever (1) holds, $\widetilde{\boldsymbol{q}}_h$ and $\widetilde{u}_h$ satisfy all the equations of the hybridized RT method. By uniqueness of solutions of the method, $\widetilde{\boldsymbol{q}}_h = \boldsymbol{q}_h$ and $\widetilde{u}_h = u_h$, so (2) follows.

(2) $\Longrightarrow$ (3): By linear superposition, $\boldsymbol{q}_h = \boldsymbol{q}_{h,\lambda_h} + \boldsymbol{q}_{h,g} + \boldsymbol{q}_{h,f}$. Moreover, $\int_e \mu [\![\boldsymbol{q}_h]\!]\, ds = 0$ for all $e \in \mathcal{E}_{i,h}$. From the implication (1) $\Longrightarrow$ (2), it follows that

$$(\boldsymbol{q}_h, u_h) = (\boldsymbol{q}_h, u_h)_{\lambda_h} + (\boldsymbol{q}_h, u_h)_g + (\boldsymbol{q}_h, u_h)_f.$$

Consequently, (2) implies that

$$(\boldsymbol{q}_h, u_h)_{\mathsf{m}-\lambda_h} = 0,$$

from which it follows that $\mathsf{m} - \lambda_h = 0$.

(3) $\Longrightarrow$ (4): Now we observe that the following identity holds for any $\mathsf{m} \in M_h$:

$$\begin{aligned} -\int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}} \cdot \boldsymbol{q}_{h,\mu}\, dx - \int_\Omega d\, u_{h,\mathsf{m}}\, u_{h,\mu}\, dx + \int_{\partial\Omega} g\, [\![\boldsymbol{q}_{h,\mu}]\!]\, ds + \int_\Omega f\, u_{h,\mu}\, dx \\ = \sum_{e\in\mathcal{E}_h} \int_e \mu\, [\![\boldsymbol{q}_{h,\mathsf{m}} + \boldsymbol{q}_{h,g} + \boldsymbol{q}_{h,f}]\!]\, ds \quad \text{for all } \mu \in M_h. \end{aligned} \tag{2.14}$$

This equality follows from the identities of Lemma 2.2. Moreover, whenever $\mathsf{m} = \lambda_h$, the last equation of the hybridized mixed method asserts that the right-hand side of (2.14) is zero. From the definition of the forms a_h and b_h in Theorem (2.1), we see that (4) follows.

(4) $\Longrightarrow$ (1): We apply (2.14) again. Whenever (4) holds, the left-hand side of (2.14) is zero. Therefore, (1) follows. This completes the proof. ∎

2.4.3. Step 3. To conclude the proof of Theorem 2.1 observe that the first assertion of the theorem follows from the equivalence of the identities (1) and (2) of Lemma 2.3. The second assertion of the theorem follows again from Lemma 2.3, this time from the equivalence of (3) and (4). This completes the proof of Theorem 2.1.

Remark 2.3. The characterization theorem we just proved states that the solution $(\boldsymbol{q}_h, u_h)$ is the sum of the lifting of $\mathsf{m} = \lambda_h + g$, $(\boldsymbol{q}_h, u_h)_{\mathsf{m}}$ and the contribution from f, $(\boldsymbol{q}_h, u_h)_f$. By the identity (ii) of Lemma 2.2, we see that these two are orthogonal with respect to the bilinear form on $(V_h \times W_h)^2$ defined by

$$\langle\!\langle\, (\boldsymbol{q}_1, u_1), (\boldsymbol{q}_2, u_2)\,\rangle\!\rangle = \int_\Omega c\, \boldsymbol{q}_1 \cdot \boldsymbol{q}_2\, dx + \int_\Omega d\, u_1\, u_2\, dx.$$

3. The matrix entries.

3.1. The local matrices. To compute the matrices $\mathbb{E}$ and $\mathbb{H}$, we can proceed in the traditional finite element way. Let e denote an interior edge of the mesh $\mathcal{T}_h$, and let $\{\, L_{e,\ell}\,\}_{\ell=0}^k$ denote a basis for the set of polynomials of degree at most k on e. For example, we can choose properly scaled Legendre polynomials. Then, we set

$$(\boldsymbol{q}_{e,\ell}, u_{e,\ell}) = (\boldsymbol{q}_h, u_h)_{L_{e,\ell}}.$$

Now, for each element K, we compute the so-called local matrices, whose entries are

$$\begin{aligned}
\mathbb{E}_{K;e,\ell;e',\ell'} &= \int_K c\, \boldsymbol{q}_{e,\ell} \cdot \boldsymbol{q}_{e',\ell'}\, dx + \int_K d\, u_{e,\ell}\, u_{e',\ell'}\, dx,\\
\mathbb{H}_{g\,K;e,\ell} &= \int_{\partial K \cap \partial\Omega} g\, \boldsymbol{q}_{e,\ell} \cdot \boldsymbol{n}\, dx,\\
\mathbb{H}_{f\,K;e,\ell} &= \int_K u_{e,\ell}\, f\, dx.
\end{aligned}$$

Then the global matrices can be easily assembled by noting that

$$\begin{aligned}
\Lambda_{e,\ell}^t \mathbb{E}\, \Lambda_{e',\ell'} &= \sum_{K \in \mathcal{T}_h} \mathbb{E}_{K;e,\ell;e',\ell'},\\
\Lambda_{e,\ell}^t\, \mathbb{H}_g &= \sum_{K \in \mathcal{T}_h} \mathbb{H}_{g\,K;e,\ell},\\
\Lambda_{e,\ell}^t\, \mathbb{H}_f &= \sum_{K \in \mathcal{T}_h} \mathbb{H}_{f\,K;e,\ell}.
\end{aligned}$$

Since the lifting $(\boldsymbol{q}_{e,\ell}, u_{e,\ell})$ is supported only on the triangles sharing the edge e, to compute the local matrices, we have to provide only the numbers $\mathbb{E}_{K;e,\ell;e',\ell'}$, $\mathbb{H}_{g\,K;e,\ell}$, and $\mathbb{H}_{f\,K;e,\ell}$ for any two edges e and e' of K and any two integers ℓ and ℓ' between 0 and k; all the remaining entries are equal to zero. This also implies that the matrix $\mathbb{E}$ is a matrix of $(k+1) \times (k+1)$ blocks which has *at most four* off-diagonal blocks in each block column.

3.2. An example. The above computations can easily be carried out for the hybridized version of the RT method of lowest order. In this example, as the subscript ℓ in $\boldsymbol{q}_{e,\ell}$ and $u_{e,\ell}$ is superfluous, we drop it.

