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Jay Gopalakrishnan
Portland State University, gjay@pdx.edu

Guido Kanschat

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J. GOPALAKRISHNAN AND G. KANSCHAT

Abstract. A variable V-cycle preconditioner for an interior penalty finite element discretization for elliptic problems is presented. An analysis under a mild regularity assumption shows that the preconditioner is uniform. The interior penalty method is then combined with a discontinuous Galerkin scheme to arrive at a discretization scheme for an advection-diffusion problem, for which an error estimate is proved. A multigrid algorithm for this method is presented, and numerical experiments indicating its robustness with respect to diffusion coefficient are reported.

1. Introduction

In this paper we show that a multigrid technique can be used for efficient solution of linear systems arising from the so-called interior penalty finite element method for second order elliptic boundary value problems. We also present a fast method for advection-diffusion equations by combining the interior penalty method with the discontinuous Galerkin method for transport equations, and applying a multigrid technique on the resulting discrete system.

Discontinuous Galerkin (DG) methods have traditionally been used in numerical solution of hyperbolic conservation laws [12, 22, 24]. Their ability to capture strong gradients in solution without spurious oscillations is well known. Recently DG methods have also been shown to be of use in solving elliptic problems [11]. It is now common to classify various earlier methods for solving elliptic problems that went by the name of "interior penalty methods" [2, 3, 28, 13, 31] under DG methods (see [1] for a unified treatment). In general, interior penalty finite element methods use discontinuous finite element functions, but penalize discontinuities of the function or its derivatives across inter-element boundaries.

Part of this paper deals with the interior penalty method considered in [2, 31]. When applied to elliptic problems, the method gives rise to

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linear systems with condition number that grows like $O(h^{-2})$ on quasiuniform grids with mesh size h. We prove (in Section 3) under weak regularity assumptions that if a variable V-cycle multigrid operator is used to precondition the linear system, then the resulting condition number is $O(1)$, i.e., bounded independently of h. The proof is an application of the abstract multigrid theory of [9] for non-inherited bilinear forms. The conjugate gradient method using this preconditioner converges in $O(N)$ operations, where N is the number of unknowns, thus yielding an asymptotically optimal solution technique. In Section 4, we confirm and illustrate the theoretical result through numerical experiments.

Preconditioners for discretizations with discontinuous spaces have been studied before [9, 29]. In [7], a multigrid analysis for a cell-centered finite difference scheme on uniform grids on square domains is available. Indeed, the interior penalty method can be interpreted as a cell-centered finite difference scheme when piecewise constant functions are used as the discretization space. Our approach here is to use interior penalty estimates directly for multigrid analysis, and we consider spaces of linear or higher order polynomials on more general grids. We note that efficient solution strategies for DG methods can also be constructed using domain decomposition techniques [15, 25].

DG schemes show their full potential in advection problems rather than elliptic problems. In Section 5, we introduce a DG scheme for an advection-diffusion equation with an arbitrarily small diffusion term. This scheme reduces to the standard DG method for advection problems when the diffusion term is zero. On the other hand, when the advection term is zero, our scheme is the interior penalty method. Before discussing efficient solution strategies for this scheme, we state an error estimate that ensures that it yields good approximate solutions. A proof of this estimate is given in Appendix A, and may be independently interesting as it provides an error estimate in a slightly stronger norm than some standard estimates (cf. [19]).

For this scheme, we also present a computational technique for fast solution that performs uniformly in both convection dominated and diffusion dominated regimes. This is inspired by the fact that a downwind Gauß–Seidel iteration is an exact solver in the case of zero diffusion. Although this is no longer true when the diffusion term is nonzero, in this case, as shown in Sections 3 and 4, multigrid works well. Therefore, we investigate the performance of a multigrid method with the downwind Gauß–Seidel iteration as smoother in Section 5. Similar ideas have appeared earlier [10, 18], although these are in connection with other discretization schemes. Two multigrid analyses for advection-diffusion equations can be found in [26, 27]. Both works deal with the streamline diffusion method and it is not clear if the properties of a downwind ordering can be exploited in these analyses.

2. Multilevel spaces and the interior penalty method

In this section we introduce notation for multilevel spaces and describe an interior penalty method. Before we define discrete spaces associated with the interior penalty method of [2, 31], we describe a model problem and state a mild regularity assumption on its solutions.

Let us first define Sobolev spaces that we will use. For a bounded connected open subset $\mathcal{D}$ of $\mathbb{R}^2$ or $\mathbb{R}$, let $L^2(\mathcal{D})$ denote the space of square integrable functions on $\mathcal{D}$, and let $(\cdot,\cdot)_{\mathcal{D}}$ and $\|\cdot\|_{0,\mathcal{D}}$ denote the innerproduct and norm on $L^2(\mathcal{D})$ (or $L^2(\mathcal{D})^2$) respectively. Denote by $H_0^1(\mathcal{D})$ the completion of compactly supported infinitely differentiable functions under the norm

$$|u|_{1,\mathcal{D}} \equiv \|\nabla u\|_{0,\mathcal{D}}.$$

Let the dual space of $H_0^1(\mathcal{D})$ be denoted by $H^{-1}(\mathcal{D})$. For $-1 < s < 0$, let $H^{-s}(\mathcal{D})$ denote the the space obtained by interpolation (by the real number method) between $H^{-1}(\mathcal{D})$ and $L^2(\mathcal{D})$. For non-negative integers m, the Sobolev space $H^m(\mathcal{D})$ is the set of functions in $L^2(\mathcal{D})$ with distributional derivatives up to order m also in $L^2(\mathcal{D})$. If s is a positive real number between non-negative integers m and $m+1$, $H^s(\mathcal{D})$ is the space obtained by interpolation between $H^m(\mathcal{D})$ and $H^{m+1}(\mathcal{D})$. The norm on $H^r(\mathcal{D})$ for any r is denoted by $\|\cdot\|_{r,\mathcal{D}}$.

For transparent presentation of multigrid analysis we will restrict ourselves to the following simple model problem: Find $U \in H_0^1(\Omega)$ such that

$$(\nabla U, \nabla\phi)_\Omega = (f,\phi)_\Omega \quad \text{for all } \phi \in H_0^1(\Omega), \tag{2.1}$$

where $\Omega \subset \mathbb{R}^2$ is a polygonal domain, $\partial\Omega$ denotes the boundary of Ω, and $f \in L^2(\Omega)$. We will make the following regularity assumption on solutions to this problem: There is an $1/2 < \alpha \leq 1$ and a constant C_Ω such that the solution U of (2.1) satisfies

$$\|U\|_{1+\alpha,\Omega} \leq C_\Omega \|f\|_{-1+\alpha,\Omega}. \tag{2.2}$$

This assumption is known to hold [23] for polygonal domains with $\alpha < \delta^{-1}$ where $\pi\delta$ is the maximum of the interior angles of Ω. For convex domains, it holds [16] with $\alpha = 1$.

Let Ω be subdivided by a "coarse" quasiuniform triangulation $\mathcal{T}_1$ of mesh-size h_1. Our multilevel spaces are based on a sequence of refinements of this mesh. We refine the triangulation $\mathcal{T}_1$ to produce $\mathcal{T}_2$ by splitting each triangle of $\mathcal{T}_1$ into four congruent triangles. The triangulation $\mathcal{T}_2$ is then quasiuniform with mesh-size $h_2 = h_1/2$. Repeating this process, we get a sequence of triangulations $\mathcal{T}_k$, $k = 1, \ldots J$, each quasiuniform with mesh-size $h_k = h_1/2^{k-1}$. Let d be a fixed integer not less than one. We define multilevel spaces $M_1 \subset M_2 \subset \ldots M_J$ by

$$M_k = \{v : v|_\tau \text{ is a polynomial of degree at most } d, \text{ for all } \tau \in \mathcal{T}_k\}.$$

We can also let $\mathcal{T}_k$ to be a mesh with quadrilateral elements and M_k to be a mapped tensor-product finite element space. Let us now describe the interior penalty method at each refinement level in terms of these spaces.

