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# Mortar estimates independent of number of subdomains

Jayadeep Gopalakrishnan\*

## Abstract

The stability and error estimates for the mortar finite element method are well established. This work examines the dependence of constants in these estimates on shape and number of subdomains. By means of a Poincaré inequality and some scaling arguments, these estimates are found not to deteriorate with increase in number of subdomains.

## 1 Introduction

This paper proves that the stability and error estimates of the mortar finite element method do not deteriorate with increase in number of subdomains, under reasonable assumptions on the subdomain partitioning.

The mortar finite element method, as introduced by Bernardi, et al. [2, 3], is a domain decomposition method which results in nonconforming approximations to the solutions of second order elliptic boundary value problems. The domain where solution is required is partitioned into subdomains, and each subdomain is independently triangulated. The mortar approximation space based on this partitioning consists of functions which when restricted to a subdomain are standard finite element functions. Although these functions are allowed to have jumps across subdomain interfaces, the jumps are constrained by conditions associated with one of the two neighboring meshes. The practical importance of the mortar method stems from the flexibility it offers by allowing sub-structures of a complicated domain to be meshed independently of each other.

In the papers introducing the mortar finite element method, it was established that the mortar finite element problem is well-posed, and error

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estimates for the method were given [1, 2, 3]. However, it is unclear from the analysis given there whether these estimates hold with constants independent of the number of subdomains. In the analysis of domain decomposition methods, it is customary to keep track of dependencies on the number of subdomains. This requires showing that a uniform, independent of the subdomain partitioning, coercivity estimate holds. This estimate is given in Section 3. The final section proves error estimates with constants independent of the number of subdomains. Most of the results here can also be found in [13].

## 2 Preliminaries

In this section we establish notation for the Sobolev spaces and state some results to be used later. We will also introduce a model problem and the mortar method.

Let  $\mathbb{R}$  denote the field of real numbers and  $\mathbb{N}$  denote the set of non-negative integers. For  $\delta = (\delta_1, \delta_2, \dots, \delta_N) \in \mathbb{N}^N$ , define  $|\delta| = \delta_1 + \delta_2 + \dots + \delta_N$ . Let  $\mathcal{O}$  denote a connected open bounded subset of  $\mathbb{R}^N$  with Lipschitz boundary [14]. For a distribution  $u$  on  $\mathcal{O}$ , we denote by  $D^\delta u$  the derivative  $(\partial/\partial x_1)^{\delta_1} \dots (\partial/\partial x_N)^{\delta_N}$ . Let  $\|\cdot\|_{0,\mathcal{O}}$  denote the norm on  $L^2(\mathcal{O})$ , the space of all Lebesgue measurable and square integrable functions on  $\mathcal{O}$ . For positive integers  $m$ , define the Sobolev seminorm  $|\cdot|_{m,\mathcal{O}}$ , and the Sobolev norm  $\|\cdot\|_{m,\mathcal{O}}$  by

$$|u|_{m,\mathcal{O}}^2 = \sum_{|\delta|=m} \|D^\delta u\|_{0,\mathcal{O}}^2 \quad \text{and} \quad \|u\|_{m,\mathcal{O}}^2 = \sum_{|\delta|\leq m} \|D^\delta u\|_{0,\mathcal{O}}^2$$

respectively. As usual, we denote the space of all functions  $u$  in  $L^2(\mathcal{O})$  for which  $\|u\|_{m,\mathcal{O}}$  is finite by  $H^m(\mathcal{O})$ . For  $s \in \mathbb{R}$ , let  $\lceil s \rceil$  denote the smallest integer greater than or equal to  $s$ . If  $s > 0$  is not an integer, then writing  $s = \sigma + (\lceil s \rceil - 1)$  for a  $\sigma \in (0, 1)$ , we define the Sobolev seminorm  $|\cdot|_{s,\mathcal{O}}$  by

$$|u|_{s,\mathcal{O}} = \left( \sum_{|\delta|=\lceil s \rceil-1} \iint_{\mathcal{O} \times \mathcal{O}} \frac{|D^\delta u(x) - D^\delta u(y)|^2}{|x - y|^{N+2\sigma}} dx dy \right)^{1/2},$$

and the Sobolev norm  $\|\cdot\|_{s,\mathcal{O}}$  by  $\|u\|_{s,\mathcal{O}} = (\|u\|_{\lceil s \rceil-1,\mathcal{O}}^2 + |u|_{s,\mathcal{O}}^2)^{1/2}$ . The space of functions  $u$  in  $L^2(\mathcal{O})$  for which  $\|u\|_{s,\mathcal{O}}$  is finite is  $H^s(\mathcal{O})$ . It is well known [5] that the space  $H^s(\mathcal{O})$  for non-integer  $s$  is the space obtained by interpolation between  $H^{\lceil s \rceil-1}(\mathcal{O})$  and  $H^{\lceil s \rceil}(\mathcal{O})$  by, for example, the real method of interpolation.

In various estimates involving Sobolev norms, we will need to keep track of the dependencies of constants on domain sizes. This is usually done by the technique of proving statements on a reference domain and scaling back to the domain under consideration. During such arguments, we will often want to bound Sobolev norms with seminorms. Proposition 2.1 below is a result in that direction. Its proof follows exactly along the lines of the proof of Deny-Lions Lemma [10, 15] or Bramble-Hilbert Lemma [6, 18], and is omitted.

**PROPOSITION 2.1** *Let  $v$  be in  $H^s(\mathcal{O})$  for some positive real number  $s$ . There exists a constant  $C(\mathcal{O})$  independent of  $v$  such that if*

$$\int_{\mathcal{O}} D^\delta v = 0 \text{ for all } \delta \in \mathbb{N}^N \text{ with } 0 \leq |\delta| \leq \lceil s \rceil - 1, \quad (2.1)$$

*then  $\|v\|_{s,\mathcal{O}} \leq C(\mathcal{O}) |v|_{s,\mathcal{O}}$ .*

Given a domain  $\mathcal{O}$  of  $\mathbb{R}^N$ , we denote by  $\hat{\mathcal{O}}$  another domain of  $\mathbb{R}^N$  for which there exists an invertible affine mapping  $F(\hat{x}) = B\hat{x} + b$ , (where  $B$  is an  $N \times N$  matrix and  $b$  is a vector in  $\mathbb{R}^N$ ) such that  $\mathcal{O} = F(\hat{\mathcal{O}})$ . For real valued functions  $v(x)$  defined for almost every  $x \in \mathcal{O}$ , we denote by  $\hat{v}$  the function defined almost everywhere on  $\hat{\mathcal{O}}$  by  $\hat{v}(\hat{x}) = v(F(\hat{x}))$ .

For  $\phi \in H^s(\mathcal{O})$ , and a segment  $\gamma$  contained in  $\overline{\mathcal{O}}$ , we will denote the trace of  $\phi$  on  $\gamma$  by  $\phi|_\gamma$ . We will often write  $\|\phi\|_{r,\gamma}$  and  $|\phi|_{r,\gamma}$  for the  $H^r(\gamma)$  norm and seminorm respectively, of the trace  $\phi|_\gamma$ .

We are interested in the dependence of constants on domain size in some well-known trace inequalities. When the domain under consideration is a triangle, such dependencies can be examined easily using affine equivalences and a standard scaling argument as the proof of the lemma below shows. By definition, triangles and edges will be open. For any triangle  $T$ , we use  $h_T$  to denote the length of the largest side of  $T$ , and  $\rho_T$  to denote the diameter of the largest ball contained in  $T$ .

