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Investigating Further Preservice Teachers' Conceptions of Multidigit Whole Numbers: Refining a Framework

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This study was designed to investigate preservice elementary school teachers' (PSTs') responses to written standard place-value-operation tasks (addition and subtraction). Previous research established that PSTs can often perform but not explain algorithms and provided a 4-category framework for PSTs' conceptions, 2 correct and 2 incorrect. Previous findings are replicated for PSTs toward the end of their college careers, and 2 conceptions are further analyzed to yield 3 categories of incorrect views of regrouped digits: (a) consistently as 1 value (all as 1 or all as 10), (b) consistently within but not across contexts (i.e., all as 10 in addition but all as 1 in subtraction), and (c) inconsistently (depending on the task).

Number is a central theme in the mathematics curriculum in many countries (National Governors Association & Council of Chief State School Officers, 2010). In the United States of America, for example, a national document, *Adding it Up,* stated, "Proficiency with numbers and numerical operations is an important foundation for further education in mathematics and in fields that use mathematics"(Kilpatrick, Swafford, & Findell, 2001, p. 1). The National Council for Teachers of Mathematics [NCTM] in the United States stated in their *Principals and Standards for School Mathematics* (2000), "Number pervades all areas of mathematics. The other four Content Standards [other than Number and Operations] as well as all five Process Standards are grounded in number." However, studies in Australia and the United States have shown that preservice elementary school teachers (PSTs[1]) struggle with understanding number and place value (Southwell & Penglase, 2005; Thanheiser, 2009), and although most PSTs and teachers in the United States can execute algorithms, many struggle when asked to explain them

[1] For the purpose of this paper, I use *PSTs* to refer to preservice elementary school teachers in the United States. Even though the literature has shown that preservice elementary school teachers struggle in various countries, the current study was conducted in the United States, and, as such, claims made about preservice elementary school teachers as a population will be restricted to those in the United States. The ideas brought forward by the refinement of the framework are, however, generalizable as an explanatory framework (Steffe & Thompson, 2000).

conceptually (Ball, 1988/1989; Ma, 1999; Thanheiser, 2009). Consider, for example, one PST's explanation for the regrouped ones in 389 + 475 (see Figures 1a & 1b).

Figure 1a. Standard algorithm for 389 + 475 used in the United States.

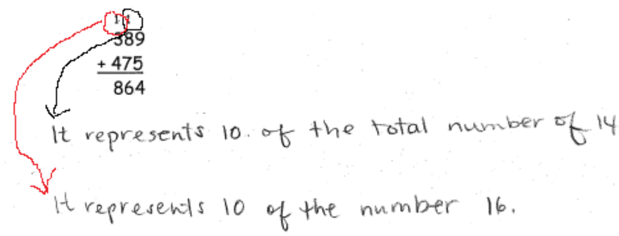


Figure 1b. One PST's explanation of the regrouped ones.

This PST interpreted both 1s as 10 ones from the sums 9 + 5 = 14 and 1 + 8 + 7 =16 (really 10 + 80 + 70 = 160). In our base-ten number system, when we read a whole number, the referent for the number is always *ones* if not stated otherwise (e.g., 387 is 387 ones). In the case of the PST's answer, the interpretation of 10 *ones* (or 1 *ten*) for the 1 above the 8 is correct; however, the interpretation of 10 *ones* (or 1 *ten*) for the 1 above the 3 is incorrect because it represents 10 *tens* or 1 *hundred*. This type of incorrect answer is typical for PSTs (Ball, 1988/1989; Ma, 1999; Thanheiser, 2009). In this paper, I further report my investigation of types of PSTs' incorrect answers to standard 3-digit addition and subtraction problems.

Literature Review

Previous research (Ball, 1988/1989; Ma, 1999; Thanheiser, 2009) has shown that PSTs struggle when asked to explain conceptually the standard algorithms for addition and subtraction used in the United States (see Figure 3 for an example of the standard algorithms used in the United States for addition and subtraction with regrouping). PSTs typically see mathematics as "meaningless memorization of basic facts, rules, and procedure" (Simonsen & Teppo, 1999, p.