The interior penalty method provides a discontinuous Galerkin approximation to the solution U of (2.1). To describe it, we will need the spaces

$$H^1(\mathcal{T}_k) = \{u \in L^2(\Omega) : u|_\tau \in H^1(\tau) \text{ for all } \tau \in \mathcal{T}_k\}. \tag{2.3}$$

Let $\mathcal{E}_k$ denote the set of edges of the triangulation $\mathcal{T}_k$. If $e \in \mathcal{E}_k$ is an interior edge, denoting by n_e one of the two unit normal vectors at e, we define jumps and averages of normal derivatives (for $x \in e$) of $u \in H^1(\mathcal{T}_k)$ by

$$\begin{aligned} [u]_e(x) &= \lim_{\delta\to 0+} \big(u(x-\delta n_e) - u(x+\delta n_e) \big), \quad \text{and} \\ \langle \partial_n u\rangle_e(x) &= \tfrac{1}{2} \lim_{\delta\to 0+} \big(n_e\cdot\nabla u(x-\delta n_e) + n_e\cdot\nabla u(x+\delta n_e) \big), \end{aligned}$$

while if $e \subseteq \partial\Omega$, we fix n_e to be the outward normal vector and let

$$[u]_e(x) = \lim_{\delta\to 0+} u(x-\delta n_e), \quad \text{and} \quad \langle \partial_n u\rangle_e = \lim_{\delta\to 0+} n_e\cdot\nabla u(x-\delta n_e).$$

Here and elsewhere, we use "$\cdot$" to denote innerproducts in Euclidean spaces. We will drop the subscript e when no confusion can arise. Define $a_k(\cdot,\cdot)$ on $H^1(\mathcal{T}_k)\times H^1(\mathcal{T}_k)$ by

$$a_k(u,v) = \sum_{\tau\in\mathcal{T}_k} (\nabla u, \nabla v)_\tau + \sum_{e\in\mathcal{E}_k} \left(\frac{\sigma}{\ell_e}([u],[v])_e - (\langle\partial_n u\rangle, [v])_e - ([u], \langle\partial_n v\rangle)_e \right).$$

Here ℓ_e denotes the length of edge e, and σ is a positive parameter to be chosen. The interior penalty method is based on the observation that the U that solves the Poisson equation (2.1) also satisfies

$$a_k(U,v) = (f,v)_\Omega \quad \text{for all } v \in H^1(\mathcal{T}_k), \tag{2.4}$$

for each $k = 1,\dots J$. The interior penalty approximation to U from M_k, namely U_k, is defined by

$$a_k(U_k, v_k) = (f, v_k) \quad \text{for all } v_k \in M_k. \tag{2.5}$$

Here and elsewhere, when the subscript indicating the domain in L^2 innerproduct is dropped, the domain is to be taken as Ω.

If σ is chosen large enough, the discrete system (2.5) is uniquely solvable [2]. This follows from the inverse inequality

$$\sum_{e\in\mathcal{E}_k} \ell_e \|\langle\partial_n u\rangle\|^2_{0,e} \le C_0 \sum_{\tau\in\mathcal{T}_k} \|\nabla u\|^2_{0,\tau}, \tag{2.6}$$

which holds for all $u \in M_k$ and for all $k = 1,\dots J$, with the constant C_0 independent of $\{h_k\}$. Indeed, whenever $\sigma > 2C_0$, it can easily be seen that

$$a_k(u,u) \ge \frac{1}{2} \sum_{\tau\in\mathcal{T}_k} \|\nabla u\|^2_{0,\tau} + (\sigma - 2C_0) \sum_{e\in\mathcal{E}_k} \frac{1}{\ell_e} \| [u] \|^2_{0,e}, \tag{2.7}$$

for all $u \in M_k$. Then $a_k(\cdot,\cdot)$ is a positive definite and symmetric bilinear form, and consequently is an innerproduct on M_k. Therefore (2.5) is uniquely solvable. The norm generated by $a_k(\cdot,\cdot)$ will be denoted by $\|\!|\cdot|\!\|_k$, i.e.,

$$\|\!|u|\!\|_k \equiv a_k(u,u)^{1/2}. \tag{2.8}$$

We will henceforth assume that $\sigma > 2C_0$. Freedom in choice of σ in actual practice can be a boon [2] or bane depending on available information.

In [2], the interior penalty method is proved to yield good approximations to U in a certain "energy norm", slightly different from $\|\!|\cdot|\!\|_k$. Nonetheless, it is easy to get estimates in $\|\!|\cdot|\!\|_k$-norm as well. Indeed, Galerkin orthogonality and the Cauchy-Schwarz inequality for $a_k(\cdot,\cdot)$ imply

$$\|\!|U - U_k|\!\|_k \leq \|\!|U - v_k|\!\|_k \quad \text{for all } v_k \in M_k. \tag{2.9}$$

In particular, we may choose for v_k an interpolant of U that is continuous on Ω and vanishes on $\partial\Omega$. Let $\mathfrak{I}_k$ denote such an interpolation operator [30], having the property that

$$|v - \mathfrak{I}_k v|_{1,\Omega} \leq C h_k^{\alpha} \|v\|_{1+\alpha,\Omega} \quad \text{for all } v \in H^{1+\alpha}(\Omega). \tag{2.10}$$

We have adopted the usual convention of denoting by C (with or without subscript) a generic constant independent of h_k. At any two different occurrences, its value may differ, but will always remain independent of mesh sizes. Since $\|\!|U - \mathfrak{I}_k U|\!\|_k = |U - \mathfrak{I}_k U|_{1,\Omega}$, we can choose $v_k = \mathfrak{I}_k U$ in (2.9) and get

$$\|\!|U - U_k|\!\|_k \leq C h_k^{\alpha} \|U\|_{1+\alpha,\Omega}. \tag{2.11}$$

Our interest is in efficiently computing the interior penalty approximation on the finest level, namely U_J. Let us define the operator $A_k : M_k \to M_k$ by

$$(A_k u, v) = a_k(u,v) \quad \text{for all } u, v \in M_k,$$

and examine its spectral properties. Let λ_k denote the maximum eigenvalue of A_k, i.e.,

$$\lambda_k = \sup_{v \in M_k} \frac{a_k(v,v)}{(v,v)}.$$

From the trace inequality

$$C \sum_{e \in \mathcal{E}_k} \frac{1}{\ell_e} \| [v] \|_{0,e}^2 \leq \sum_{\tau \in \mathcal{T}_k} h_k^{-2} \|v\|_{0,\tau}^2 + |v|_{1,\tau}^2, \quad \text{for all } v \in H^1(\mathcal{T}_k), \tag{2.12}$$

the inverse inequality

$$\sum_{\tau \in \mathcal{T}_k} \|\nabla v\|_{0,\tau}^2 \leq C h_k^{-2} \|v\|_{0,\Omega}^2, \quad \text{for all } v \in M_k, \tag{2.13}$$

and (2.6), we get that

$$\lambda_k \leq C(\sigma + C_0) h_k^{-2}. \tag{2.14}$$

Remark 2.1. The minimum eigenvalue of A_k is bounded from below by a constant independent of h_k. To see this, note that (2.7) implies

$$a_k(u,u) \geq C \min(1/2, \sigma - 2C_0)\, \widetilde{a}_k(u,u), \quad \text{where} \tag{2.15}$$
$$\widetilde{a}_k(u,u) = \sum_{\tau\in\mathcal{T}_k} \|\nabla u\|^2_{0,\tau} + \sum_{e\in\mathcal{E}_k} \frac{1}{\ell_e} \| [u] \|^2_{0,e},$$

for all $u \in M_k$. The assertion on minimum eigenvalue follows from the Poincaré inequality

$$\widetilde{a}_k(u,u) \geq C\|u\|^2_{0,\Omega}. \tag{2.16}$$

This is proved for convex domains in [2, Lemma 2.1]. That it holds for for general polygonal domains can be seen using [29, Theorem 3.1] and the inf-sup condition for Raviart-Thomas pair of spaces.

From (2.14) and Remark 2.1, we see that the condition number of the discrete system (2.5) is bounded by Ch^{-2}. This estimate is sharp. Therefore, iterative solution of (2.5) for large problems calls for a good preconditioner. In the next section, we show that a multigrid technique can be used to precondition (2.5) effectively.

Remark 2.2. It is not difficult to show that $a_k(u,u) \leq C\sigma\, \widetilde{a}_k(u,u)$. This, together with (2.15), imply that any preconditioner for the form $\widetilde{a}_k(u,u)$ also gives a preconditioner for $a_k(u,u)$. In [29], an overlapping Schwarz preconditioner for $\widetilde{a}_k(u,u)$ is analyzed. This preconditioner can very well be used to precondition our discrete system (2.5).

