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Multisymplectic Theory of Balance Systems, I

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MULTISYMPLECTIC THEORY OF BALANCE SYSTEMS, I

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Abstract. In this paper we are presenting the theory of balance equations of the Continuum Thermodynamics (balance systems) in a geometrical form using Poincare-Cartan formalism of the Multisymplectic Field Theory. A constitutive relation $\mathcal{C}$ of a balance system $\mathcal{B}_{\mathcal{C}}$ is realized as a mapping between a (partial) 1-jet bundle of the configurational bundle $\pi : Y \to X$ and the extended dual bundle similar to the Legendre mapping of the Lagrangian Field Theory. Invariant (variational) form of the balance system $\mathcal{B}_{\mathcal{C}}$ is presented in three different forms and the space of admissible variations is defined and studied. Action of automorphisms of the bundle π on the constitutive mappings $\mathcal{C}$ is studied and it is shown that the symmetry group $Sym(\mathcal{C})$ of the constitutive relation $\mathcal{C}$ acts on the space of solutions of balance system $\mathcal{B}_{\mathcal{C}}$. Suitable version of Noether Theorem for an action of a symmetry group is presented with the usage of conventional multimomentum mapping. Finally, the geometrical (bundle) picture of the RET in terms of Lagrange-Liu fields is developed and the entropy principle is shown to be equivalent to the holonomicy of the current component of the constitutive section.

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Contents

1. Introduction.

This paper is the first part of a work where we are presenting the theory of balance equations of Thermodynamics of Continuum in the framework of the variational, Multisymplectic Field Theory ([1, 11, 8, 14, 24, 29]). In doing so we pursue, in this, first, part of the work the following main goals. The first is to formulate the theory of balance equations (balance systems) possibly closer to the classical

Lagrangian Field Theory in order to be able to use an extensive variety of tools developed in this theory for the study of balance systems. The second goal is to have a united mathematical scheme of several variants of Irreversible Thermodynamics that differs by the type of the domain of the constitutive relations of this theory. On one side is the scheme where the state space of basic fields for which the balance equations are present is chosen to be as small as possible and a constitutive relations of the theory is allowed to depend on all first derivatives of these fields (in this part of the work we consider only first order theories). See for instance [35] for the presentation of entropy principle in such a case. On the other end there is the Rational Extended Thermodynamics where all the necessary derivatives of the basic fields are included into the state space and the constitutive relations depend on the fields but not on their derivatives ([32, 31]). In between these two extreme positions there is a variety of situations where some derivatives of basic fields are included into the state space and some are not [17]. Quite often the choice of the derivatives (gradients of basic fields or their time derivatives) is related to the symmetry group of the described physical situation or to the covariance group required for the system of balance equations.

The second part of this work (in preparation, see also [44]) will be devoted to the study of the "entropy principle" - requirement that any solution of the balance system that includes all but the entropy balances satisfies to the entropy balance (such a requirement place a serious constraints to the constitutive relations of the balance system).

In the third part of the work we study the covariance principle for the balance systems - condition that the balance system would be covariant with respect to a (finite or infinite dimensional) Lie group. Such study was pioneered in the Green-Naghdy-Rivlin Theorem and later on studied by J.Marsden and T.Hughes, see [28, 54] and M.Silhavy ([51]).

This work was originated at the Conference Thermoconn 2005 in Messina after the lecture of Professor T. Ruggeri on the Rational Extended Thermodynamics and the discussion that I had there with Professor W. Muschik about the Entropy Principle.

Rational Extended Thermodynamics, (RET) which occupies an important place in this part of our work, was initiated in the works of I.Liu and I.Muller and developed by the T.Ruggeri and I.Muller (see [32, 45, 46, 47]). The formalism of RET "is elegant and appealing" ([17]) and it was very tempting to present it in a geometrical form following the framework of a Classical Field Theory ([1, 11, 43, 13]) and to implement the principal structures of RET in a natural geometrical way. Thus, we present in Appendix II a sketch of the formalism of RET.

In Section 3 we recall the basic structures of Multisymplectic Field Theory following ([24, 8]). The only new material here is the subsection 4.5 on the vertical contact structure in the space W_0 of the united multisymplectic scheme and the characterization of Legendre mappings generated by the Lagrangians in terms of this structure.

In Section 4 we define the partial 1-jet bundles $J^1_p(\pi)$ for a configurational bundle $\pi : Y^{(n+1)+m} \rightarrow X^{n+1}$ of m basic fields $y^i \in U$ over the physical or material space-time X^{n+1}. We discuss two examples of such jet bundles. One, $J^1_K(\pi)$, defined by a distribution $K \subset T(X)$ (or, with more details, by an almost product structure $T(X) = K \oplus K'$) on the space X, another, $J^1_S(\pi)$, for a case where

$K\oplus K' =<\partial_t> \oplus T(B)$, B being the material or physical space and the space U of the basic fields splits as the product corresponding to the type of (first order) derivatives that enters the constitutive relations. A study of more general types of partial jet-bundles including the jet bundles of higher order will be pursued in the second part of this work. In Section 6 we define and study the partial Cartan structure on the 1-jet bundles $J^1_p(\pi)$ of these two types.

In Sections 7 and 8 we study the prolongation of vector fields and connections to the partial 1-jet bundles following the similar prolongation procedures for the conventional 1-jet bundles ([48, 19, 24]).

In Section 9 we define a general constitutive relation $\mathcal{C}$ as a smooth mapping between the partial 1-jet bundle $J^1(\pi)$ and the (total) dual space $\tilde{Z}=\Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y$

$$\mathcal{C}(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)),$$

containing the current part $F^\mu_i dy^i\wedge\eta_\mu$ and the source part $\Pi_i dy^i\wedge\eta$. We introduce the covering constitutive relation $\widetilde{\mathcal{C}}$ defined by $\mathcal{C}$, extending the Legendre transformations defined by a Lagrangian form $L\eta$. We define the Poincare-Cartan form $\Theta_{\mathcal{C}} = F^\mu_i dy^i\wedge\eta_\mu + \Pi_i dy^i\wedge\eta$ of a constitutive relation $\mathcal{C}$, the Poincare-Cartan form $\Theta_{\widetilde{\mathcal{C}}}$ of a covering relation $\widetilde{\mathcal{C}}$ and give several examples of types of constitutive relations: Lagrange Type $\mathcal{C}_L$, mixed type with a Lagrangian current part and the source term given by a dissipative potential ($L+D$ type), and vector-potential type.

In Section 10 we discuss three variational ways to get to the balance system $\mathcal{B}_{\mathcal{C}}$ corresponding to a constitutive relation $\mathcal{C}$ (i.e. using variations $\xi\in X(J^1_p(\pi)$ and the differential of the Poincare-Cartan form Θ_C). In doing this a traditional way, i.e. requiring that $j^1(s)^* di_{\xi^1}\Theta_{\mathcal{C}}=0$ or $j^1(s)^* i_{\xi^1} d\Theta_{\widetilde{\mathcal{C}}}=0$ we have, in general, to put the condition(s) $F^\mu_i D_\mu\xi^i=0$ on the variations ξ of the Poincare-Cartan form. Locally there are always enough of such $\mathcal{C}$-admissible variations ξ to separate balance equations (Proposition 15) but globally this may not be true. In a case of semi-Lagrangian constitutive relations (see Sec.9) close to the conventional Lagrangian field theory or in the case of RET constitutive relations no limitations on the admissible variations ξ are present.

That is why we present the third way, using the restricted horizontal differential $\hat{d}$ (see Appendix III) instead of the conventional de-Rham differential d for the invariant formulation of a balance system. In this case one does not need to restrict variations ξ. In a case of Lagrangian constitutive relation $\mathcal{C}_L$ the balance system coincide with the Euler-Lagrange system of equations defined by the Lagrangian L in the traditional way.