517), and their often superficial understanding of place value leads to their mere manipulation of algorithms (McClain, 2003; Tirosh & Graeber, 1989). One PST explained,

> I learned [at the beginning of my elementary mathematics methods class] that there was a lot more to the concept [of number and place value] than I was aware of. I am able to use math effectively in my everyday life, such as balancing my checkbook, but when I was presented with questions as to why I carry out such procedures as *carrying*[2] and *borrowing* in addition and subtraction, I was stuck. I could not explain why I followed any of these procedures or rules. I just knew how to do them. This came as a huge shock to me considering I did well in most of my math classes. I felt terrible that I could not explain simple addition and subtraction. [PST in elementary mathematics methods course]

PSTs often approach mathematics from a calculational (cf. Thompson, Philipp, Thompson, & Boyd, 1994) perspective (McClain, 2003). Although they are able to successfully solve multidigit-addition and -subtraction problems and are able to demonstrate a procedure, they struggle when asked to explain the mathematics underlying the procedure. However, before being explicitly asked to explain, the PSTs are likely to take their own (or children's) procedural fluency as evidence of conceptual understanding (Graeber, 1999). With the recent calls for a focus on having children in the United States develop conceptual understanding (Kilpatrick, et al., 2001; Lundin & Burton, 1998; NCTM, 2000), mathematics teacher educators in the United States need to focus on how to help their PSTs develop such knowledge so that they are poised to assist their future students to do so. The research reported in this study grows out of a rich cognitive-science paradigm focused upon children's prior knowledge in learning situations, a consideration that is equally important in work with adults (Bransford, Brown, & Cocking, 1999).To help teachers (and PSTs) to develop an understanding of the mathematics that enables them to teach for conceptual understanding, teacher educators need to build on the PSTs' currently held conceptions. "Understanding and supporting the student's [in our case, PST's]

[2] Note that students in the United States often term regrouping in the context of addition *carrying* and regrouping in the context of subtraction *borrowing.*

reasoning are acknowledged by many as important to successful instruction" (Graeber, 1999, p. 192). ~~To help PSTs develop conceptual understanding of the mathematics~~In the context of whole numbers, teacher educators need to build upon the PSTs' initial conceptions of numbers, which both determine what the PSTs understand when looking at a numbers and serve as a basis for building more sophisticated conceptual structures of number. Thus, it is imperative that teacher educators have a good understanding of PSTs' currently held conceptions and have a framework for making sense of PSTs' responses in various mathematical contexts.

In previous work (Thanheiser, 2009), I identified and categorized PSTs' conceptions of multidigit whole numbers into four major groups: thinking in terms of (a) *reference units*, (b) *groups of ones*, (c) *concatenated-digits plus*, and (d) *concatenated-digits only*. See Table 1 for the definition and distribution of the conceptions among the PSTs in that study.

Table 1

Definition and Distribution of Conceptions in the Context of the Standard Algorithm for the 15 U.S. PSTs in Thanheiser's Study (p. 263)

Conception	# of PSTs
1. *Reference units.* PSTs with this conception reliably[3] conceive of the reference units for each digit and relate reference units to one another, seeing the 3 in 389 as 3 *hundreds* or 30 *tens* or 300 ones, the 8 as 8 *tens* or 80 *ones*, and the 9 as 9 *ones*. They can reconceive of 1 *hundred* as 10 *tens*, and so on.	3
2. *Groups of ones*. PSTs with this conception reliably conceive of all digits correctly in terms of groups of ones (389 as 300 *ones*, 80 *ones*, and 9 *ones*) but not in terms of reference units; they do not relate reference units (e.g., 10 *tens* to 1 *hundred*).	2
3. *Concatenated-digits plus.* PSTs with this conception conceive of *at least one* digit as an incorrect unit type at least on occasion. They struggle when relating values of the digits to one another (e.g., in 389, 3 is 300 ones but the 8 is only 8 ones).	7
4. *Concatenated-digits only*. PSTs holding this conception conceive of *all* digits in terms of *ones* (e.g., 548 as 5 *ones*, 4 *ones*, and 8 *ones*).	3

[3] *Reliably* in these definitions means that after the PSTs were first able to draw on a conception in their explanations in a context, they continued to do so in that context.