3. A variable V-cycle preconditioner

In this section, we define and analyze a variable V-cycle operator. Our main theorem will show that this operator provides a uniform preconditioner for (2.5). The multigrid analysis we present here is an application of the abstract theory of multigrid algorithms in [6, 9].

Let us first write (2.5) in matrix form. Let N_k be the number of degrees of freedom of M_k. Let $\{\phi^i_k\}_{i=1}^{N_k}$ be the nodal basis for M_k. Let A_k be the matrix whose (i,j)-th entry is $a_k(\phi^j_k, \phi^i_k)$. If u_J is the vector of coefficients in the expansion of U_J in the nodal basis, then (2.5) implies that

$$\mathsf{A}_J \mathsf{u}_J = \mathsf{b}_J,$$

where b_J is a vector whose i-th component equals (f, ϕ^i_J). We are interested in efficiently solving this equation. Our intention is to define a matrix B_J such that $\mathsf{B}_J\mathsf{A}_J$ is well conditioned and solve $\mathsf{B}_J\mathsf{A}_J\mathsf{u}_J = \mathsf{B}_J\mathsf{b}_J$ instead.

A main ingredient of multigrid algorithms are "smoother" matrices, which we denote by R_k. Let L_k, U_k, and D_k be strictly lower triangular, strictly upper triangular, and diagonal matrices respectively such that $\mathsf{A}_k = \mathsf{L}_k + \mathsf{D}_k + \mathsf{U}_k$. Define J_k and G_k by

$$\mathsf{J}_k = \gamma \mathsf{D}_k^{-1} \quad \text{and} \quad \mathsf{G}_k = (\mathsf{D}_k + \mathsf{L}_k)^{-1},$$

where γ is an appropriate scaling factor. We will set R_k to equal either J_k or G_k. Then the iteration

$$\mathsf{x}^{i+1} = \mathsf{x}^i + \mathsf{R}_k(\mathsf{b}_k - \mathsf{A}_k\mathsf{x}^i), \quad i = 1, 2, \ldots$$

with some initial guess x^0, is either a scaled Jacobi or a Gauß–Seidel iteration, and both are well known [17] to be smoothing iterations.

Multigrid algorithms also require intergrid transfer operators, often called "prolongation" and "restriction". These are given naturally in our application by the imbeddings $M_1 \subset M_2 \ldots \subset M_J$. Because of these imbeddings, there are numbers $\{\alpha_{ij}^k\}$ such that

$$\phi_{k-1}^i = \sum_{i=1}^{N_k} \alpha_{ij}^k \phi_k^i.$$

Let C_{k-1} be a matrix whose (i,j)-th entry is the α_{ij}^k above. Then C_{k-1} and its transpose C_{k-1}^t are the restriction and prolongation matrices, respectively.

We can now define the variable V-cycle matrix B_J, following [6, 9]. Instead of giving the entries of the matrix B_J, the algorithm below defines B_J by defining its action on any vector. Thus, B_J (which is a full matrix) need not be assembled in an implementation. Let $m(k)$, $k = 1, \ldots, J$ be a sequence of positive integers.

Algorithm 3.1. Set $\mathsf{R}_k^{(l)} = \mathsf{R}_k$ if l is odd, and $\mathsf{R}_k^{(l)} = \mathsf{R}_k^t$ if l is even (R_k^t is the transpose of R_k). Also set $\mathsf{x}^0 = 0$ and $\mathsf{B}_1 = \mathsf{A}_1^{-1}$. For $k \geq 2$ and any vector $\mathsf{d}_k \in \mathbb{R}^{N_k}$, $\mathsf{B}_k\mathsf{d}_k$ can be computed by the following four steps, assuming that B_{k-1} is already defined.

(1) Compute x^l for $l = 1, \ldots m(k)$:

$$\mathsf{x}^l = \mathsf{x}^{l-1} + \mathsf{R}_k^{(l+m(k))}(\mathsf{d}_k - \mathsf{A}_k\mathsf{x}^{l-1}).$$

(2) Set $\mathsf{y}^{m(k)} = \mathsf{x}^{m(k)} + \mathsf{C}_{k-1}^t\mathsf{B}_{k-1}\mathsf{C}_{k-1}(\mathsf{d}_k - \mathsf{A}_k\mathsf{x}^{m(k)})$.

(3) Compute y^l for $l = m(k)+1, \ldots, 2m(k)$:

$$\mathsf{y}^l = \mathsf{y}^{l-1} + \mathsf{R}_k^{(l+m(k))}(\mathsf{d}_k - \mathsf{A}_k\mathsf{y}^{l-1}).$$

(4) Set $\mathsf{B}_k\mathsf{d}_k = \mathsf{y}^{2m(k)}$.

This algorithm thus defines B_J recursively. We assume that the number of smoothings $m(k)$ increases as k decreases in such a way that

$$\beta_0 m(k) \leq m(k-1) \leq \beta_1 m(k), \tag{3.1}$$

with $1 < \beta_0 \leq \beta_1$. A typical choice is $\beta_0 = \beta_1 = 2$, i.e., $m(k) = 2^{J-k}$. In this case the cost of the algorithm is comparable to that of the W-cycle. We can now give the main theorem.

Theorem 3.1. *Suppose that the regularity assumption* (2.2) *holds and the number of smoothings satisfy* (3.1). *Then* B_J *is a symmetric and positive*

definite matrix and there is a constant c_σ such that

$$\zeta^{-1} \leq \frac{(\mathsf{B}_J\mathsf{A}_J\mathsf{x})\cdot(\mathsf{A}_J\mathsf{x})}{(\mathsf{A}_J\mathsf{x})\cdot\mathsf{x}} \leq \zeta, \quad \textit{for all } \mathsf{x}\in\mathbb{R}^{N_J},$$

with $\zeta = (c_\sigma + m(J)^{\alpha/2})/m(J)^{\alpha/2}$. Here c_σ is independent of mesh sizes $\{h_k\}$. Consequently, the spectral condition number of $\mathsf{B}_J\mathsf{A}_J$, namely $\kappa(\mathsf{B}_J\mathsf{A}_J)$, satisfies $\kappa(\mathsf{B}_J\mathsf{A}_J)\leq\zeta^2$.

From this theorem, it follows that B_J is well suited for use in a preconditioned conjugate gradient iteration. Indeed, because of the bound on $\kappa(\mathsf{B}_J\mathsf{A}_J)$, such an iteration will converge in a fixed number of steps independently of how many levels of refinement were used to obtain the fine mesh, i.e., independently of h_J. Note that even with one smoothing step at the finest level ($m(J)=1$) we get a uniform preconditioner. Increasing $m(J)$ gives a better condition number estimate according to Theorem 3.1, but it also increases the cost of applying B_J.

The proof of Theorem 3.1, by virtue of the abstract theory given in [9] (or see [6, Theorem 4.6]), reduces to verification of two conditions. Define $P_{k-1}: M_k \to M_{k-1}$ by

$$a_{k-1}(P_{k-1}u, v_{k-1}) = a_k(u, v_{k-1}) \quad \text{for all } u\in M_k,\ v_{k-1}\in M_{k-1}. \tag{3.2}$$

The two conditions are as follows:

(1) There exists an $0<\omega<2$ such that for all $k=2,\dots,J$,

$$\frac{\omega}{\lambda_k}\mathsf{M}_k^{-1}\mathsf{x}\cdot\mathsf{x} \leq 2\mathsf{R}_k\mathsf{x}\cdot\mathsf{x} - \mathsf{A}_k\mathsf{R}_k\mathsf{x}\cdot\mathsf{R}_k\mathsf{x}, \quad \text{for all } \mathsf{x}\in\mathbb{R}^{N_k}, \tag{3.3}$$

where M_k is the mass matrix, i.e., its (i,j)-th entry is (ϕ_k^i,ϕ_k^j).

(2) There is a $0<\eta\leq 1$ and $C_P>0$ such that

$$|a_k(u-P_{k-1}u,u)| \leq C_P\left(\frac{\|A_ku\|_k^2}{\lambda_k}\right)^\eta a_k(u,u)^{1-\eta} \tag{3.4}$$

holds for all $u\in M_k$, and for any $k=2,\dots,J$.