We begin by computing the lifting $\mathsf{m} \mapsto (\boldsymbol{q}_h, u_h)_{\mathsf{m}}$. Let m take the constant value λ_i on the edge e_i of the triangle K, $i = 1, 2, 3$. Then, on K,

$$(\boldsymbol{q}_h, u_h)_{\mathsf{m}} = \sum_{i=1}^{3} (\boldsymbol{q}_{e_i}, u_{e_i})\, \lambda_i,$$

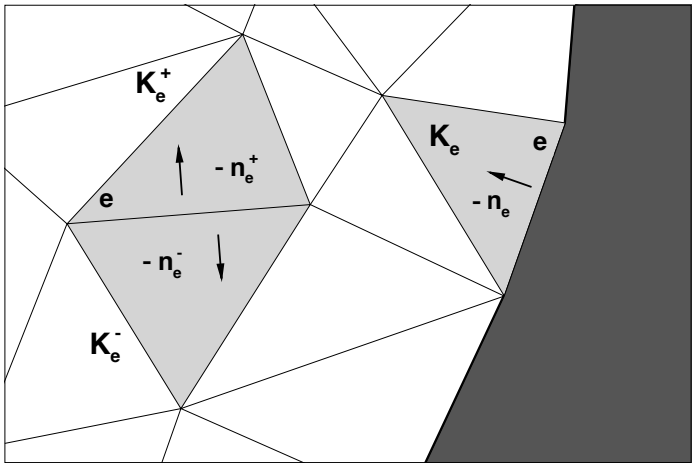

FIG. 1. *A representation of the lifting* $\boldsymbol{q}_{h,\mathsf{m}}$ *when* $\mathsf{m} = \chi_e$ *for interior and boundary edges* e. *In this case, we have taken* $d = 0$ *and* $c = Id$.

where, for $\boldsymbol{x} \in K$ and $i = 1, 2, 3$,

$$\boldsymbol{q}_{e_i}(\boldsymbol{x}) = -\frac{|\, e_i \,|}{|\, K \,|}\,(\bar{c})^{-1}\boldsymbol{n}_i - \frac{1}{2}\left(\frac{\bar{d}\,\rho_i}{1+\bar{d}\,\mathsf{h}^2}\right)(\boldsymbol{x}-\boldsymbol{B}), \qquad u_{e_i}(\boldsymbol{x}) = \frac{\rho_i}{1+\bar{d}\,\mathsf{h}^2},$$

and

$$\bar{c} = \frac{1}{|\, K \,|}\int_K c\,dx, \qquad \bar{d} = \frac{1}{|\, K \,|}\int_K d\,dx, \qquad \rho_i = \frac{|\, e_i \,|\,(\boldsymbol{m}_i - \boldsymbol{B})\cdot\boldsymbol{n}_i}{2\,|\, K \,|},$$
$$\mathsf{h}^2 = \frac{1}{4\,|\, K \,|}\int_K c\,(\boldsymbol{x}-\boldsymbol{B})\cdot(\boldsymbol{x}-\boldsymbol{B})\,dx, \qquad \boldsymbol{B} = (\bar{c})^{-1}\frac{\int_K c\,\boldsymbol{x}\,dx}{|\, K \,|}.$$

Here $\boldsymbol{m}_i$ denotes the midpoint of the edge e_i and $\boldsymbol{n}_i$ its outward unit normal. Note that if c is the identity, $\boldsymbol{B}$ is the barycenter of the triangle; if, moreover, K is an equilateral triangle of diameter h, then $\rho_i = 1/3$ and $\mathsf{h}^2 = h^2/48$. The case of c equaling the 2×2 identity and vanishing d is illustrated in Figure 1.

Now, it is easy to compute the entries of the local matrices:

$$\mathbb{E}_{K;e_i;e_j} = \frac{|\, e_i \,|\,|\, e_j \,|}{|\, K \,|}\,\boldsymbol{n}_j\cdot(\bar{c})^{-1}\boldsymbol{n}_j + \frac{\bar{d}\,|\, K \,|}{1+\bar{d}\,\mathsf{h}^2}\,\rho_i\,\rho_j,$$
$$\mathbb{H}_{g\,K;e_i} = -\sum_{e_j\subset\partial K\cap\partial\Omega}\left(\frac{|\, e_i \,|}{|\, K \,|}(\bar{c})^{-1}\boldsymbol{n}_i\cdot\boldsymbol{n}_j + \frac{|\, K \,|}{|\, e_j \,|}\left(\frac{\bar{d}}{1+\bar{d}\,\mathsf{h}^2}\right)\rho_i\,\rho_j\right)\int_{e_j} g(s)\,ds,$$
$$\mathbb{H}_{f\,K;e_i} = \frac{\rho_i}{1+\bar{d}\,\mathsf{h}^2}\int_K f(x)\,dx.$$

3.3. The reference element. A convenient implementation results if the local matrices for the multiplier can be computed by using quadratures on a reference element alone.

To achieve it, we need to define local mappings on the reference element and map spaces on the reference element $\widehat{K}$ to corresponding ones on any triangle K. Let $\widehat{K}$ be mapped one-to-one onto K by the standard affine mapping

$$\boldsymbol{x} = D_K\,\widehat{\boldsymbol{x}} + b_K,$$

and let us set

$$\widehat{u}(\widehat{\boldsymbol{x}}) \equiv u(\boldsymbol{x}), \quad \widehat{\boldsymbol{q}}(\widehat{\boldsymbol{x}}) \equiv |\det D_K|\, D_K^{-1}\, \boldsymbol{q}(\boldsymbol{x})$$

for scalar-valued functions u and vector-valued functions $\boldsymbol{q}$, respectively. Finally, set

$$\begin{aligned}
\widehat{V} &= P^k(\widehat{K}) \times P^k(\widehat{K}) + \boldsymbol{x}\, P^k(\widehat{K}),\\
\widehat{W} &= P^k(\widehat{K}),\\
\widehat{M} &= \{\widehat{\mathsf{m}} \in L^2(\partial\widehat{K}) : \quad \widehat{\mathsf{m}}|_{\widehat{e}} \in P^k(\widehat{e}) \quad \text{for all } \ \widehat{e} \in \partial\widehat{K}\}.
\end{aligned}$$

Now, suppose we are given a symmetric positive definite 2×2 matrix function $\widehat{C}$ and a scalar nonnegative function $\widehat{D}$ on $\widehat{K}$. For each $\widehat{\mathsf{m}}$ in $\widehat{M}$, we define the element $(\widehat{\boldsymbol{Q}}_{\widehat{\mathsf{m}}}, \widehat{U}_{\widehat{\mathsf{m}}}) \in \widehat{V} \times \widehat{W}$ by requiring that

$$\begin{aligned}
&\int_{\widehat{K}} \widehat{C}\, \widehat{\boldsymbol{Q}}_{\widehat{\mathsf{m}}} \cdot \boldsymbol{V}\, d\widehat{\boldsymbol{x}} - \int_{\widehat{K}} \widehat{U}_{\widehat{\mathsf{m}}} \widehat{\nabla} \cdot \boldsymbol{V}\, d\widehat{\boldsymbol{x}} = -\sum_{\widehat{e} \in \partial\widehat{K}} \int_{\widehat{e}} \widehat{\mathsf{m}}\, [\![\boldsymbol{V}]\!]\, d\widehat{s},\\
&\int_{\widehat{K}} W \widehat{\nabla} \cdot \widehat{\boldsymbol{Q}}_{\widehat{\mathsf{m}}}\, d\widehat{\boldsymbol{x}} + \int_{\widehat{K}} \widehat{D}\, \widehat{U}_{\widehat{\mathsf{m}}}\, W\, d\widehat{\boldsymbol{x}} = 0
\end{aligned}$$

hold for all $(\boldsymbol{V}, W) \in \widehat{V} \times \widehat{W}$. Then, we have the following result.

