

Portland State University

PDXScholar

Electrical and Computer Engineering Faculty
Publications and Presentations

Electrical and Computer Engineering

11-1-1996

Generalized Beam Matrices. III. Application to Diffraction Analysis

Lee W. Casperson
Portland State University

Anthony A. Tovar

Follow this and additional works at: https://pdxscholar.library.pdx.edu/ece_fac



Part of the [Electrical and Computer Engineering Commons](#)

Let us know how access to this document benefits you.

Citation Details

Anthony A. Tovar and Lee W. Casperson, "Generalized beam matrices. III. Application to diffraction analysis," *J. Opt. Soc. Am. A* 13, 2239-2246 (1996).

This Article is brought to you for free and open access. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Publications and Presentations by an authorized administrator of PDXScholar. Please contact us if we can make this document more accessible: pdxscholar@pdx.edu.

Generalized beam matrices. III. Application to diffraction analysis

Anthony A. Tovar

Department of Physics and Astronomy, Murray State University, P.O. Box 9, Murray, Kentucky 42071-0009

Lee W. Casperson

Department of Electrical Engineering, Portland State University, P.O. Box 751, Portland, Oregon 97207-0751

Received February 5, 1996; accepted May 14, 1996

In analogy with Huygen's wavelets a new method based on Gaussian beamlets is used to develop a conventional diffraction integral formalism for paraxial optical systems representable by complex 2×2 *ABCD* Gaussian beam matrices. This method, along with a new phase parameter transformation, is then used to produce a new diffraction integral for studying the propagation of light beams with arbitrary spatial profiles through much more general misaligned complex optical systems representable by 3×3 *ABCDGH* beam matrices.
© 1996 Optical Society of America.

1. INTRODUCTION

A diffraction integral based on the work of Christian Huygens (1629–1695) may be used to study the propagation of arbitrarily profiled light beams in free space. This integral is a continuous summation of spherical waves, which are solutions to the Helmholtz equation for free space. For this continuous summation to be performed, each of the spherical waves must have a vanishingly small amplitude. Hence these spherical waves are sometimes referred to as Huygen's wavelets. In a seminal work on laser resonator theory Fox and Li used this diffraction integral to examine the beam modes of high-diffraction-loss resonators.¹ The Fox and Li method simulates actual beam mode formation in the lasing process and is still an important tool in studying laser resonator modes.

Gaussian beam theory was popularized by Kogelnik, who showed that, like paraxial rays, Gaussian-profiled beams of light could be traced through a wide variety of aligned optical systems by simple 2×2 *ABCD* matrix multiplication.² In 1969 Baues combined the beam matrix theory of Kogelnik and the resonator analyses of Fox and Li.^{3,4} Baues used geometric optics ideas and Huygen's principle to obtain an integral equation valid for any optical system represented by Kogelnik's beam matrices. The systems considered by Baues have two perpendicular planes of symmetry, and the extension of these results to more general asymmetric systems was reported by Collins in 1970.⁵

Kogelnik's original beam matrices were strictly real valued and were identical in form to ray matrices. Later, several other optical elements were added to Kogelnik's beam matrix formalism; some of these elements, such as the complex lenslike medium⁶ and the Gaussian aperture,⁷ were represented by complex-valued matrices. A Green-function propagation method applicable to complex beam matrices was described by Arnaud in 1971.⁸ In 1982 Nazarathy and Shamir developed a canonical op-

erator theory to show that Baues's *ABCD* diffraction integral was valid for complex-valued matrices as well as real-valued matrices.⁹ In 1985 Siegman showed that Baues's integral is valid for complex beam matrices by examining individual complex-valued matrices and then showing that the results may be cascaded by matrix multiplication.¹⁰ In 1987 Wright and Garrison rederived the complex form of Baues's integral by using Feynman path integrals.¹¹ Each of these derivations of the complex *ABCD* diffraction integral has its advantages and drawbacks.

In 1995 Kogelnik's Gaussian beam formalism was generalized to include complex optical systems that are also misaligned.¹² New complex matrices for exponential variable-reflectivity mirrors, laser amplifiers with linear gain profiles, and several other optical elements were also introduced. These optical systems can all be represented by generalized 3×3 complex *ABCDGH* matrices. One of the purposes of this paper is to extend the Huygen's diffraction integral so that it can be applied to these more general optical systems. This extension is developed by expanding the initial field at an input plane into a distribution of small off-axis Gaussian beams and then using the *ABCDGH* formalism to propagate these beams through the system. The overall output field is obtained as the sum of these individual output beams. Besides the single-pass propagation of fields through complex systems, the results may also be used to perform Fox and Li mode calculations on more complicated laser resonators than those on which it has been possible in the past.

The basic formulas that govern the propagation of off-axis beams in complex misaligned systems are summarized in Section 2, and they include a new transformation for the phase parameters of the beams. The use of these formulas to obtain the conventional *ABCD* diffraction integral formalism is described in Section 3. In Section 4 this new approach is applied again in obtaining a more general diffraction integral formalism for complex mis-

aligned systems describable in terms of $ABCDGH$ matrices. Some specific practical applications of this formalism are discussed in Section 5.