In Section 11 we discuss the properties of $\mathcal{C}$-admissible vector fields, prove that $\mathcal{C}$-admissible vector fields form a Lie algebra with respect to the brackets of vector fields in the $L+D$-case and study the form of $\mathcal{C}$-admissible vector fields in the case of a model (2+2)-balance system (two fields and one space dimension) and for the five fields model of fluid thermodynamical system with generic constitutive relations (see Sec.2).

In Section 12 we discuss the action of extended geometrical (lifted from Y) transformations on the constitutive relations $\mathcal{C}$ and on the corresponding Poincare-Cartan form Θ_C, define the symmetry group $Sym(\mathcal{C})$ of a constitutive relation $\mathcal{C}$ and prove that this symmetry group acts on the space of solutions $Sol(\mathcal{B}_{\mathcal{C}})$ of the balance system $\mathcal{B}_{\mathcal{C}}$. Using a connection ν in the configurational bundle $\pi: Y\to X$

we define the ν-homogeneous constitutive relations corresponding to a case where $\mathcal{C}$ depends on the fields and their derivatives but not on the points of space-time X explicitly.

In Section 13 we prove the Noether Theorem for a balance system $\mathcal{B}_{\mathcal{C}}$ under an action of a symmetry Lie group $G \subset Aut(\pi)$ using the multimomentum mapping of a multisymplectic field theory [25, 29]. The Noether Theorem leads to the family of the balance equations which reduces to the conservation laws for special (or absent) source terms. For a semi-Lagrangian constitutive relations or for the case where constitutive relations do not depend on the derivatives of the basic fields (RET case) this theorem is essentially equivalent to the conventional Noether Theorems of Lagrangian field Theory, for the general constitutive relations our version of Noether Theorem has more limited character.

In Section 14 we discuss the type of the balance system $\mathcal{B}_{\mathcal{C}}$ as a system of PDE and show it is a combination of interacting hyperbolic, parabolic and stationary parts.

In Section 15 we present the dual bundle picture of the RET balance systems in terms of LL-multipliers. We prove that the fulfillment of the entropy principle here reduces to the holonomity of the total constitutive section of the 1-jet bundle $J^1(\Lambda, \Omega^3(X))$ of λ^i-fields with values in the space of semi-basic 3-forms.

In Appendices I-IV we collect the information on the properties of partial volume forms η_ν, used in the text, present the basic formalism of the Rational Extended Thermodynamics, recall the definition of the Iglesias differential [16] and definitions and principal properties of the horizontal differential d_H and its restricted version $\hat{d}$.

Results of this work were presented at the Seventh International Seminar on Geometry, Continua and Microstructure that took place at the University of Lancaster, UK in September 2006. Short exposition of the the work will be published in the Proceedings of this conference.

Notations.

For a manifold M we will denote by

- $\mathcal{X}(M)$ - the Lie algebra of vector fields on M,
- $\Lambda^k M$ - the space of exterior k-forms on M,
- $\Lambda M = \oplus_{k=0}^{\infty} \Lambda^k M$ - the exterior algebra of the manifold M,
- $J^1(\pi)$ - the 1-jet bundle of a bundle $\pi : Y \to X$.

Chapter I. Preliminaries.

2. Settings.

In this section we present the bundle settings of the classical field theory in the form suited for later presentation. For deeper exposition we refer to the monographs [1, 11].

2.1. **Space-time base manifold.** A state of material body will be described by the collection of the fields $\{y^i, i = 1, \dots, m\}$ defined in a domain $X = B\times I \subset R^{n+1}$ of the physical or material space-time R^{n+1} . Here $I \subset R_t$ is an interval of time and $B \subset R^n$ is a domain in the n-dim physical or material (reference) space with or without boundary. In the first case denote by ∂B the boundary of domain B. That makes $\bar{X}$ the manifold with the boundary $\partial X = \partial B\times I \cup B\times\partial I$. We assume that the pseudo-Riemannian metric G is defined in X that can be extended to the boundary of X if such does exist. An example of such a metric is the Euclidian metric $G = dt^2 + h$, h being the canonical Euclidian metric in the physical space R^{n+1}, but having in mind application of our scheme to material manifold or relativistic systems we prefer to keep G more general.

In this part of the work we will consider B to be an open subset of R^n.

We will use local coordinates $x^\mu, \mu = 1, 2, \dots, n$ and the time variable $t = x^0$ in X. We will be using Greek indices for the space-time variables and large Latin indices for space variables only.

Denote by η the volume n-form $\eta = \sqrt{|G|}dx^0 \wedge dx^2\ \dots\ \wedge dx^n$ corresponding to the metric G. We will be using the n-forms

$$\eta_\mu = i_{\partial_{x^\mu}}\eta,\ \mu = 0, 1, 2, \dots, n,$$

for instance $\eta_0 = \sqrt{|G|}dx^1 \wedge dx^2\ \dots\ \wedge dx^n$. Necessary properties of these forms are presented in Appendix I.

For separating of space and time we employ the flat connection κ in the bundle $X \to R_t : (t = x^0, x^A) \to t$. This defines the product structure (see [23]) in the space X: $T(X) = T(R_t) \oplus T(B)$ and the corresponding decomposition in the exterior algebra $\Lambda^*(X)$. In particular, we have, in the fiber over each point $x \in X$ the following decomposition

$$\Lambda^{n+1}_x(X) = \mathbb{R}\eta_{0\ x} \oplus dt \wedge \Lambda^{n-1}_x(B^n). \tag{2.1}$$

2.2. **Configurational (state) bundle.** Basic fields of a continuum thermodynamical theory y^i (except of the entropy that will be included later) take values in the space $U \subset \mathbb{R}^m$ which we will call, a **basic state space** of the system (see [33] for a discussion about possible choices of the basic state space and the consequences for the structure of corresponding thermodynamical theory).

Following the framework of a classical field theory (see [1, 11]) we organize these fields in the bundle

$$\pi_U : Y \to X,\ X = I \times B,$$

with the base X. In simple cases $Y = X \times U$ is the cylinder $\mathbb{R} \times B$ with the base X and the fiber U.

Denote by Z the 1-jet bundle of the bundle π: $Z = J^1(\pi)$. Thus, we get the double bundle $Z \xrightarrow{\pi_{10}} Y \xrightarrow{\pi} X$ with the composition mapping $\pi^1 = \pi \circ \pi_{10}$ defining the bundle $Z \xrightarrow{\pi^1} X$.

To formulate balance equations in terms of exterior forms we denote by $\Lambda^k(X)$ the bundle of k-forms on X and by

$$\Lambda^{n+(n+1)}(X) = \Lambda^n(X) \oplus \Lambda^{n+1}(X)$$

the bundle of exterior forms in X of degree $n+(n+1)$. Space of sections of this bundle has as its basis the forms $\eta_\mu, \mu = 0,1,2,\ldots,n;\eta$.

Taking the pullback of the bundle $\Lambda^{n+(n+1)}(X) \to X$ to Y (or, what is the same, construct the fiber product of π and $\pi_{\Lambda X}$ we get the following commutative diagram

$$\begin{array}{ccccc} U \times \Lambda^{n+(n+1)} & \longrightarrow & Y \underset{X}{\times} \Lambda^{n+(n+1)} & \longrightarrow & \Lambda^{n+(n+1)}(X) \\ {\scriptstyle \pi_{\Lambda U}} \downarrow & & {\scriptstyle \pi_{\Lambda Y}} \downarrow & & {\scriptstyle \pi_{\Lambda X}} \downarrow \\ U & \longrightarrow & Y & \xrightarrow{\pi} & X \end{array} . \tag{2.2}$$

Left column of this diagram represents a typical fiber of bundle $\pi_{\Lambda Y}$ over a point $x \in X$. Sections of the bundle $\pi \circ \pi_{\Lambda Y} : Y \underset{X}{\times} \Lambda^{n+(n+1)} \to X$ are called "semibasic" (n+(n+1)) exterior forms on the total space Y of the bundle π, see [23], Sec.4.2.

In the same way, taking, for arbitrary k the pullback of k-forms on X with respect to the projection π^1 we get the bundle of π^1-semi basic k-forms on $Z = J^1(\pi)$.