PROPOSITION 3.1. *Let K be any triangle and e be one of its edges. Set*

$$\widehat{C}(\widehat{\boldsymbol{x}}) = |\det D_K|^{-1}\, D_K^t\, c(\boldsymbol{x})\, D_K \quad \textit{and} \quad \widehat{D}(\widehat{\boldsymbol{x}}) = |\det D_K|\, d(\boldsymbol{x}).$$

Then, the lifting $(\boldsymbol{q}_{e,\ell}, u_{e,\ell})$ on K when mapped to $\widehat{K}$ satisfies

$$\widehat{\boldsymbol{q}}_{e,\ell} = \widehat{\boldsymbol{Q}}_{\widehat{L_{e,\ell}}} \qquad \textit{and} \qquad \widehat{u}_{e,\ell} = \widehat{U}_{\widehat{L_{e,\ell}}}.$$

Moreover,

$$\begin{aligned}
\mathbb{E}_{K;e,\ell;e',\ell'} &= -\int_{\widehat{e}} \widehat{L_{e,\ell}}\, \widehat{\boldsymbol{Q}}_{\widehat{L_{e',\ell'}}} \cdot \widehat{\boldsymbol{n}}\, d\widehat{s},\\
\mathbb{H}_{g\,K;e,\ell} &= \int_{\widehat{e}} \widehat{g}\, \widehat{\boldsymbol{Q}}_{\widehat{L_{e,\ell}}} \cdot \widehat{\boldsymbol{n}}\, d\widehat{s},\\
\mathbb{H}_{f\,K;e,\ell} &= \int_{\widehat{K}} \widehat{U}_{\widehat{L_{e,\ell}}}\, \widehat{f}\, |\det D_K|\, d\widehat{\boldsymbol{x}}.
\end{aligned}$$

This result can be easily proved by a straightforward change of variables and application of Lemma 2.2. Note that if the functions $\widehat{C}$ and $\widehat{D}$ are constant, there is no need to use quadrature rules to find the matrix entries.

4. Comparison with the hybridized BDM method. Now we compare the multipliers given by the hybridized version of the RT method and those given by the hybridized version of the corresponding BDM method.

4.1. Statement of the results. To state our comparison results, we first introduce the hybridized BDM method. The approximate solution given by this method, $(\boldsymbol{q}_h^{\mathrm{BDM}}, u_h^{\mathrm{BDM}}, \lambda_h^{\mathrm{BDM}})$, is sought in the finite element space $V_h^{\mathrm{BDM}} \times W_h^{\mathrm{BDM}} \times M_h$ given by

$$\begin{aligned}
V_h^{\mathrm{BDM}} &= \{\boldsymbol{v} \in L^2(\Omega) \times L^2(\Omega) : \quad \boldsymbol{v}|_K \in P^k(K) \times P^k(K) \quad \text{for all } \ K \in \mathcal{T}_h\},\\
W_h^{\mathrm{BDM}} &= \{w \in L^2(\Omega) : \quad w|_K \in P^{k-1}(K) \quad \text{for all } \ K \in \mathcal{T}_h\},
\end{aligned}$$

where $k \geq 1$, and is defined by requiring that, for all $(\boldsymbol{v}, u, \mu) \in V_h^{\mathrm{BDM}} \times W_h^{\mathrm{BDM}} \times M_h$,

$$\int_\Omega c\, \boldsymbol{q}_h^{\mathrm{BDM}} \cdot \boldsymbol{v}\, dx - \sum_{K \in \mathcal{T}_h} \int_K u_h^{\mathrm{BDM}} \nabla \cdot \boldsymbol{v}\, dx + \sum_{e \in \mathcal{E}_{i,h}} \int_e \lambda_h^{\mathrm{BDM}} [\![\boldsymbol{v}]\!]\, ds = -\int_{\partial\Omega} g\, [\![\boldsymbol{v}]\!]\, ds,$$

$$\sum_{K \in \mathcal{T}_h} \int_K w \nabla \cdot \boldsymbol{q}_h^{\mathrm{BDM}}\, dx + \int_\Omega d\, u_h^{\mathrm{BDM}}\, w\, dx = \int_\Omega f\, w\, dx,$$

$$\sum_{e \in \mathcal{E}_{i,h}} \int_e \mu\, [\![\boldsymbol{q}_h^{\mathrm{BDM}}]\!]\, ds = 0.$$

Note that the approximate solution of the BDM method satisfies exactly the same weak formulation as the approximate solution of the RT method; the only difference is the choice of the finite element spaces. As a consequence, the characterization theorem (Theorem 2.1) holds for the hybridized version of the BDM method. This is the key fact that allows us to compare the hybridized versions of the RT and the BDM methods.

In comparing the RT and BDM methods, for the sake of readability, we shall superscript the notation previously introduced in connection with the RT method by "RT." When superscripted by "BDM," such notation is to be understood as defined exactly as before except that the RT spaces are replaced by the BDM spaces. For example, $(\boldsymbol{q}_h, u_h)_{\mathsf{m}}^{\mathrm{BDM}} \equiv (\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}, u_{h,\mathsf{m}}^{\mathrm{BDM}}) \in V_h^{\mathrm{BDM}} \times W_h^{\mathrm{BDM}}$ is defined by requiring that

$$\int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} \cdot \boldsymbol{v}\, dx - \sum_{K \in \mathcal{T}_h} \int_K u_{h,\mathsf{m}}^{\mathrm{BDM}} \nabla \cdot \boldsymbol{v}\, dx = -\sum_{e \in \mathcal{E}_h} \int_e \mathsf{m}\, [\![\boldsymbol{v}]\!]\, ds,$$

$$\sum_{K \in \mathcal{T}_h} \int_K w \nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\, dx + \int_\Omega d\, u_{h,\mathsf{m}}^{\mathrm{BDM}}\, w\, dx = 0$$

hold for all $(\boldsymbol{v}, w) \in V_h^{\mathrm{BDM}} \times W_h^{\mathrm{BDM}}$.