2. GAUSSIAN BEAM THEORY

In this section many of the basic procedures for analyzing beam propagation in complex misaligned systems are reviewed. These procedures form a basis for the following diffraction integral formulation. The electric field for a linearly polarized electromagnetic beam that obeys the scalar paraxial wave equation may be written as

$$\mathbf{E}(x, y, z, t) = \text{Re}[E'(x, y, z)\exp(i\omega t)] \times (\mathbf{i}_x \cos \nu + \mathbf{i}_y \sin \nu), \quad (1)$$

where the notation $\text{Re}(\cdot)$ designates the real part of a complex function. In a wide variety of media, including complex lenslike media, the complex amplitude of the electric field in Eq. (1) for a Gaussian beam with x and y symmetry axes is governed by

$$E'(x, y, z) = E'_0 \exp\left(-i \int_0^z k_0 dz\right) \exp\left\{-i \left[\frac{Q_x(z)x^2}{2} + \frac{Q_y(z)y^2}{2} + S_x(z)x + S_y(z)y + P(z) \right]\right\}. \quad (2)$$

In particular, if the complex propagation constant is of the form

$$k^2(x, y, z) = k_0^2(z) - k_0(z)[k_{1x}(z)x + k_{2x}(z)x^2 + k_{1y}(z)y + k_{2y}(z)y^2], \quad (3)$$

then the z -dependent parameters satisfy the following differential equations¹³:

$$Q_x^2 + k_0(z) \frac{dQ_x}{dz} + k_0(z)k_{2x}(z) = 0, \quad (4)$$

$$Q_y^2 + k_0(z) \frac{dQ_y}{dz} + k_0(z)k_{2y}(z) = 0, \quad (5)$$

$$Q_x S_x + k_0(z) \frac{dS_x}{dz} + \frac{k_0(z)k_{1x}(z)}{2} = 0, \quad (6)$$

$$Q_y S_y + k_0(z) \frac{dS_y}{dz} + \frac{k_0(z)k_{1y}(z)}{2} = 0, \quad (7)$$

$$\frac{dP}{dz} = -\frac{i(Q_x + Q_y)}{2k_0(z)} - \frac{S_x^2 + S_y^2}{2k_0(z)} - \frac{i}{2k_0(z)} \frac{dk_0}{dz}. \quad (8)$$

If the input and output planes are in a lossless medium (such as free space) or if the low gain (or loss) per wavelength approximation ($k_0 \approx \beta_0$) is made,¹⁴ then the beam

parameters are related to the beam's spot size ($1/e$ amplitude radius) in the x direction, w_x , and the radius of curvature of the phase fronts in the x -direction, R_x , by

$$\frac{1}{q_x} = \frac{1}{R_x} - i \frac{2}{\beta_0 w_x^2}, \quad (9)$$

where $\beta_0 = 2\pi n_0/\lambda$ is the real part of k_0 . Also, the displacement parameters are related to the position d_{xa} and the slope d'_{xa} of the beam by¹²

$$\frac{S_x}{\beta_0} = -\frac{d_{xa}}{q_x} + d'_{xa}. \quad (10)$$

In Eqs. (9) and (10) the beam parameter Q_x has been related to the often employed parameter q_x by the definition

$$\frac{1}{q_x} \equiv \frac{Q_x}{k_0}. \quad (11)$$

The real and imaginary parts of the phase parameter $P(z)$ represent axial phase and amplitude corrections. For these equations and the rest of the equations in this section each x -subscripted equation implies a corresponding y -subscripted equation, which for conciseness is usually not written out explicitly.

Generalized beam matrices may be used to determine the changes of the beam and displacement parameters as the beam propagates through an astigmatic misaligned complex optical system. The matrix for each optical element may be written in the form¹²

$$\begin{bmatrix} u_{x2} \\ (1/q_{x2})u_{x2} \\ S_{x2}u_{x2} \end{bmatrix} = \begin{bmatrix} A_x & B_x & 0 \\ C_x & D_x & 0 \\ G_x & H_x & 1 \end{bmatrix} \begin{bmatrix} u_{x1} \\ (1/q_{x1})u_{x1} \\ S_{x1}u_{x1} \end{bmatrix}, \quad (12)$$

where the 1 and 2 subscripts indicate the input and output parameters of the optical element, respectively. The A_x, B_x, C_x, D_x elements in Eq. (12) are the usual complex matrix elements for an aligned system, while G_x and H_x allow the inclusion of displacements and misalignments in the formalism. The matrix representation for a given optical element is obtained by solving the differential equations (4)–(7). These matrices have been obtained for a wide variety of optical elements.¹² Optical systems may be analyzed by multiplying the matrix representations of the optical elements in the reverse of the order in which those elements are encountered by the incident light beam.