2.3. **Balance Equations.** Here we define the balance equations of a conventional first order field theory. Fields y^i are to be determined as solutions of the field equations having the form of **balance equations** for the currents F_i^μ, (where often $F_i^0 = y^i$)

$$F^\mu_{i,\mu} = F^0_{i,t} + F^A_{i,x^A} = \Pi_i,\ i = 1,\ldots,m. \tag{2.3}$$

Here the functions $\Pi_i(x^\mu, y^i, y^i_{,x^\mu})$ are called the **production and source** of the components y^i and $\sum_{\nu=1}^n F_i^A(x^\mu, y^i, y^i_{,x^\mu})\frac{\partial}{\partial x^A}$ - the **flow** of the component y^i. These quantities, in general, are assumed to be function of the fields y^i, of the point $x^\mu \in X$ and of (all or some of) the derivatives $y^i_{,x^\mu}$. To shorten notations we will be using z as the short notation of all arguments $(x^\mu, y^i, y^i_{,x^\mu})$.

In the Rational Extended Thermodynamics where all the derivatives entering constitutive relations are included in between the fields y^i densities, currents and sources of balance laws depend on x^μ, y^i only (see discussion of different types of balance systems below in Sec. 14.

As it is customary in the classical field theory, the balance laws could be rewritten by introducing the exterior forms:

(n+1)-form of the flows

$$\mathcal{F}_i = F_i^\mu(z)\eta_\mu,\ i = 1,\ldots,m, \tag{2.4}$$

and the

(n+2)-form of the production and source

$$\Pi_i = \Pi_i(z)\eta. \tag{2.5}$$

Then the balance laws (2.4) takes the form

$$d\mathcal{F}_i = \Pi_i,\ i = 1,\ldots,m, \tag{2.6}$$

or

$$\tilde{d}(\mathcal{F}_i + \Pi_i) = 0$$

if we employ the Iglesias differential $\tilde{d}$ introduced in [16] (see Appendix III).

These relations should be fulfilled for the fields $y^i = s^i(x)$.

Example 1. Five fields thermodynamical system - fluid (5F-fluid).

As an example we consider a "five fields" thermodynamical system describing a fluid ([34, 31]. Such a system has 5 basic fields: mass density ρ, velocity vector field v^A and the absolute temperature ϑ. Correspondingly there are five balance laws in this system - mass balance (conservation) law for the density ρ, linear momentum balance law for $p_I = \rho h_{IJ} v^J$ (h being the standard Euclidian metric in R^3) and the total energy balance law for the total energy $e = \epsilon + \frac{1}{2}|v|^2$ (sum of internal (ϵ) and kinetic energy per unit of mass). To each of these balance laws there corresponds the flux form F and the source (+production) form Π:

$$\begin{cases} \mathcal{F}_\rho = \rho\eta_0 + \rho v^A \eta_A;\ \Pi_\rho = 0 \\ \mathcal{F}_{\rho v^B} = \rho v^B \eta_0 + (\rho v^B v^A - t^{BA})\eta_A;\ \Pi_{v^B} = \rho f_B \eta, \\ \mathcal{F}_e = \rho(\epsilon + \frac{1}{2}|v|^2)\eta_0 + [\rho(\epsilon + \frac{1}{2}|v|^2)v^A - t^A_B v^B + q^A]\eta_A;\ \Pi_\vartheta = \rho f_A v^A + r. \end{cases} \tag{2.7}$$

Production term is zero for mass balance law, equal to the density of body forces ρf_B for the linear momentum balance law and equals to the power of the body forces f_A plus the heat source density for the energy balance law.

One can introduce the internal energy ϵ as the basic variable instead of the temperature ϑ. In this case

$$F_\epsilon = \rho\epsilon\eta_0 + (\rho\epsilon v^A + q^A)\eta_A,\ \Pi_\epsilon = (t^A_B \frac{\partial v^B}{\partial x^A} + r)\eta,$$

see [32], Sec.5.3.

Denote by $\sigma_i = F_i + \Pi_i$ the $(n+1)+(n+2)$-form of the corresponding balance law.

Constitutive relations of this system determine, in addition to the components explicitly defined above, the stress tensor t^{AB}, heat flux q^A, internal energy ϵ, force covector f_A and the volume heat source density r as functions of basic fields ρ, v^A, ϑ and *some of their derivatives*. In addition to this, force f_A and volume heat source density may explicitly depend on the position and time x^μ.

A fundamental physical requirements known as "material axioms" put restrictions on the character of dependence of density, flux and source components of the balance laws on the basic fields and their derivatives ([52, 34, 40]. One of these material axioms - material indifference or, more generally, a transformation properties of a balance system under the change of observer, leads to the independence of the heat flux q^A and the stress tensor t^{AB} on the velocity v^A and on the antisymmetric part of the velocity gradient.

Other material axioms - material symmetries, II law of thermodynamics (see [34],Ch.6) further restricts the form of constitutive relations. Geometrical form of these restriction will be studied in the continuation of this work.

In the simplest variant the constitutive relations of a 5F-fluid system depend on the spacial gradient of temperature $\nabla\vartheta$ and symmetrized gradient of velocity $\nabla\mathbf{v}_{sym}$ only. Next level of complexity is represented by the fluid with the *short memory* where constitutive relations may depend on the rate of change of temperature $\dot{\vartheta}$ (see [52, 31]).

As a result, the domain of constitutive relations (the **state space**) of 5F-fluid system consists of the fields $(\rho, v^A, \vartheta; \nabla\vartheta, \nabla\mathbf{v}_{sym})$.

An example of specific constitutive relations of a 5F-fluid system is the Navier-Stokes-Fourier fluid where

$$\begin{cases} t^A_B = -p(\rho,\vartheta)\delta^A_B + \nu(\rho,\vartheta)Tr(\nabla \mathbf{v}_{sym})\delta^A_B + 2\mu(\rho,\vartheta)(\nabla \mathbf{v}_{sym})^A_B; \\ q^A = -\kappa(\rho,\vartheta)\nabla^A \vartheta; \\ \epsilon = \epsilon_0(\rho,\vartheta). \end{cases} \tag{2.8}$$

Here p is the pressure scalar field and ν, μ, κ are scalar coefficients of viscosity (the first two) and the heat conductivity respectively.

Notice also that there is another, more fundamental 5-fields thermodynamical system - $5F - solid$, where the basic fields are: mass density ρ, embedding $\phi : B^3 \to E^3$ of the material manifold B to the physical Euclidian space (E^3, h) and the absolute temperature ϑ. Constitutive relations of $5F - solid$ system typically (for instance, in thermoelasticity) depends on the spacial derivatives of embedding mapping ϕ (deformation gradient, or, with the use of material indifference axiom Cauchy deformation tensor $C(\phi) = \phi^* h$), its time derivative (velocity), temperature ϑ and its spacial gradient $\nabla\vartheta$. Adding of the time derivative of Cauchy deformation tensor, or, equivalently, the symmetrized velocity gradient $\nabla \mathbf{v}_{sym}$ (containing second derivatives of basic fields!) allows to take into account effects of viscoelastc behavior. The model $5F - fluid$ represents a *reduction* of the $5F - solid$ system related to the usage of the largest possible material symmetry group $SL(3, R)$ for the fluids (see [53]).

Remark 1. In the geometrical theory of differential equations (see, for instance, [21]) it is customary to extend given system of differential equations to include all the differential equations that are consequences of ones in a given system. It would be equally interesting to complete the system (2.3) of the balance laws of a given thermodynamical system by all the balance laws that are their consequences. In the second part of this work we study such "secondary balance laws" of a given balance system (of zero or first order by the degree of derivatives of basic fields included into the constitutive relations) that have the *same domain as the initial balance laws*. Higher order balance laws that are consequences of a given balance system will be studied elsewhere.