To state our first comparison theorem, we need the following additional notation: Denote by $\mathcal{P}_k$ the L^2-orthogonal projection into the space of functions which are piecewise polynomials of degree k on each triangle $K \in \mathcal{T}_h$. Let $\mathcal{R}f = \mathcal{P}_k f - \mathcal{P}_{k-1} f$ for all $f \in L^2(\Omega)$. Define the form

$$b_{h,\mathcal{R}f}^{\mathrm{RT}}(\mu) = \int_\Omega \mathcal{R}f\, u_{h,\mu}^{\mathrm{RT}}\, dx \quad \text{for all } \mu \in M_h$$

and the function $\rho_h \equiv \rho_h(\mathcal{R}f) \in M_h$ by

$$a_h^{\mathrm{RT}}(\rho_h, \mu) = b_{h,\mathcal{R}f}^{\mathrm{RT}}(\mu) \quad \text{for all } \mu \in M_h.$$

Also set

$$\Psi(\mu, g, f) = (\mathbf{0}, \mathcal{R}u_{h,\mu}^{\mathrm{RT}}) + (\mathbf{0}, \mathcal{R}u_{h,g}^{\mathrm{RT}}) + (\mathbf{0}, \mathcal{R}u_{h,\mathcal{P}_{k-1}f}^{\mathrm{RT}}) \quad \text{and}$$
$$\Upsilon(\mathcal{R}f) = (\boldsymbol{q}_h, u_h)_{\rho_h}^{\mathrm{RT}} + (\boldsymbol{q}_h, u_h)_{\mathcal{R}f}^{\mathrm{RT}}.$$

Now we can state the theorem.

THEOREM 4.1 (comparison of the RT and BDM methods for $d = 0$).
Assume that $d(\boldsymbol{x}) = 0$ almost everywhere in Ω.

1. *Suppose $f \in L^2(\Omega)$ is such that $(\mathcal{P}_k - \mathcal{P}_{k-1})f = 0$. Then, the Lagrange multiplier components of the RT and BDM solutions coincide:*

$$\lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}}.$$

2. *If $f \in L^2(\Omega)$ is arbitrary, then the following statements hold:*

$$
\begin{aligned}
&(\alpha) \quad a_h^{\mathrm{RT}}(\mathsf{m},\mu) = a_h^{\mathrm{BDM}}(\mathsf{m},\mu) && \textit{for all } \mathsf{m},\mu \in L^2(\mathcal{E}_h).\\
&(\beta) \quad b_h^{\mathrm{RT}}(\mu) = b_h^{\mathrm{BDM}}(\mu) + b_{h,\mathcal{R}f}^{\mathrm{RT}}(\mu) && \textit{for all } \mu \in L^2(\mathcal{E}_h).\\
&(\gamma) \quad \lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}} + \rho_h(\mathcal{R}f).\\
&(\delta) \quad (\boldsymbol{q}_h^{\mathrm{RT}}, u_h^{\mathrm{RT}}) = (\boldsymbol{q}_h^{\mathrm{BDM}}, u_h^{\mathrm{BDM}}) + \Psi(\lambda_h^{\mathrm{BDM}}, g, f) + \Upsilon(\mathcal{R}f).
\end{aligned}
$$

Before proving the theorem, let us discuss the result and some of its consequences.

Remark 4.1. Statements (α) and (β) can be easily rewritten in matrix form as follows:

$$
\begin{aligned}
&(\alpha') \quad \mathbb{E}^{\mathrm{RT}} = \mathbb{E}^{\mathrm{BDM}},\\
&(\beta') \quad \mathbb{H}^{\mathrm{RT}} = \mathbb{H}^{\mathrm{BDM}} + \mathbb{H}_{\mathcal{R}f}^{\mathrm{RT}},
\end{aligned}
$$

where the matrix $\mathbb{E}^{\mathrm{N}}$ is the stiffness matrix associated to the bilinear form $a_h^{\mathrm{N}}(\cdot,\cdot)$, and the matrix $\mathbb{H}^{\mathrm{N}}$ is the right-hand side matrix associated to the linear form $b_h^{\mathrm{N}}(\cdot)$ for $\mathrm{N} \in \{\mathrm{RT}, \mathrm{BDM}\}$. The matrix $\mathbb{H}_{\mathcal{R}f}^{\mathrm{RT}}$ is, of course, the right-hand side matrix associated to the linear form $b^{\mathrm{RT}}(\cdot)_{h,\mathcal{R}f}$. This means that, when $d = 0$, the stiffness matrices of the multipliers of both methods *coincide*. However, the right-hand side matrices differ. But they differ by a matrix which vanishes when $\mathcal{R}f = 0$—hence the coincidence of the Lagrange multipliers whenever $\mathcal{R}f = 0$.

Remark 4.2. The coincidence of Lagrange multipliers asserted by the theorem in the case $\mathcal{R}f = 0$ (and $d = 0$) appears to have gone unnoticed hitherto even numerically. This case occurs, e.g., when f is a polynomial of degree $k-1$ on every element of the mesh and in several applications of practical interest, e.g., incompressible flow in porous media (where $f = 0$ usually). The condition $\mathcal{R}f = 0$ is not only sufficient but also necessary for such a coincidence: From the characterization theorem, it is clear that the only part of f that determines λ_h^{RT} is $\mathcal{P}_k f$, while the only part of f that determines λ_h^{BDM} is $\mathcal{P}_{k-1} f$. Therefore, setting f to a polynomial of degree k for which $0 = \mathcal{P}_{k-1} f \neq \mathcal{P}_k f$, we can make $\lambda_h^{\mathrm{RT}} \neq \lambda_h^{\mathrm{BDM}}$.

Remark 4.3. Statement (δ) of the theorem shows how the solution components other than the multipliers are related. Obviously, $\Upsilon(\mathcal{R}f)$ depends linearly on $\mathcal{R}f$. Therefore, when $\mathcal{R}f = 0$, the solution $(\boldsymbol{q}_h^{\mathrm{RT}}, u_h^{\mathrm{RT}})$ differs from $(\boldsymbol{q}_h^{\mathrm{BDM}}, u_h^{\mathrm{BDM}})$ only by $\Psi(\lambda_h^{\mathrm{BDM}}, g, f)$, a function that can be computed locally element by element. In particular, this means that it is possible to implement the less expensive BDM method and locally recover the RT solution u_h^{RT}, which is one order higher in accuracy (under certain regularity assumptions). In this sense, (δ) can be thought of as yielding a postprocessing technique. Of course, one can then further postprocess the RT solution by the technique of [1] and gain one further order in accuracy.

Remark 4.4. It is well known that the Lagrange multiplier of both the RT and the BDM methods approximates the traces of the exact solution u on mesh edges. Specifically, under certain regularity assumptions, [5, Lemma 4.1] asserts that when $\mathcal{T}_h$ is a quasi-uniform mesh with mesh size h, as $h \to 0$,

$$
\|\lambda_h^{\mathrm{BDM}} - \mathcal{P}_{M_h} u\|_{\mathcal{E}_h} = O(h^{k+3/2-\delta_{k1}}), \tag{4.1}
$$

where $\mathcal{P}_{M_h}$ denotes the L^2-orthogonal projection onto M_h, $\|\cdot\|_{\mathcal{E}_h}$ denotes the $L^2(\mathcal{E}_h)$-norm, and δ_{k1} is zero for all k except for $k = 1$, in which case it equals one. The analogous estimate for the RT method [1, Corollary 1.5] is

$$
\|\lambda_h^{\mathrm{RT}} - \mathcal{P}_{M_h} u\|_{\mathcal{E}_h} = O(h^{k+3/2}).
$$

However, obviously whenever $\lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}}$, there can be no difference in the convergence rates. Therefore, when $k = 1$ and $\mathcal{R}f = 0$, by virtue of Theorem 4.1, we conclude that although (4.1) provides for only $O(h^{3/2})$-convergence, in fact the convergence rate is at least $O(h^{5/2})$.