**LEMMA 2.1** *Let  $T$  be a triangle and  $L$  be one of its edges. Denote by  $r_T$  the ratio  $h_T/\rho_T$ . Then, there exist positive constants  $C_1, C_2$  and  $C_3$  independent of  $T$  such that*

$$|u|_{1/2,L} \leq C_1 r_T |u|_{1,T}, \quad (2.2)$$

$$\|u\|_{0,L}^2 \leq C_2 r_T^2 \left( \rho_T^{-1} \|u\|_{0,T}^2 + h_T |u|_{1,T}^2 \right) \quad \text{and}, \quad (2.3)$$

$$\|u - \bar{u}_L\|_{0,L} \leq C_3 r_T h_T^{1/2} |u|_{1,T}, \quad (2.4)$$

for all  $u$  in  $H^1(T)$ . In the last inequality,  $\bar{u}_L$  denotes the average of the trace of  $u$  on  $L$ .

PROOF. There is an affine correspondence between  $T$  and the reference triangle  $\hat{T}$  bounded by the co-ordinate axes and the line  $x + y = 1$  in the  $(x, y)$  plane. Let  $\hat{L}$  be the image of  $L$  under the affine transformation. To prove the first inequality of the lemma, we start by applying a well-known trace inequality [14] on the reference domain to get

$$|\hat{u}|_{1/2, \hat{L}} \leq \hat{C}_1 \|\hat{u}\|_{1, \hat{T}}.$$

If  $\bar{u}$  denotes the average of  $\hat{u}$  on  $\hat{T}$ , then using the above inequality for  $u - \bar{u}$  we get

$$|\hat{u}|_{1/2, \hat{L}} \leq \hat{C}_1 \|\hat{u} - \bar{u}\|_{1, \hat{T}} \leq \hat{C}_1' |\hat{u}|_{1, \hat{T}}.$$

The last inequality followed from Proposition 2.1. Here, the constant  $\hat{C}_1'$  is obviously independent of  $T$ . The analogous inequality for  $u$  now follows using the standard scaling argument. This proves (2.2). The proofs of the other inequalities proceed similarly.  $\square$

Now we introduce the model problem. Consider a bounded, connected and open subset  $\Omega$  of  $\mathbb{R}^2$  with a polygonal boundary  $\partial\Omega$ . Let  $\partial\Omega_D$  be a closed subset of  $\partial\Omega$  with positive measure, and denote by  $\partial\Omega_N$  the remainder  $\partial\Omega \setminus \partial\Omega_D$ . Denote by  $H_D^1(\Omega)$  the subspace of  $H^1(\Omega)$  consisting of functions whose trace on  $\partial\Omega_D$  is zero.

Let  $A(\cdot, \cdot)$  be the bilinear form on  $H_D^1(\Omega) \times H_D^1(\Omega)$  defined by

$$A(u, v) = \int_{\Omega} \nabla u \cdot \nabla v \, dx,$$

and  $(\cdot, \cdot)$  denote the  $L^2(\Omega)$ -innerproduct. We seek an approximate solution to the following problem.

PROBLEM 2.1 Find  $U \in H_D^1(\Omega)$  such that

$$A(U, \phi) = (f, \phi) \quad \text{for all } \phi \in H_D^1(\Omega),$$

for a given  $f$  in  $L^2(\Omega)$ .

This problem has a unique solution [15, 18]. This is the variational form of the problem of finding  $U$  that satisfies  $-\Delta U = f$  on  $\Omega$  with the boundary conditions

$$U = 0 \text{ on } \partial\Omega_D, \quad \text{and} \quad \frac{\partial U}{\partial n} = 0 \text{ on } \partial\Omega_N.$$

Here and elsewhere,  $\partial U/\partial n$  will denote the directional derivative of  $U$  in the direction of the outward normal vector  $n$ , i.e.,  $\partial U/\partial n = \nabla U \cdot n$ . Although our results are stated for this model problem, extension to more general second order elliptic partial differential equations with more general boundary conditions are straightforward.

We conclude this section by describing the mortar finite element method. Consider a decomposition of  $\Omega$  into disjoint open polygonal subdomains  $\Omega_i$  with  $\bar{\Omega} = \bigcup_{i=1}^K \bar{\Omega}_i$ . Define the space  $\tilde{\mathcal{V}}$  by

$$\tilde{\mathcal{V}} = \{u : u|_{\Omega_i} \in H^1(\Omega_i)\}.$$

Associated with a decomposition of  $\Omega$  is a set  $Z$  of interface segments defined as follows. Each member of  $Z$  is an open straight line segment contained in  $\partial\Omega_i \cap \partial\Omega_j$  for some  $i$  and  $j$ . It will be convenient to have a notation for adjacent subdomains of a  $\gamma \in Z$ . Out of the two subdomains that lie adjacent to  $\gamma$ , pick one arbitrarily, and call it the *mortar* subdomain of  $\gamma$ . Call the other the *nonmortar* subdomain of  $\gamma$ . Also, let  $\mathfrak{m}(\gamma)$  and  $\mathfrak{nm}(\gamma)$  denote indices such that  $\Omega_{\mathfrak{m}(\gamma)}$  and  $\Omega_{\mathfrak{nm}(\gamma)}$  denote the mortar and nonmortar subdomains respectively. For a function  $u \in \tilde{\mathcal{V}}$ , let  $u_\gamma^{\mathfrak{m}}$  and  $u_\gamma^{\mathfrak{nm}}$  denote the traces on  $\gamma$  of the restrictions of  $u$  to  $\Omega_{\mathfrak{m}(\gamma)}$  and  $\Omega_{\mathfrak{nm}(\gamma)}$  respectively, i.e.,  $u_\gamma^{\mathfrak{m}}$  is the trace from the mortar side and  $u_\gamma^{\mathfrak{nm}}$  is the trace from the nonmortar side. The jump of  $u$  across the interface  $\gamma$  will be denoted by  $[u]_\gamma$ , i.e.,  $[u]_\gamma = u_\gamma^{\mathfrak{m}} - u_\gamma^{\mathfrak{nm}}$ .

Denote by  $\mathcal{T}_i$  a triangulation of  $\Omega_i$ . We assume that  $\mathcal{T}_i$  are such that endpoints of a  $\gamma \in Z$  are vertices of both the triangulations on the subdomains adjacent to  $\gamma$ .

To define the mortar finite element space, first let  $\tilde{M}(\Omega_i)$  denote the space of functions on  $\Omega_i$  that are continuous and are polynomials of degree at most  $d_i \geq 1$  when restricted to a triangle of  $\mathcal{T}_i$ . We now define two spaces  $S(\gamma)$  and  $W(\gamma)$  associated with a  $\gamma \in Z$ .  $W(\gamma)$  is the space of functions on  $\gamma$  that vanish at the endpoints of  $\gamma$ , and are traces on  $\gamma$  of functions in  $\tilde{M}(\Omega_{\mathfrak{nm}(\gamma)})$ . The space  $S(\gamma)$  consists of traces on  $\gamma$  of functions in  $\tilde{M}(\Omega_{\mathfrak{nm}(\gamma)})$  that are polynomials of degree  $d_{\mathfrak{nm}(\gamma)} - 1$  on the two end sub-intervals of  $\gamma$ . Let  $\tilde{M}$  denote the space of functions on  $\Omega$  whose restrictions to  $\Omega_i$  are in  $\tilde{M}(\Omega_i)$ . Then the mortar finite element space  $M$  is defined by

$$M = \left\{ v \in \tilde{M} : \int_\gamma \nu [v]_\gamma ds = 0, \text{ for all } \nu \in S(\gamma), \text{ for all } \gamma \in Z \right\}.$$

The “mortaring” is done by constraining the jump across interfaces by the integral constraint above.