2.4. **Entropy condition.** Entropy density h^0, entropy flux $h^A,\ A = 1, 2, \ldots, n$ and the entropy production Σ are typically assumed to be a functions of the the same variables $x^\mu, y^i, y^i_{,x^\mu}$ as the coefficients of the balance laws (2.3). II law of thermodynamics requires that entropy satisfies to the balance law

$$d(h^\mu \eta_\mu) = \Sigma, \tag{2.9}$$

with the production 4-form

$$\Sigma = \Sigma(x^\mu, y^i, y^i_{,x^\mu})\eta, \sigma \geqq 0, \tag{2.10}$$

being positive on the solutions of the balance system (2.3).

Entropy principle ([31, 32]) requires that any solution of the balance equations (2.3) would also satisfy to the equation (2.9) and that the production σ. This requirement places serious restrictions to the form of the balance equations (2.3).

To close system of equations (2.3) (or (2.3+2.9)) for y^i one has to choose the **constitutive relation** C of the thermodynamical system, i.e. to choose the densities,flows and production forms as functions of x^μ, y^i and the appropriate derivatives of fields y^i . In particular, one have to choose the domain of the constitutive relation which is typically the full or partial jet-bundle of the configurational bundle π of dynamical variables. By definition, the Rational Extended Thermodynamics (see Appendix II for short exposition of the formalism of this theory) is the zero order theory in that the domain of its constitutive relation is the space Y. In this article we consider the cases of constitutive relations of zero and first order only. Constitutive relations depend also ont the background fields (metric G in X or a connection ν in the bundle $\pi : Y \rightarrow X$ in this paper). As we will see in the part II of this work (in preparation, see also [44]), utilizing of the entropy condition allows to effectively reduce this process to a choice of smaller number of constitutive fields.

3. Multisymplectic Field Theory.

In this section we recall briefly the Poincare-Cartan formalism of the Multisymplectic Field Theory with the modifications necessary for the formulation of the covariant theory of balance systems. We will follow ([24, 13, 11]).

3.1. **The 1-jet bundle.** Given a frame bundle $\pi : Y \rightarrow X$ we say that two sections $s, s' : U \rightarrow Y$ defined in a neighborhood U of a point $x \in X$ define the same 1-jet $j^1 s(x)$ if $s(x) = s'(x),\ s_{*x} = s_{*x} : T_x(X) \rightarrow T_{s(x)}(Y)$. This defines an equivalence relation on the set of locally (near the point x) defined sections of π. Space of equivalence classes of such local sections is defined $J^1_x(\pi)$.

The total space $J^1(\pi) = \bigcup_{x\in X} J^1_x(\pi)$ cam be endowed with a smooth structure such that the mappings $J^1(\pi) \rightarrow Y \rightarrow X$ are fibrations. The fibration π_{10} : $J^1(\pi) \rightarrow Y$ is the affine bundle modeled in the vector bundle $\pi^*(T^*(X)) \otimes V(\pi)$, where $V(\pi) \subset T(Y)$ is the vertical subbundle of the bundle π.

Let $(x^\mu, y^i; \mu = 1, \ldots, n+1 = dim(X); i = 1, \ldots, m)$ be an adopted local coordinate system in Y. Then the local coordinate system $(x^\mu, y^i, z^i_\mu; \mu = 1, \ldots, n; i = 1, \ldots, m)$ can be defined in $J^1(\pi)$ by the condition

$$z^i_\mu(j^1_x s) = \frac{\partial y^i}{\partial x^\mu}(x).$$

3.2. **Lagrangian picture: Poincaré-Cartan Form.** The volume form $\eta = \sqrt{|G|}d^{n+1}x$ permits to construct the **vertical endomorphism**

$$S_\eta = (dy^i - z^i_\mu dx^\mu) \wedge \eta_\nu \otimes \frac{\partial}{\partial z^i_\nu} \tag{3.1}$$

which is a tensor field of type $(1, n+1)$ on the 1-jet bundle space $Z = J^1(\pi)$ of the configurational bundle $\pi : Y \rightarrow X$. Here $\eta_\mu = i_{\partial_{x^\mu}}\eta$, see Appendix I.

For a Lagrangian (n+1)-form $L\eta$, L being a (smooth) function on the manifold $Z = J^1(\pi)$ the Poincaré-Cartan $(n+1)$ and $(n+2)$-forms are defined as follows:

$$\Theta_L = L\eta + S^*_\eta(dL),\ \Omega_L = -d\Theta_L, \tag{3.2}$$

where S^*_η is the adjoint operator of S_η. In coordinates we have

$$\Theta_L = (L - z^i_\mu \frac{\partial L}{\partial z^i_\mu})\eta + \frac{\partial L}{\partial z^i_\mu} dy^i \wedge \eta_\mu, \tag{3.3}$$

$$\Omega_L = -d(L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) \wedge \eta - d(\frac{\partial L}{\partial z^i_\mu}) \wedge dy^i \wedge \eta_\mu -$$
$$- \frac{\partial L}{\partial z^i_\mu}((-1)^\mu \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu}) \wedge dy^i \wedge \eta = -(dy^i - z^i_\mu dx^\mu) \wedge \left(\frac{\partial L}{\partial y^i}\eta - d\left(\frac{\partial L}{\partial z^i_\mu}\right) \wedge \eta_\mu \right). \tag{3.4}$$

Remark 2. If the space manifold B has the boundary and some boundary conditions are prescribed for the sections $\phi : X \rightarrow Y$ of the configurational bundle in the form

$$\phi(x) \in \mathcal{B}c,$$

where $\mathcal{B}c$ is a subbundle of the bundle Y, then, in order to correlate boundary conditions with the variation of Poincaré-Cartan form it is reasonable to require that the restriction of Ω_L to the boundary subbundle $\mathcal{B}c$ is exact: there exists a n-form θ on the subbundle $\mathcal{B}c$ such that

$$i_{\mathcal{B}c}\Theta_L = d\theta. \tag{3.5}$$

We refer to [1], Chapter 7 or to the paper [2] for more details.

Recall ([24]) that the couple (Z, Ω_L) is a multisymplectic manifold provided the Lagrangian L is regular, i.e. the matrix $L_{z^i_\mu z^j_\nu}$ is **nondegenerate**.

An *extremal* of L is a section of π_{XY} such that for any vector field ξ_Z on Z,

$$(j^1\phi)^*(i_{\xi_Z} d\Theta_L) = 0, \tag{3.6}$$

where $j^1\phi$ is the first jet prolongation of ϕ.

A section ϕ is an extremal of L if and only if it satisfies the Euler-Lagrange Equation (see, for instance, [1, 11])

$$(j^1\phi)^*\left(\frac{\partial(L\sqrt{|G|})}{\partial y^i} - \frac{d}{dx^\mu}\left(\frac{\partial(L\sqrt{|G|})}{\partial z^i_\mu}\right)\right) = 0,\ 1 \leqq i \leqq m. \tag{3.7}$$

There is an operator $\mathcal{EL} : \Gamma(\pi) \to \Gamma(V^*(\pi))$ (Euler-Lagrange operator) that has the local form

$$\mathcal{EL}(\phi) = \left(\frac{\partial(L\sqrt{|G|})}{\partial y^i} \circ j^1\phi - \frac{d}{dx^\mu}\left(\frac{\partial(L\sqrt{|G|})}{\partial z^i_\mu} \circ j^1\phi\right)\right) \otimes dy^i. \tag{3.8}$$

In terms of this operator the Euler-Lagrange equations looks simply

$$\mathcal{EL}(\phi) = 0. \tag{3.9}$$

3.3. **Canonical multisymplectic bundles Λ^k_r.** Denote by $V(Y) \to Y$ the subbundle of **vertical tangent vectors** of the tangent bundle $T(Y)$.