When $d \neq 0$, the liftings $\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}$ and $\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}$ are no longer divergence-free on each element, in general. In fact, as we show later, there are multipliers m for which $\mathcal{R}u_{h,\mathsf{m}}^{\mathrm{RT}} \neq 0$. This property implies that the statements of Theorem 4.1 do not hold in general. In particular, the following theorem provides a case wherein statement (α) does not hold. Obviously, whenever statement (α) fails to hold, one cannot expect coincidence of RT and BDM Lagrange multipliers.

THEOREM 4.2 (comparison of the RT and BDM methods for $d \geq 0$). *Assume that $c(\boldsymbol{x})$ and $d(\boldsymbol{x})$ are constant on each element of the mesh. Then, whenever $d(\boldsymbol{x})$ is positive on at least one element, statement (α) of Theorem 4.2 does not hold.*

Now, we prove Theorems 4.1 and 4.2.

4.2. Proof of Theorem 4.1. The proof proceeds by establishing a connection between the BDM and RT liftings and applying the characterization theorem. Two properties of the finite element spaces of the RT and BDM methods play a crucial role. The first is simply that the multipliers of both methods share the same space. The second is that the elements of V_h^{RT} whose divergence on each element is a polynomial of degree $k-1$ also belong to the space V_h^{BDM}. Let us begin by proving the latter property.

LEMMA 4.3. *The following containment holds:*

$$\{\boldsymbol{q}_h \in V_h^{\mathrm{RT}} : \nabla \cdot \boldsymbol{q}_h|_K \in P^{k-1}(K) \ \text{ for all } \ K \in \mathcal{T}_h\} \subset V_h^{\mathrm{BDM}}.$$

Proof. If $\boldsymbol{q}_h \in V_h^{\mathrm{RT}}$, then $\boldsymbol{q}_h|_K = \boldsymbol{v}_k + \boldsymbol{x}\widetilde{p}_k$ for some $\boldsymbol{v}_k \in P^k(K) \times P^k(K)$ and some homogeneous polynomial $\widetilde{p}_k(\boldsymbol{x})$ of degree k on K. Taking the divergence, we find that

$$\nabla \cdot (\boldsymbol{q}_h|_K) = \nabla \cdot \boldsymbol{v}_k + (k+2)\widetilde{p}_k.$$

Therefore, $\nabla \cdot (\boldsymbol{q}_h|_K) \in P^{k-1}(K)$ implies that $\widetilde{p}_k = 0$, and consequently $\boldsymbol{q}_h|_K \in P^k(K) \times P^k(K)$. Hence $\boldsymbol{q}_h \in V_h^{\mathrm{BDM}}$. This completes the proof. ∎

The next result uses the above lemma and the definition of the local mappings to establish key relations between the local mappings of the RT and BDM methods.

LEMMA 4.4. *Assume that $d = 0$. Then, for all $\mathsf{m} \in L^2(\mathcal{E}_h)$ and $f \in L^2(\Omega)$,*

$$\begin{aligned}
&\text{(i)} \quad \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} = \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}, && u_{h,\mathsf{m}}^{\mathrm{RT}} = u_{h,\mathsf{m}}^{\mathrm{BDM}} + \mathcal{R}u_{h,\mathsf{m}}^{\mathrm{RT}},\\
&\text{(ii)} \quad \boldsymbol{q}_{h,f}^{\mathrm{RT}} = \boldsymbol{q}_{h,f}^{\mathrm{BDM}} + \boldsymbol{q}_{h,\mathcal{R}f}^{\mathrm{RT}}, && u_{h,f}^{\mathrm{RT}} = u_{h,f}^{\mathrm{BDM}} + \mathcal{R}u_{h,\mathcal{P}_{k-1}f}^{\mathrm{RT}} + u_{h,\mathcal{R}f}^{\mathrm{RT}}.
\end{aligned}$$

Proof. Let us begin by proving (i). From (2.9), we have

$$\int_\Omega c\,\big(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\big)\cdot \boldsymbol{v}\,dx - \sum_{K\in\mathcal{T}_h}\int_K \big(u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\big)\,\nabla\cdot\boldsymbol{v}\,dx = 0 \quad \text{for all } \ \boldsymbol{v} \in V_h^{\mathrm{BDM}},$$

and from (2.10), $\nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}|_K = 0$ and $\nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}|_K = 0$ for all $K \in \mathcal{T}_h$. Since by Lemma 4.3 $\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} \in V_h^{\mathrm{BDM}}$, we can take $\boldsymbol{v} = \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}$ in the first equation of this proof to get that $\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} = \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}$. It immediately follows that

$$-\sum_{K\in\mathcal{T}_h}\int_K \big(u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\big)\,\nabla\cdot\boldsymbol{v}\,dx = 0 \qquad \text{for all } \ \boldsymbol{v} \in V_h^{\mathrm{BDM}},$$

which implies that $\mathcal{P}_{k-1}\,u_{h,\mathsf{m}}^{\mathrm{RT}} = u_{h,\mathsf{m}}^{\mathrm{BDM}}$. This proves (i).

Now, let us prove (ii). It suffices to show that for $p = \mathcal{P}_{k-1} f$,

$$\boldsymbol{q}_{h,p}^{\mathrm{RT}} = \boldsymbol{q}_{h,p}^{\mathrm{BDM}} \quad \text{and} \quad u_{h,p}^{\mathrm{RT}} = u_{h,p}^{\mathrm{BDM}} + \mathcal{R} u_{h,p}^{\mathrm{RT}}. \tag{4.2}$$

Indeed, once we have (4.2), by linearity and the obvious equality

$$(\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{\mathcal{P}_{k-1} f} = (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{f},$$

we get that

$$\begin{aligned}(\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{f} &= (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{p} + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{\mathcal{R} f} \\ &= (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{p} + (\boldsymbol{0}, \mathcal{R} u_{h,p}^{\mathrm{RT}}) + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{\mathcal{R} f},\end{aligned}$$

and (ii) follows. To show (4.2), first observe that since $p|_K = (\mathcal{P}_{k-1} f)|_K \in P^{k-1}(K)$, (2.12) implies that

$$\nabla \cdot \boldsymbol{q}_{h,p}^{\mathrm{RT}}|_K = p \quad \text{and} \quad \nabla \cdot \boldsymbol{q}_{h,p}^{\mathrm{BDM}}|_K = p \quad \text{for all } K \in \mathcal{T}_h.$$

