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Electromagnetic modes of an inhomogeneous sphere*

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Vector wave solutions are derived for the electromagnetic modes of a sphere having radial variations of the refractive index and gain. For some inhomogeneity models, confined-mode solutions exist. The results are applicable to the refracting atmospheres of stars and planets.

Index Headings: Refractive index; Atmospheric optics.

The propagation of electromagnetic beams has been analyzed for various types of inhomogeneous media. Two familiar examples include propagation along graded-slab and cylindrical dielectric structures. The slab geometry is of practical interest because of its application to semiconductor-junction lasers. More recently, slab and rectangular waveguide channels have been analyzed for integrated optics. Substantial effort has also gone into the analysis of beam propagation along cylindrically symmetric media, and optical fibers of this type are leading candidates for single-mode optical communication.¹ Similar profiles occur in laser media; in the simplest cases of quadratic index or gain variations, the resultant beam modes are characterized by Hermite-gaussian or Laguerre-gaussian functions.² A few other geometries have also been considered. The purpose of this work is to derive the electromagnetic modes of a sphere that has radial inhomogeneities. The initial equations are exact and general and may be applied to all spherical configurations. However, in the detailed solutions, emphasis is placed on situations that involve a spherical refracting or amplifying shell with a radius that is large compared to the shell's thickness.

There are several applications for these results. For example, unconventional lasers can be visualized in which the amplifying medium is localized in a spherical shell. This might be an efficient arrangement when a point-like pump source, such as a fusion reaction is available. The principal example in this work, however, involves the electromagnetic modes of stellar atmospheres. The gaseous atmosphere of a star or planet always exhibits large-scale optical inhomogeneity. The decrease of density of a neutral gas with height above a stellar surface causes a decrease of refractive index; this tends to confine radiation. On the other hand, the occurrence of free electrons in the atmospheric plasma acts to depress the refractive index, according to $n \approx 1 - \lambda^2 N_e e^2 / (2\pi m c^2)^{-1}$, where N_e is the electron density, λ is the wavelength of the electromagnetic field, e is the electron charge, m is the electron mass, and c is the speed of light. Unless the electrons are concentrated in an ionosphere, they tend to refract radiation out of the atmosphere. Another confining effect that is important with highly condensed stellar objects is the relativistic deflection of light by the gravitational field. This effect is well known theoretically and from solar-eclipse data. For example, for weak gravitational fields the effective correction of the index of refraction is $n_g = 2GM/(c^2 r)^{-1}$, where G is the uni-

versal gravitation constant, M is the mass, c is the velocity of light, and r is the distance from the center of the gravitating object.³

In addition to these effects, it is expected that the high levels of optical and collisional excitation may lead to a radially varying gain. Sunlight is used as an efficient optical-pump source for earth-bound lasers,⁴ and OH and H₂O maser action in the nebulae surrounding stars is also well known.⁵ Other unconventional plasma-pumping techniques have also been proposed for obtaining stimulated emission near a magnetized star.⁶ Thus the presence of gain within an atmosphere is not unexpected. Such gain can have a profound effect on the electromagnetic modes, and laser action with feedback may be possible.

In Sec. I, the exact vector equations governing the field components are reduced rigorously to a set of ordinary differential equations. The simplest beam-mode solutions consist of uniform traveling waves circulating in the ϕ direction about the equator of the spherical object. The field variations in the θ direction are governed by Hermite-gaussian functions. In Sec. II, the radial equation is solved for certain plausible forms for the radial variation of the refractive index and the gain.