The mortar constraint makes the size of the jump across interfaces small. More precisely, we have the following result.

PROPOSITION 2.2 *For all  $v \in M$ ,*

$$\| [v]_\gamma \|_{0,\gamma} \leq \inf_{\chi \in S(\gamma)} \| [v]_\gamma - \chi \|_{0,\gamma}.$$

PROOF. The result follows immediately from

$$\int_\gamma [v]_\gamma^2 ds = \int_\gamma [v]_\gamma ([v]_\gamma - \chi) ds$$

and Cauchy-Schwarz inequality.  $\square$

Note also that there is a projection operator  $\Pi_\gamma : L^2(\gamma) \rightarrow W(\gamma)$  that is associated with the integral constraints. It can be shown [3] that for a  $u \in L^2(\Omega)$  there exists a unique  $v \in W(\gamma)$  satisfying

$$\int_\gamma v \chi ds = \int_\gamma u \chi ds \quad \text{for all } \chi \in S(\gamma).$$

This  $v$  is defined to be  $\Pi_\gamma u$ . Clearly, if  $w$  is a function that satisfies the integral constraints (i.e.,  $w \in M$ ), then  $\Pi_\gamma [w]_\gamma$  is zero. The stability of this projection with respect to  $H_0^1(\gamma)$  and  $L^2(\gamma)$  norms are also known [3, 20]. Consequently, there exists a constant  $C_\Pi$  such that

$$\| \Pi_\gamma u \|_{H_{00}^{1/2}(\gamma)} \leq C_\Pi \| u \|_{H_{00}^{1/2}(\gamma)} \quad \text{for all } u \in H_{00}^{1/2}(\gamma), \quad (2.5)$$

where  $\| \cdot \|_{H_{00}^{1/2}(\gamma)}$  denotes the norm on  $H_{00}^{1/2}(\gamma)$ , the space half-way in the interpolation scale between  $L^2(\gamma)$  and  $H_0^1(\gamma)$  (the latter normed with  $|\cdot|_{1,\gamma}$ ). Although we will use (2.5) only when each of the meshes  $\mathcal{T}_i$  are quasiuniform, we point out that this result is known to hold true under much weaker assumptions on meshes [20]. Note that  $C_\Pi$  can be chosen independent of the size of  $\gamma$ , as a simple scaling argument readily shows.

The discrete mortar problem can now be described as the problem of finding a Galerkin approximation to  $U$  from  $M$ :

PROBLEM 2.2 Find  $U_M \in M$  such that

$$\tilde{A}(U_M, w) = (f, w) \quad \text{for all } w \in M.$$

Here  $\tilde{A}(\cdot, \cdot)$  is a bilinear form on  $\tilde{\mathcal{V}} \times \tilde{\mathcal{V}}$  defined by

$$\tilde{A}(u, v) = \sum_{i=1}^K \int_{\Omega_i} \nabla u \cdot \nabla v \, dx. \quad (2.6)$$

As we will in the next section, there is a unique  $U_M$  that solves Problem 2.2.

### 3 A Poincaré inequality

We provide a Poincaré inequality for some nonconforming spaces in this section. Of particular interest to us will be the dependence of the constant in such an inequality. For  $v \in \tilde{\mathcal{V}}$ , let

$$|v|_{\Sigma}^2 = \sum_{i=1}^K |v|_{1, \Omega_i}^2 \quad \text{and} \quad \|v\|_{\Sigma}^2 = \|v\|_{0, \Omega}^2 + |v|_{\Sigma}^2. \quad (3.1)$$

In general,  $|\cdot|_{\Sigma}$  may not be a norm on  $\tilde{\mathcal{V}}$ .

Consider the space

$$\mathcal{V} = \left\{ v \in \tilde{\mathcal{V}} : v|_{\partial\Omega_D} = 0, \text{ and } \int_{\gamma} [v]_{\gamma} \, ds = 0 \text{ for all } \gamma \in Z \right\}. \quad (3.2)$$

This space arises naturally in the analysis of mortar finite elements, as all mortar finite element spaces (based on the same partitioning  $\{\Omega_i\}$ ) are subspaces of  $\mathcal{V}$ . We shall provide a Poincaré inequality for the space  $\mathcal{V}$ . Such inequalities have been proved before [2] using a contradiction argument involving compact imbedding of  $H^1(\Omega_i)$  in  $L^2(\Omega_i)$ . However, those analyses give no indication of the dependence of the constant on subdomain shape, size and number. Under the following fairly general condition on the subdomain partitioning, we will show that the constant in the Poincaré inequality can be taken independent of the partitioning.

**ASSUMPTION 3.1** There is a triangulation  $\mathcal{T}$  corresponding to the partitioning which satisfies the following conditions:

1. The triangulation is locally quasiuniform, i.e., the minimal angle of any triangle in  $\mathcal{T}$  is greater than or equal to some positive constant  $\mathcal{C}_*$ .
2. The triangles align with the partitioning of  $\Omega$  in the sense that each  $\bar{\Omega}_i$  is the union of the closures of triangles in  $\mathcal{T}$  and each interface in  $Z$  is an edge of some triangle in  $\mathcal{T}$ .



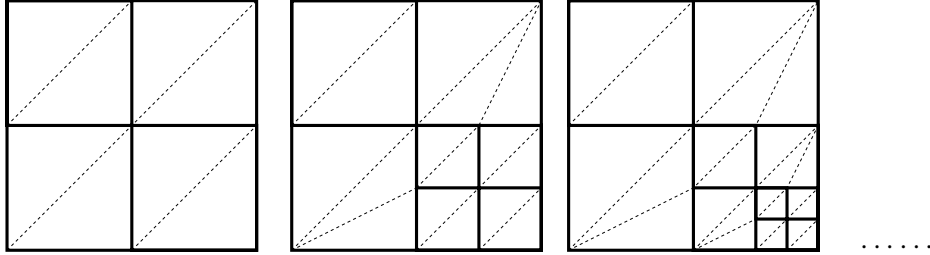


Fig. 1: A sequence of decompositions of the unit square satisfying Assumption 3.1. (The thick lines delineate subdomains, while the dotted lines show the associated triangulations. Here  $\mathcal{C}_* = \pi/4 - \arctan 1/2$ .)

3. The triangles align with  $\partial\Omega_D$  in that  $\partial\Omega_D$  is a union of edges (and their end points) of triangles in  $\mathcal{T}$ .

Clearly, Assumption 3.1 constrains the angles of the polygonal subdomains. One may consider a sequence of partitionings of  $\Omega$  for which the number of subdomains tend to infinity (see Figure 1 for an example). For each partitioning, it may be easy to construct a triangulation that satisfies the last two conditions of Assumption 3.1. But, for the assumption to hold for all the partitionings in the sequence, the minimal angles of all such triangulations must be uniformly bounded away from zero. We now state the Poincaré inequality for  $\mathcal{V}$ .

**THEOREM 3.1** *Let  $\mathcal{V}$  be the space defined by (3.2) for a decomposition of  $\Omega$  that satisfies Assumption 3.1. Then there exists a constant  $C_0$  depending only on  $\Omega$ ,  $\partial\Omega_D$ , and  $\mathcal{C}_*$  such that*

$$\|v\|_{0,\Omega} \leq C_0 |v|_{\Sigma} \quad (3.3)$$

for all  $v \in \mathcal{V}$ . In particular,  $C_0$  is independent of the number of subdomains.

In the proofs and elsewhere, it will be convenient to denote by  $\mathcal{C}$  a generic constant that is independent of the number of the subdomains. Its value at different occurrences may differ.