The first condition pertains only to smoothers. In our application, this follows from the locality of basis functions. Indeed it can immediately be seen that the limited interaction and stable decomposition hypotheses in [8] hold in our case. It follows from [8, Theorem 3.2] that (3.3) holds for $\mathsf{R}_k=\mathsf{G}_k$ and from [8, Theorem 3.1] that there are values for γ such that (3.3) holds for $\mathsf{R}_k=\mathsf{J}_k$. Note that the theory in [8] can also be used to conclude (3.3) for block Jacobi and Gauß–Seidel smoothers. Indeed these are the smoothers that we will use in our numerical experiments in later sections.

Therefore, to prove Theorem 3.1, it suffices to prove (3.4). In the remainder of this section, we will prove (3.4) with $\eta=\alpha/2$, where α is as in the regularity assumption (2.2). The following lemma estimates the difference between u and $P_{k-1}u$, and will be useful while proving Theorem 3.1.

Lemma 3.1. *Assume that the regularity assumption* (2.2) *holds. Then, for all* $u \in M_k$, $(k = 2, \ldots, J)$

$$\|\!|u - P_{k-1}u|\!\|_k \leq C(\sigma)\, h_k^{\alpha}\, \|A_k u\|_{-1+\alpha,\Omega},$$

where $C(\sigma) = C\sigma/(\sigma - 2C_0)$.

Proof. We start with

$$\|\!|u - P_{k-1}u|\!\|_k \leq \|\!|u - w|\!\|_k + \|\!|w - P_{k-1}u|\!\|_k, \tag{3.5}$$

where $w \in H^{1+\alpha}(\Omega)$ satisfies

$$-\Delta w = A_k u \quad \text{on } \Omega, \quad w = 0 \quad \text{on } \partial\Omega.$$

The proof now proceeds by estimating the two terms on the right hand side of (3.5).

From the consistency of the interior penalty method, we have that

$$a_k(w, v) = (A_k u, v) \quad \text{for all } v \in H^1(\mathcal{T}_k).$$

By the definition of $A_k u$, we also have that

$$a_k(u, v_k) = (A_k u, v_k) \quad \text{for all } v_k \in M_k.$$

Together, these equalities imply that u is the interior penalty approximation to w from M_k. Thus, by the error estimate (2.11),

$$\|\!|u - w|\!\|_k \leq C h_k^{\alpha} \|w\|_{1+\alpha,\Omega}. \tag{3.6}$$

It now remains only to estimate the last term in (3.5). As before, we have

$$a_{k-1}(w, v) = (A_k u, v) \quad \text{for all } v \in H^1(\mathcal{T}_{k-1}).$$

In view of (3.2), this equation implies that $P_{k-1}u$ is the interior penalty approximation of w from M_{k-1}. Thus, by (2.11),

$$\|\!|w - P_{k-1}u|\!\|_{k-1} \leq C h_{k-1}^{\alpha} \|w\|_{1+\alpha,\Omega}. \tag{3.7}$$

We will now use (3.7) to estimate $\|\!|w - P_{k-1}u|\!\|_k$. Observe that $[w - P_{k-1}u]_e$ is zero for edges $e \in \mathcal{E}_k$ that are not subsets of a coarse edge in $\mathcal{E}_{k-1}$. Therefore,

$$\sum_{e \in \mathcal{E}_k} \frac{1}{\ell_e} \| [w - P_{k-1}u] \|_{0,e}^2 = \sum_{e \in \mathcal{E}_{k-1}} \frac{2}{\ell_e} \| [w - P_{k-1}u] \|_{0,e}^2.$$

This implies that

$$\|\!|w - P_{k-1}u|\!\|_k^2 \leq \|\!|w - P_{k-1}u|\!\|_{k-1}^2 + \sigma \sum_{e \in \mathcal{E}_{k-1}} \frac{1}{\ell_e} \| [w - P_{k-1}u] \|_{0,e}^2. \tag{3.8}$$

Note that $[w]_e = 0$ for all $e \in \mathcal{E}_{k-1}$. Consider the continuous interpolant $\mathcal{I}_{k-1}w$ of w. Then $[\mathcal{I}_{k-1}w]_e$ is zero as well. Hence,

$$\begin{aligned} \sigma \sum_{e \in \mathcal{E}_{k-1}} \frac{1}{\ell_e} \| [w - P_{k-1}u] \|_{0,e}^2 &= \sigma \sum_{e \in \mathcal{E}_{k-1}} \frac{1}{\ell_e} \| [\mathcal{I}_{k-1}w - P_{k-1}u] \|_{0,e}^2 \\ &\leq \frac{\sigma}{\sigma - 2C_0} \|\!|\mathcal{I}_{k-1}w - P_{k-1}u|\!\|_{k-1}, \end{aligned}$$

where the last inequality was because of (2.7). Now since

$$|||\mathfrak{I}_{k-1}w - P_{k-1}u|||_{k-1} \le |||w - \mathfrak{I}_{k-1}w|||_{k-1} + |||w - P_{k-1}u|||_{k-1},$$

and since $|||w - \mathfrak{I}_{k-1}w|||_{k-1} = |w - \mathfrak{I}_{k-1}w|_{1,\Omega}$, (2.10) and (2.11) imply that

$$\sigma \sum_{e\in\mathcal{E}_{k-1}} \frac{1}{\ell_e} \| [w - P_{k-1}u] \|_{0,e}^2 \le C \left(\frac{\sigma}{\sigma - 2C_0}\right) h_{k-1}^{\alpha} \|w\|_{1+\alpha,\Omega}.$$

Consequently, from (3.8) and (3.7), we get that

$$|||w - P_{k-1}u|||_k \le \frac{C\sigma}{\sigma - 2C_0} h_{k-1}^{\alpha} \|w\|_{1+\alpha,\Omega}. \tag{3.9}$$

Combining this with (3.5), and using the regularity assumption (2.2) for w, we have the result. □

Proof of Theorem 3.1. From the discussion before, it suffices to prove (3.4) with $\eta = \alpha/2$. We will start with the result of Lemma 3.1:

$$|||u - P_{k-1}u|||_k \le C(\sigma) h_k^{\alpha} \|A_k u\|_{-1+\alpha,\Omega} \le C(\sigma) h_k^{\alpha} \|A_k u\|_{-1,\Omega}^{1-\alpha} \|A_k u\|_{0,\Omega}^{\alpha}, \tag{3.10}$$

where the last inequality was because $H^{-1+\alpha}(\Omega)$ is in the scale of intermediate spaces between $L^2(\Omega)$ and $H^{-1}(\Omega)$. Let us now observe that if we show that

$$\|A_k u\|_{-1,\Omega} \le C |||u|||_k, \tag{3.11}$$

then (3.4) follows. Indeed, when (3.10) is combined with (3.11) and (2.14), we have that

$$\begin{aligned} a_k(u - P_{k-1}u, u) &\le |||u - P_{k-1}u|||_k |||u|||_k \\ &\le C(\sigma) h_k^{\alpha} |||u|||_k^{2-\alpha} \|A_k u\|_{0,\Omega}^{\alpha} \\ &\le C(\sigma)\sigma^{\alpha/2} \left(\frac{\|A_k u\|_{0,\Omega}^2}{\lambda_k}\right)^{\alpha/2} a_k(u,u)^{1-\alpha/2}, \end{aligned}$$

from which (3.4) follows with $\eta = \alpha/2$ and $C_P = C(\sigma)\sigma^{\alpha/2}$.

Thus, it remains only to prove (3.11). For this we again use the continuous interpolant $\mathfrak{I}_k$ of [30]. It is proved in [30] that for all $\psi \in H_0^1(\Omega)$,

$$|\mathfrak{I}_k \psi|_{1,\Omega} \le C |\psi|_{1,\Omega} \tag{3.12}$$

$$\|\psi - \mathfrak{I}_k \psi\|_{0,\Omega} \le C h_k |\psi|_{1,\Omega}. \tag{3.13}$$

Now, to estimate $\|A_k u\|_{-1,\Omega}$, note that

$$\begin{aligned} \|A_k u\|_{-1,\Omega} &= \sup_{\psi\in H_0^1(\Omega)} \frac{(A_k u, \psi)}{|\psi|_{1,\Omega}} \\ &\le \sup_{\psi\in H_0^1(\Omega)} \frac{(A_k u, \psi - \mathfrak{I}_k\psi)}{|\psi|_{1,\Omega}} + \sup_{\psi\in H_0^1(\Omega)} \frac{a_k(u, \mathfrak{I}_k\psi)}{|\psi|_{1,\Omega}}. \end{aligned}$$

Since $\|\!|\mathcal{I}_k\psi|\!\|_k = |\mathcal{I}_k\psi|_{1,\Omega}$, by (3.12) we have $\|\!|\mathcal{I}_k\psi|\!\|_k \le C|\psi|_{1,\Omega}$. Using this, (3.13), and the Cauchy–Schwarz inequality, we get that

$$\|A_k u\|_{-1,\Omega} \le Ch_k \|A_k u\|_{0,\Omega} + C\|\!|u|\!\|_k. \tag{3.14}$$

Since A_k is a symmetric positive definite operator, we also have

$$\|A_k u\|_{0,\Omega}^2 \le \lambda_k(A_k u, u).$$

When this combined with (2.14) is used in (3.14), we get (3.11), thus completing the proof. □

4. Numerical results for Poisson equation

We report numerical results obtained using the finite element library `deal.II` by W. Bangerth and G. Kanschat [4, 5]. All computations use meshes with square grid cells. On each cell the finite element space consists of tensor products of polynomials in each coordinate variable of degree at most d (namely Q_d). Multilevel meshes are obtained by a uniform refinement procedure that breaks each coarse cell into four congruent cells.