I. DERIVATION OF MODE EQUATIONS

The vector electromagnetic modes of an inhomogeneous sphere may be found as solutions of Maxwell's equations

$$\nabla \times \vec{E} = \mu_0 \frac{\partial \vec{H}}{\partial t} - \mu_0 \frac{\partial \vec{M}}{\partial t}, \tag{1}$$

$$\nabla \times \vec{H} = \epsilon_0 \frac{\partial \vec{E}}{\partial t} + \frac{\partial \vec{P}}{\partial t} + \sigma \vec{E}. \tag{2}$$

If a harmonic time dependence is assumed in the form

$$\vec{E} = |\vec{E}'| \cos(\omega t + \phi_E) = \text{Re } \vec{E}' \exp(i\omega t),$$

$$\vec{H} = |\vec{H}'| \cos(\omega t + \phi_H) = \text{Re } \vec{H}' \exp(i\omega t), \text{ etc.},$$

then the equations for the complex field amplitudes reduce to

$$\nabla \times \vec{E}' = -i\omega \mu \vec{H}', \tag{3}$$

$$\nabla \times \vec{H}' = i\omega \epsilon \vec{E}'. \tag{4}$$

Here the permittivity $\epsilon(r)$ is assumed to be complex to account for the conductivity σ and also because in an absorbing or amplifying medium the polarization \vec{P} is not in phase with the electric field \vec{E} . Similarly the

permeability $\mu(r)$ is complex to account for out-of-phase components of the magnetization \vec{M} . In free space, ϵ and μ reduce, respectively, to ϵ_0 and μ_0 . For a general inhomogeneous medium Eqs. (3) and (4) may be combined to yield wave equations for the electric and magnetic fields; the results are

$$\nabla \times \nabla \times \vec{E}' - \omega^2 \mu \epsilon \vec{E}' = (\nabla \mu / \mu) \times \nabla \times \vec{E}', \quad (5)$$

$$\nabla \times \nabla \times \vec{H}' - \omega^2 \mu \epsilon \vec{H}' = (\nabla \epsilon / \epsilon) \times \nabla \times \vec{H}'. \quad (6)$$

Equation (5) is really three coupled equations for the components of the vector electric field. For the problem of a spherical atmosphere, the permittivity and permeability vary only with radius; this vector wave equation may be reduced to a scalar equation by means of the substitution⁷

$$\vec{E}'^{(1)} = \nabla \times (\vec{i}_r r \psi^{(1)}), \quad (7)$$

where \vec{i}_r is a unit vector in the radial direction. The result may be written

$$\nabla^2 \psi^{(1)} + \omega^2 \mu \epsilon \psi^{(1)} = \frac{1}{\mu} \frac{d\mu}{dr} \frac{\partial \psi^{(1)}}{\partial r}. \quad (8)$$

For this set of fields, $\vec{E}'^{(1)}$ has no radial component and the corresponding vector $\vec{H}'^{(1)}$ may be found from Eq. (3). In a similar way, we may use the substitution

$$\vec{H}'^{(2)} = \nabla \times (\vec{i}_r r \psi^{(2)}) \quad (9)$$

in Eq. (6) to obtain a set of fields $\vec{E}'^{(2)}$ and $\vec{H}'^{(2)}$ in which the radial component of the magnetic vector vanishes. The scalar equation for this case is

$$\nabla^2 \psi^{(2)} + \omega^2 \mu \epsilon \psi^{(2)} = \frac{1}{\epsilon} \frac{d\epsilon}{dr} \frac{\partial \psi^{(2)}}{\partial r}. \quad (10)$$

Related scalar equations have been used to describe scattering from spheres.⁸

Equation (8) may be separated into three ordinary differential equations by use of the substitution $\psi^{(1)} = \Phi(\phi) \Theta(\theta) R(r)$. The results are

$$\frac{d^2 \Phi}{d\phi^2} + m^2 \Phi = 0, \quad (11)$$

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left[l(l+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta = 0, \quad (12)$$

$$\frac{1}{r} \frac{d}{dr} \left(rR \right) - \frac{1}{\mu} \frac{d\mu}{dr} \frac{dR}{dr} + \left[\omega^2 \mu \epsilon - \frac{l(l+1)}{r^2} \right] R = 0. \quad (13)$$

The lowest-order modes would be expected to involve fields that simply propagate around some great circle of the sphere. Therefore, we choose for Φ the traveling-wave solution

$$\Phi = \Phi_0 \exp(-im\phi), \quad (14)$$

where m is an integer. Other sets of modes that describe θ -directed propagation, for example, are also possible.