Dividing the second row of Eq. (12) by its first row yields the Kogelnik transformation, which was originally known as the $ABCD$ law²:

$$\frac{1}{q_{x2}} = \frac{C_x + D_x/q_{x1}}{A_x + B_x/q_{x1}}. \quad (13)$$

Similarly, the displacement transformation is obtained by dividing the third row of Eq. (12) by its first row¹²:

$$S_{x2} = \frac{S_{x1}}{A_x + B_x/q_{x1}} + \frac{G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}. \quad (14)$$

The Gaussian factor in Eq. (2) contains a phase parameter $P(z)$. With Eqs. (13) and (14) Eq. (8) may be integrated to yield the phase transformation

$$\begin{aligned}
 P_2 - P_1 = & \frac{i}{2} \ln(A_x D_x - B_x C_x) - \frac{i}{2} \ln(A_x + B_x/q_{x1}) \\
 & - \frac{i}{2} \ln(A_y + B_y/q_{y1}) \\
 & - \frac{B_x}{2k_{01}} \frac{(S_{x1} + G_x + H_x/q_{x1})^2}{A_x + B_x/q_{x1}} \\
 & - \frac{B_y}{2k_{01}} \frac{(S_{y1} + G_y + H_y/q_{y1})^2}{A_y + B_y/q_{y1}} \\
 & + \frac{H_x}{2k_{01}} (2S_{x1} + G_x + H_x/q_{x1}) \\
 & + \frac{H_y}{2k_{01}} (2S_{y1} + G_y + H_y/q_{y1}) \\
 & + \frac{1}{2k_{01}} \int_0^z \left(G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right) dz' \\
 & + \frac{1}{2k_{01}} \int_0^z \left(G_y \frac{dH_y}{dz'} - H_y \frac{dG_y}{dz'} \right) dz', \quad (15)
 \end{aligned}$$

where we have also used the relationships $C_x = dA_x/dz$, $D_x = dB_x/dz$, $C_y = dA_y/dz$, $D_y = dB_y/dz$, $G_x = -\frac{1}{2} \int_0^z k_{1x}(z') A_x(z') dz'$, $G_y = -\frac{1}{2} \int_0^z k_{1y}(z') A_y(z') dz'$, $H_x = -\frac{1}{2} \int_0^z k_{1x}(z') B_x(z') dz'$ and $H_y = -\frac{1}{2} \int_0^z k_{1y}(z') B_y(z') dz'$.¹² The result given in Eq. (15) may be checked by direct substitution into Eq. (8). Phase parameter transformations of this type are important in interferometry, for example. Hereafter, only the special case in which both the input medium and the output medium are identical is considered. Thus $k_{01} = k_0 = \beta_0 = 2\pi n_0/\lambda$, and $A_x D_x - B_x C_x = 1$.

If the optical system is aligned, then G_x and H_x are zero, and the Kogelnik transformation in Eq. (13) remains unchanged. However, in this case the displacement and phase parameter transformations given in Eqs. (14) and (15) simplify to

$$S_{x2} = \frac{S_{x1}}{A_x + B_x/q_{x1}}, \quad (16)$$

$$\begin{aligned}
 P_2 - P_1 = & -\frac{i}{2} \ln(A_x + B_x/q_{x1}) - \frac{i}{2} \ln(A_y + B_y/q_{y1}) \\
 & - \frac{B_x}{2\beta_0} \frac{S_{x1}^2}{A_x + B_x/q_{x1}} - \frac{B_y}{2\beta_0} \frac{S_{y1}^2}{A_y + B_y/q_{y1}}. \quad (17)
 \end{aligned}$$

The real part of the right-hand side of Eq. (17) is the axial phase shift experienced by an input beam after propagat-

ing through an optical system. If one is interested in the gain in the axial field magnitude, it may be more convenient to rewrite this equation as

$$\begin{aligned}
 & \exp(-iP_2) \\
 & = \frac{\exp(-iP_1) \exp\left[\frac{i}{2\beta_0} \left(\frac{B_x S_{x1}^2}{A_x + B_x/q_{x1}} + \frac{B_y S_{y1}^2}{A_y + B_y/q_{y1}} \right) \right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}}. \quad (18)
 \end{aligned}$$

These results provide the basis for the diffraction calculations that follow.

3. ABCD DIFFRACTION INTEGRAL

In this section we develop a method for expanding an arbitrary electromagnetic field at one reference surface in a series of small Gaussian beams or beamlets. As shown in Fig. 1, these beams are propagated to the output surface by the use of a matrix formalism and then summed to yield the output field distribution. As noted in Section 1, the results can be interpreted as a generalization of an early diffraction integral approach that incorporated ABCD matrix elements for the propagation medium.