Following [24, 13] let $\Lambda^k_r Y$ denote the subbundle of the vector bundle $\Lambda^k Y$ of exterior k-forms on Y consisting of those forms that vanish when r of their arguments are vertical (with respect to the fibration $\pi : Y \to X$)

$$\Lambda^k_r Y = \{\sigma \in \Lambda^k Y\ | i_{\xi_1} \dots i_{\xi_r}\sigma = 0, \forall\ \xi_i \in V(\pi).\}$$

The manifold $\Lambda^k Y$ carries a canonical k-form Θ^k_0 define as follows:

$$\Theta_0(\omega)(\xi_1, \dots \xi_k) = \omega(\pi_{\Lambda^k}(\omega))(\pi_{\Lambda^k\ *}(\xi_1), \dots \pi_{\Lambda^k\ *}(\xi_k)), \tag{3.10}$$

where $\omega \in \Lambda^k Y$, $\xi_i \in T_\omega(\Lambda^k Y)$, and $\pi_{\Lambda^k} : \Lambda^k Y \to Y$ is the canonical bundle projection.

By restriction, this form induces an k-form Θ^k_r on the manifold $\Lambda^k_r Y$. We denote $\Omega^k_r = -d\Theta^k_r$.

3.3.1. *Case k=n,n+1.* We will use the construction above for $k = n+1, n+2; r = 1, 2$, or, more specifically, for $k = 4, 5$.

In particular, The bundle $\Lambda^k_2(Y)$ of the exterior forms on Y which are annulated if 2 of its arguments are vertical:

$$\omega^k \in \Lambda^k_2(Y) \Leftrightarrow i_\xi i_\eta \omega = 0,\ \xi, \eta \in V(Y). \tag{3.11}$$

Elements of the space $\Lambda^{n+1}_1 Y$ are semibasic n-forms locally expressed as $p(x, y)\eta$.

Elements of the space $\Lambda^{n+1}_2 Y$ have, in local adapted coordinates (x^μ, y^i) the form

$$p(x,y)\eta + p^{\mu}_i dy^i \wedge \eta_\mu.$$

This introduces coordinates (x^μ, y^i, p) on the manifold $\Lambda^{n+1}_1 Y$ and (x^μ, y^i, p, p^μ_i) on the manifold $\Lambda^{n+1}_2 Y$.

Taking the case $k = n+2$ we see that the forms $dy^i \wedge \eta$ form the basis of $\Lambda^{n+2}_2(Y)$ while the bundle $\Lambda^{n+2}_1(Y)$ is *zero bundle*.

Introduce also the notations

$$\Lambda_2^{(n+1)+(n+2)}(Y) = \Lambda^{n+1}_2(Y) \oplus \Lambda^{n+2}_1(Y), \Lambda_1^{(n+1)+(n+2)}(Y) = \Lambda^{n+1}_1(Y) \oplus \Lambda^{n+2}_1(Y)$$

for the direct sum of the bundles on the right side.

It is clear that $\Lambda^k_1(Y) \subset \Lambda^k_2(Y)$ is the subbundle of the larger bundle. Therefore we have the embedding of subbundles

$$\Lambda_1^{(n+1)+(n+2)}(Y) \subset \Lambda_2^{(n+1)+(n+2)}(Y). \tag{3.12}$$

For the Poincare-Cartan forms (3.10) in this case $(k = n+1, r = 2)$ we have local expressions

$$\Theta^{n+1}_2 = p\eta + p^\mu_i dy^i \wedge \eta_\mu, \tag{3.13}$$

$$\Omega^{n+1}_2 = -dp \wedge \eta - dp^\mu_i \wedge dy^i \wedge \eta_\mu - (-1)^\mu p^\mu_i \frac{\partial \lambda_G}{\partial x^\mu} \wedge dy^i \wedge \eta,$$

where $\lambda_G = ln(\sqrt{|G|})$

3.3.2. *Dual MS-picture: Hamiltonian systems.* Basic for the Hamiltonian form of multisymplectic field theory is the bundle: $\Lambda^{n+1}_2 Y$ endowed with the canonical MS-form Θ^{n+1}_2 and its factor bundle over Y (*polysymlectic bundle*)

$$Z^* = \Lambda^{n+1}_2 Y / \Lambda^{n+1}_1 Y,\ q : \Lambda^{n+1}_2 Y \to Z^*.$$

Corresponding to the local adopted chart (x^μ, y^i) the manifold Z^* has the (local) coordinates (x^μ, y^i, p^i_μ).

Pairing

$$J^1(\pi) \underset{Y}{\times} Z^* \to 1_Y :\ (z,p) \to z^i_\mu p^\mu_i \tag{3.14}$$

identifies the bundle $Z^* \to Y$ with the linear dual to the affine bundle $\pi^1_0 : J^1(\pi) \to Y$. In the similar way, bundle $\Lambda^{n+1}_2(Y) \to Y$ can be identified with the affine dual to the bundle $\pi^1_0 : J^1(\pi) \to Y$ (see [22, 29]).

Remark 3. Another way to defined Z^* is to take

$$Z^* = \pi^*(T(X)) \otimes V^*(\pi) \otimes \pi^*(\Lambda^{n+1}(X)), \tag{3.15}$$

see [22, 29].

A **Hamiltonian** is, in this approach, a section h of the projection q. Having it available, we define $\Theta_h = h^*\Theta^{n+1}_2,\ \Omega_h = h^*\Omega^{n+1}_2$.

A section $\sigma : X \to Z^*$ is said to satisfy the Hamilton equation (for a given Hamiltonian h) if

$$\sigma^*(i_\xi \Omega_h) = 0,$$

for all vector fields ξ on Z^*,

In local coordinates (x^μ, y^i, z^i_μ) a Hamiltonian h is represented by a local function H:

$$p = -H(x^\mu, y^i, z^i_\mu).$$

Then,

$$\Theta_h = -H\eta + p^\mu_i dy^i \wedge \eta_\mu, \tag{3.16}$$

$$\Omega_h = -d\Theta_h = -dH \wedge \eta + dp^\mu_i \wedge dy^i \wedge \eta_\mu + (-1)^\mu p^\mu_i \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu} dy^i \wedge \eta, \tag{3.17}$$

and the Hamilton equations for a section $\sigma = (x^\mu, \sigma^i(x), \sigma^\mu_i(x))$ take the form:

$$\begin{cases} \frac{\partial \sigma^i}{\partial x^\mu} = \frac{\partial H}{\partial \sigma^\mu_i}, \\ div_G(p^\mu_i) = \sum_\mu [\frac{\partial \sigma^\mu_i}{\partial x^\mu} + \sigma^\mu_i \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu}] = -\frac{\partial H}{\partial y^i}. \end{cases} \tag{3.18}$$

In difference to the bundle $\Lambda^{n+1}(Y)$ the bundle Z^* does not have a canonically defined form of the Poincare-Cartan type (see for instance, Sec. below where it is shown that under the transformation induced by an automorphism of the bundle π the (locally defined) for). Locally though, we can define the form

$$\Theta^* = p^\mu_i dy^i \wedge \eta_\mu \tag{3.19}$$

Under the action of an adopted transformation $\phi \in Aut(\pi)$ considered as a change of variables and lifted to Z^* to the tensorially transformed form the term of the form $Q\eta$ is added (see below, Section 11.1). As a result, the form Θ^* is not defined canonically, **but its class** $mod(\Lambda^{n+1}_1(Z^*))$ **is**. Taking $mod(\Lambda^{n+1}_1(Z^*))$ we get **canonically defined element of the bundle** $\Lambda^{n+1}_2 Z^*/\Lambda^{n+1}_1 Z^*$ on Z^*.

One may consider this class as defining the canonical $V^*(\pi)$-valued semi-basic n-form on Y.

Below we will see that it is sufficient for the separating components of a balance system to the individual balance laws with the help of independent vertical variations.