The solutions of Eq. (12) involve the associated Legendre functions, provided that l is an integer that satisfies $l \geq |m|$. Although these solutions are correct, more-useful expressions can be obtained for the limit in which $|m|$ and l are much greater than unity. This limit is appropriate for a typical star, where the di-

mensions are much larger than any wavelength of interest. If the energy is localized near the equator, a useful change of variables in Eq. (12) is $\theta' = \pi/2 - \theta$, so that latitude is measured from the equator. Thus, the equation reduces to

$$\frac{d^2 \Theta}{d\theta'^2} - \tan \theta' \frac{d\Theta}{d\theta'} + \left[l(l+1) - \frac{m^2}{\cos^2 \theta'} \right] \Theta = 0. \quad (15)$$

To obtain confined modes, we may try

$$\Theta(\theta') = A(\theta') \exp(-iQ_\theta \theta'^2/2), \quad (16)$$

which leads to

$$\begin{aligned} \frac{d^2 A}{d\theta'^2} - (\tan \theta' + 2iQ_\theta \theta') \frac{dA}{d\theta'} \\ + \left[l(l+1) - \frac{m^2}{\cos^2 \theta'} - iQ_\theta - Q_\theta^2 \theta'^2 \right] A = 0. \end{aligned} \quad (17)$$

For small values of θ' , the trigonometric functions may be expanded, keeping only the leading terms. The result is

$$\begin{aligned} \frac{d^2 A}{d\theta'^2} - (1 + 2iQ_\theta \theta') \frac{dA}{d\theta'} \\ + \left[l(l+1) - m^2(1 + \theta'^2) - iQ_\theta - Q_\theta^2 \theta'^2 \right] A = 0. \end{aligned} \quad (18)$$

If we set the quadratic terms in θ' equal to zero, we obtain

$$\frac{d^2 A}{d\theta'^2} - (1 + 2iQ_\theta \theta') \frac{dA}{d\theta'} + [l(l+1) - m^2 - iQ_\theta] A = 0 \quad (19)$$

and the constraint $Q_\theta = -|m|i$. Then, from Eq. (16) it follows that the $1/e$ radian beam width (spot size) of the fundamental mode ($l = |m|$) is $w_\theta = (2/|m|)^{1/2}$.

Higher-order modes can also be found. Using $w_\theta \ll 1$ and introducing the new variable $x = 2^{1/2} \theta' / w_\theta$ reduces Eq. (19) to

$$\frac{d^2 A}{dx^2} - 2x \frac{dA}{dx} + \frac{l(l+1) - |m|(|m|+1)}{|m|} A = 0. \quad (20)$$

This may be written, approximately, as

$$\frac{d^2 A}{dx^2} - 2x \frac{dA}{dx} + 2nA = 0, \quad (21)$$

where we have introduced $n = l - |m|$ and used again $|m| \gg 1$. The solutions of Eq. (21) are the Hermite polynomials H_n . Thus the θ' dependence of the fields is expressible in terms of the Hermite-gaussian functions

$$\Theta(\theta') = \Theta_0 H_n(2^{1/2} \theta' / w_\theta) \exp(-\theta'^2 / w_\theta), \quad (22)$$

which are analogous to the modes of conventional lasers and lens waveguides.²

The θ' intensity dependence of some typical low-order modes is shown schematically in Fig. 1. The intensity is found from $I = (\epsilon/\mu)^{1/2} \vec{E}' \cdot \vec{E}'^*/2$. The modes are normalized with respect to total power; in this example, the spot size is $w_\theta = 7.07^\circ = 0.123$ rad, corresponding to the value $m \approx 131$. This means, in effect, that the circumference of the star is approximately 131 wavelengths; in most situations the value of m would be larger.

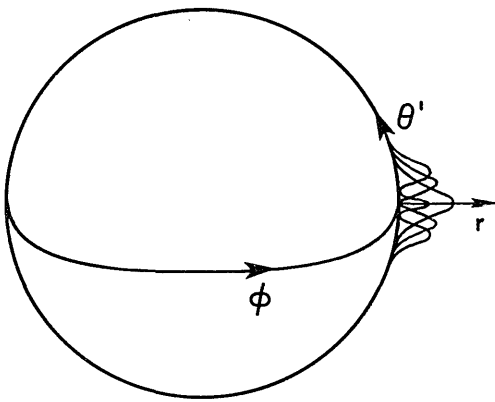


FIG. 1. Schematic drawing showing the θ' variation of normalized Hermite-gaussian modes propagating around the equator of a star. The modes shown correspond to the index values $n=0, 1, 2$.