With  $\mathcal{T}$  as given by Assumption 3.1, consider the space  $\mathcal{V}_{\mathcal{T}}$  defined exactly as in (3.2) but with respect to the partitioning

$$\bar{\Omega} = \bigcup_{\tau \in \mathcal{T}} \bar{\tau},$$

with  $Z$  equal to the set of interior edges of  $\mathcal{T}$ . For the remainder of this section we let  $|\cdot|_\Sigma$  denote the seminorm \* resulting from this partitioning. Note that  $|\cdot|_\Sigma$  coincides with the seminorm defined by (3.1) when applied to functions in  $\mathcal{V}$ . Moreover,  $\mathcal{V}_\mathcal{T}$  contains  $\mathcal{V}$ . Therefore, to prove Theorem 3.1, it suffices to prove (3.3) for  $v \in \mathcal{V}_\mathcal{T}$ .

Define the discrete spaces

$$\begin{aligned}\tilde{V}_\mathcal{T} &= \{v : v \text{ is linear on each } \tau \in \mathcal{T} \text{ and } v = 0 \text{ on } \partial\Omega_D\} \quad \text{and,} \\ \bar{V}_\mathcal{T} &= \{v \in \tilde{V}_\mathcal{T} : v \text{ is continuous on } \Omega\},\end{aligned}$$

and the norm  $\|\cdot\|$  on  $\mathcal{V}_\mathcal{T}$  by

$$\|v\| = \left( \sum_{\tau \in \mathcal{T}} h_\tau^{-2} \|v\|_{0,\tau}^2 \right)^{1/2}.$$

The proof of Theorem 3.1 is based on the following lemma.

LEMMA 3.1 *For every  $v \in \mathcal{V}_\mathcal{T}$ , there exists a  $v_\mathcal{T} \in \bar{V}_\mathcal{T}$  such that*

$$\|v - v_\mathcal{T}\| \leq \mathcal{C} |v|_\Sigma, \quad \text{and} \quad (3.4)$$

$$|v_\mathcal{T}|_{1,\Omega} \leq \mathcal{C} |v|_\Sigma. \quad (3.5)$$

PROOF. We start by noting that there exists  $\tilde{v}_\mathcal{T} \in \tilde{V}_\mathcal{T}$  such that on each triangle  $\tau$  of  $\mathcal{T}$ ,

$$|v - \tilde{v}_\mathcal{T}|_{s,\tau} \leq \mathcal{C} h_\tau^{1-s} |v|_{1,\tau} \quad \text{for } s = 0 \text{ and } 1. \quad (3.6)$$

For example, we can take  $\tilde{v}_\mathcal{T}|_\tau$  to be the  $L^2(\tau)$  orthogonal projection for triangles whose edges do not intersect  $\partial\Omega_D$  and  $\tilde{v}_\mathcal{T}|_\tau = 0$  for the remaining triangles. It immediately follows that

$$\|v - \tilde{v}_\mathcal{T}\| \leq \mathcal{C} |v|_\Sigma. \quad (3.7)$$

Because of (3.7), (3.4) will follow if we construct  $v_\mathcal{T}$  satisfying

$$\|v_\mathcal{T} - \tilde{v}_\mathcal{T}\| \leq \mathcal{C} |v|_\Sigma. \quad (3.8)$$

Let  $\{x_i\}$  denote the vertices of the triangulation  $\mathcal{T}$ . At each  $x_i$ ,  $\tilde{v}_\mathcal{T}$  generally has multiple values, each being a limit from one of the triangles with vertex  $x_i$ . Pick one triangle which has  $x_i$  as a vertex and denote it by  $\tau^i$ . If

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\*We will revert to the previous definitions of  $|\cdot|_\Sigma$  and  $Z$  in the next section.

$\mathbf{x}_i \in \partial\Omega_D$ , we always choose a triangle which has an edge contained in  $\partial\Omega_D$  and ending at  $\mathbf{x}_i$ . Define  $v_{\mathcal{T}} \in \overline{V}_{\mathcal{T}}$  by

$$v_{\mathcal{T}}(\mathbf{x}_i) = \tilde{v}_{\mathcal{T}} \Big|_{\tau^i}(\mathbf{x}_i).$$

Fix  $\tau \in \mathcal{T}$ . We clearly have that

$$h_{\tau}^{-2} \|v_{\mathcal{T}} - \tilde{v}_{\mathcal{T}}\|_{0,\tau}^2 \leq \mathcal{C} \sum_{\mathbf{x}_i \in \overline{\tau}} (v_{\mathcal{T}}(\mathbf{x}_i) - \tilde{v}_{\mathcal{T}}(\mathbf{x}_i))^2.$$

Let  $\mathcal{E}(i)$  denote the set of all edges of the triangulation which are contained in  $\Omega$  and have  $\mathbf{x}_i$  as an endpoint. Let  $[\tilde{v}_{\mathcal{T}}]_e(\mathbf{x}_i)$ , for an edge  $e$  in  $\mathcal{E}(i)$  denote the difference of values of  $\tilde{v}_{\mathcal{T}}(\mathbf{x}_i)$  from the triangles adjacent to  $e$ . Since the value of  $v_{\mathcal{T}}$  coincides with the value of  $\tilde{v}_{\mathcal{T}}$  on  $\tau^i$  at  $\mathbf{x}_i$ , we can write  $(v_{\mathcal{T}} - \tilde{v}_{\mathcal{T}})(\mathbf{x}_i)$  as a sum of a few of the differences  $[\tilde{v}_{\mathcal{T}}]_e(\mathbf{x}_i)$ ,  $e \in \mathcal{E}(i)$ . This is true even for boundary nodes. Thus,

$$(v_{\mathcal{T}}(\mathbf{x}_i) - \tilde{v}_{\mathcal{T}}(\mathbf{x}_i))^2 \leq (\mathfrak{n}(i) - 1) \sum_{e \in \mathcal{E}(i)} [\tilde{v}_{\mathcal{T}}]_e^2(\mathbf{x}_i),$$

where  $\mathfrak{n}(i)$  is the cardinality of  $\mathcal{E}(i)$ . By the angle condition,  $\mathfrak{n}(i)$  can be bounded above in terms of  $\mathcal{C}_*$  (independently of  $i$ ). In addition, if  $\tau_1$  and  $\tau_2$  are two triangles which meet at the vertex  $\mathbf{x}_i$  then

$$ch_{\tau_1} \leq h_{\tau_2} \leq Ch_{\tau_1}$$

holds with constants  $c$  and  $C$  which only depend on  $\mathcal{C}_*$ . Thus,

$$\begin{aligned} h_{\tau}^{-2} \|v_{\mathcal{T}} - \tilde{v}_{\mathcal{T}}\|_{0,\tau}^2 &\leq \mathcal{C} \sum_{\mathbf{x}_i \in \overline{\tau}} \sum_{e \in \mathcal{E}(i)} [\tilde{v}_{\mathcal{T}}]_e^2(\mathbf{x}_i) \\ &\leq \mathcal{C} h_{\tau}^{-1} \sum_e \int_e [\tilde{v}_{\mathcal{T}}]_e^2 ds \\ &\leq \mathcal{C} h_{\tau}^{-1} \sum_e \left[ \int_e [v - \tilde{v}_{\mathcal{T}}]_e^2 ds + \int_e [v]_e^2 ds \right], \end{aligned} \tag{3.9}$$

where the last two sums run over the interior edges which have one of the vertices of  $\tau$  as an endpoint. For each such edge  $e$  there are two triangles  $\tau_1$  and  $\tau_2$  which have  $e$  as an edge. Then,

$$\int_e [v - \tilde{v}_{\mathcal{T}}]_e^2 ds \leq 2 \|v^{\tau_1} - \tilde{v}_{\mathcal{T}}^{\tau_1}\|_{0,e}^2 + 2 \|v^{\tau_2} - \tilde{v}_{\mathcal{T}}^{\tau_2}\|_{0,e}^2.$$