Thus, I found that 2/3 of the PSTs in that study saw the digits in a number incorrectly in terms of ones at least some of the time. Although this framework provides initial information for understanding how PSTs think about number in the context of algorithms, further study of the incorrect conceptions is needed to understand how PSTs who fall into the third and fourth categories may be thinking. That investigation is the focus of this study.

Methods

The data analyzed are drawn from a survey administered to 33 PSTs in an elementary mathematics methods course at a large state university prior to a discussion of the teaching of place-value ideas. Because elementary school teachers teach a variety of subjects in the United States, PSTs are typically required to complete various content courses (focused on, for example, mathematics) and methods courses (focused on pedagogy, for example, the teaching of mathematics). The content courses often precede the methods courses. The requirements pertaining to these two types of course for mathematics vary widely across the United States. Some universities offer and require completion of up to three mathematics content courses designed and developed for elementary school teachers before allowing a PST to enter an elementary education program; others require no special mathematics courses.

The PSTs in this study were in the 4th year of a 5-year program at the end of which they would receive a credential to teach elementary school as well as an educational master’s degree. Students were required to earn 60 credits[4] in liberal-arts coursework, including two college-level mathematics courses of their choice (not necessarily designed for elementary school teachers),

[4] A course at this institution is typically 3 credits for a course that meets for 3 hours each week for a 16-week period (15 weeks of instruction and 1 week of final exams). Undergraduate students are expected to take a minimum of 12 credits for full-time status; graduate students are expected to take a minimum of 9 credits for full-time status.

prior to being admitted to the program. During their 4th year and the following summer, the PSTs take various methods courses and do fieldwork. During the 5th year, they student teach and take additional courses. At the time of the study (at the beginning of the methods course in the 4^{th} year), the PSTs had completed their content course requirements (i.e., they had taken all the mathematics they would take before becoming teachers). The PSTs completed the survey at the beginning of the methods course during class time before discussing place value or algorithms. They were told that this assessment would be used to inform instruction but would not affect their grades.

The survey given consisted of four questions (see Appendix). The two tasks analyzed for this paper focused on their explaining the value of the regrouped digits in the addition algorithm (see Figure 2) and comparing the value of the regrouped digits in addition and subtraction (see Figure 3). These tasks were developed on the basis of the framework described above. My goal was to address and further examine the two concatenated-digits conceptions. To analyze the Addition Task, I examined which values the PSTs assigned to the regrouped 1s. For example, PSTs with a concatenated-digits-only conception might see both as 1; PSTs with a concatenated-digits-plus conception might see both as 10 (which would be correct for the 1 above the 8 but not for the 1 above the 3). The rationale for using the Ones Task was to see whether PSTs considered the location of the regrouped digit, its source, or both. Because of each regrouped digit's location in the ten's place, in Problem A the 1 represents 1 ten (or 10 ones) and in Problem B the 1 represents the ten's place of the ten's place, thus 10 tens (or 1 hundred or 100 ones). Considering the digit's source indicates that in Problem A the value of the 1 is 1 hundred (or 10 tens or 100 ones) from the 1 hundred regrouped into the ten's place and in Problem B the value of the 1 is 10 ones (or 1 ten) as 10 ones regrouped from the one's place. PSTs with a concatenated-digits-only conception might interpret both regrouped 1s as 1. PSTs with a concatenated-digits-plus conception might interpret both digits as 10, which is correct for Problem A and incorrect for Problem B. The numbers for these problems were selected to require the need for regrouping and to ensure that communication is clear (i.e., the regrouped 1s in the Addition Task are not both above an 8, etc.).

To establish intercoder reliability, each survey was coded by two independent researchers. Intercoder reliability was established at 86%, the differences in coding were resolved through discussion. In another study (Thanheiser, in preparation) the survey was validated with 91% agreement between interpretations drawn on the basis of the survey and those drawn on the basis of an individual interview. In that study, all PSTs completed the survey, and I subsequently conducted a 15-minute interview with each. These interviews were based on my previous work (Thanheiser, 2009). Those PSTs who correctly interpreted all digits in terms of their values on the survey were subsequently categorized as holding one of the correct conceptions. Those who interpreted at least one digit on the survey incorrectly in terms of its value were subsequently almost all categorized as holding one of the incorrect conceptions. Thus the survey gives a good indication of whether PSTs hold a correct or an incorrect conception of number.