Recall (see [6, 13]) that given an **Ehresmann connection** $\nu : Y \to J^1(\pi)$ on the bundle π with the vertical projector

$$P_v = \partial_{y^i} \otimes (dy^i + \Gamma^i_\mu dx^\mu),$$

defines naturally the linear section $\delta_\nu : Z^* \to \Lambda^{n+1}_2 Y$ given by

$$\delta_\nu(F^\mu_i dy^i \wedge \eta_\mu) = (F^\mu_i \Gamma^i_\mu)\eta + F^\mu_i dy^i \wedge \eta_\mu. \tag{3.20}$$

Section δ_ν defines the pullback of the form Θ^{n+1}_2:

$$\delta^*_\nu \Theta^{n+1}_2 = (F^\mu_i \Gamma^i_\mu)\eta + F^\mu_i dy^i \wedge \eta_\mu. \tag{3.21}$$

Form $\Theta^{n+1}_\nu = \delta^*_\nu \Theta^{n+1}_2$ is defined correctly on the manifold Z^*.

3.3.3. *Bundle $\tilde{Z}$ for the balance systems.* To present the system of balance laws in the multisymplectic form we will need to use the vector bundles $\Lambda_i^{(n+1)+(n+2)}Y = \Lambda_i^{n+1}Y \oplus \Lambda_i^{n+2}Y$, where $i = 1, 2$ and the vector bundle

$$\tilde{Z} = \Lambda_2^{(n+1)+(n+2)}Y/\Lambda_1^{(n+1)+(n+2)}Y = \Lambda_2^{(n+1)}Y/\Lambda_1^{(n+1)}Y \oplus \Lambda_2^{(n+2)}Y/\Lambda_1^{(n+2)}Y, \tag{3.22}$$

Notice that the first term in the sum on the right is Z^*.

Locally, elements of the factor bundle $\tilde{Z}$ can be presented in the form

$$p_i^\mu dy^i \wedge \eta_\mu + q_i dy^i \wedge \eta. \tag{3.23}$$

Canonical forms Θ_i^k for $k = n+1, n+2; i = 1, 2$ induce on the bundle $\tilde{Z}$ the class of $(n+1)+(n+2)$ form

$$\tilde{\Theta} = p_i^\mu dy^i \wedge \eta_\mu + q_i dy^i \wedge \eta \ mod\ \Lambda_1^* \tilde{Z} \tag{3.24}$$

where $n+1$ and $n+2$ components of this form are lifted from the canonical forms on the components $\Lambda_2^k Y/\Lambda_1^k Y$ for $k = n+1, n+2$. *Class mod $\Lambda_1^*\tilde{Z}$ of this form is defined canonically* (see above). In examples below we will be using these constructions for $n = 3$.

3.4. **Legendre Transformation.** Let L be a Lagrangian function. We define the fiber mapping over Y

$$leg_L : Z \to \Lambda_2^{n+1}Y,$$

as follows:

$$leg_L(j_x^1\phi))(X_1, \ldots, X_{n+1}) = (\Theta_L)_{j_x^1\phi)}(\tilde{X}_1, \ldots, \tilde{X}_{n+1}),$$

where $j_x^1\phi \in Z, X_i \in T_{\phi(x)}Y$ and $\tilde{X}_i \in T_{j_x^1\phi(X)}Z$ are such that $\pi_*(\tilde{X}_i) = X_i$.

Notice that addition of a constant to the Lagrangian L leads to the constant shift (in p) of the image of Legendre mapping in $\Lambda_2^{n+1}Y$ (which is a submanifold of codimension 1 if the Lagrangian is regular). Thus, the space $\Lambda_2^{n+1}Y$ is foliated by these shifts.

In local coordinates, we have

$$leg_L(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, p = L - z_\mu^i \frac{\partial L}{\partial z_\mu^i}, p_i^\mu = \frac{\partial L}{\partial z_\mu^i}).$$

The Legendre transformation $Leg_L : Z \to Z^*$ is defined as the composition $Leg_L = q \circ leg_L$. Locally

$$Leg_L(x^\mu, y^i, z_\mu^i) = (x^\mu, y^i, p_i^\mu = \frac{\partial L}{\partial z_\mu^i}).$$

Recall [24] that the Legendre transformation $Leg_L : Z \to Z^*$ is a local diffeomorphism if and only if L is regular.

If, in addition, the Lagrangian L is hyperregular (i.e. if Leg_L is a global diffeomorphism), one can define a **Hamiltonian** $h : Z^* \to \Lambda_2^{n+1}Y$ by setting

$$h = leg_L \circ Leg_L^{-1}.$$

Then

$$Leg_L^*\Theta_h = \Theta_L,\ Leg_L^*\Omega_h = \Omega_L.$$

In this case $Leg_L : (Z, \Omega_L) \to (Z^*, \Omega_h)$ is a multisymplectomorphism.

3.5. **Unified formalism.** In this subsection we describe the united geometrical setting of a classical field theory developed in ([24],[10]) as the generalization of Skinner-Rusk (Dirac) geometrical mechanics.

Introduce the fiber product $W_0 = Z \times_Y \Lambda^{n+1}_2 Y$ with canonical projectors pr_i to the i-th factor. We consider canonical coordinates $(x^\mu, y^i, z^i_\mu, p, p^\mu_i)$ on W_0.

Define the $(n+1)$-forms $\Theta = pr_1^* \Theta^{n+1}_2$ and $\Theta^L = pr_2^* \Theta_L$, and the corresponding $(n+2)$-forms $\Omega = d\Theta,\ \Omega^L = d\Theta^L$. In addition, we introduce the differences $\Theta^* = \Theta^L - \Theta,\ \Omega^* = \Omega^L - \Omega$. In local coordinates we have

$$\Theta^* = \left[L - z^i_\mu \frac{\partial L}{\partial z^i_\mu} - p\right]\eta + \left[\frac{\partial L}{\partial z^i_\mu} - p^\mu_i\right] dy^i \wedge \eta_\mu. \tag{3.25}$$

Define the submanifold $W_1 \subset W_0$ as the graph of the Legendre mapping $Leg_L : Z \to \Lambda^{n+1}_2 Y$, see [10] and [24]. Locally it is given by equations

$$p^\mu_i - \frac{\partial L}{\partial z^i_\mu} = 0. \tag{3.26}$$

Then we have

$$\Theta^*|_{W_1} = \left[L - z^i_\mu \frac{\partial L}{\partial z^i_\mu} - p\right]\eta. \tag{3.27}$$

Introduce the function H_0 on W_0 as follows ([24]):

$$H_0 = p + p^\mu_i z^i_\mu - pr_2^* L. \tag{3.28}$$

This function allows to define the form

$$\Omega_{H_0} = \Omega + dH_0 \wedge \eta = d(\Theta + H_0 \eta), \tag{3.29}$$

which takes the following local expression

$$\Omega_{H_0} = -(dp + (-1)^\mu p^\mu_i \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu} dy^i) \wedge \eta - dp^\mu_i \wedge dy^i \wedge \eta_\mu + dH_0 \wedge \eta. \tag{3.30}$$

We can develop the above local expression to obtain the following

$$\begin{aligned} \Omega_{H_0} \quad = \quad & -dp^\mu_i \wedge dy^i \wedge \eta_\mu + p^\mu_i dz^i_\mu \wedge \eta + z^i_\mu dp^\mu_i \wedge \eta - \frac{\partial L}{\partial y^i} dy^i \wedge \eta - \\ & -\frac{\partial L}{\partial z^i_\mu} dz^i_\mu \wedge \eta - (-1)^\mu p^\mu_i \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu} dy^i \wedge \eta. \end{aligned} \tag{3.31}$$

Along W_1 the second and fifth terms in expression for Ω_{H_0} cancel each other and one get

$$\Omega_{H_0}|_{W_1} = -dp^\mu_i \wedge dy^i \wedge \eta_\mu + z^i_\mu dp^\mu_i \wedge \eta - [\frac{\partial L}{\partial y^i} + (-1)^\mu p^\mu_i \frac{\partial ln(\sqrt{|G|})}{\partial x^\mu}] dy^i \wedge \eta. \tag{3.32}$$

Notice that this form on W_1 does not depend on p.

Consider now the submanifold $W_2 \subset W_1$ defined by equation $H_0 = 0$, i.e.

$$p = -(p^\mu_i z^i_\mu - L). \tag{3.33}$$

which defines, for a given Lagrangian the *hamiltonian* section of $\mu : \Lambda^{n+1}_2 Y \to Z^*$ (a "local energy", see above).