The solutions of the radial equations depend on the forms of $\epsilon(r)$ and $\mu(r)$. Numerical solutions of Eq. (13) can always be obtained for arbitrary radial variations, but it is useful to consider also some specific analytical results. We treat first the behavior of the fields at large radii, where no spatial variations exist. Then Eq. (13) reduces to

$$r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} + [r^2 k_0^2 - l(l+1)] R = 0, \quad (23)$$

where $k_0^2 = \omega^2 \mu \epsilon$ is a constant. In terms of the dimensionless variable $\rho = k_0 r$, Eq. (23) is

$$\rho^2 \frac{d^2 R}{d\rho^2} + 2\rho \frac{dR}{d\rho} + [\rho^2 - l(l+1)] R = 0, \quad (24)$$

which is the standard form of the spherical Bessel equation.⁹ Thus the solutions for outward-traveling waves involve the spherical Hankel functions

$$h_l^{(2)}(\rho) = (\pi/2\rho)^{1/2} H_{l+1/2}^{(2)}(\rho). \quad (25)$$

For values of ρ greater than l , the Hankel functions may be replaced by their asymptotic values

$$\lim_{\rho \rightarrow \infty} h_l^{(2)}(\rho) = \rho^{-1} \exp[-i(\rho - l\pi/2 - \pi/2)]. \quad (26)$$

Because the energy density is proportional to $\vec{E}' \cdot \vec{E}'^*$, it follows from Eq. (26) that the energy varies inversely as the square of the radius. The behavior of the phase fronts follows from Eqs. (14) and (26), and the total phase is

$$P = m\phi + \rho - l\pi/2 - \pi/2. \quad (27)$$

Thus a surface of constant phase is described by

$$\phi = -\rho/m + \text{const.} \quad (28)$$

A plot of some typical phase fronts from Eq. (28) is shown in Fig. 2. These archimedean spirals are not extended inside of $\rho = m$, because there the asymptotic form of the Hankel functions is not valid, and also a detailed description of the energy source is required. An interesting consequence of these results is that, to an observer at a distance large compared to $\rho = m$, the

apparent position of the sphere is displaced by m from its actual position. In the next section, analytic solutions of the radial equation are obtained for the fields in the vicinity of a stellar surface for specific assumed profiles of the gain and refractive index. From the solutions of the scalar equation the vector field components follow immediately by means of Eqs. (7) and (9).

II. SOLUTION OF THE RADIAL EQUATION

Many profiles are possible for the refractive index and gain in the vicinity of a star. Under some conditions, these profiles can lead to confined electromagnetic modes. The simplest test for optical confinement follows from a ray analysis for radially inhomogeneous media.¹⁰ Ray confinement at a radius r_0 is assured by the condition

$$\left. \frac{dn}{dr} \right|_{r_0} \leq -\frac{n(r_0)}{r_0}. \quad (29)$$

These rays are attenuated due to leakage of radiation into the surrounding medium.¹¹ For highly condensed objects, such as dwarf stars, the gas density gradient may lead to such ray confinement. Also for neutron stars and black holes, the relativistic effects become severe. In addition, inhomogeneities of the gain or loss can strongly affect the behavior of electromagnetic fields. In regions of spatially varying gain, ray analysis is not useful.

As an example of an object that has strong refraction effects, we consider briefly the planet Venus. Based on results of the Venera 4 and Mariner 5 space flights, the index of refraction of the lower atmosphere of Venus is given by $n - 1 = 4.82 \times 10^{-3} \exp(-0.066h)$ where h is the height in kilometers.¹² Thus, the value of the gradient at $h=0$ is $dn/dh = -3.18 \times 10^{-4} \text{ km}^{-1}$. But $n(r)/r$ at the surface of Venus ($r_0 = 6052 \text{ km}$) has the value $1.66 \times 10^{-4} \text{ km}^{-1}$, so that, from Eq. (29), refraction trapping of rays occurs and low-loss modes may be possible.