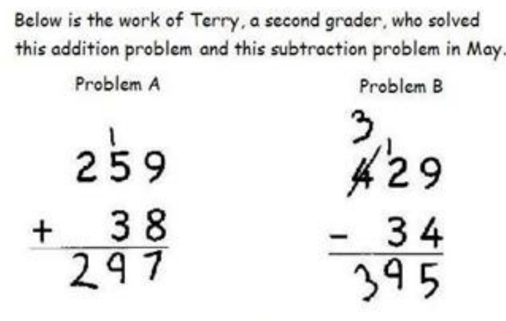
Please consider the regrouped ones in the problem below:

```
  1 1
  389
+ 475
  864
```

- What does the 1 above the 8 represent?
- What does this 1 above the 3 represent?
- Compare the two 1s. Are they the same or are they different? Please be as specific as you can.

Figure 2. Addition Task.

Please answer the questions below:



- Does the 1 in each of these problems represent the same amount? Please explain your answer.
- Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.

Figure 3. Ones Task (Philipp, Schappelle, Siegfried, Jacobs, & Lamb, 2008, p. 32).

Results and Discussion

Most PSTs struggled in explaining the values of the regrouped 1s on the survey. In the context of regrouping in addition (see Figure 2), only 8 of the 33 PSTs explained the values of the regrouped 1s in terms of the appropriate values (10 and 100). Thirteen PSTs saw each digit as representing 10, and 12 PSTs saw each digit as representing 1. In comparing the regrouped 1s in addition and subtraction (Ones Task shown in Figure 3), only 4 PSTs explained them in terms of the correct values (10 in addition and 100 in subtraction). Twelve PSTs saw each digit as representing 10, 8 PSTs saw each digit as representing 1, 8 PSTs saw the digit in addition as 1 and the digit in subtraction as 10, and one PST saw the digit in addition as 10 and the digit in subtraction as 1. (For a summary of the PSTs' conceptions, see Table 2).

Table 2

Conceptions of the 33 PSTs in the Context of Standard Algorithms

Context	Addition Task Explain the values of the regrouped 1s in 3-digit addition (see Figure 2).	Ones Task Compare the regrouped 1s in addition and subtraction and explain their values (see Figure 3).
Correct value for each digit	*8*	*4*
Each digit is 10	*13*	*12*
Each digit is 1	*12*	*8*
Digit in addition is 1; in subtraction is 10		*8*
Digit in addition is 10; in subtraction is 1		*1*

Thus, in the context of addition, 25 of 33 PSTs saw the digits incorrectly in terms of ones at least on occasion. In the context of comparing, 29 of 33 PSTs saw the digits incorrectly in terms of ones at times. This result replicates the earlier finding Thanheiser, (2009) that most PSTs do see the digits incorrectly in terms of ones at least some of the time[5] and, further, shows that the result holds even *toward the end* of PSTs' college careers. With such a large number of PSTs falling into the two concatenated-digits categories, further refinement of those categories is in order.

To better understand the PSTs' thinking, I compared their answers across contexts. Across the two tasks only 3 of 33 PSTs identified the value of the regrouped 1s correctly in both contexts (for a summary of all PSTs' answers, see Table 3). One of these, Felicitas[6], explained that in the Addition Task the 1 above the 8 represented "one ten" and the 1 above the 3

[5] Thanheiser (2009) examined PSTs at a different institution and at a different point in their education, and whereas the framework from that study was derived from interviews with PSTs before their first mathematics content course (typically in their first or second year at the university), the current study was conducted with PSTs after they had completed all their content requirements (typically in their 4th year at the university).
[6] All names are pseudonyms.

represented “one hundred.” For the Ones Task, she explained, “The one in Problem A represents one ten and the one in Problem B represents one hundred or ten tens taken from the hundreds column.”