Notice that the form Θ^* (and therefore, Ω^*) vanish when they restrict to W_2. This allows to identify W_2 with the graph of the Legendre mapping $Leg_L : Z \to Z^*$.

3.6. **Vertical contact structure.** Notice that the dimensions of the fiber of the bundle $\pi^1_0 : Z = J^1(\pi) \to Y$ and that of the dual bundle $Z^* = (\Lambda^{n+1}_2 Y/\Lambda^{n+1}_1 Y) \to Y$ are the same. In addition to the $(n+1)$ and $(n+2)$-forms introduced above, the bundle $\pi_{YW_0} : W_0 \to Y$ has one more geometrical structure, namely the *contact structure in the fibers* $W_{0\ y}$ *of the bundle* $\pi_{YW_0} : W_0 \to Y$. Indeed, fibers of this bundle are endowed with the canonical contact 1-form

$$\tilde{\theta} = dp + z^i_\mu dp^\mu_i = d(p + z^i_\mu p^\mu_i) - p^\mu_i dz^i_\mu. \tag{3.34}$$

This "vertical contact structure" allows to distinguish the Legendre mappings defined by some lagrangian $L \in C^\infty(Z)$ from the general bundle mappings $\hat{C} : Z \to Z^*$, namely,

Proposition 1. (1) *Let* $L\eta$ *be a Lagrangian defined on the space* $Z = J^1(\pi)$. *Then the intersection of the graph* Γ_L *of Legendre transformation* $leg_L : J^1(\pi) \to \Lambda^{n+1}_2 Y$ *with the fibers* $W_{0\ y}$ *of the bundle* $W_0 \to Y$ *are the* ***Legendre submanifolds*** *of the fibers* $W_{0\ y}$.

(2) *Let* $\hat{C} : J^1(\pi) \to \Lambda^{n+1}_2 Y$ *be any smooth bundle morphism (over* Y*) given by*

$$\hat{C}(x, y; z^i_\mu) = (x, y; p(x, y; z^i_\mu), p^\mu_i(x, y; z^i_\mu)).$$

Then the intersection of the graph $\Gamma_{\hat{C}} \subset W_0 =$ *of this morphism with the fibers* $W_{0\ y}$ *of the bundle* $W_0 \to Y$ *are the* ***Legendre submanifolds*** *of the (contact) fibers* $W_{0\ y}$ *if and only there exists a (locally defined) function* $L \in C^\infty(J^1(\pi))$ *such that*

$$p^\nu_j = L_{z^j_\nu},\ p = L - z^i_\mu L_{z^i_\mu}.$$

Proof. For the first statement, notice that we have in local coordinates

$$leg_L(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i, p = L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p^\mu_i = \frac{\partial L}{\partial z^i_\mu}).$$

Thus, along Γ_L we have

$$\tilde{\theta}|_{\Gamma_L} = d_v((L - z^i_\mu \frac{\partial L}{\partial z^i_\mu}) + z^i_\mu \frac{\partial L}{\partial z^i_\mu}) - \frac{\partial L}{\partial z^i_\mu} dz^i_\mu = 0.$$

For the second part we notice that the restriction of the 1-form $\tilde{\theta}$ on the fiber $W_{0\ y}$ to the graph of $\hat{C}$ has the form

$$p_{,z^j_\nu} dz^j_\nu + z^i_\mu p^\mu_{i,z^j_\nu} dz^j_\nu = \partial_{z^j_\nu}(p + z^i_\mu p^\mu_i) - p^\nu_j.$$

Introducing function $L = p + z^i_\mu p^\mu_i$ we immediately get the necessary expressions for components of mapping $\hat{C}$. □

Introduce the smooth submanifold (fiberwise quadric) $W_r \subset W_0$ (reduced) defined by the condition

$$W_r = \{w = (x, y, z^i_\mu, p, p^\mu_i) \in W_0 | p + z^i_\mu p^\mu_i = 0\}.$$

Remark 4. Submanifold W_r is the abstract analog of the quadric C in the phase space of the homogeneous thermodynamics which contains, due to the homogeneity requirement) all the (Legendre) constitutive surfaces of different homogeneous TD systems with given thermodynamical phase space, see [4].

Restriction of the form $\tilde{\theta}$ to the fibers over Y of the subbundle W_r of the bundle $W_0 \to Y$ has the form

$$\tilde{\theta}|_{W_r} = -p^\mu_i dz^i_\mu.$$

Consider the projection $W_0 \to Z \times Z^* = J^1(\pi) \times \Lambda^{n+1}_2 Y/\Lambda^{n+1}_1 Y$ (see Sec. 3.5). Restriction of this projection to the submanifold W_r is the diffeomorphism of the bundles over Y with the inverse j given by

$$(x, y, z^i_\mu, p^\mu_i) \to (x, y, z^i_\mu, p = -z^i_\mu p^\mu_i, p^\mu_i).$$

Consider the fiberwise pullback $\theta^* = j^*\tilde{\theta}|_{W_r}$. This 1-form has the same type $\theta^* = -p^\mu_i dz^i_\mu$.

Lift it to the bundle $Z \times \tilde{Z} \to Y$ via the projection to the first factor keeping the same notation for this form.

Let now

$$C : J^1(\pi) \to Z^* : z \to z^* = F^\mu_i(x, y, z)dy^i \wedge \eta_\mu$$

be *any smooth mapping of the bundles* over Y, let $\widehat{C} : J^1(\pi) \to \Lambda^{n+1}_2(Y)$ be its lift to the mapping into the bundle $\Lambda^{n+1}_2(Y)$ defined by $p(z) = z^i_\mu F^\mu_i(z)$ and let C^g be the lift of the mapping $\hat{C}$ to the embedding $Z \to W_0 = Z \times \Lambda^{n+1}_2(Y)$. Then we define the 1-form on Z

$$\theta_C = C^*(\theta^*) = F^\mu_i(x, y, z)dz^i_\mu \tag{3.35}$$

on the fibers of the 1-jet bundle $\pi_{10} : J^1(\pi) \to Y$.

Chapter II. Partial 1-jet bundles.

4. Partial jet-bundles $Z_p = J^1_p(\pi)$.

4.1. **State spaces and the partial jet-bundles.** State fields bundle Y (with the fiber U and field variables y^i) over the base X: $\pi_{YX} : Y \to X$, introduced in Section 2 represents the first floor of the construction of a bundle that serves as a domain of general constitutive relations (shortly CR) of a balance system (2.3).

Following types of the bundles serving as the domain for constitutive relations are most widespread:

(1) Minimal state space (case of the total 1-jet bundle): No derivatives of physical fields are included into the space U. Constitutive relations are defined on the full 1-jet bundle $J^1(\pi)$ (first order theory), or on the full 2-jet bundle (second order theory), see [52]). Elasticity theory is an example of such a case. Notice that the base manifold X here can be taken ss the product of the time line T and the **material manifold** B (Lagrange picture) rather then the physical space-time. Similar bundle picture is used in astrophysics (see. [5]).
(2) Optimal (in physical sense) state space: some fields are included into the state space U with some of their derivatives or only these derivatives are included into the state space. For instance, it is customary to include velocity field v (which is defined by the time derivatives of the deformation embedding of the material manifold into the physical space) in the list of basic fields and write down the balance law of the linear momentum, corresponding to the velocity field. This is the generic case. Five field model (2.7) is an example. In such scheme one can distinction between the fields entering the constitutive relations alone from those entering CR with the time derivative, with the spacial gradients or with both. Such a distinction is important if one would like to preview the type of a PDE-system that corresponds to a given balance system - is it hyperbolic, dissipative, or some, definite mix of both, does it have a stationary, possibly elliptic components etc.
(3) Maximal state space (Rational Extended Thermodynamics - RET): In the Rational Extended Thermodynamics all the derivatives of physical fields entering the constitutive relations (CR) (temperature gradients, rates of strain tensors, etc.) are included into to the bundle U of the basic fields of the theory. It gives a tremendous technical advantage to write constitutive relations in terms of fields y^i only (without any derivatives included), to have first derivatives only while differentiating the constitutive relations and to have a simple duality picture (see Appendix II). On the other hand it makes the whole scheme somewhat too cumbersome: one has to include into the system the balance laws for the derivatives of physical fields, such derivatives being listed in between the basic fields in the space U.

