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Nonstandard PL Embeddings in Codimension Two

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Abstract

This paper will explore the following question: If K is a compact k-polyhedron and M is a PL m-manifold, under which cases will the embedded polyhedra have regular neighborhoods which are not homeomorphic?

An important topological question, raised by W.S. Massey in 1959, asks whether the normal vector bundle of a compact, orientable, differentiable, k-manifold is determined by the embedding, or is it intrinsically determined by the manifold itself [Mas59].

Definition. [Rol76] An M-MANIFOLD M is a metric space in which each point $x \in M$ has a neighborhood homeomorphic to $\mathbb{R}^n$ or $\mathbb{R}^n_+$. Those points which have neighborhoods homeomorphic to $\mathbb{R}^n$ comprise the INTERIOR of M (denoted $\mathring{M}$, while the points with neighborhoods homeomorphic to $\mathbb{R}^n_+$ comprise the BOUNDARY of M (denoted ∂M).

Definition. [Kaw96] A SIMPLICIAL COMPLEX is a set K of simplices in $\mathbb{R}^n$ which satisfies the following conditions:

1. For each pair $A_1, A_2 \in K$, the intersection $A_1 \cap A_2$ is a face of A_1 and of A_2 (unless it is $\emptyset$).

2. All faces of each $A \in K$ are contained in K.

3. For each $A \in K$, there are finitely many elements of K meeting A.

The union of all simplices in K is called the POLYHEDRON of K (denoted $|K|$).

Definition. [Rol76] A manifold pair $K^k \subset M^m$ is LOCALLY FLAT at $x \in K$ if there exists a closed neighborhood N of x in M such that $(N, K \cap N)$ is homeomorphic with the standard ball pair (B^m, B^k).

Throughout this paper, we will be working in the non-locally-flat PL category, thus each space will be assumed to be a simplicial complex and each map will be assumed to be piecewise-linear, that is all maps send simplexes linearly onto simplexes (after some subdivision), but we will allow embeddings which are not locally flat. For the definition of a regular neighborhood, see [RS72]. Since we are dealing with compact polyhedra embedded in PL manifolds, then the following criteria will be used:

Proposition 1 [RS72, p.34]. *Suppose $K \subset M$ is a compact polyhedron in the manifold M. A regular neighborhood N of K in M is ca compact manifold with boundary. If $K \subset \mathring{M}$ then $\partial N = |\dot{N}|$, where $\dot{N}$ denotes the frontier of N in M.*

Theorem 1 (Collapsing criterion for regular neighborhoods)[RS72]. *Let N be a neighborhood of K in $\mathring{M}$. Then N is regular if and only if*

(i) N is a compact manifold with boundary,

(ii) N collapses to K.

The Massey problem later motivated the following question:[CRS07]

> Find all pairs (m, k) such that if K is a compact k-polyhedron and M a PL m-manifold then
>
> $$R_M(fK) \cong R_M(gK) \qquad (1)$$
>
> for each two homotopic PL embeddings $f, g : K \to M$.

In the case of $M = S^m$, [CRS07] produced the following results:

Theorem 2 [CRS07]. *For $k \geq 2$ and given embedding $S^k \subset S^{k+2}$ set $N = R_{S^{k+2}}(S^k)$. Then $G = \pi_1(S^{k+1} - S^{k-1})$ maps into $\pi_1(\partial N)$ in each of the following cases:*

(a) *the embedding $S^k \subset S^{k+2}$ is the suspension over a locally flat PL knot $S^{k-1} \subset S^{k+1}$.*

(b) *$k \geq 3$ and the embedding $S^k \subset S^{k+2}$ has an isolated non-locally flat point with the singularity $S^{k-1} \subset S^{k+1}$.*

Theorem 3 [CRS07]. *$R_{S^{k+2}}(S^k)$ is not homeomorphic to $S^k \times D^2$ for any $k \geq 2$ and*

(a) *any PL sphere $S^k \subset S^{k+2}$ which is the suspension over a locally flat PL knot $S^{k-1} \subset S^{k+1}$ such that $G = \pi_1(S^{k+1} - S^{k-1}) \ncong \mathbb{Z}$; or*

(b) *any PL sphere $S^k \subset S^{k+2}$ having an isolated non-locally flat point with the singularity $S^{k-1} \subset S^{k+1}$ such that $G = \pi_1(S^{k+1} - S^{k-1}) \ncong \mathbb{Z}$.*

Proof. Let $S^k \subset S^{k+2}$ be a suspension over a locally-flat PL knot $K = S^{k-1} \subset S^{k+1}$ such that $G = \pi_1(S^{k+1} - S^{k-1}) \ncong \mathbb{Z}$, and let $N = R_{S^{k+2}}(S^k)$. Suppose, by contradiction, $N \cong S^k \times D^2$. Since $\pi_1(S^{k+2} - \Sigma K)$ is isomorphic to $\pi_1(S^{k+1} - K)$ [Rol76, p.85], then by Theorem 1.a, there exists a monomorphism

$$h : \pi_1(S^{k+1} - K) \to \partial N.$$

Since $\pi_1(\partial N) \cong \mathbb{Z}$, then $\pi_1(S^{k+1} - K)$ is abelian. But the only knot with an abelian knot group is the trivial knot , so K is trivial, a contradiction. □

In [CRS07], the following conjecture was made:

Conjecture 1. *For the pair (m, k), where K is a compact k polyhedron and M is a PL m-manifold, there exist homotopic PL embeddings f and g then $R_M(fK) \ncong R_M(gK)$ whenever $M = S^{2k}$ and $K = \Sigma(S^{k-1} \sqcup S^{k-1})$.*

Specifically, they conjectured that the proof would follow if we let f and g be the suspensions of the trivial link and the Hopf link. Unfortunately, although the fundamental group of a knot complement is isomorphic to that of its suspension, the same does not hold for links [Rol76].

Definition. [Sti93] If M_1 and M_2 are $n-$manifolds, then the CONNECTED SUM $M_1 \# M_2$ is obtained by removing an open $n-$ball B_i from each manifold and then identifying $(M_1 - B_1)$ and $(M_2 - B_2)$ along their boundary $(n-1)$-spheres.

One possible route to a proof of the conjecture came through the observation that the trefoil link can be expressed as the connected sum of the components of the Hopf link, whereas any connected sum of the components of the trivial link will yield only the trivial knot. However, the connected sum is not a well-defined operation. In fact, it is also possible to obtain the unknot by connecting the components of the Hopf link, so the aforementioned surgery may instead lead to a counterexample to the above conjecture, rather than a proof.

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