Table 3

Conceptions of the 33 PSTs in the Context of Standard Algorithms (Detailed)

Interpretation of regrouped 1		Number of PSTs
Addition Task	Ones Task	
Appropriate values	Appropriate values	3
Appropriate values	Both as 1	2*
Appropriate values	Both as 10	3
Both as 10	Both as 1	1*
Both as 10	Both as 10	7
Both as 10	1 in addition, 10 in subtraction	3*
Both as 10	Appropriate values	1
Both as 10	10 in addition, 1 in subtraction	1
Both as 1	1 in addition, 10 in subtraction	5
Both as 1	Both as 1	5
Both as 1	Both as 10	2*

* The interpretation was inconsistent.

The remaining 30 PSTs explained at least some of the regrouped 1s incorrectly—thus exhibiting one of the concatenated-digits conceptions. Two major categories were found in the PSTs’ incorrect explanations: value of digits *consistently* misinterpreted (numbers without an asterisk in Table 3) and value of digits *inconsistently* misinterpreted (numbers with an asterisk in Table 3). The former category is further subdivided into (a) all regrouped digits have the same value and (b) the value of the digit depends on the context. These categories are explained further below.

Incorrect but consistent explanations

All regrouped digits have the same value. Twelve PSTs saw the values of all regrouped digits consistently either as 10 (7 PSTs) or as 1 (5 PSTs) in both tasks (addition and comparison). Erica, for example, saw all regrouped digits consistently as 10: In the Addition Task, for both 1s, she said that "10 is represented by the 1." In the Ones Task her answer was

> Yes. They both represent ten, but in one problem, ten is being taken away (subtraction problem) and in the other ten is being added to the next column. In the subtraction problem ten is borrowed because one cannot subtract negative numbers.

Lara saw all digits consistently as 1: In the Addition Task, she explained that the 1 above the 8 is "the one carried over to be added to 8 and 7," and the 1 above the 3 is "the one carried over to be added to the 3 and 4." For the Ones Task, she explained, "In A you are adding the one to the other numbers, in B, the one makes 2 a 12 so you can subtract."

When introduced to regrouping in the context of addition, students often see the regrouped digit in terms of the sum of the single digits they are adding. In the Addition Task with the sum of 14 in the one's column, they may see the regrouped 1 as either the 10 of the 14 (groups-of-ones conception for 2-digit numbers) or the 1 of the 14 (concatenated-digits conception for 2-digit numbers). Although the first of these conceptions seems correct in regrouping from the ones to the tens, it will not generalize as such for regrouping larger values. Even though the sum of the digits in the ten's place is 16, the 16 now refers to 16 tens or 160 ones; thus, one regroups 10 tens or 100 ones, each of which has a value of 100. So that students develop proficiency with the algorithms, they are often taught to view the columns individually. If the sum of the digits is less than 10, it can simply be written below the column; if it is larger than 9, regrouping is often explained by phrases such as “we put the 10 over here in the next column” or “we move the 1 to the next column.” Although this practice enables one to learn the procedure, it does not build a conceptual understanding of what the 1 or 10 represents and why the amount can be “moved to the next column” and be represented there as a 1. The PSTs in this group held an incorrect conception of the value of the regrouped digit; however, they did so consistently across contexts. Other PSTs also saw the value of each digit consistently, but the value of the digit depended on the context.

The value of the digits depends on the context (addition or subtraction). Five PSTs saw the regrouped digits as 1 for regrouping in the context of addition and as 10 in the context of subtraction. Amy, for example, explained the 1 above the 8 as “the 1 from the 14 when 9 + 5 was carried out,” and the 1 above the 3 as “the 1 from the 16 when 8 + 1 + 7 was computed.” In the Ones Task she explained,

> They are different because the problem B the borrowing is of a 10 from the 100s column that is needed to compute the next step. On the other hand the problem A the 1 is “extra” that was created during the process of adding.

This type of explanation is typical for one who views a digit in terms of its value in the column. In adding, one regroups the digit as a 1 into the column to the left, and in subtracting, one adds 10 to the column to the right. Without considering the place values (or units to which the digits refer), one sees the digits as 1s and 10s.