First we consider the case when $\Omega = (-1,1)^2$ and $d = 1$. The coarse grid $\mathcal{T}_1$ consists of the single cell Ω. In Table 4.1, we present spectral condition numbers with and without preconditioning. Here we used a block Gauß–Seidel smoother, where the block partitioning is such that degrees of freedom within an element are grouped together. Estimates for both $\kappa(\mathsf{A}_J)$ and $\kappa(\mathsf{B}_J\mathsf{A}_J)$ (second and third columns in the table) are in accordance with theoretical estimates. The advantage of preconditioning is clear.

Often V-cycle is used as a solver by itself, rather than as a preconditioner. In this case it is important to see if $\mathsf{I} - \mathsf{B}_J\mathsf{A}_J$ is a contraction independent of the mesh size. (Here I is the identity matrix.) Although we do not have a theoretical result in this direction, our experiments seem to indicate that the spectral radius of $\mathsf{I} - \mathsf{B}_J\mathsf{A}_J$, namely $\varrho(\mathsf{I} - \mathsf{B}_J\mathsf{A}_J)$, remains bounded with refinement (see third column of Table 4.1).

The above-mentioned results are obtained with the number of smoothing steps $m(k) = 2^{J-k}$, set in accordance with Assumption (3.1). The last two columns of Table 4.1 are included so we can compare these with the case when only one smoothing step before and after coarse grid correction is done ($m(k) = 1$). The former is found to give a slightly better preconditioner.

In Table 4.2, we present results for the variable V-cycle with block Jacobi smoothing. A relaxation parameter of 0.95 (which appeared to be the best computationally) was used. The condition numbers are not as good as the Gauß-Seidel case. However, as the last two columns indicate, with a slight increase in number of smoothing steps, the preconditioner with Jacobi smoother also yields good condition numbers.

Table 4.3 shows that the multigrid preconditioner works well for higher order finite element spaces.

We also investigated the variation of extremal eigenvalues of A_J and $\mathsf{B}_J\mathsf{A}_J$ with σ. We found that the minimum eigenvalue of A_J is independent of σ

		$m(k) = 2^{J-k}$		$m(k) = 1$	
J	$\kappa(\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I} - \mathsf{B}_J\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I} - \mathsf{B}_J\mathsf{A}_J)$
2	10	1.36	0.19	1.36	0.19
3	22	1.71	0.26	1.72	0.27
4	79	1.97	0.36	2.11	0.42
5	312	2.08	0.41	2.39	0.54
6	1246	2.11	0.42	2.56	0.62
7	4981	2.11	0.42	2.66	0.66
8	19921	2.12	0.42	2.73	0.69

TABLE 4.1. Condition numbers and contraction numbers, when Gauß-Seidel smoother, and Q_1 elements are used. $\Omega = (-1,1)^2$, $\sigma = 3$.

	$m(k) = 2^{J-k}$		$m(k) = 2^{J-k+1}$	
J	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I} - \mathsf{B}_J\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I} - \mathsf{B}_J\mathsf{A}_J)$
2	1.62	0.30	1.14	0.10
3	2.23	0.40	1.36	0.17
4	2.72	0.54	1.49	0.21
5	2.95	0.64	1.54	0.22
6	3.02	0.67	1.56	0.22
7	3.04	0.68	1.56	0.22

TABLE 4.2. Condition numbers and contraction numbers when Jacobi smoother and Q_1 elements are used. $\Omega = (-1,1)^2$, $\sigma = 3$.

	Q_2 elements ($\sigma = 8$)			Q_3 elements ($\sigma = 22$)		
J	$\kappa(\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I}-\mathsf{B}_J\mathsf{A}_J)$	$\kappa(\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I}-\mathsf{B}_J\mathsf{A}_J)$
2	23	2.07	0.36	77	2.97	0.53
3	69	2.11	0.39	269	2.88	0.50
4	263	2.14	0.40	1061	2.90	0.52
5	1041	2.16	0.41	4235	2.92	0.52
6	4154	2.16	0.41	16934	2.92	0.52
7	16605	2.16	0.41	67731	2.92	0.52

TABLE 4.3. Condition numbers of A_J and $\mathsf{B}_J\mathsf{A}_J$ when Gauß-Seidel smoother and biquadratic (Q_2) and bicubic (Q_3) shape functions are used. $\Omega = (-1,1)^2$.

as long as the discretization is stable, while its maximum eigenvalue grows linearly with σ. In contrast the minimum eigenvalue of $\mathsf{B}_J\mathsf{A}_J$ decreases with increasing σ and its maximum eigenvalue seems bounded with increasing σ.

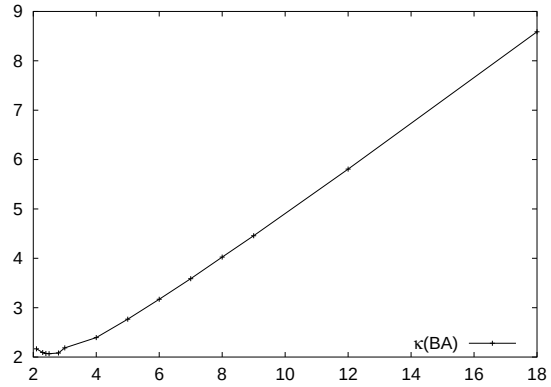

FIGURE 1. Variation of the condition number of $\mathsf{B}_6\mathsf{A}_6$ (ordinate) with σ (abscissa).

	Ω_L		Ω_S	
L	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I}-\mathsf{B}_J\mathsf{A}_J)$	$\kappa(\mathsf{B}_J\mathsf{A}_J)$	$\varrho(\mathsf{I}-\mathsf{B}_J\mathsf{A}_J)$
2	1.70	0.28	1.70	0.31
3	1.96	0.38	1.92	0.35
4	2.08	0.41	2.06	0.40
5	2.11	0.42	2.10	0.41
6	2.11	0.42	2.11	0.42
7	2.12	0.42	2.12	0.42

TABLE 4.4. Condition numbers and contraction numbers for L-shaped and slit domains using bilinear shape functions and Gauß-Seidel smoothing.

Figure 1 shows that the condition number of $\mathsf{B}_J\mathsf{A}_J$ grows linearly in σ for sufficiently large σ.

In the analysis, we assumed a mild regularity assumption (2.2), that holds for polygonal domains with re-entrant corners. We now investigate the performance of the preconditioner on an L-shaped domain $\Omega_L = \Omega \setminus [0,1)^2$ and a domain with a slit $\Omega_S = \Omega \setminus \{0\} \times [0,1)$. While (2.2) holds with $\alpha < 2/3$ on Ω_L, it does not hold for Ω_S. We conclude from Table 4.4 that the algorithm yields a good preconditioner in both cases.

5. Advection-diffusion problems

The subject of this section is the advection-diffusion problem

$$-\varepsilon\Delta V + \beta\cdot\nabla V = f \quad \text{on } \Omega, \tag{5.1}$$

$$V = g \quad \text{on } \partial\Omega. \tag{5.2}$$

Here $\varepsilon > 0$ is a constant that may be arbitrarily small, β is a constant vector (of $O(1)$ magnitude), and Ω is convex. It is well known that a standard finite element method is inappropriate for this problem [21]. We will introduce a discontinuous Galerkin scheme for this problem, give an error estimate, and report numerical experiments with a multigrid technique that makes good heuristic sense.