The radial field distribution of the confined modes can always be found from Eq. (13) but, for complicated radi-

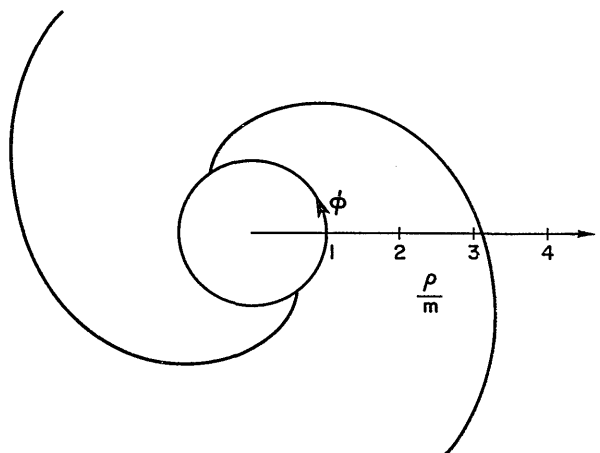


FIG. 2. Polar view of typical phase fronts of an equatorial mode.

al inhomogeneities, numerical methods are required. However, considerable insight can be obtained from analytical solutions for a general class of refraction profiles. The refractive behavior of the stellar atmosphere can always be expanded in a Taylor series in the vicinity of the field maximum. Thus the complex propagation constant $k^2 = \omega^2 \mu \epsilon$ near the radius r_0 can be expressed as

$$k^2 = k_0[k_0 - k_1(r - r_0) - k_2(r - r_0)^2], \quad (30)$$

where higher-order terms are assumed to be unimportant. This profile corresponds also to the case of an amplifying layer in the stellar atmosphere, because k may be complex. With this substitution, Eq. (13) reduces to

$$\frac{d^2 B}{dr^2} + k_0[k_0 - k_1(r - r_0) - k_2(r - r_0)^2]B - \frac{l(l+1)}{r^2}B = 0, \quad (31)$$

where $B = rR$ and the permeability is constant. We now introduce $r' = r - r_0$ and require that the thickness of the mode be much less than the radius of the star. Then the last term in Eq. (31) may be expanded to second order in r' ; with $l \gg 1$ we obtain

$$\frac{d^2 B}{dr'^2} + \left[\left(k_0^2 - \frac{l^2}{r_0^2} \right) - \left(k_0 k_1 - \frac{2l^2}{r_0^3} \right) r' - \left(k_0 k_2 + \frac{3l^2}{r_0^4} \right) r'^2 \right] B = 0. \quad (32)$$

In analogy with conventional gaussian-beam formalism, a useful substitution in Eq. (32) is¹³

$$B(r') = G(r') \exp[-i(Qr'^2/2 + Sr')]. \quad (33)$$

Equating the terms that are linear and quadratic in r' leads to the separation

$$Q^2 = -k_0 k_2 - 3l^2/r_0^4, \quad (34)$$

$$QS = -k_0 k_1/2 + l^2/r_0^3, \quad (35)$$

$$\frac{d^2 G}{dr'^2} - 2i(S + Qr') \frac{dG}{dr'} + \left(k_0^2 - \frac{l^2}{r_0^2} \right) G - (S^2 + iQ)G = 0. \quad (36)$$

With the change of variables

$$\rho = (iQ)^{1/2}(r' + S/Q), \quad (37)$$

Eq. (36) becomes

$$\frac{d^2 G}{d\rho^2} - 2\rho \frac{dG}{d\rho} - \left[S^2 + iQ + \frac{l^2}{r_0^2} - k_0^2 \right] \frac{G}{iQ} = 0. \quad (38)$$