Three PSTs described the regrouped digits in terms of the appropriate values in the Addition Task but explained both in terms of 10 in the Ones Task. Summer, for example, correctly explained the 1 above the 8 as "the ten in 14" and the 1 above the 3 as "the one hundred in 160." On the Ones Task, however, she explained, "They both represent the same amount ten, but for different numbers. In A, the 1 is for 10 out of 17 (carried over from ones). In B, the 1 is for 10 out of 12 (carried over/borrowed from hundreds.)" This type of explanation may derive from the fact that the PSTs interpret the value of the digit in terms of the value of its place. This practice is correct for addition (just read the digit with its place), but for subtraction one must read the regrouped 1 as a 10 in its new place (i.e., a 1 regrouped from the hundred's place is *10* tens in the ten's place); the regrouped digit in subtraction is regrouped into the ten's place and, thus, may be seen as 10 by the PSTs

Two PST saw the regrouped digits as 10 for regrouping in the context of addition but differed in their explanation for the regrouped 1 in the context of subtraction. Nadine saw the regrouped digit in the context of subtraction as 1, Jess as 100. Nadine explained the 1 above the 8 in the Addition Task as "the 10 in 14. The 4 goes in the solution and the 10 gets carried over." She explained the 1 above the 3 as "8+7+1 is 16. [The] 6 goes in the solution and [the] 10 is above the 3." In the context of the Ones Task she explained "in addition we are carrying over, so we give the extra '10' to the next number … in subtraction, we borrow from the next number so we put the one over the number who is borrowing." In the context of addition, this type of explanation is consistent with focusing on single digit addition in each column. In the context of subtraction an interpretation of the regrouped digit as 1 indicates a focus on the face value of the regrouped digit while an interpretation of the digit as 100 indicates a consideration of place it came from. Jess, for example, explained regrouping in the context of subtraction as "you need to exchange the hundred for tens," thus considering where the regrouped digit came from (hundred's place).

Incorrect and inconsistent explanations

Consider the interpretation of the regrouped digit in the ten's place of the addition problem in both tasks. For the Addition Task, PSTs are asked to identify the values of the regrouped 1s, that is, to identify the value of the regrouped 1 in the ten's place as well as the regrouped 1 in the hundred's place. For the Ones Task, PSTs compare the values of the regrouped 1s in the ten's places of addition and subtraction. Thus, PSTs are asked twice to make sense of the regrouped digit in the ten's place in the context of addition (see Figure 4).

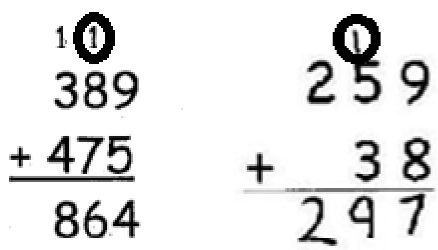


Figure 4. Comparing the regrouped digit in the ten's place in the context of addition.

The digits are interpreted inconsistently (the same digit is interpreted in multiple ways depending on the task). Eight PSTs interpreted the regrouped digit in the ten's place of the addition problems differently in the two tasks (see Table 4 numbers with an asterisk), either as 10 in the Addition Task and as 1 in the Ones Task (6 PSTs) or as 1 in the Addition Task and as 10 in the Ones Task (2 PSTs). Marie, for example, explained that in the Addition Task, the 1 above the 8 is "adding another group of ten." In the Ones Task, she explained, "Problem A is carrying over a 1 from 17 in the ones column; Problem B is borrowing a 1 from the hundreds column to create 12 – 3 for the tens column." Thus, Marie explained the regrouped 1 in the ten's place in the context of addition as 10 in the Addition Task and as 1 in the Comparison task.

Karin, explained, the 1 above the 8 as follows: “9 + 5 = 14 the one goes above the 8.” In the Ones Task she explained, “Yes they are both added on to the 10s column and so represent a 10 being added.” These inconsistent interpretations of the regrouped digit in the ten’s place in the context of addition indicate not only an incorrect conception of the values of the digits (and regrouped digits) but also an inconsistency of value across questions/tasks. Although these 8 PSTs interpreted the regrouped 1 correctly as 10 in one context, they interpreted it incorrectly as 1 in the other. The fact that the PSTs’ interpretations of the value of the digit in the same context changed with the question posed shows that they view the value of the digit as question dependent rather than as a quantity in a context.