A. Gaussian Beamlet Concept

Huygen's wavelets are off-axis spherical waves that start with vanishingly small radii of curvature. In the conventional Gaussian beam formalism of Eq. (9) such waves can be represented by the limit

$$\frac{1}{q_{x1}} \rightarrow \infty, \quad (19)$$

with a similar equation for $1/q_{y1}$. We refer to the beam in this limit as a beamlet. At a given plane the electric field of an off-axis Gaussian beamlet can be written as

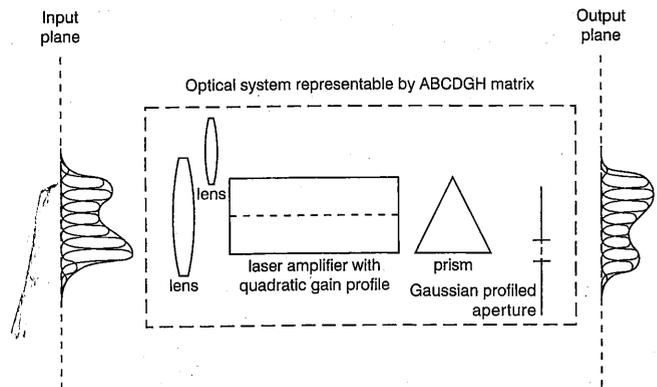


Fig. 1. An input light beam is decomposed into an infinite number of Gaussian beams with infinitely small width (referred to herein as beamlets). These beamlets are then propagated through a system that is represented in terms of complex ABCDGH matrices and summed to yield the output field distribution.

$$E'_{in} = E'_1 \exp \left\{ -i \left[\frac{Q_{x1}}{2} (x - x_0)^2 \right] - i \left[\frac{Q_{y1}}{2} (y - y_0)^2 \right] \right\} \quad (20)$$

$$= E'_1 \exp \left[-i \left(\frac{Q_{x1}}{2} x^2 - Q_{x1} x_0 x + \frac{Q_{x1}}{2} x_0^2 \right) - i \left(\frac{Q_{y1}}{2} y^2 - Q_{y1} y_0 y + \frac{Q_{y1}}{2} y_0^2 \right) \right], \quad (21)$$

where E'_1 incorporates all of the phase and amplitude terms at that plane. If this result is compared with Eq. (2), one finds that the displacement and phase parameters of the beamlet can be written as

$$S_{x1} = -\frac{\beta_0 x_0}{q_{x1}}, \quad (22)$$

$$P_1 = \frac{\beta_0 x_0^2}{2q_{x1}} + \frac{\beta_0 y_0^2}{2q_{y1}}. \quad (23)$$

B. Gaussian Beamlet Propagation

The propagation of conventional Gaussian beams, includ-

$$E'_{out} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \frac{\exp \left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y} \right) \right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} dx_0 dy_0. \quad (30)$$

ing the Gaussian beamlets of interest here, is governed by the Kogelnik transformation given above as Eq. (13). In the limit of large $1/q_{x1}$ Eq. (13) becomes

$$\frac{1}{q_{x2}} = \frac{D_x}{B_x}. \quad (24)$$

Combining Eqs. (16) and (22), one finds that the displacement parameter transformation for an off-axis Gaussian beam can be written as

$$S_{x2} = -\frac{\beta_0 x_0/q_{x1}}{A_x + B_x/q_{x1}}. \quad (25)$$

In the limit of large $1/q_{x1}$ Eq. (25) becomes

$$S_{x2} = -\frac{\beta_0 x_0}{B_x}. \quad (26)$$

Similarly, Eqs. (18), (22), and (23) may be combined to produce the phase parameter transformation

$$\exp(-iP_2) = \frac{\exp \left[-i \frac{\beta_0}{2} \left(\frac{x_0^2}{q_{x1}} + \frac{y_0^2}{q_{y1}} \right) \right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} \times \exp \left[i \frac{\beta_0}{2} \left(\frac{x_0^2}{q_{x1}} \frac{1}{1 + q_{x1} A_x/B_x} + \frac{y_0^2}{q_{y1}} \frac{1}{1 + q_{y1} A_y/B_y} \right) \right]. \quad (27)$$

It is now helpful to expand binomially the arguments of the second exponential in Eq. (27). When this is done in the limit of large $1/q_{x1}$ and $1/q_{y1}$, the leading terms in the exponentials cancel and the equation becomes

$$\exp(-iP_2) = \frac{\exp \left[-i \frac{\beta_0}{2} \left(\frac{x_0^2 A_x}{B_x} + \frac{y_0^2 A_y}{B_y} \right) \right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}}. \quad (28)$$

Equations (2), (24), (26), and (28) may be combined to produce the non-plane-wave part of the output field that is due to a single beamlet:

$$E'_2 = E'_1 \frac{\exp \left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y} \right) \right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}}. \quad (29)$$

The plane-wave part of the beamlet will be restored at the conclusion of this calculation. As expected, the output field given by Eq. (29) for a single beamlet is vanishingly small, since the denominator approaches infinity.