The discretization scheme we will consider is obtained by discretizing the term $-\varepsilon\Delta V$ by the interior penalty method, and the transport term $\beta\cdot\nabla V$ by the discontinuous Galerkin method. Let the inflow part of the boundary $\partial\Omega$ be denoted by $\partial\Omega_-$, i.e. $\partial\Omega_- = \{x \in \partial\Omega : n(x)\cdot\beta < 0\}$ (here $n(x)$ is the outward normal at x) and for all $u \in H^1(\mathcal{T}_k)$ let

$$u^+(x) = \lim_{\delta\to 0+} u(x+\delta\beta), \quad u^-(x) = \lim_{\delta\to 0+} u(x-\delta\beta).$$

The scheme is motivated by the fact that if the solution V of (5.1) is in $H^s(\Omega)$ with $s > 3/2$, then it satisfies

$$\varepsilon a_k(V,v) + b_k(V,v) = F(v) \quad \text{for all } v \in H^1(\mathcal{T}_k), \tag{5.3}$$

where $H^1(\mathcal{T}_k)$ and $a_k(\cdot,\cdot)$ are as defined earlier, and

$$b_k(u,v) = \sum_{\tau\in\mathcal{T}_k} (\beta\cdot\nabla u, v)_\tau + \sum_{e\in\mathcal{E}_k^0} (u^+ - u^-, v^+|n\cdot\beta|)_e + \sum_{e\in\mathcal{E}_k^-} (u^+, v^+|n\cdot\beta|)_e,$$

$$F(v) = (f,v) + \sum_{e\in\mathcal{E}_k^\partial} \left(\frac{\varepsilon\sigma}{\ell_e}(g,[v])_e - \varepsilon(g,\langle\partial_n v\rangle)_e \right) + \sum_{e\in\mathcal{E}_k^-} (g, v^+|n\cdot\beta|)_e.$$

Here and elsewhere we use the following subsets of $\mathcal{E}_k$:

$$\mathcal{E}_k^0 = \{e \in \mathcal{E}_k : e \subseteq \Omega\}, \qquad \mathcal{E}_k^- = \{e \in \mathcal{E}_k : e \subseteq \partial\Omega_-\},$$
$$\mathcal{E}_k^\partial = \{e \in \mathcal{E}_k : e \subseteq \partial\Omega\}, \qquad \mathcal{E}_k^+ = \{e \in \mathcal{E}_k : e \subseteq \partial\Omega \setminus \partial\Omega_-\}.$$

Our scheme computes an approximation $V_k \in M_k$ that satisfies

$$\varepsilon a_k(V_k,v) + b_k(V_k,v) = F(v) \quad \text{for all } v \in M_k. \tag{5.4}$$

It can be shown [22] that

$$\begin{aligned} b_k(u,u) =& \frac{1}{2}\sum_{e\in\mathcal{E}_k^-} (u^+, u^+|n\cdot\beta|)_e + \frac{1}{2}\sum_{e\in\mathcal{E}_k^+} (u^-, u^-|n\cdot\beta|)_e \\ &+ \frac{1}{2}\sum_{e\in\mathcal{E}_k^0} (u^+ - u^-, (u^+ - u^-)|n\cdot\beta|)_e. \end{aligned} \tag{5.5}$$

for all $u \in H^1(\mathcal{T}_k)$. Thus, $b_k(u,u) \geq 0$, and (5.4) has a unique solution. Moreover, the following error estimate holds.

Theorem 5.1. *Suppose V and V_k solve* (5.3) *and* (5.4) *respectively. Assume that $V \in H^s(\Omega)$ with $3/2 < s \leq d+1$. Then in the norm $|||\cdot|||_{\varepsilon,k}$ defined by*

$$|||u|||_{\varepsilon,k}^2 = \varepsilon a_k(u,u) + b_k(u,u) + \sum_{\tau\in\mathcal{T}_k} h_k \|\beta\cdot\nabla u\|_{0,\tau}^2, \tag{5.6}$$

the following error estimate holds:

$$|\!|\!| V - V_k |\!|\!|_{\varepsilon,k} \le C_1 \max(\sqrt{\varepsilon} h_k^{s-1},\, h_k^{s-1/2})\, |V|_{s,\Omega}.$$

Here C_1 is independent of ε and h_k.

We give a proof in Appendix A. Although the theorem is stated for quasiuniform meshes (which we assumed primarily for multigrid analysis), similar estimates on locally refined meshes may be obtained. A more exhaustive analysis with *hp*-estimates for a different DG method for advection-diffusion problems can be found in [19]. However, our error estimate is in a norm slightly stronger than in [19], due to the final term in (5.6) (and we do not have stabilizing terms like in [20]). This term seems to provide more control of gradient of error in the direction of β as ε tends to zero.

Having ensured that the method yields good approximations, we now look for efficient methods for solving the linear system arising from (5.4) for $k = J$. Let T_k be the matrix with its (i,j)-th entry equal to $\varepsilon a_k(\phi_k^j, \phi_k^i) + b_k(\phi_k^j, \phi_k^i)$. The following result shows that the B_J defined by Algorithm 3.1 can be used to precondition T_J. For $\mathsf{x}, \mathsf{y} \in \mathbb{R}^{N_J}$, we let $[\mathsf{x}, \mathsf{y}] = \mathsf{B}_J^{-1}\mathsf{x} \cdot \mathsf{y}$.

Corollary 5.1. *The convergence rate of GMRES in the $[\cdot,\cdot]^{1/2}$-innerproduct applied to the preconditioned system $\mathsf{B}_J\mathsf{T}_J\mathsf{v}_J = \mathsf{B}_J\mathsf{f}$ is bounded (independently of $\{h_k\}$) by*

$$1 - \left(\zeta^{-2} \frac{\varepsilon}{\varepsilon + C|\beta|}\right)^2.$$

Proof. The result essentially follows from the two inequalities

$$\varepsilon \mathsf{A}_J\mathsf{x} \cdot \mathsf{x} \le \mathsf{T}_J\mathsf{x} \cdot \mathsf{x} \quad \text{and} \tag{5.7}$$

$$\mathsf{T}_J\mathsf{x} \cdot \mathsf{y} \le (\varepsilon + C|\beta|)\, (\mathsf{A}_J\mathsf{x} \cdot \mathsf{x})^{1/2} (\mathsf{A}_J\mathsf{y} \cdot \mathsf{y})^{1/2}, \tag{5.8}$$

for all $\mathsf{x}, \mathsf{y} \in \mathbb{R}^{N_J}$. The first inequality is obvious, and the second follows from

$$\begin{aligned} b_J(u,v) &\le \sum_{\tau \in \mathcal{T}_J} \|\beta \cdot \nabla u\|_{0,\tau} \|v\|_{0,\tau} + \sum_{e \in \mathcal{E}_J^0 \cup \mathcal{E}_J^-} (h_J^{-1/2} \|[u]\|_{0,e})(h_J^{1/2} \left\|v^+ |n \cdot \beta|\right\|_{0,e}) \\ &\le C|\beta| \Bigg(\sum_{\tau \in \mathcal{T}_J} \|\nabla u\|_{0,\tau}^2 + \sum_{e \in \mathcal{E}_J} \frac{1}{\ell_e} \| [u] \|_{0,e}^2 \Bigg)^{1/2} \|v\|_{0,\Omega} \\ &\le C|\beta|\, \widetilde{a}_J(u,u)^{1/2} \|v\|_{0,\Omega} \;\le\; C|\beta|\, a_J(u,u)^{1/2} a_J(v,v)^{1/2}, \end{aligned}$$

where we have used (2.15) and the Poincaré inequality (2.16). Now, it can easily be shown that the estimate of Theorem 3.1 implies

$$\zeta^{-1}\, \mathsf{B}_J^{-1}\mathsf{x} \cdot \mathsf{x} \le \mathsf{A}_J\mathsf{x} \cdot \mathsf{x} \le \zeta\, \mathsf{B}_J^{-1}\mathsf{x} \cdot \mathsf{x}.$$

This with (5.7) and (5.8) yields

$$\varepsilon\zeta^{-1}[\mathsf{x},\mathsf{x}] \le [\mathsf{B}_J\mathsf{T}_J\mathsf{x},\mathsf{x}] \quad \text{and}$$
$$[\mathsf{B}_J\mathsf{T}_J\mathsf{x},\mathsf{y}] \le (\varepsilon + C|\beta|)\zeta\,[\mathsf{x},\mathsf{x}]^{1/2}[\mathsf{y},\mathsf{y}]^{1/2}.$$

Combining this with well-known results in [14] we have the result. □

This result gives an efficient method for solving (5.4) whenever ε is of the same order of magnitude as β. Indeed, it says that GMRES converges in a fixed number of steps, no matter how highly refined the meshes are. Note that GMRES in the $[\cdot,\cdot]^{1/2}$-innerproduct can be realized computationally without multiplications by B_J^{-1}. Although this gives a good preconditioner when ε is $O(1)$, the estimate of Corollary 5.1 deteriorates as ε approaches zero. We will now show how we can arrive at a method that performs independently of ε as well.