Explicit expressions for the parameters Q and S follow immediately from Eqs. (34) and (35). These exponential terms in Eqs. (33) imply a gaussian beam with its amplitude center displaced from r_0 by the amount $d_a = -S/iQ$, and its phase center displaced by $d_p = -S_r/Q_r$. Here the subscripts denote, respectively, the real and imaginary parts of the parameters S and Q . The real part of Q implies that the phase fronts are curved in the r' direction; in general, we can show that the real and imaginary parts of Q are related to the exponential beam spot size w and the phase-front curvature R by

$$Q = Q_r + iQ_i = \frac{k_0}{R} - i \frac{2}{w^2}. \quad (39)$$

The implications of these results and the solution of Eq. (38) are best illustrated by considering a special

case. For weak spatial variations, it follows from Eq. (30) that the propagation constant is

$$k = k_0 - k_1(r - r_0)/2 - k_2(r - r_0)^2/2. \quad (40)$$

If $|k_0 k_2| \gg 3l^2/r_0^4$ and $k_1 = 0$, Eqs. (34) and (35) reduce to

$$Q^2 = -k_0 k_2, \quad (41)$$

$$QS = l^2/r_0^3, \quad (42)$$

Following convention, we separate the components of the propagation constant into their real and imaginary parts according to $k = \beta + i\alpha$. The gain or loss per wavelength is small ($\beta_0 \gg \alpha_0$); for this example, we assume that there is no profile of the gain ($\alpha_2 = 0$). The solutions of Eqs. (41) and (42) are now

$$Q = -i(\beta_0 \beta_2)^{1/2}, \quad (43)$$

$$S = i l^2 r_0^{-3} (\beta_0 \beta_2)^{-1/2}. \quad (44)$$

From Eqs. (39) and (43) it follows that the phase-front curvature is infinite, and that the spot size is $w = 2^{1/2} (\beta_0 \beta_2)^{-1/4}$.

Equation (38) can also be readily solved. With the previous results this equation is

$$\frac{d^2 G}{d\rho^2} - 2\rho \frac{dG}{d\rho} - \left[\frac{-l^4}{r_0^6 \beta_0 \beta_2} + (\beta_0 \beta_2)^{1/2} + \frac{l^2}{r_0^2} - \beta_0^2 \right] (\beta_0 \beta_2)^{-1/2} G = 0 \quad (45)$$

and from Eq. (37) the radial coordinate is

$$\rho = (\beta_0 \beta_2)^{1/4} [r' + l^2 r_0^{-3} (\beta_0 \beta_2)^{-1}].$$

The solutions of Eq. (45) are the Hermite polynomials $H_p(\rho)$, provided that the integer p satisfies the condition

$$p = 2^{-1/2} \left[-\frac{l^4}{r_0^6 \beta_0 \beta_2} + (\beta_0 \beta_2)^{1/2} + \frac{l^2}{r_0^2} - \beta_0^2 \right] (\beta_0 \beta_2)^{-1/2}. \quad (46)$$

This equation is a constraint on l . If we assume $l \approx \beta_0 r_0$ in the l^4 term (to be checked later), we find

$$l^2 \approx \beta_0^2 r_0^2 [1 + \beta_0 r_0^{-2} \beta_2^{-1} + (2p - 1) \beta_0^{-2} (\beta_0 \beta_2)^{1/2}]$$

or

$$l \approx \beta_0 r_0 [1 + 2^{-1} \beta_0 \beta_2^{-1} r_0^{-2} + (p - \frac{1}{2}) \beta_0^{-2} (\beta_0 \beta_2)^{1/2}]. \quad (47)$$

Thus a quadratic index profile near the stellar surface leads to confined Hermite-gaussian modes that are similar to the waveguide modes of conventional quadratic lens-like media.² Because the index of refraction cannot actually extend to minus infinity, some radiation loss occurs at the surface of the guiding region. This leaking energy supplies the observable output fields shown in Fig. 2. The exact level of radiation loss depends on the details of the assumed refraction profile and is not of interest here. For a truncated parabolic profile the loss calculations can be carried out in much the same way as for curved optical fibers.¹⁴