Here  $v^{\tau^i}$  and  $\tilde{v}_{\mathcal{T}}^{\tau^i}$  denote the values of  $v$  and  $\tilde{v}_{\mathcal{T}}$  taken from  $\tau^i$ . Using Lemma 2.1 and (3.6), we have

$$\left\| v^{\tau^i} - \tilde{v}_{\mathcal{T}}^{\tau^i} \right\|_{0,e}^2 \leq \mathcal{C}(h_{\tau^i}^{-1} \|v - \tilde{v}_{\mathcal{T}}\|_{0,\tau^i}^2 + h_{\tau^i} |v - \tilde{v}_{\mathcal{T}}|_{1,\tau^i}^2) \leq \mathcal{C}h_{\tau^i} |v|_{1,\tau^i}^2. \quad (3.10)$$

To bound the second term of the last inequality in (3.9), we note that the averages of  $v^{\tau_1}$  and  $v^{\tau_2}$  are the same on  $e$ . Let  $\bar{v}_e$  denote this number. Then

$$\begin{aligned} \int_e [v]_e^2 ds &\leq 2 \|v^{\tau_1} - \bar{v}_e\|_{0,e}^2 + 2 \|v^{\tau_2} - \bar{v}_e\|_{0,e}^2 \\ &\leq \mathcal{C} \left( h_{\tau_1} |v|_{1,\tau_1}^2 + h_{\tau_2} |v|_{1,\tau_2}^2 \right), \end{aligned}$$

where the second inequality followed from Lemma 2.1. Combining the above inequalities and summing over  $\tau \in \mathcal{T}$  proves (3.8) and hence (3.4).

It only remains to prove (3.5). By the triangle inequality and a local inverse inequality,

$$\begin{aligned} |v_{\mathcal{T}}|_{1,\Omega} &\leq |v_{\mathcal{T}} - \tilde{v}_{\mathcal{T}}|_{\Sigma} + |\tilde{v}_{\mathcal{T}}|_{\Sigma} \\ &\leq \mathcal{C} \|v_{\mathcal{T}} - \tilde{v}_{\mathcal{T}}\| + |\tilde{v}_{\mathcal{T}}|_{\Sigma}. \end{aligned}$$

Application of (3.8) and (3.6) now finishes the proof.  $\square$

**REMARK 3.1** It is well-known [8, 19] that for every  $v \in H_D^1(\Omega)$ , there exists a  $v_{\mathcal{T}} \in \mathcal{V}_{\mathcal{T}}$  such that

$$\sum_{\tau \in \mathcal{T}} h_{\tau}^{-2} \|v - v_{\mathcal{T}}\|_{0,\tau}^2 \leq \mathcal{C} |v|_{1,\Omega}^2.$$

Since  $H_D^1(\Omega) \subset \mathcal{V}_{\mathcal{T}}$ , Lemma 3.1 is a generalization of this result.

**PROOF OF THEOREM 3.1.** For  $v \in \mathcal{V}$ , let  $v_{\mathcal{T}}$  be as given by Lemma 3.1. By the triangle inequality,

$$\|v\|_{0,\Omega} \leq \|v_{\mathcal{T}}\|_{0,\Omega} + \|v - v_{\mathcal{T}}\|_{0,\Omega}. \quad (3.11)$$

The standard Poincaré inequality on  $H_D^1(\Omega)$  gives a constant  $C_{\Omega}$  such that

$$\|v_{\mathcal{T}}\|_{0,\Omega} \leq C_{\Omega} |v_{\mathcal{T}}|_{1,\Omega}. \quad (3.12)$$

From Lemma 3.1, we have that  $|v_{\mathcal{T}}|_{1,\Omega} \leq \mathcal{C} |v|_{\Sigma}$ . Therefore it suffices to verify that

$$\|v - v_{\mathcal{T}}\|_{0,\Omega} \leq \mathcal{C} |v|_{\Sigma}. \quad (3.13)$$

Since the obvious inequality  $\|v - v_{\mathcal{T}}\|_{0,\Omega} \leq \mathcal{C}\|v - v_{\mathcal{T}}\|$  and Lemma 3.1 implies (3.13), the proof is complete.  $\square$

REMARK 3.2 Consider the case of partitioning  $\overline{\Omega} = \cup_{\tau \in \mathcal{T}} \bar{\tau}$  with  $Z$  equal to the set of interior edges, and let

$$\mathcal{V}'_{\mathcal{T}} = \left\{ v : \begin{array}{l} v|_{\tau} \in H^1(\tau) \text{ for triangles } \tau \text{ of } \mathcal{T}, \int_e v ds = 0 \text{ for} \\ \text{edges } e \subseteq \partial\Omega_D \text{ of } \mathcal{T}, \text{ and } \int_{\gamma} [v]_{\gamma} ds = 0 \text{ for } \gamma \in Z. \end{array} \right\} \quad (3.14)$$

Assume that the triangulation  $\mathcal{T}$  satisfies the conditions of Assumption 3.1. It is possible to modify the proof of Lemma 3.1 to show that its conclusion holds for functions in  $\mathcal{V}'_{\mathcal{T}}$ . Thus Theorem 3.1 holds for  $\mathcal{V}'_{\mathcal{T}}$ .

REMARK 3.3 Consider the  $P_1$  [9] or  $P_2$  [12] nonconforming finite element space based on a quasiuniform triangulation  $\mathcal{T}$  of mesh size  $h$ . Such a space is a subspace of  $\mathcal{V}'_{\mathcal{T}}$  defined by (3.14). Let  $u$  be the continuous solution and  $u_h$  be its nonconforming finite element approximation. By (3.3) and Remark 3.2,

$$\|u - u_h\|_{0,\Omega} \leq \mathcal{C} |u - u_h|_{\Sigma} \leq \mathcal{C}h^l |u|_{l+1,\Omega}.$$

The second inequality above is, with  $l = 1$  or  $2$ , the standard error estimate for the nonconforming method. Thus, the finite element error estimate for the nonconforming method in the discrete energy norm at least implies a (weak) error estimate in  $L^2(\Omega)$  without any further regularity assumptions on the problem.

We close this section by stating an application of the Poincaré inequality to the mortar finite element method. The following result is an immediate consequence of Theorem 3.1.

THEOREM 3.2 *Suppose that Assumption 3.1 holds. Then Problem 2.2 has a unique solution  $U_M$ , and  $U_M$  satisfies the following a priori stability estimate:*

$$\|U_M\|_{\Sigma}^2 \leq C_0^2(1 + C_0^2) \|f\|_{0,\Omega}^2.$$

*Thus, the stability of the mortar finite element method does not deteriorate as the number of subdomains increases.*

The stability result above can also be deduced from a another work [21] independent of ours. There, a Poincaré inequality is proved for the discrete mortar finite element spaces with a constant independent of the subdomain diameters. Our assumption and techniques are different from those in [21],

and we arrive at a Poincaré inequality on the infinite dimensional space  $\mathcal{V}$ . Moreover, our assumption yields error estimates independent of the number of subdomains, as the next section shows.

## 4 Error estimates

In this section we prove that under the previous assumption on the subdomain partitioning, the error estimates for the mortar finite element solution hold with constants independent of the number of subdomains.