When $\varepsilon = 0$, as pointed out in [24], a downwind ordering of mesh elements makes T_k a block triangular matrix. The block partitioning is such that each block corresponds to nodes of one element. Specifically, if $\tau_1, \tau_2, \dots \tau_L$ is an ordering of elements of $\mathcal{T}_k$ such that for any i, the inflow part of $\partial\tau_i$ is either a subset of $\partial\Omega_-$ or of the outflow part of $\partial\tau_j$ for some $j < i$, then in this ordering T_k is $L \times L$ block triangular. Now we observe that because T_k is block triangular, a block Gauß–Seidel iteration in this ordering solves a system involving T_k in one step. If ε is nonzero but small, T_k is a perturbation of a block triangular matrix, and we can still expect this iteration to work well. Finally, when ε is $O(1)$, a multigrid cycle with a block Gauß–Seidel smoother (in any ordering) works well.

These observations lead us to modify Algorithm 3.1 by substituting the above-mentioned downwind block Gauß–Seidel matrix (which we denote by G_k^β) for R_k. Also, since T_k is nonsymmetric anyway, the symmetrizing post-smoothing operation (Step 3 of Algorithm 3.1) may be removed. The resulting preconditioner B_J^β is given by the following algorithm.

Algorithm 5.1. Set $\mathsf{x}^0 = 0$ and $\mathsf{B}_1^\beta = \mathsf{A}_1^{-1}$. For $k \ge 2$ and any vector $\mathsf{d} \in \mathbb{R}^{N_k}$, the vector $\mathsf{B}_k^\beta\mathsf{d}$ is computed as follows:

(1) Compute x^l for $l = 1, \dots m(k)$:
$$\mathsf{x}^l = \mathsf{x}^{l-1} + \mathsf{G}_k^\beta(\mathsf{d} - \mathsf{T}_k\mathsf{x}^{l-1}).$$
(2) Set $\mathsf{B}_k^\beta\mathsf{d} = \mathsf{x}^{m(k)} + \mathsf{C}_{k-1}^t\mathsf{B}_{k-1}^\beta\mathsf{C}_{k-1}(\mathsf{d} - \mathsf{T}_k\mathsf{x}^{m(k)})$.

We now report a representative numerical experiment using B_J^β, with $m(k) = 2^{J-k}$. At the outset, we note that in all our experiments, the computational cost of reordering the unknowns downwind was little, compared with that of the floating point operations involved in solution process. With $\beta = (0.5, 0.866)^t$, we choose f and g such that

$$u(x,y) = -\arctan(8(0.5y - 0.866x))$$

$\log_2 \varepsilon$ \ $J = 2 - \log_2 h_J$	2	3	4	5	6	7	8	9	10
$-\infty$	1	1	1	1	1	1	1	1	1
-18	2	3	3	3	4	4	5	7	9
-16	3	3	3	4	4	5	7	10	15
-14	3	3	4	4	5	7	10	15	17
-12	3	4	5	6	7	10	14	15	14
-10	4	5	6	8	11	14	14	14	14
-8	5	6	8	11	13	14	15	16	17
-6	6	9	12	14	16	17	18	19	19
-4	7	12	15	17	18	19	20	20	20
-2	8	14	17	19	20	21	21	20	20
0	9	15	18	21	21	21	21	21	20
2	9	15	19	21	21	21	21	21	20
∞	9	15	18	21	21	21	21	21	20

TABLE 5.1. Iteration counts depending on ε and refinement level J.

on the square $\Omega = (-1, 1)^2$. (This $u(x, y)$ solves (5.1) with $\varepsilon = 0$ and $f = 0$. There are no boundary layers when $\varepsilon > 0$.) We use Q_1 elements, and solve for V_J defined by (5.4). GMRES with B_J^β as preconditioner and with zero as initial guess is used, and iterations were stopped when residual norm was reduced by 10^{-10}. The number of iterations required for convergence as a function of ε and mesh size is reported in Table 5.1. The iteration counts at the upper left corner of the table reflect the fact that the smoother is almost an exact solver. At the lower right part, diffusion dominates, and we recover convergence rates of the elliptic problem. For comparison, we have included iteration counts for pure advection problem, i.e., $\varepsilon = 0$ (see row labeled $-\infty$), and pure elliptic problem, i.e., $\beta = 0$ and $\varepsilon = 1$ (see row labeled ∞). Clearly, the iteration counts remain bounded in all the ranges of ε and refinement levels considered.

6. Conclusion

We have presented multigrid techniques for two DG schemes: one for elliptic problems and another for singularly perturbed advection-diffusion problems. For the latter DG scheme we have also given an error estimate.

In the former case, our analysis predicts convergence rates of the multigrid method independent of the mesh size, and numerical experiments bear this out. It is clear from the analysis that second order elliptic boundary value problems more general than (2.1) can be considered. Although our main theorem was for the V-cycle, results for W-cycle with "sufficiently many"

smoothings can also be obtained [6, 17]. In practical computations, a V-cycle with only one smoothing at all levels may be adequate, if it is used as a preconditioner. Our numerical experiments in Section 3 support this.

For the advection-diffusion equation (5.1), we presented a method that is stable and accurate in diffusion dominated as well as convection dominated regime. We clarified the behavior of the multigrid method applied to this problem, and presented results with a modified multigrid method that behaves independently of the diffusion coefficient. When β is not a constant, the multigrid technique presented still applies whenever a downwind ordering can be found.

Appendix A

We give a proof of Theorem 5.1. Here the multilevel nature of the spaces and meshes are immaterial. To simplify notation, let us define

$$\langle u, v\rangle_S = \sum_{e\in S} (u, v|n\cdot\beta|)_e,$$

where S is any of the sets $\mathcal{E}_k^0$, $\mathcal{E}_k^-$, $\mathcal{E}_k^\partial$, or $\mathcal{E}_k^+$. Also, for $u \in H^1(\mathcal{T}_k)$, let $u_\beta \in L^2(\Omega)$ be defined by

$$u_\beta|_\tau = \beta\cdot\nabla u \quad \text{for all } \tau \in \mathcal{T}_k.$$

We begin with a stability result, which can also be thought of as an inf-sup condition.

Lemma A.1. *There exists a $C_2 > 0$ such that for all $u \in M_k$,*

$$C|\!|\!| u |\!|\!|_{\varepsilon,k}^2 \le \varepsilon a_k(u, C_2u + h_k u_\beta) + b_k(u, C_2u + h_k u_\beta).$$

Proof. We will prove that for any C_2, there exist constants $c_2', c_2'' > 0$ such that

$$b_k(u, C_2u + h_k u_\beta) \ge C_2 b_k(u,u) - c_2' b_k(u,u) + h_k\|u_\beta\|_{0,\Omega}^2/2, \tag{A.1}$$

$$a_k(u, C_2u + h_k u_\beta) \ge C_2 a_k(u,u) - c_2'' a_k(u,u), \tag{A.2}$$

for any $u \in M_k$. This is sufficient, because we can choose $C_2 = 1 + \max(c_2', c_2'')$ in (A.1) and (A.2), and complete the proof.

To prove (A.1), we start with

$$\begin{aligned} b_k(u, C_2u + h_k u_\beta) = {} & C_2 b_k(u,u) + h_k\|u_\beta\|_{0,\Omega}^2 \\ & + h_k\langle u^+ - u^-, (u_\beta)^+\rangle_{\mathcal{E}_k^0} + h_k\langle u^+, (u_\beta)^+\rangle_{\mathcal{E}_k^-}, \end{aligned}$$

and apply Cauchy-Schwarz inequality, a trace theorem, and a scaling argument. Then, for any $\delta > 0$,

$$\begin{aligned} b_k(u, C_2u + h_k u_\beta) \ge {} & C_2 b_k(u,u) + h_k\|u_\beta\|_{0,\Omega}^2 - \delta C h_k\|u_\beta\|_{0,\Omega}^2 \\ & - \frac{1}{2\delta}\left(\langle u^+ - u^-, u^+ - u^-\rangle_{\mathcal{E}_k^0} + \langle u^+, u^+\rangle_{\mathcal{E}_k^-}\right). \end{aligned}$$

Choosing δ appropriately, and using (5.5), we get (A.1).