C. The Diffraction Integral

The total output electric field from a continuous distribution of beamlets is

In general, the form of the distribution function $\rho(x_0, y_0)$ is not known in advance. However, this function is derivable from the initial field distribution. The relationship may be obtained by using the definition of the input field:

$$E'_{in} \equiv E'_{out} (A_x = A_y = A_0 \rightarrow 1, B_x = B_y = B_0 \rightarrow 0, D_x = D_y = D_0 \rightarrow 1). \quad (31)$$

With this substitution the input field distribution may be written in terms of $\rho(x_0, y_0)$:

$$E'_{in} = \frac{1}{(A_0 + B_0/q_{x1})^{1/2} (A_0 + B_0/q_{y1})^{1/2}} \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \exp \left\{ -i \frac{\beta_0}{2B_0} [(x_0 - x)^2 + (y_0 - y)^2] \right\} dx_0 dy_0. \quad (32)$$

In this limit of small B_0 the function $\rho(x_0, y_0)$ may be removed from the integral, and Eq. (32) reduces to

$$\begin{aligned}
 E'_{in} &= \frac{1}{(A_0 + B_0/q_{x1})^{1/2}(A_0 + B_0/q_{y1})^{1/2}} \rho(x, y) \\
 &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left\{-i \frac{\beta_0}{2B_0} [(x_0 - x)^2 \right. \\
 &\left. + (y_0 - y)^2]\right\} dx_0 dy_0 \\
 &= \frac{1}{(A_0 + B_0/q_{x1})^{1/2}(A_0 + B_0/q_{y1})^{1/2}} \rho(x, y) \\
 &\times \left(-i \frac{2\pi B_0}{\beta_0}\right). \tag{33}
 \end{aligned}$$

Solving Eq. (33) for $\rho(x, y)$ and substituting the result into Eq. (30) yields

$$\begin{aligned}
 E'_{out} &= \frac{(A_0 + B_0/q_{x1})^{1/2}(A_0 + B_0/q_{y1})^{1/2}}{(A_x + B_x/q_{x1})^{1/2}(A_y + B_y/q_{y1})^{1/2}} \left(i \frac{\beta_0}{2\pi B_0}\right) \\
 &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E'_{in}(x_0, y_0) \\
 &\times \exp\left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} \right. \right. \\
 &\left. \left. + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y}\right)\right] dx_0 dy_0. \tag{34}
 \end{aligned}$$

In the limit of large $1/q_{x1}$ and $1/q_{y1}$, Eq. (34) reduces to

$$\begin{aligned}
 E'_{out} &= i \frac{\beta_0}{2\pi(B_x B_y)^{1/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E'_{in}(x_0, y_0) \\
 &\times \exp\left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} \right. \right. \\
 &\left. \left. + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y}\right)\right] dx_0 dy_0. \tag{35}
 \end{aligned}$$

This result can also be written in the more compact form

$$\begin{aligned}
 E'_{out}(x, y) &= \exp(-i\phi) \\
 &\times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} K_{ABCD}(x_0, y_0, x, y) \\
 &\times E'_{in}(x_0, y_0) dx_0 dy_0, \tag{36}
 \end{aligned}$$

where the exponential factor is included to restore the plane-wave part of the propagation and the diffraction kernel is

$$\begin{aligned}
 K_{ABCD}(x_0, y_0, x, y) &= \frac{i}{\lambda_m (B_x B_y)^{1/2}} \\
 &\times \exp\left[-i \frac{\pi}{\lambda_m} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} \right. \right. \\
 &\left. \left. + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y}\right)\right]. \tag{37}
 \end{aligned}$$

In writing this equation, we have used the relationship between the wave number and the wavelength in the medium, $\beta_0 = 2\pi/\lambda_m$. Equations (36) and (37) are equivalent to Eq. (30) of Ref. 3, but the more direct derivation given here is also more general in the sense that there is no restriction to real matrix elements.

4. ABCDGH DIFFRACTION INTEGRAL

The same method that has been described in Section 3 for analyzing the propagation of arbitrary paraxial field distributions through aligned complex optical systems can be generalized to misaligned systems by means of the recently described ABCDGH matrix formalism.¹² The Kogelnik transformation [Eq. (13)] is unchanged for misaligned systems, and in the limit of large $1/q_{x1}$ the beam parameter is again given by Eq. (24). Combining Eqs. (14) and (22), one finds that the displacement parameter transformation for an off-axis beam can be written as

$$S_{x2} = \frac{-\beta_0 x_0/q_{x1} + G_x + H_x/q_{x1}}{A_x + B_x/q_{x1}}. \tag{38}$$

In the limit of large $1/q_{x1}$ Eq. (38) becomes

$$S_{x2} = \frac{H_x - \beta_0 x_0}{B_x}. \tag{39}$$

Similarly, Eqs. (15), (22), and (23) may be combined to produce the phase parameter transformation