We assume, throughout this section, that the mesh on  $\Omega_i$ , namely  $\mathcal{T}_i$ , is quasiuniform with mesh size  $h_i$ , i.e, the ratios  $h/h_\tau$  for any triangle  $\tau$  of any of the triangulations  $\mathcal{T}_i$  are bounded above and below by fixed constants (independent of  $\tau$  and  $i$ ). We will also assume that the solution to Problem 2.1, namely  $U$ , is in  $H^{3/2+\epsilon}(\Omega)$  for some  $\epsilon > 0$ .

Before proving the error estimates, let us note a consequence of Assumption 3.1 involving some extension operators. Let  $\hat{L}$  be an edge of the reference triangle  $\hat{T}$ . It is a well-known result (cf. [16, Chapter 2, Theorem 5.7]) that there is an extension operator  $\hat{\mathcal{R}} : H_{00}^{1/2}(\hat{L}) \rightarrow H^1(\hat{T})$  such that

$$\left| \hat{\mathcal{R}}\hat{\nu} \right|_{1,\hat{T}} \leq \hat{C}_{\mathcal{R}} \|\hat{\nu}\|_{H_{00}^{1/2}(\hat{L})} \quad \text{for all } \hat{\nu} \in H_{00}^{1/2}(\hat{L}). \quad (4.1)$$

Moreover, the trace of  $\hat{\mathcal{R}}\nu$  on  $\partial\hat{T} \setminus \hat{L}$  is zero. Using  $\hat{\mathcal{R}}$ , we can define extension operators  $\mathcal{R}_\gamma : H_{00}^{1/2}(\gamma) \rightarrow H^1(\Omega_{\text{nm}(\gamma)})$  for all  $\gamma \in \mathcal{Z}$ . Indeed, if  $T \subseteq \Omega_{\text{nm}(\gamma)}$  is the triangle from the triangulation  $\mathcal{T}$  given by Assumption 3.1 having  $\gamma$  as an edge, and  $F$  is the affine map that takes  $\hat{T}$  one-one onto  $T$  (with  $\hat{L} = F^{-1}(\gamma)$ ), then for  $\nu \in H_{00}^{1/2}(\gamma)$ , define  $\mathcal{R}_\gamma\nu$  almost everywhere on  $\Omega_{\text{nm}(\gamma)}$  by

$$\mathcal{R}_\gamma\nu(x) = \begin{cases} \hat{\mathcal{R}}\hat{\nu}(F^{-1}(x)) & \text{if } x \in T, \text{ and} \\ 0 & \text{if } x \in \Omega_{\text{nm}(\gamma)} \setminus T. \end{cases}$$

It is then immediate from (4.1) and the way  $\|\cdot\|_{H_{00}^{1/2}(\gamma)}$  and  $|\cdot|_{1,T}$  scale, that

$$|\mathcal{R}_\gamma\nu|_{1,\Omega_{\text{nm}(\gamma)}} \leq C_{\mathcal{R}} \|\nu\|_{H_{00}^{1/2}(\gamma)}, \quad (4.2)$$

with  $C_{\mathcal{R}}$  independent of  $\gamma$ .

The existence of discrete extension operators also follows. Let  $\mathcal{J}_i : H^1(\Omega_i) \rightarrow \widetilde{M}(\Omega_i)$  denote the averaging interpolant operator defined in [19]. Then Theorem 3.1 there gives constants  $C_{\mathcal{J}}(i)$  depending only on the minimal angle of  $\mathcal{T}_i$  such that

$$|\mathcal{J}_i u|_{1,\Omega_i} \leq C_{\mathcal{J}}(i) |u|_{1,\Omega_i} \quad \text{for all } u \in H^1(\Omega_i). \quad (4.3)$$

For each  $\gamma \in Z$ , define a discrete extension operator  $R_\gamma : W(\gamma) \rightarrow \widetilde{M}(\Omega_{\text{nm}(\gamma)})$  by

$$R_\gamma \nu = \mathcal{J}_{\text{nm}(\gamma)} \mathcal{R}_\gamma \nu.$$

$R_\gamma$  is indeed an extension operator, since  $\mathcal{J}_{\text{nm}(\gamma)}$ , by construction, leaves traces that are continuous piecewise polynomials of degree  $d_{\text{nm}(\gamma)}$  invariant. By (4.2) and (4.3), we have the following result.

**PROPOSITION 4.1** *If Assumption 3.1 holds, then there exist extension operators  $R_\gamma : W(\gamma) \rightarrow \widetilde{M}(\Omega_{\text{nm}(\gamma)})$ , such that for all  $\nu \in W(\gamma)$ , the traces of  $R_\gamma \nu$  on  $\gamma$  and  $\partial\Omega_{\text{nm}(\gamma)} \setminus \gamma$  are  $\nu$  and zero respectively, and there exists a constant  $C_R$  such that*

$$|R_\gamma \nu|_{1, \Omega_{\text{nm}(\gamma)}} \leq C_R \|\nu\|_{H_{00}^{1/2}(\gamma)}. \quad (4.4)$$

*In particular,  $C_R$  is independent of  $\gamma$ .*

We now provide some auxiliary results which help in error analysis. Recall that Proposition 2.2 estimated the jump of functions in  $M$  by a best approximation error. The error in best approximation by a function in  $S(\gamma)$  can be estimated, under the current assumptions on meshes, by well-known techniques. Using also the familiar scaling argument, we conclude that the constant in such estimates can be chosen independent of the size of  $\gamma$ .

**PROPOSITION 4.2** *If  $q$  denotes the  $L^2$  orthogonal projection into  $S(\gamma)$ , then for any  $0 < \alpha \leq d_{\text{nm}(\gamma)}$ ,*

$$\|\nu - q(\nu)\|_{0, \gamma} \leq \mathcal{C} h_{\text{nm}(\gamma)}^\alpha |\nu|_{\alpha, \gamma} \quad \text{for all } \nu \in H^\alpha(\gamma).$$

The importance of the next result also lies in the independence of the constant involved on the subdomain size.

**LEMMA 4.1** *Let Assumption 3.1 hold and consider a  $\gamma \in Z$ . If the restriction of  $U$  to  $\Omega_{\text{nm}(\gamma)}$  is in  $H^s(\Omega_{\text{nm}(\gamma)})$  for  $3/2 < s \leq d_{\text{nm}(\gamma)} + 1$ , then*

$$\left\| \frac{\partial U}{\partial n} - q\left(\frac{\partial U}{\partial n}\right) \right\|_{0, \gamma} \leq \mathcal{C} h_{\text{nm}(\gamma)}^{s-3/2} |U|_{s, \Omega_{\text{nm}(\gamma)}}.$$

**PROOF.** Let  $T \subseteq \Omega_{\text{nm}(\gamma)}$  be the triangle of  $\mathcal{T}$  (the triangulation guaranteed by Assumption 3.1), which has  $\gamma$  as an edge. Also let  $F(\hat{x}) = B\hat{x} + b$  be the affine mapping that takes  $\hat{T}$  one-one onto  $T$ , and let  $\hat{\gamma} = F^{-1}(\gamma)$ .