To prove (A.2), it suffices to prove that there exists $c_2'' > 0$ such that

$$|a_k(u, h_k u_\beta)| \le c_2'' a_k(u,u). \tag{A.3}$$

This requires bounding the four sums on the right hand side of

$$a_k(u, h_k u_\beta) = \sum_{\tau\in\mathcal{T}_k} (\nabla u, h_k \nabla u_\beta)_\tau + \sum_{e\in\mathcal{E}_k} \left(\frac{\sigma}{\ell_e} ([u], [h_k u_\beta])_e - (\langle \partial_n u\rangle, [h_k u_\beta])_e - ([u], h_k \langle \partial_n u_\beta\rangle)_e \right),$$

by $a_k(u,u)$. It can be seen that such a bound does hold for each sum by using the inverse inequalities (2.6) and (2.13), the trace inequality (2.12), and (2.15). For example, consider the last sum:

$$\begin{aligned}\sum_{e\in\mathcal{E}_k} ([u], h_k\langle \partial_n u_\beta\rangle)_e &\le \left(\sum_{e\in\mathcal{E}_k} h_k^{-1} \| [u] \|_{0,e}^2 \right)^{1/2} \left(\sum_{\tau\in\mathcal{T}_k} h_k^2 \|\nabla u_\beta\|_{0,\tau}^2 \right)^{1/2} \\ &\le C\, \widetilde{a}_k(u,u)^{1/2} \|u_\beta\|_{0,\Omega} \le C\, a_k(u,u)^{1/2} \|u_\beta\|_{0,\Omega}.\end{aligned}$$

□

Proof of Theorem 5.1. Let $\widetilde{V}_k$ be an interpolant [30] of V that is continuous on Ω and satisfies the approximation property

$$\|V - \widetilde{V}_k\|_{0,\Omega} + h_k |V - \widetilde{V}_k|_{1,\Omega} \le C h_k^s |V|_{s,\Omega}. \tag{A.4}$$

Using a trace inequality, it is easy to see that this implies that

$$|\!|\!| V - \widetilde{V}_k |\!|\!|_{\varepsilon,k} \le C(\sqrt{\varepsilon} h^{s-1} + h^{s-1/2}) |V|_{s,\Omega}. \tag{A.5}$$

Now, if we let $E_k = V_k - \widetilde{V}_k$, then

$$|\!|\!| V - V_k |\!|\!|_{\varepsilon,k} \le |\!|\!| V - \widetilde{V}_k |\!|\!|_{\varepsilon,k} + |\!|\!| E_k |\!|\!|_{\varepsilon,k}.$$

Thus, it suffices to establish a bound for E_k similar to that in (A.5).

Because of the consistency of the method as given by (5.3), we have

$$\varepsilon a_k(V - V_k, w_k) + b_k(V - V_k, w_k) = 0 \quad \text{for all } w_k \in M_k. \tag{A.6}$$

Therefore, denoting $E_k^\beta = C_2 E_k + h_k (E_k)_\beta$, we have by Lemma A.1,

$$\begin{aligned} C |\!|\!| E_k |\!|\!|_{\varepsilon,k}^2 &\le \varepsilon a_k(E_k, E_k^\beta) + b_k(E_k, E_k^\beta) \\ &= \varepsilon a_k(V - \widetilde{V}_k, E_k^\beta) + b_k(V - \widetilde{V}_k, E_k^\beta). \end{aligned} \tag{A.7}$$

Now we will use continuity properties of $a_k(\cdot,\cdot)$ and $b_k(\cdot,\cdot)$ to estimate the quantities on the right hand side above.

To estimate the first term on the right hand side of (A.7), we first use Cauchy-Schwarz inequality:

$$a_k(V-\widetilde{V}_k,E_k^\beta) \le |V-\widetilde{V}_k|_{1,\Omega}\Big(\sum_{\tau\in\mathfrak{T}_k}\|\nabla E_k^\beta\|_{0,\tau}^2\Big)^{1/2} +$$

$$\Big(\sum_{e\in\mathcal{E}_k} h_k\|\langle\partial_n(V-\widetilde{V}_k)\rangle\|_{0,e}^2 + \sum_{e\in\mathcal{E}_k^\partial} h_k^{-1}\|V-\widetilde{V}_k\|_{0,e}^2\Big)^{1/2}\Big(\sum_{e\in\mathcal{E}_k} h_k^{-1}\|\left[E_k^\beta\right]\|_{0,e}^2\Big)^{1/2}.$$

Now, since the interpolant preserves polynomials of degree $d\ge 1$, a Bramble-Hilbert argument (for fractional order spaces) readily shows that

$$h_k\sum_{e\in\mathcal{E}_k}\|\langle\partial_n(V-\widetilde{V}_k)\rangle\|_{0,e}^2 \le h_k^{2s-2}|V|_{s,\Omega}^2.$$

Local inverse inequalities imply

$$\sum_{\tau\in\mathfrak{T}_k}\|\nabla E_k^\beta\|_{0,\tau}^2 \le C\sum_{\tau\in\mathfrak{T}_k}\big(\|\nabla E_k\|_{0,\tau}^2+\|(E_k)_\beta\|_{0,\tau}^2\big), \quad \text{and}$$

$$\sum_{e\in\mathcal{E}_k} h_k^{-1}\|\left[E_k^\beta\right]\|_{0,e}^2 \le C\sum_{e\in\mathcal{E}_k} h_k^{-1}\|\left[E_k\right]\|_{0,e}^2 + C\sum_{\tau\in\mathfrak{T}_k}\|(E_k)_\beta\|_{0,\tau}^2.$$

These estimates together with a trace theorem, (A.4), and (2.15), yield

$$\varepsilon a_k(V-\widetilde{V}_k,E_k^\beta) \le C\,\varepsilon h_k^{s-1}|V|_{s,\Omega}\,a_k(E_k,E_k)^{1/2}. \tag{A.8}$$

Now we estimate the second term on the right hand side of (A.7). It can be seen using the integration by parts formula

$$b_k(u,v) = (u,-v_\beta) + \langle u^-, v^- - v^+\rangle_{\mathcal{E}_k^0} + \langle u^-, v^-\rangle_{\mathcal{E}_k^+},$$

that for any $u\in H^1(\mathfrak{T}_k)$ and $v\in M_k$,

$$b_k(u,C_2v+h_kv_\beta) \le C\Big(h_k^{-1/2}\|u\|_{0,\Omega} + \langle u^-,u^-\rangle_{\mathcal{E}_k^0}^{1/2} + \big(\sum_{\tau\in\mathfrak{T}_k} h_k\|u_\beta\|^2\big)^{1/2} + b_k(u,u)^{1/2}\Big)\,|\!|\!|v|\!|\!|_{\varepsilon,k}.$$

Using this with $u=V-\widetilde{V}_k$ and $v=E_k$, a trace inequality, and (A.4), we get that

$$b_k(V-\widetilde{V}_k,E_k^\beta) \le Ch_k^{s-1/2}|V|_{s,\Omega}|\!|\!|E_k|\!|\!|_{\varepsilon,k}. \tag{A.9}$$

Combining (A.8) and (A.9) and using it in (A.7), we have that

$$|\!|\!|E_k|\!|\!|_{\varepsilon,k} \le C(\sqrt{\varepsilon}h_k^{s-1}+h_k^{s-1/2})|V|_{s,\Omega}.$$

This together with (A.5) proves the theorem. □

Remark A.1. The proofs of this appendix continue to hold when $\beta\equiv\beta(x)$ is a function on Ω such that $\beta(x)|_\tau$ is linear and $\operatorname{div}(\beta|_\tau)=0$, for all $\tau\in\mathfrak{T}_k$.

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Department of Mathematics, University of Florida, Gainesville, FL 32611.
E-mail address: jayg@math.ufl.edu

Institut für Angewandte Mathematik, Universität Heidelberg, INF 293/294, D-69121 Heidelberg.
E-mail address: guido.kanschat@na-net.ornl.gov