$$\begin{aligned}
 \exp(-iP_2) &= \frac{\exp[-i(\beta_0/2)(x_0^2/q_{x1} + y_0^2/q_{y1})]}{(A_x + B_x/q_{x1})^{1/2}(A_y + B_y/q_{y1})^{1/2}} \\
 &\times \exp\left[\frac{i}{2\beta_0 q_{x1}} \frac{\beta_0^2 x_0^2 - 2\beta_0 x_0(G_x q_{x1} + H_x) + (G_x^2 q_{x1}^2 + 2G_x H_x q_{x1} + H_x^2)}{1 + A_x q_{x1}/B_x}\right] \\
 &\times \exp\left[\frac{i}{2\beta_0 q_{y1}} \frac{\beta_0^2 y_0^2 - 2\beta_0 y_0(G_y q_{y1} + H_y) + (G_y^2 q_{y1}^2 + 2G_y H_y q_{y1} + H_y^2)}{1 + A_y q_{y1}/B_y}\right] \\
 &\times \exp\left[-i \frac{H_x}{2\beta_0} (-2\beta_0 x_0/q_{x1} + G_x + H_x/q_{x1})\right] \exp\left(-i \frac{J_x}{2\beta_0}\right) \\
 &\times \exp\left[-i \frac{H_y}{2\beta_0} (-2\beta_0 y_0/q_{y1} + G_y + H_y/q_{y1})\right] \exp\left(-i \frac{J_y}{2\beta_0}\right), \tag{40}
 \end{aligned}$$

where J_x is defined by

$$J_x \equiv \int_0^z \left(G_x \frac{dH_x}{dz'} - H_x \frac{dG_x}{dz'} \right) dz', \quad (41)$$

with a similar equation for J_y . It is now helpful to expand binomially the denominators of the arguments of the second and third exponentials on the right-hand side of Eq. (40). In the limit of large $1/q_{x1}$ and $1/q_{y1}$ the leading terms in the exponentials cancel, and the equation becomes

$$\exp(-iP_2) = \frac{\exp\left\{-i \frac{\beta_0}{2} \left[\frac{(x_0 - H_y/\beta_0)^2 A_x}{B_x} + \frac{(y_0 - H_x/\beta_0)^2 A_y}{B_y} \right]\right\}}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} \times \exp[-i(G_x x_0 + G_y y_0)] \exp\left(-i \frac{G_x H_x + G_y H_y}{2\beta_0}\right) \exp\left(-i \frac{J_x + J_y}{2\beta_0}\right). \quad (42)$$

Equations (2), (24), (39), and (42) may be combined to produce the non-plane-wave part of the output field that is due to a single beamlet:

which is the same as Eq. (32). It follows by analogy with

the discussion in Section 3 of aligned $ABCD$ systems that the diffraction integral for general complex misaligned $ABCDGH$ systems can be written as

$$E'_2 = E'_1 \frac{\exp\left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y} \right)\right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} \exp\left[-i \left(\frac{H_x x}{B_x} + \frac{H_y y}{B_y} \right)\right] \times \exp\left[-i \left(x_0 - \frac{H_x}{2\beta_0} \right) \left(\frac{B_x G_x - A_x H_x}{B_x} \right) - i \left(y_0 - \frac{H_y}{2\beta_0} \right) \left(\frac{B_y G_y - A_y H_y}{B_y} \right)\right] \exp\left(-i \frac{J_x + J_y}{2\beta_0}\right). \quad (43)$$

Again, the plane-wave part of the beamlet will be restored at the conclusion of this calculation.

The total output electric field from a continuous distribution of beamlets is

$$E'_{\text{out}}(x, y) = \exp(-i\phi') \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \times K_{ABCDGH}(x_0, y_0, x, y) \times E'_{\text{in}}(x_0, y_0) dx_0 dy_0, \quad (47)$$

$$E'_{\text{out}} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \rho(x_0, y_0) \frac{\exp\left[-i \frac{\beta_0}{2} \left(\frac{D_x x^2 - 2x_0 x + A_x x_0^2}{B_x} + \frac{D_y y^2 - 2y_0 y + A_y y_0^2}{B_y} \right)\right]}{(A_x + B_x/q_{x1})^{1/2} (A_y + B_y/q_{y1})^{1/2}} \exp\left[-i \left(\frac{H_x x}{B_x} + \frac{H_y y}{B_y} \right)\right] \times \exp\left[-i \left(x_0 - \frac{H_x}{2\beta_0} \right) \left(\frac{B_x G_x - A_x H_x}{B_x} \right) - i \left(y_0 - \frac{H_y}{2\beta_0} \right) \left(\frac{B_y G_y - A_y H_y}{B_y} \right)\right] \exp\left(-i \frac{J_x + J_y}{2\beta_0}\right) dx_0 dy_0. \quad (44)$$

The distribution function $\rho(x_0, y_0)$ is derivable from the initial field distribution. The relationship may again be obtained by using the definition of the input field:

$$E'_{\text{in}} \equiv E'_{\text{out}}(A_x = A_y = A_0 \rightarrow 1, B_x = B_y = B_0 \rightarrow 0, D_x = D_y = D_0 \rightarrow 1, G_x = G_y = G_0 \rightarrow 0, H_x = H_y = H_0 \rightarrow 0). \quad (45)$$

where the new diffraction kernel is

$$K_{ABCDGH}(x_0, y_0, x, y) = K_{ABCD}(x_0, y_0, x, y) \exp\left[-i \left(\frac{H_x x}{B_x} + \frac{H_y y}{B_y} \right)\right] \times \exp\left[-ix_0 \left(\frac{B_x G_x - A_x H_x}{B_x} \right) - iy_0 \left(\frac{B_y G_y - A_y H_y}{B_y} \right)\right] \quad (48)$$