The standard estimate for  $L^2$  projection when applied on  $\hat{\gamma}$  gives

$$\left\| \frac{\partial \hat{U}}{\partial \hat{n}} - \hat{q} \left( \frac{\partial \hat{U}}{\partial \hat{n}} \right) \right\|_{0, \hat{\gamma}} \leq \hat{C}_q \hat{h}_n^{s-3/2} \left\| \frac{\partial \hat{U}}{\partial \hat{n}} \right\|_{s-3/2, \hat{\gamma}}.$$

Here, as before,  $\hat{h} = h_{\text{nm}(\gamma)} / |\gamma|$ ,  $\hat{U} = U \circ F$ , and  $\hat{n} = B^{-1}n / \|B^{-1}n\|_{\ell^2}$ . A trace theorem now gives

$$\left\| \frac{\partial \hat{U}}{\partial \hat{n}} - \hat{q} \left( \frac{\partial \hat{U}}{\partial \hat{n}} \right) \right\|_{0, \hat{\gamma}} \leq \hat{C}_{q,1} \hat{h}^{s-3/2} \|\hat{U}\|_{s, \hat{T}}.$$

Since  $s \leq d_{\text{nm}(\gamma)} + 1$ , polynomials on  $\gamma$  of degree  $\lceil s \rceil - 2$  are in  $S(\gamma)$ . If  $p$  is a polynomial of degree at most  $\lceil s \rceil - 1$ , then  $(\partial p / \partial \hat{n})|_{\hat{\gamma}}$  is a polynomial of degree at most  $\lceil s \rceil - 2$  on  $\hat{\gamma}$ , and  $\hat{q}$  preserves it. Hence,

$$\begin{aligned} \left\| \frac{\partial \hat{U}}{\partial \hat{n}} - \hat{q} \left( \frac{\partial \hat{U}}{\partial \hat{n}} \right) \right\|_{0, \hat{\gamma}} &= \left\| \frac{\partial(\hat{U} + p)}{\partial \hat{n}} - \hat{q} \left( \frac{\partial(\hat{U} + p)}{\partial \hat{n}} \right) \right\|_{0, \hat{\gamma}} \leq \hat{C}_{q,1} \hat{h}^{s-3/2} \|\hat{U} + p\|_{s, \hat{T}} \\ &\leq \hat{C}_{q,2} \hat{h}^{s-3/2} \|\hat{U}\|_{s, \hat{T}}, \end{aligned} \quad (4.5)$$

where in the last step we have used Proposition 2.1. Noting that  $(\partial U / \partial n)(x)$  is equal to  $\|B^{-1}n\|_{\ell^2} (\partial \hat{U} / \partial \hat{n})(\hat{x})$ , the proof can now be finished easily using a scaling argument.  $\square$

The error analysis of the mortar method uses interpolation error estimates. Let  $U_I \in \bar{M}$  denote the finite element interpolant of  $U$ . The next lemma states some estimates involving  $U_I$  in a form that will be of use later.

**LEMMA 4.2** *Suppose Assumption 3.1 holds. Also assume that  $U|_{\Omega_i} \in H^{s_i}(\Omega_i)$  with  $3/2 < s_i \leq d_i + 1$ . Then, for a  $\gamma \in Z$ , with  $n = \text{nm}(\gamma)$  and  $m = \text{m}(\gamma)$ , we have*

$$\left\| (U - U_I)_\gamma^{\text{m}} \right\|_{H_{00}^{1/2}(\gamma)} \leq \mathfrak{C} h_m^{s_m - 1} |U|_{s_m, \Omega_m}, \quad \text{and} \quad (4.6)$$

$$\left\| (U - U_I)_\gamma^{\text{nm}} \right\|_{H_{00}^{1/2}(\gamma)} \leq \mathfrak{C} h_n^{s_n - 1} |U|_{s_n, \Omega_n}. \quad (4.7)$$

**PROOF.** We prove (4.6) using standard estimates for the interpolant and a scaling argument. The proof of (4.7) is similar. As before, we let  $T \subseteq \Omega_m$  be the triangle of  $\mathcal{T}$  with  $\gamma$  as an edge. Also let  $E = (U - U_I)|_T$ .



Two standard estimates for error in interpolation [7, 11] are

$$\left| \hat{E} \right|_{1, \hat{\gamma}} \leq \hat{C}'_E \hat{h}^{s_m-3/2} \left| \hat{U} \right|_{s_m-1/2, \hat{\gamma}} \quad \text{and} \quad \left\| \hat{E} \right\|_{0, \hat{\gamma}} \leq \hat{C}''_E \hat{h}^{s_m-1/2} \left| \hat{U} \right|_{s_m-1/2, \hat{\gamma}},$$

where  $\hat{h} = h_m/|\gamma|$ . Interpolation of operators, and a trace inequality [14, Theorem 1.5.2.8] gives that

$$\left\| \hat{E} \right\|_{H_{00}^{1/2}(\hat{\gamma})} \leq \hat{C}'''_E \hat{h}^{s_m-1} \left\| \hat{U} \right\|_{s_m-1, \hat{T}}.$$

Now, if  $p$  is any polynomial of degree at most  $\lceil s_m \rceil - 1 \leq d_m$ , the interpolant of  $\hat{U} + p$  is  $\hat{U}_I + p$ . So,

$$\left\| \hat{E} \right\|_{H_{00}^{1/2}(\hat{\gamma})} \leq \hat{C}'''_E \hat{h}^{s_m-1} \left\| \hat{U} + p \right\|_{s_m, \hat{T}} \leq \hat{C}_E \hat{h}^{s_m-1/2} \left| \hat{U} \right|_{s_m, \hat{T}},$$

where we have used Proposition 2.1. The scaling argument now finishes the proof.  $\square$

We now prove error estimates for the mortar method that do not deteriorate with increase in number of subdomains.

**THEOREM 4.1** *Suppose that Assumption 3.1 holds. If the restriction of  $U$  to  $\Omega_i$  is in  $H^{s_i}(\Omega_i)$  for an  $s_i$  satisfying  $3/2 < s_i \leq d_i + 1$ , then*

$$|U - U_M|_{\Sigma}^2 \leq C_4 \sum_{i=1}^K \left( h_i^{s_i-1} |U|_{s_i, \Omega_i} \right)^2.$$

Here,  $C_4$  is a constant independent of the number of subdomains and of mesh sizes  $h_i$ .

**PROOF.** The proof is based on the ideas in [2]. However, in contrast to [2], we will eliminate dependencies on subdomain sizes in the constants in our estimates. As usual, we first write the error as a sum of an ‘‘approximation error’’ term and a ‘‘consistency error’’ term, as in the proof of the so-called Second Strang Lemma [18]. Each of these terms are then separately estimated.

For any  $z \in M$  ( $z \neq U_M$ ) we clearly have that

$$|U - U_M|_{\Sigma}^2 \leq 2|U - z|_{\Sigma}^2 + 2|U_M - z|_{\Sigma}^2 \quad (4.8)$$

Since

$$\tilde{A}(U_M - z, U_M - z) = \tilde{A}(U - z, U_M - z) - \tilde{A}(U, U_M - z) + (f, U_M - z),$$

integration by parts and Cauchy-Schwarz inequality gives

$$|U_M - z|_\Sigma \leq |U - z|_\Sigma + \frac{1}{|U_M - z|_\Sigma} \sum_{\gamma \in Z} \left| \int_\gamma \frac{\partial U}{\partial n} [U_M - z]_\gamma ds \right|.$$

This with (4.8), gives that

$$|U - U_M|_\Sigma^2 \leq 4e_A^2 + 2e_C^2, \quad (4.9)$$

where

$$e_A = \inf_{w \in M} |U - w|_\Sigma \quad \text{and} \quad e_C = \sup_{w \in M} \frac{1}{|w|_\Sigma} \sum_{\gamma \in Z} \left| \int_\gamma \frac{\partial U}{\partial n} [w]_\gamma ds \right|.$$