Therefore, using Lemma 4.3 again, $\boldsymbol{q}_{h,p}^{\mathrm{RT}} \in V_h^{\mathrm{BDM}}$. Now, (2.11) yields

$$\int_\Omega c \left(\boldsymbol{q}_{h,p}^{\mathrm{RT}} - \boldsymbol{q}_{h,p}^{\mathrm{BDM}} \right) \cdot \boldsymbol{v} \, dx - \sum_{K \in \mathcal{T}_h} \int_K \left(u_{h,p}^{\mathrm{RT}} - u_{h,p}^{\mathrm{BDM}} \right) \nabla \cdot \boldsymbol{v} \, dx = 0 \quad \text{for all } \boldsymbol{v} \in V_h^{\mathrm{BDM}}.$$

Since we have shown that $\boldsymbol{q}_{h,p}^{\mathrm{RT}} \in V_h^{\mathrm{BDM}}$, we can choose $\boldsymbol{v} = \boldsymbol{q}_{h,p}^{\mathrm{RT}} - \boldsymbol{q}_{h,p}^{\mathrm{BDM}}$ above. Then the first equality of (4.2), namely, $\boldsymbol{q}_{h,p}^{\mathrm{RT}} = \boldsymbol{q}_{h,p}^{\mathrm{BDM}}$, immediately follows. The second equality of (4.2) also follows, because by (i) we have $\mathcal{P}_{k-1} \, u_{h,p}^{\mathrm{RT}} = u_{h,p}^{\mathrm{BDM}}$. This completes the proof. ∎

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. First, observe that the first conclusion of the theorem, namely, $\lambda_h^{\mathrm{RT}} = \lambda_h^{\mathrm{BDM}}$ whenever $\mathcal{R} f = 0$, follows from statement (γ), because $\rho_h(\mathcal{R} f)$ depends linearly on $\mathcal{R} f$. Therefore, we shall prove only statements (α)–(δ).

Applying the characterization theorem with $d = 0$, we find that

$$a_h^{\mathrm{N}}(\mathsf{m}, \mu) = \int_\Omega c \, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{N}} \cdot \boldsymbol{q}_{h,\mu}^{\mathrm{N}} \quad \text{for } \mathrm{N} \in \{\mathrm{RT}, \mathrm{BDM}\},$$

so the equality of (α) follows from the first equality of Lemma 4.4(i). Similarly, since

$$b_h^{\mathrm{N}}(\mu) = \int_{\partial\Omega} g \, [\![\boldsymbol{q}_{h,\mu}^{\mathrm{N}}]\!] \, ds + \int_\Omega f \, u_{h,\mu}^{\mathrm{N}} \, dx \quad \text{for } \mathrm{N} \in \{\mathrm{RT}, \mathrm{BDM}\},$$

the second equality, namely, (β), also follows from Lemma 4.4(i). Now, statement (γ) obviously follows from (α), (β), and the definitions of the multipliers.

To prove statement (δ), we start again from the characterization theorem, apply statement (γ), and use the identities of Lemma 4.4 in succession:

$$\begin{aligned}(\boldsymbol{q}_h^{\mathrm{RT}}, u_h^{\mathrm{RT}}) &= (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{\lambda_h^{\mathrm{RT}}} + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{g} + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{f} \\ &= (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{(\lambda_h^{\mathrm{BDM}} + \rho_h)} + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{g} + (\boldsymbol{q}_h, u_h)^{\mathrm{RT}}_{f} \\ &= (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{\lambda_h^{\mathrm{BDM}}} + (\boldsymbol{0}, \mathcal{R} u_{h,\lambda_h^{\mathrm{BDM}}}^{\mathrm{RT}} + u_{h,\rho_h}^{\mathrm{RT}}) + (\boldsymbol{q}_{h,g}^{\mathrm{BDM}}, u_{h,g}^{\mathrm{BDM}} + \mathcal{R} u_{h,g}^{\mathrm{RT}}) \\ &\qquad + (\boldsymbol{q}_{h,f}^{\mathrm{BDM}} + \boldsymbol{q}_{h,\mathcal{R} f}^{\mathrm{RT}}, \; u_{h,f}^{\mathrm{BDM}} + \mathcal{R} u_{h,\mathcal{P}_{k-1} f}^{\mathrm{RT}} + u_{h,\mathcal{R} f}^{\mathrm{RT}}) \\ &= (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{\lambda_h^{\mathrm{BDM}}} + (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{g} + (\boldsymbol{q}_h, u_h)^{\mathrm{BDM}}_{f} \\ &\qquad + \Psi(\lambda_h^{\mathrm{BDM}}, g, f) + \Upsilon(\mathcal{R} f).\end{aligned}$$

The first three terms on the right-hand side of the last equality sum to the BDM solution. This completes the proof of Theorem 4.1. ∎

4.3. Proof of Theorem 4.2. To prove this result, we begin by studying the local spaces of the RT method, namely,

$$\begin{aligned} V_K^{\mathrm{RT}} &= P^k(K) \times P^k(K) + \boldsymbol{x}\, P^k(K), \\ W_K^{\mathrm{RT}} &= P^k(K), \\ M_K^{\mathrm{RT}} &= \{\mu \in L^2(\partial K) : \mu|_e \in P^k(e) \text{ for each edge of } K\}. \end{aligned}$$

It turns out that, when $d > 0$, the lifting maps induce a natural orthogonal decomposition of the local space V_K^{RT}, namely,

$$V_K^{\mathrm{RT}} = V_K^0 \oplus V_K^{\perp},$$

where

$$\begin{aligned} V_K^0 &= \{\boldsymbol{v} \in V_K^{\mathrm{RT}} : \boldsymbol{v} \cdot \boldsymbol{n}_e = 0 \text{ on each edge } e \text{ of } K\}, \\ V_K^{\perp} &= \{\boldsymbol{v} \in V_K^{\mathrm{RT}} : (\!(\boldsymbol{v}, \boldsymbol{q})\!) = 0 \ \text{ for all } \boldsymbol{q} \in V_K^0\}, \end{aligned}$$

and

$$(\!(\boldsymbol{v}_1, \boldsymbol{v}_2)\!) = \int_K c\, \boldsymbol{v}_1 \cdot \boldsymbol{v}_2\, dx + \int_K \frac{1}{d} \nabla \cdot \boldsymbol{v}_1\, \nabla \cdot \boldsymbol{v}_2\, dx.$$

Indeed, the following result states that the local space $V_K^{\perp}$ is nothing but the image of the lifting map $\mathsf{m} \mapsto \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}$.