With this substitution Eq. (44) can be written as

and the old kernel K_{ABCD} is still given by Eq. (37). The new phase exponential in Eq. (47) is

$$\begin{aligned} \exp(-i\phi') = & \exp\left[i\frac{H_x}{2\beta_0}\left(\frac{B_x G_x - A_x H_x}{B_x}\right)\right. \\ & \left.+ i\frac{H_y}{2\beta_0}\left(\frac{B_y G_y - A_y H_y}{B_y}\right)\right] \\ & \times \exp\left(-i\frac{J_x + J_y}{2\beta_0}\right)\exp(-i\phi). \quad (49) \end{aligned}$$

Equations (47)–(49) govern the propagation of an arbitrary input field through any misaligned complex optical system that is representable by 3×3 $ABCDGH$ beam matrices.

5. DISCUSSION

Diffraction integrals were originally used to study the propagation of arbitrarily profiled light beams through optical systems that consist of a thin optical element with some spatial transmission function and free space. These types of diffraction calculations are still pervasive, and the Fresnel and the Fraunhofer regions are of particular importance. In 1969 Baues generalized the diffraction integral method to include any optical system representable by an $ABCD$ matrix. In the present paper the diffraction integral for the Fresnel region has been obtained to study the propagation of arbitrarily profiled light beams through optical systems that consist of a thin optical element with arbitrary spatially varying magnitude and phase transmission functions and an extended optical system representable by $ABCDGH$ matrices. It has been noted that Huygens's integral in the Fresnel approximation and the paraxial wave equation represent the same mathematical and physical approximations.¹⁵ Thus the same optical elements that are found to propagate Gaussian beams in the paraxial approximation must also be usable in the diffraction method. Examples of such elements are lenses, prisms, laser amplifiers with linear or quadratic gain profiles, Gaussian-profiled apertures, and exponential-profiled apertures.¹²

In addition to optical systems, diffraction integrals are used to study laser resonator modes.¹ Conventional Gaussian beam analyses are used for laser resonators that contain only optical elements representable by $ABCD$ matrices. However, some important resonators incorporate Fresnel zone plates, phase plates, holographic diffractive optical elements, axicons, waxicons, variable-reflectivity mirrors, etc. The most common application of the diffraction integral to laser resonator mode studies involves finite-aperture effects. Indeed, an empty resonator with finite-aperture mirrors was studied originally by Fox and Li.¹ The diffraction integral obtained here may be used to study the modes of laser resonators with any combination of misaligned complex optical elements representable by $ABCDGH$ matrices.

Mode summation is an alternative strategy to study these optical systems and laser resonators. In mode-summation analyses an input light beam is decomposed into polynomial-Gaussian beams by the use of the orthogonality properties of these higher-order modes.

Since the transformation properties of these polynomial-Gaussian beams are known,¹⁶ the output properties of each mode may be obtained. The total output beam is the superposition of each of the individual output modes. In concept, this procedure is somewhat similar to the diffraction integral method studied here. However, in practice, one method may have advantages over the other one, depending on the form of the input beam and the optical system configuration.

Some of the novelty of the results mentioned here involves the extension of the previous results to misaligned optical systems. By misaligned, we mean optical systems that are representable by an $ABCDGH$ matrix but not by an $ABCD$ matrix (i.e., an optical system that does not have $G_x = G_y = 0$ and $H_x = H_y = 0$). As an example, consider the exponential aperture. A Gaussian light beam incident upon an exponential aperture is displaced by some amount depending on the spot size of the input beam.¹² Thus there is no globally defined axis, even when the aperture is aligned in the conventional sense. Thus, by our definition, the exponential aperture is inherently misaligned—it has an $ABCDGH$ matrix representation but no $ABCD$ matrix representation.

The diffraction integral method may be interpreted as a Fourier transform. Thus Eq. (47) may be rewritten as

$$\begin{aligned} E'_{\text{out}}(x, y) = & \frac{\exp(-i\phi')}{\lambda_m(B_x B_y)^{1/2}} \exp\left[-i\left(\frac{\pi D_x}{\lambda_m B_x} x^2 + \frac{H_x}{B_x} x\right)\right] \\ & \times \exp\left[-i\left(\frac{\pi D_y}{\lambda_m B_y} y^2 + \frac{H_y}{B_y} y\right)\right] \\ & \times \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} E'_{\text{in}}(x_0, y_0) \\ & \times \exp\left[-i\frac{\pi}{\lambda_m}\left(\frac{A_x}{B_x} x_0^2 + \frac{A_y}{B_y} y_0^2\right)\right] \\ & \times \exp\left[-i\left(\frac{B_x G_x - A_x H_x}{B_x} - \frac{2\pi x}{\lambda_m B_x}\right)x_0\right] \\ & \times \exp\left[-i\left(\frac{B_y G_y - A_y H_y}{B_y} - \frac{2\pi y}{\lambda_m B_y}\right)y_0\right] dx_0 dy_0. \quad (50) \end{aligned}$$