As a reasonable numerical example of the previous results, we let the radius of the object be $r_0 = 10^8$ m; then, for a wavelength of $1 \mu\text{m}$, $\beta_0 \approx 2\pi/\lambda = 2\pi \times 10^6 \text{ m}^{-1}$. If the thickness of the focusing layer is about 10^6 m, it follows from Eq. (40) that $\beta_2 \approx 8 \times 10^{-12} \beta_0 = 1.6\pi \times 10^{-5} \text{ m}^{-3}$, and the approximation preceding Eq. (41) is valid. Therefore, from Eq. (47), the parameter l for the fundamental mode ($p = 0$) has the value

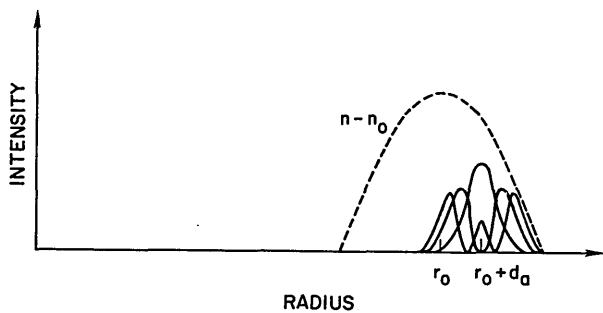


FIG. 3. Qualitative sketch of the normalized Hermite-gaussian radial intensity distribution for a parabolic index profile centered at the radius r_0 .

$$l \approx 2\pi \times 10^{14} [1 + 10^{-5}/1.6 - 2^{-1/2} \pi^{-1} \times 10^{-12}] .$$

Because the correction terms are small, the approximation preceding Eq. (47) is also justified for this example and $l \approx \beta_0 r_0$. From Sec. I, l is approximately equal to the azimuthal index m , and we get the reasonable result that the distance between the phase fronts at the radius r_0 of the index maximum is $2\pi/\beta_0 = \lambda$. The displacement of the beam is $d_a = -S_i/Q_i = \beta_0 r_0^{-1} \beta_2^{-1} = 1.25 \times 10^3$ m. Thus, the amplitude center of the beam is located at a larger radius than the maximum of the index profile. The spot size is $w = 2^{1/2} (\beta_0 \beta_2)^{-1/4} = 2^{-3/4} \pi^{-1/2}$ m so that the energy is concentrated in a region that is thin compared to the index maximum, and the Taylor-series expansion is valid.

The preceding results are shown qualitatively in Fig. 3. The Hermite-gaussian radial modes are pushed outward from the center of the parabolic refraction profile in the same way that a circulating particle would be held by centrifugal force against the outside of a parabolic potential well. The correspondence between ray and particle trajectories has recently been considered in detail.¹⁵ It follows from Eq. (7) that the beam modes are primarily linearly polarized in the θ direction. Another set of vector modes can be obtained from the solution of Eq. (10). The term $d\epsilon/dr$ can be ignored as long as the changes of permittivity per wavelength are small. Thus, these other scalar wave functions are the same as in the preceding case, but now the magnetic field has only a θ component.

Deflection of light by a refraction profile is a familiar phenomenon. Only refraction effects have been considered in the foregoing. However, a light beam can also be deflected in a medium where the refractive index is constant but the gain varies with position.¹³ Thus we can consider the existence of high-loss electromagnetic modes in a stellar atmosphere that contains an amplifying layer, even if the real refraction profile is insuf-

ficient for mode confinement. The preceding analysis, through Eq. (42), is valid for this case as well, but the detailed calculations are more cumbersome when both gain and refraction profiles are present.

III. CONCLUSION

The electromagnetic modes of a radially inhomogeneous sphere have been analyzed, and the results have been applied to refraction profiles of the type expected in stellar atmospheres. In the simplest cases, the radiation propagates around a great circle of the star with the phase fronts spiraling outward to a distant observer. Depending on the details of the atmospheric composition, conditions may be favorable for stimulated emission in the gaseous medium, and the result is laser oscillation in the electromagnetic mode. The refraction effects described here provide a mechanism for positive feedback. Laser or maser oscillation can have a drastic effect on the spectral and temporal properties of the observed stellar emission lines. Possible effects include emission line narrowing, optical relaxation oscillations, and mode-lock pulsations.

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