LEMMA 4.5. *Assume that $d(\boldsymbol{x})$ is a positive constant on an element $K \in \mathcal{T}_h$, and let $\{\mathsf{m}_i\}_{i=1}^{3(k+1)}$ be a basis of M_K^{RT}. Then, $\{\boldsymbol{q}_{h,\mathsf{m}_i}^{\mathrm{RT}}\}_{i=1}^{3(k+1)}$ is a basis of $V_K^{\perp}$.*

Proof. By (2.10), we have for any $\mathsf{m} \in M_K^{\mathrm{RT}}$

$$u_{h,\mathsf{m}}^{\mathrm{RT}} = -\frac{1}{d} \nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}. \tag{4.3}$$

Substituting this expression for $u_{h,\mathsf{m}}^{\mathrm{RT}}$ into (2.9), we see that (2.9) can be rewritten as follows:

$$(\!(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}, \boldsymbol{v})\!) = -\sum_{e \in \partial K} \int_e \mathsf{m}\, [\![\boldsymbol{v}]\!]\, ds.$$

As a consequence

$$\{\boldsymbol{q}_{h,\mathsf{m}_i}^{\mathrm{RT}}\}_{i=1}^{3(k+1)} \subset V_K^{\perp}.$$

Since the dimension of $V_K^{\perp}$ is $3(k+1)$, it remains only to show that the elements of the set $\{\boldsymbol{q}_{h,\mathsf{m}_i}^{\mathrm{RT}}\}_{i=1}^{3(k+1)}$ are linearly independent. So, assume that there are scalars α_i such that

$$\sum_{i=1}^{3(k+1)} \alpha_i\, \boldsymbol{q}_{h,\mathsf{m}_i}^{\mathrm{RT}} = 0.$$

By the linearity of the lifting, this implies that $\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} = 0$, where

$$\mathsf{m} = \sum_{i=1}^{3(k+1)} \alpha_i\, \mathsf{m}_i.$$

However, since by (4.3)

$$u_{h,\mathsf{m}}^{\mathrm{RT}} = -\frac{1}{d}\nabla\cdot\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} = 0,$$

we have that the lifting $(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}}, u_{h,\mathsf{m}}^{\mathrm{RT}})$ is zero. Consequently, $\mathsf{m} = 0$. Since $\{\mathsf{m}_i\}_{i=1}^{3(k+1)}$ is a basis for M_K^{RT}, it follows that $\alpha_i = 0$ for all $i = 1, \ldots, 3(k+1)$. This completes the proof. $\square$

Now, we use the above result to show that the lifting $u_{h,\mathsf{m}}^{\mathrm{RT}}$ is not always a polynomial of degree at most $k-1$ on all the elements of the triangulation.

LEMMA 4.6. *Assume that $c(\boldsymbol{x})$ is constant on an element $K \in \mathcal{T}_h$. Also assume that $d(\boldsymbol{x})$ is a positive constant on K. Then there is a function $\mathsf{m} \in M_K^{\mathrm{RT}}$ such that*

$$\mathcal{R}u_{h,\mathsf{m}}^{\mathrm{RT}} \neq 0 \qquad \text{on } K.$$

Proof. Since by (4.3)

$$u_{h,\mathsf{m}}^{\mathrm{RT}} = -\frac{1}{d}\nabla\cdot\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}},$$

an application of Lemma 4.3 shows that $\mathcal{R}u_{h,\mathsf{m}}^{\mathrm{RT}} = 0$ if and only if

$$\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} \in P^k(K)\times P^k(K) \quad \text{for all } \mathsf{m} \in M_K^{\mathrm{RT}}. \tag{4.4}$$

We claim that this is not possible for all $\mathsf{m} \in M_K^{\mathrm{RT}}$. Indeed, if this were the case, $V_K^{\perp} \subset P^k(K)\times P^k(K)$. This implies that the orthogonal complement of $P^k(K)\times P^k(K)$ in V_K^{RT} with respect to the inner product $(\!(\cdot,\cdot)\!)$, which we denote by W_K, satisfies

$$W_K \subset V_K^0. \tag{4.5}$$

However, as we shall now see, this implies that $W_K = \{\mathbf{0}\}$, which is a contradiction.

In the orthogonality relation

$$\int_K c\boldsymbol{\phi}\cdot\boldsymbol{v}\,dx + \frac{1}{d}\int_K (\nabla\cdot\boldsymbol{\phi})(\nabla\cdot\boldsymbol{v})\,dx = 0 \quad \text{for all } \boldsymbol{v} \in P^k(K)\times P^k(K),$$

let us choose $\boldsymbol{v} = c^{-T}\nabla\eta$ for some $\eta \in P^{k+1}(K)$ (where c^{-T} denotes the inverse of the transpose of c):

$$\int_K \boldsymbol{\phi}\cdot\nabla\eta\,dx + \frac{1}{d}\int_K (\nabla\cdot\boldsymbol{\phi})(\nabla\cdot c^{-T}\nabla\eta)\,dx = 0. \tag{4.6}$$

By (4.5) and integration by parts

$$\int_K \boldsymbol{\phi}\cdot\nabla\eta\,dx + \int_K \eta\nabla\cdot\boldsymbol{\phi}\,dx = \int_{\partial K}\eta(\boldsymbol{\phi}\cdot\boldsymbol{n})\,ds = 0. \tag{4.7}$$

Subtracting (4.6) from (4.7), we have

$$\int_K \left(\eta - \frac{1}{d}\nabla\cdot c^{-T}\nabla\eta\right)(\nabla\cdot\boldsymbol{\phi})\,dx = 0 \quad \text{for all } \eta \in P^{k+1}(K). \tag{4.8}$$

Now we show that (4.8) implies that $\nabla\cdot\boldsymbol{\phi} = 0$. Choosing $\eta \in P^1(K)$ in (4.8), we conclude that $\nabla\cdot\boldsymbol{\phi}$ is $L^2(K)$-orthogonal to $P^1(K)$. For $k \geq 2$, if $\nabla\cdot\boldsymbol{\phi}$ is $L^2(K)$-orthogonal to $P^{k-1}(K)$, then choosing $\eta \in P^k(K)$ we find that $\nabla\cdot\boldsymbol{\phi}$ is $L^2(K)$-orthogonal to $P^k(K)$ as well, because $\nabla\cdot(c^{-T}\nabla\eta) \in P^{k-2}(K)$. Thus, by induction, $\nabla\cdot\boldsymbol{\phi}$ is zero.

It follows from $\nabla \cdot \boldsymbol{\phi} = 0$ that $\boldsymbol{\phi} = 0$: Indeed, any $\boldsymbol{\phi} \in W$ can be written as $\boldsymbol{\phi}(\boldsymbol{x}) = \boldsymbol{x}\widetilde{p}_k - \mathcal{Q}_k(\boldsymbol{x}\widetilde{p}_k)$, where $\mathcal{Q}_k$ is the orthogonal projection onto $P^k(K) \times P^k(K)$ in the $(\!(\cdot,\cdot)\!)$-inner product, and $\widetilde{p}_k$ is a homogeneous polynomial of degree k. Therefore,

$$0 = \nabla \cdot \boldsymbol{\phi} = (k+2)\widetilde{p}_k - \nabla \cdot \mathcal{Q}_k(\boldsymbol{x}\widetilde{p}_k).$$

Since the latter term is in $P^{k-1}(K)$, we conclude that $\widetilde{p}_k = 0$, so $\boldsymbol{\phi} = \mathbf{0}$. Thus (4.4) does not hold, and the lemma is proved. $\square$

The next result establishes an equivalent criterion for statement (α) of Theorem 4.1 in terms of the liftings.