This may be written as a Fourier transform:

$$\begin{aligned} E'_{\text{out}}(x, y) = & \frac{\exp(-i\phi')}{\lambda_m(B_x B_y)^{1/2}} \exp\left[-i\left(\frac{\pi D_x}{\lambda_m B_x} x^2 + \frac{H_x}{B_x} x\right)\right] \\ & \times \exp\left[-i\left(\frac{\pi D_y}{\lambda_m B_y} y^2 + \frac{H_y}{B_y} y\right)\right] \\ & \times \mathcal{F}\left\{E'_{\text{in}}(x_0, y_0)\right\} \\ & \times \exp\left[-i\frac{\pi}{\lambda_m}\left(\frac{A_x}{B_x} x_0^2 + \frac{A_y}{B_y} y_0^2\right)\right], \quad (51) \end{aligned}$$

where, by definition of the Fourier transform,

$$\mathcal{F}\{f(x_0, y_0)\} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x_0, y_0) \exp[-i2\pi(f_x x_0 + f_y y_0)] dx_0 dy_0. \quad (52)$$

The spatial frequencies f_x and f_y in Eq. (52) are defined by

$$f_x = \frac{B_x G_x - A_x H_x}{B_x} - \frac{x}{\lambda B_x}, \quad (53)$$

$$f_y = \frac{B_y G_y - A_y H_y}{B_y} - \frac{y}{\lambda B_y}. \quad (54)$$

From Eq. (51) it may be seen that an optical system is purely Fourier transforming when $A_x = A_y = 0$, $D_x = D_y = 0$, and $H_x = H_y = 0$. One may also rewrite the diffraction integral in cylindrical coordinates and obtain a Hankel-type transform. However, the system misalignments of interest here would remove the desired cylindrical symmetry.

6. SUMMARY

The Gaussian beam matrix method is a simple and elegant way to analyze and design complex optical systems and laser resonators. However, many recent optical designs are incorporating diffractive optical elements that do not fit within the beam matrix formalism. Examples of such elements include Fresnel zone plates, phase plates, holographic diffractive optical elements, axicons, waxicons, variable-reflectivity mirrors, and apertures. A single diffraction integral developed here [Eq. (47)] can be used to study diffraction from one of these elements into any optical system representable by $ABCDGH$ matrices. This includes optical systems whose individual optical elements are complex and misaligned.

ACKNOWLEDGMENT

This work was supported in part by the National Science Foundation under grant ECS-9014481.

REFERENCES

1. A. G. Fox and T. Li, "Resonant modes in a maser interferometer," *Bell Syst. Tech. J.* **40**, 453-458 (1961).
2. H. Kogelnik, "Imaging of optical modes—resonators with internal lenses," *Bell Syst. Tech. J.* **44**, 455-494 (1965).
3. P. Baues, "Huygens' principle in inhomogeneous, isotropic media and a general integral equation applicable to optical resonators," *Opto-Electronics* **1**, 37-44 (1969).
4. P. Baues, "The connection of geometrical optics with the propagation of Gaussian beams and the theory of optical resonators," *Opto-Electronics* **1**, 103-118 (1969).
5. S. A. Collins, "Lens-system diffraction integral written in terms of matrix optics," *J. Opt. Soc. Am.* **60**, 1168-1177 (1970).
6. L. W. Casperson and A. Yariv, "The Gaussian mode in optical resonators with a radial gain profile," *Appl. Phys. Lett.* **12**, 355-357 (1968).
7. L. W. Casperson and S. D. Lunnam, "Gaussian modes in high loss laser resonators," *Appl. Opt.* **14**, 1193-1199 (1975).
8. J. A. Arnaud, "Mode coupling in first-order optics," *J. Opt. Soc. Am.* **61**, 751-758 (1971).
9. M. Nazarathy and J. Shamir, "First-order optics: operator representation for systems with loss or gain," *J. Opt. Soc. Am.* **72**, 1398-1408 (1982).
10. A. E. Siegman, *Lasers* (University Science Books, Mill Valley, Calif. 1985), pp. 792-797.
11. E. M. Wright and J. C. Garrison, "Path-integral derivation of the complex $ABCD$ Huygens integral," *J. Opt. Soc. Am. A* **4**, 1751-1755 (1987).
12. A. A. Tovar and L. W. Casperson, "Generalized beam matrices: Gaussian beam propagation in misaligned complex optical systems," *J. Opt. Soc. Am. A* **12**, 1522-1533 (1995).
13. L. W. Casperson, "Gaussian light beams in inhomogeneous media," *Appl. Opt.* **2**, 2434-2441 (1973).
14. A. A. Tovar and L. W. Casperson, "Gaussian beam optical systems with high gain or high loss media," *IEEE Trans. Microwave Theory Tech.* **43**, 1857-1862 (1995).
15. Ref. 10, p. 634.
16. A. A. Tovar and L. W. Casperson, "Generalized beam matrices. II. Mode selection in lasers and periodic misaligned complex optical systems," *J. Opt. Soc. Am. A* **13**, 90-96 (1996).