Refining the framework

Most PSTs’ responses to these tasks are in the concatenated-digits-plus category. The results of this paper can be used to further refine that category. Incorrect responses in which not all digits are explained in terms of ones (and thus not in the concatenated-digits-only conception) can be categorized as (a) digits consistently explained as 10 (regardless of the context), (b) digits explained consistently depending on context (i.e., 10 in subtraction, 1 in addition), and (c) changed interpretations of the digit depending on the question posed (i.e., regrouped 1 in the ten’s place in the context of addition as 10 or 1 in different tasks).

Conclusions

The results of this study are a detailed analysis of PSTs’ varied explanations for the regrouped digits in multidigit addition and subtraction. Whereas previous research (a) established the fact that PSTs struggle when asked to explain the mathematics underlying the algorithms before they take content courses at the university level and (b) provided a conceptual framework for four broad categories of conceptions, the current study (a) showed that PSTs also struggle after completing their content courses and (b) was focused on refining the categories of incorrect conceptions PSTs hold. The incorrect answers described above are consistent with one of the previously identified concatenated-digits conception (seeing all or some of the digits incorrectly in terms of an incorrect unit type). The categories that emerged in this analysis show that educators can further refine the concatenated-digits-plus conception by considering PSTs' consistency and inconsistency in their interpretation of digits. Further, because the tasks used in this study comprise an efficient instrument with which mathematics educators can assess their PSTs' understanding of the regrouped digits in the ubiquitous multidigit-addition and -subtraction algorithms (without conducting individual interviews), they can ascertain their students' initial conceptions and build on those conceptions in their courses.

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Appendix

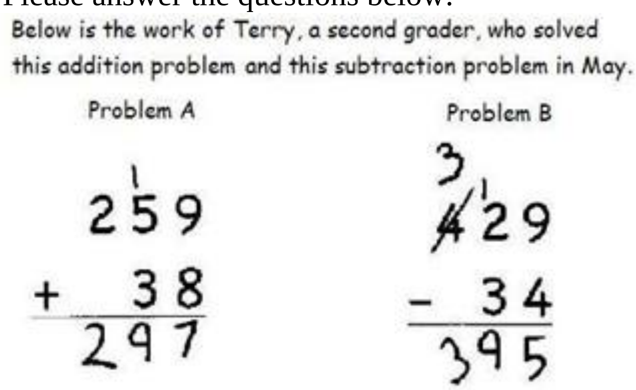
Survey questions

1. One of the main topics in 2nd grade is place value. What in your mind are the essential elements of place value a 2nd grader should understand? Please be as specific as you can be.
2. Another main topic in 2nd grade is addition and subtraction of 2-digit numbers. What in your mind does a 2nd grader need to understand to be able to add and subtract 2-digit numbers? Please be as specific as you can be.
3. Please consider the regrouped ones in the problem below:

$$\begin{array}{r} {\scriptstyle 1\;1}\;\; \\ 389 \\ \underline{+\,475} \\ 864 \end{array}$$

   3.1. What does the 1 above the 8 represent?
   3.2. What does the[7] 1 above the 3 represent?
   3.3. Compare the two 1s. Are they the same or are they different? Please be as specific as you can be.

4. Please answer the questions below:[8]



   4.1. Does the 1 in each of these problems represent the same amount? Please explain your answer.
   4.2. Explain why in addition (as in Problem A) the 1 is added to the 5, but in subtraction (as in Problem B) 10 is added to the 2.

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[7] The survey used for the study used t*his* instead of the. For further use of this survey I suggest a consistent use of *the* for both 3.1 and 3.2.

[8] From *The effects of professional development on the mathematical content knowledge of K–3 teachers* by R. A. Philipp, B. P. Schappelle, J. M. Siegfried, V. Jacobs, and L. C. Lamb, 2008 presented at the annual meeting of the American Educational Research Association, New York, NY. Reprinted with permission of the authors.