LEMMA 4.7. *Assume that $d(\boldsymbol{x})$ is constant on every element of the mesh. Let $\mathsf{m} \in L^2(\partial K)$. Then*

$$a_h^{\mathrm{RT}}(\mathsf{m},\mathsf{m}) = a_h^{\mathrm{BDM}}(\mathsf{m},\mathsf{m})$$

if and only if

$$\begin{array}{lll} \text{(i)} & \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} = \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} & \textit{on } \Omega \textit{ and} \\ \text{(ii)} & u_{\mathsf{m}}^{\mathrm{RT}} = u_{\mathsf{m}}^{\mathrm{BDM}} & \textit{on all elements } K \in \mathcal{T}_h, \textit{ where } d > 0. \end{array}$$

Proof. Set $\mathbb{J}(\mathsf{m}) = a_h^{\mathrm{RT}}(\mathsf{m},\mathsf{m}) - a_h^{\mathrm{BDM}}(\mathsf{m},\mathsf{m})$. Since, by Theorem 2.1, we have, for $\mathrm{N} \in \{\mathrm{RT}, \mathrm{BDM}\}$,

$$a_h^{\mathrm{N}}(\mathsf{m},\mathsf{m}) = \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{N}} \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{N}}\, dx + \int_\Omega d\, u_{h,\mathsf{m}}^{\mathrm{N}}\, u_{h,\mathsf{m}}^{\mathrm{N}}\, dx,$$

a straightforward use of the identity $a^2 - b^2 = (a-b)^2 + 2\,b\,(a-b)$ allows us to write $\mathbb{J}(\mathsf{m}) = \Theta(\mathsf{m}) + \mathbb{D}(\mathsf{m})$, where

$$\Theta(\mathsf{m}) = \int_\Omega c\,(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}) \cdot (\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}})\, dx + \int_\Omega d\,(u_{\mathsf{m}}^{\mathrm{RT}} - u_{\mathsf{m}}^{\mathrm{BDM}})^2\, dx$$

and

$$\mathbb{D}(\mathsf{m}) = 2 \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} \cdot \left(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\right)\, dx + 2 \int_\Omega d\, u_{h,\mathsf{m}}^{\mathrm{BDM}} \left(u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\right)\, dx.$$

We will now show that $\mathbb{D}(\mathsf{m}) = 0$. Consider the first term in the definition of $\mathbb{D}(\mathsf{m})$. By (2.9),

$$\begin{aligned} \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} \cdot \left(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\right)\, dx &= \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\, dx - \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\, dx \\ &= -\sum_{K \in \mathcal{T}_h} \int_K \left(u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\right) \nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\, dx. \end{aligned}$$

Hence, using also (2.10), we have

$$\begin{aligned} \int_\Omega c\, \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}} \cdot \left(\boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{RT}} - \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\right)\, dx &= -\sum_{K \in \mathcal{T}_h} \int_K \left(\mathcal{P}_{k-1} u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\right) \nabla \cdot \boldsymbol{q}_{h,\mathsf{m}}^{\mathrm{BDM}}\, dx \\ &= -\int_\Omega d\, u_{h,\mathsf{m}}^{\mathrm{BDM}} \left(\mathcal{P}_{k-1} u_{h,\mathsf{m}}^{\mathrm{RT}} - u_{h,\mathsf{m}}^{\mathrm{BDM}}\right)\, dx. \end{aligned}$$

Inserting this expression in the definition of $\mathbb{D}$, we get

$$\mathbb{D}(\mathsf{m}) = 2\int_\Omega d\, u_{h,\mathsf{m}}^{\mathrm{BDM}} \left(u_{h,\mathsf{m}}^{\mathrm{RT}} - \mathcal{P}_{k-1} u_{h,\mathsf{m}}^{\mathrm{RT}} \right) dx = 0,$$

because d is constant on each element of the mesh. In other words, $\mathbb{J}(\mathsf{m}) = \Theta(\mathsf{m})$.

This implies that $\mathbb{J}(\mathsf{m}) = 0$ if and only if $\Theta(\mathsf{m}) = 0$. The lemma follows from the definition of $\Theta(\mathsf{m})$ and the fact that $c(\boldsymbol{x})$ is positive definite. $\square$

We are now ready to prove Theorem 4.2.

Proof of Theorem 4.2. Since there is at least one element $K \in \mathcal{T}_h$ wherein d is a positive constant, Lemma 4.6 asserts the existence of at least one function $\mathsf{m} \in M_K^{\mathrm{RT}}$ for which $\mathcal{R} u_{h,\mathsf{m}}^{\mathrm{RT}} \neq 0$ on K. This implies that $u_{h,\mathsf{m}}^{\mathrm{RT}} \neq u_{h,\mathsf{m}}^{\mathrm{BDM}}$ on K. Therefore, by Lemma 4.7, for any $\mu \in M_h$ such that $\mu|_{\partial K} = \mathsf{m}$, we have $a_h^{\mathrm{RT}}(\mu,\mu) \neq a_h^{\mathrm{BDM}}(\mu,\mu)$. Consequently, statement (α) of Theorem 4.1 does not hold. This completes the proof. $\square$

5. Concluding remarks. The characterization theorem obtained in this paper for the hybridized RT method on triangular meshes also holds for various other methods. For example, it holds for the RT method on simplicial meshes in any space dimension as well as on rectangular and cubic meshes. It also holds for the hybridized versions of the mixed methods of Brezzi, Douglas, and Marini [5, 6] on rectangles, the method of Brezzi et al. [3] on tetrahedra and bricks, and the method of Brezzi et al. [4] on triangles, rectangles, tetrahedra, and bricks.

As a consequence, the matrix entries for the multipliers of all of the above-mentioned methods can be computed as described in section 3. Moreover, a result similar to the comparison theorem (Theorem 4.1) holds for the RT and BDM methods on multidimensional simplices. However, when the elements are rectangles or bricks, the subspace of divergence-free members of the RT and BDM spaces on an element are not identical. Therefore, in general, we cannot expect a result analogous to Theorem 4.1 to hold in this case.

Other applications of the characterization theorem are studied elsewhere. Indeed, in [11] it is shown how to use the characterization theorem to construct a Schwarz preconditioner for the multiplier equation of the RT and BDM hybridized mixed methods of *any* order. Such preconditioners were known only for the lowest order hybridized RT method. Also, in a forthcoming paper, we show how to use the characterization result to obtain error estimates for the multipliers *without* relying on error estimates on the other variables, as is customarily done.

Finally, let us briefly comment on the relation between the hybridized mixed methods and the discontinuous Galerkin methods. It is not difficult to see that the counterpart of the multiplier λ_h is nothing but the so-called numerical trace of the approximation u_h given by the discontinuous Galerkin method. How to exploit this link to achieve a better theoretical understanding of *both* methods remains a challenging open problem; see [9].

Acknowledgment. The authors would like to thank Wolfgang Dahmen, whose stimulating visit to the I.M.A., University of Minnesota, in the Spring of 2001, prompted them to explore ways of characterizing stiffness matrices of mixed methods by using discontinuous test functions. This paper is the unexpected outcome of such exploration.

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