To estimate  $e_A$ , we choose a  $w$  in  $M$  that approximates  $U$ . Note that although  $U_I$  approximates  $U$ , it is not in  $M$ . We let  $w = U_I + \sum_{\gamma \in Z} R_\gamma \Pi_\gamma [U_I]_\gamma$ . Here  $\Pi_\gamma$  is as in (2.5) and  $R_\gamma$  is as given by Proposition 4.1. Clearly,  $w$  is in  $M$ , and

$$U - w = (U - U_I) - \sum_{\gamma \in Z} R_\gamma \Pi_\gamma [U_I]_\gamma.$$

When restricted to a triangle of  $\mathcal{T}$ , the sum in the last term above has at most three nonzero summands. Summing the squares of  $H^1$ -seminorms triangle by triangle, we have

$$\begin{aligned} |U - w|_\Sigma^2 &\leq 4|U - U_I|_\Sigma^2 + 4 \sum_{\gamma \in Z} \left| R_\gamma \Pi_\gamma [U_I]_\gamma \right|_{1, \Omega_{\text{nm}}(\gamma)}^2 \\ &\leq 4|U - U_I|_\Sigma^2 + 4C_R^2 C_\Pi^2 \sum_{\gamma \in Z} \left\| [U_I]_\gamma \right\|_{H_0^{1/2}(\gamma)}^2. \end{aligned} \quad (4.10)$$

Let  $\tau$  be a triangle in  $\mathcal{T}_i$ . Then by standard estimates for interpolation error [7, Theorem 3.1.6], there is a constant, say  $C_I$ , depending only on  $s_i$ ,  $d_i$  and the minimal angle of  $\mathcal{T}_i$  such that  $|U - U_I|_{1, \tau} \leq C_I h_i^{s_i-1} |U|_{s_i, \tau}$ . Summing, we have

$$|U - U_I|_\Sigma^2 \leq C_I^2 \sum_{i=1}^K \left( h_i^{s_i-1} |U|_{s_i, \Omega_i} \right)^2.$$

To complete the estimation of  $e_A$ , it now suffices to estimate the last term in (4.10). But since  $[U_I]_\gamma$  is equal to  $[U_I - U]_\gamma$ , the triangle inequality and Lemma 4.2 estimates this term as needed.

It now only remains to estimate  $e_C$ . Since

$$\int_\gamma \frac{\partial U}{\partial n} [w]_\gamma ds = \int_\gamma \left( \frac{\partial U}{\partial n} - \chi \right) [w]_\gamma ds, \quad \text{for all } \chi \in S(\gamma),$$

by Cauchy-Schwarz inequality and Proposition 2.2, we have that

$$\int_{\gamma} \frac{\partial U}{\partial n} [w]_{\gamma} ds \leq \left( \inf_{\chi \in \mathcal{S}(\gamma)} \left\| \frac{\partial U}{\partial n} - \chi \right\|_{0,\gamma} \right) \left( \inf_{\chi \in \mathcal{S}(\gamma)} \left\| [w]_{\gamma} - \chi \right\|_{0,\gamma} \right).$$

The first infimum on the right hand side can be bounded using Lemma 4.1 and the second using Proposition 4.2. These together with the discrete Cauchy-Schwarz inequality gives that

$$\sum_{\gamma \in \mathcal{Z}} \left| \int_{\gamma} \frac{\partial U}{\partial n} [w]_{\gamma} ds \right| \leq \mathfrak{C} \left( \sum_{i=1}^K h_i^{2s_i-2} |U|_{s_i, \Omega_i}^2 \right)^{1/2} \left( \sum_{\gamma \in \mathcal{Z}} |[w]_{\gamma}|_{1/2, \gamma}^2 \right)^{1/2}.$$

Estimating the summands of the last sum above using the first inequality of Lemma 2.1, we find that  $e_C$  is bounded as required. This completes the proof.  $\square$

Finally, we provide an error estimate in  $L^2(\Omega)$ -norm. Note that  $L^2$  error estimates for the mortar finite element method were proved before in [1] and [4]. What is new in our theorem is the independence of the constant in the error estimate on number of subdomains.

**THEOREM 4.2** *In addition to the assumptions of Theorem 4.1, if Problem (2.1) admits  $H^2$ -regularity of solutions (see (4.11)),  $U$  is in  $H^s(\Omega)$  for  $2 \leq s \leq \min_i d_i + 1$ , and the meshes in all subdomains are quasiuniform with same mesh size  $h$ , then*

$$\|U - U_M\|_{0,\Omega} \leq C_5 h^s |U|_{s,\Omega}$$

where  $C_5$  is a constant independent of  $h$  and the number of subdomains.

**PROOF.** The argument is analogous to the well-known Aubin-Nitsche duality argument [17, 18]. For any  $g \in L^2(\Omega)$ , let  $U^g$  and  $U_M^g$  solve Problem (2.1) and Problem (2.2) respectively, with  $g$  replacing  $f$  on the right hand side. Regularity of solutions implies the existence of a constant  $C$  such that

$$\|U^g\|_{2,\Omega} \leq C \|g\|_{0,\Omega}. \quad (4.11)$$

We start by observing that for  $e = U^f - U_M^f$ ,

$$\|e\|_{0,\Omega} = \sup_{g \in L^2(\Omega)} \frac{(U^f, g) - (U_M^f, g)}{\|g\|_{0,\Omega}} = \sup_{g \in L^2(\Omega)} \frac{A(U^g, U^f) - \tilde{A}(U_M^g, U_M^f)}{\|g\|_{0,\Omega}}.$$

Now if we let

$$e_1 = \tilde{A}(U^g - U_M^g, e),$$

$$e_2 = \tilde{A}(U_M^g, U^f) - (g, U_M^f), \quad \text{and} \quad e_3 = \tilde{A}(U_M^f, U^g) - (f, U_M^g),$$

then

$$A(U^g, U^f) - \tilde{A}(U_M^g, U_M^f) = e_1 + e_2 + e_3.$$

So, if we show that

$$e_i \leq \mathcal{C}h^s |U^g|_{2,\Omega} |U^f|_{s,\Omega}, \quad (4.12)$$

for  $i = 1, 2$ , and  $3$ , the proof will be complete by virtue of (4.11). Theorem 4.1 readily gives this estimate for  $e_1$ . Obviously, if we prove (4.12) for  $e_2$ , the same will hold for  $e_3$  also.

To estimate  $e_2$ , we first do an integration by parts to get

$$|\tilde{A}(U^g, U_M^f) - (g, U_M^f)| \leq \sum_{\gamma \in Z} \left| \int_{\gamma} \frac{\partial U^g}{\partial n} [U_M^f]_{\gamma} ds \right|$$

Now, as in the proof of Theorem 4.1, applying Cauchy-Schwarz inequality and Proposition 2.2 gives

$$\begin{aligned} \left| \int_{\gamma} \frac{\partial U^g}{\partial n} [U_M^f]_{\gamma} ds \right| &\leq \left( \inf_{\chi \in S(\gamma)} \left\| \frac{\partial U^g}{\partial n} - \chi \right\|_{0,\gamma} \right) \left( \inf_{\chi \in S(\gamma)} \left\| [U_M^f]_{\gamma} - \chi \right\|_{0,\gamma} \right) \\ &\leq \mathcal{C}h |U^g|_{2,\Omega} |[U_M^f]_{\gamma}|_{1/2,\gamma}. \end{aligned}$$

Here, as before, the last inequality is obtained by estimating the first infimum using Lemma 4.1, and the second using Proposition 4.2. Replacing  $[U_M^f]_{\gamma}$  by  $[U_M^f - U^f]_{\gamma}$  and applying Lemma 2.1 we have

$$\sum_{\gamma \in Z} \left| \int_{\gamma} \frac{\partial U^g}{\partial n} [U_M^f]_{\gamma} ds \right| \leq \mathcal{C}h |U^g|_{2,\Omega} |U^f - U_M^f|_{\Sigma}.$$

Theorem 4.1 now yields (4.12) for  $e_2$  and finishes the proof.  $\square$

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