Here we present a construction of two types of the partial jet bundles $J^1_p(\pi)$ of a fiber bundle $\pi : Y \to X$ that will be used in the paper. One, denoted as $J^1_K(\pi)$ is related to a subbundle K of the tangent bundle $T(X)$ (or, more exact, with an almost product structure $T(X) = K \oplus K'$ (see [23]), the other, denoted as $J^1_S(\pi)$, is defined by the decomposition S of the state space U into the direct sum of subspaces with different set of derivatives entering constitutive relations. In the second part of the work we define and study more general type of partial first and higher higher order jet bundles.

4.2. **Partial 1-jet bundles $J^1_K(\pi)$.**

Definition 1. *Let $\pi_{XY} : Y \to X$ be a fiber bundle and let $K(X) \subset T(X)$ be a subbundle of the tangent bundle of manifold X.*

(1) *Let ϕ_1, ϕ_2 be a two sections of the bundle π_{XY} such that $\phi_1(x) = \phi_2(x)$. We say that these two sections are K**-equivalent of order 1 at a point** $x \in X$ and denote this as $\phi_1 \sim_{K_x} \phi_2$ if the restrictions of the tangent mappings $\phi_{i*x} : T(X) \to T_{\phi_i(x)}(Y)$ to the subspace $K_x(X)$ coincide.*
(2) *Space of classes of K_x-equivalence of order 1 will be called a K-partial 1-jet (of a section) at a point x. Space of K-partial 1-jets at a point $x \in X$ will be denoted by $J^1_{K\ x}(\pi_{XY})$.*
(3) *Union $\cup_{x\in X} J^1_{K\ x}(\pi_{XY})$ will be denoted by $J^1_K(\pi_{XY})$ and will be called the space of K-partial 1-jets of sections of the bundle π_{XY}.*

Example 2. (1) For $K = \emptyset$, the bundle $J^1_K(\pi) = \{0\}_Y$ - affine bundle with zero-dimensional fiber (this is the case of RET).
(2) Let $\mathcal{F}$ be a foliation of the manifold X and let $K = T(\mathcal{F})$ be the distribution tangent to the foliation. Then, $J^1_K(\pi)$ is the bundle of sections of $\pi : Y \to X$ and their derivatives **along the foliation** $\mathcal{F}$.
(3) For $K = T(X)$ the bundle $J^1_K(\pi) = J^1(\pi)$ is the conventional 1-jet bundle.

Proposition 2. (1) *Space $J^1_K(\pi)$ of K-partial 1-jets of sections of the bundle π_{XY} has a natural structure of affine bundle $\pi_{10\ K}$ over Y based on the vector bundle $\pi^*(K^*) \otimes V(\pi) \to Y$ and of the fiber bundle over X. Here $K^*(X)$ is the vector bundle dual to the subbundle $K(X) \subset T(X)$.*
(2) *There is a canonical surjection of affine bundles*

$$w_K : J^1(\pi) \to J^1_K(\pi),$$

associating with any class of equivalent sections of order 1 of the bundle π containing a section ϕ the class of K_x-equivalent of order 1 sections of π containing section ϕ.
(3) *Let $T(X) = K(X) \oplus K'(X)$ be a decomposition of the tangent bundle of X into the direct sub of vector subbundles (an almost product structure (AP)), then the commutative diagram*

$$\begin{array}{ccc} J^1(\pi) & \xrightarrow{w_K} & J^1_K(\pi) \\ {\scriptstyle w_{K'}}\downarrow & & \downarrow{\scriptstyle \pi_{10\ K}} \\ J^1_{K'}(\pi) & \xrightarrow{\pi_{10\ K'}} & Y \end{array}$$

is the diagram determining $J^1(\pi)$ as the fiber product of partial affine 1-jet bundles with respect to K and K' over Y.
(4) *Let $K_1 \subset K_2 \subset T(X)$ be two subbundles of the tangent bundle $T(X)$. Then there is defined the canonical surjection $w_{21} : J^1_{K_2}(\pi) \to J^1_{K_1}(\pi)$ such that $w_{K_1} = w_{21} \circ w_{K_2}$.*

(5) *(Functoriality) Let the lower square of the diagram*

$$\begin{array}{ccc} J^1_K(\pi_1) & \xrightarrow{j^1(f)_K} & J^1_K(\pi_2)\\ {\scriptstyle \pi^1_1}\downarrow & & {\scriptstyle \pi^2_1}\downarrow \\ Y_1 & \xrightarrow{f} & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ X & = & X \end{array}$$

represents a morphism of the bundles $f : Y_1 \to Y_2$ over the manifold B. Then there exists the morphism of the bundles $j^1(f)_K : J^1_K(\pi_1) \to J^1_K(\pi_2)$ such that the diagram above is commutative.

Proof. Almost all statements of this proposition are simple and their proof just repeat the proof of similar statements for the full 1-jet bundle $J^1(\pi)$. To prove the last statement of the proposition notice that for any section s of the bundle π_1 and any (local) section ξ of the subbundle $K \subset T(X)$ we have for the section $s' = f \circ s$ of the bundle π_2:

$$\xi \cdot s'^j(x) = \frac{\partial f^j}{\partial y^i}(\xi \cdot s^i)(x)$$

and thus, the derivatives of components of a section s' in the directions of the subbundle K are defined by the linear mapping of the derivatives of components of the sections s in the same direction. Therefore the mapping sending the point $(x, y, z^i_{\xi_k}$, ξ_k being a local basis of the bundle K to the point $(x, y' = f(x, y), z'^j_{\xi_k} = \frac{\partial f^j}{\partial y^i}(x, y)z^i_{\xi_k}$ is defined correctly (independent on a choice of a section s) and determine the mapping $\hat{f}_K$. □

Remark 5. If a subbundle $K \subset T(X)$ is chosen, the G-orthogonal complement to K: $K' = K^{\perp_G}$ can be taken as the complemental subbundle K' in the AP structure $T(X) = K \oplus K'$.

Let now a section ν of the bundle $(J^1(\pi), \pi_{10}, Y)$ over Y (**a jet field or Ehresmann connection on** π) is chosen ([13, 19]). The section ν of the full 1-jet bundle determines, by composition with the surjection w_K the section of the bundle $J^1_K(\pi)$. Thus, in the affine fibers $J^1_y(\pi)$ of π^1_0 over Y (respectively $\pi^1_{K\ 0} : J^1_K \to Y$) *a point $\nu(y)$ is chosen.*

This defines an identification of affine space $J^1_y(\pi)$ with the vector space $V_y \otimes T^*_{\pi(y)}(X)$.

$$I_\nu : J^1_y(\pi) \simeq V_y \otimes T^*_{\pi(y)}(X), \tag{4.1}$$

and similar identification of the partial 1-jet bundle

$$J^1_{K\ y}(\pi) \simeq V_y \otimes \pi^*(K^*(X)), \tag{4.2}$$

where $K^*(X) \equiv T^*_x(X)/K^\perp(X)$.

Thus, a choice of a connection ν identifies (noncanonically) 1-jet bundles $J^1_K(\pi)$ with the **vector bundles**: $J^1_{K\ \nu}(\pi) \simeq V_\pi \otimes \pi^*(K^*(X))$, see [19], Sec.17.2.

If an almost product structure $T(X) = K \oplus K'$ is chosen, by definition of the fiber product there is the bijection between the **pairs of sections** of the bundles

$J^1_K\ (\pi), J^1_{K'}(\pi)$ over Y and the sections of the bundle $J^1(\pi)$. Thus, a choice of connection ν defines the sections in the bundles $J^1_K\ (\pi), J^1_{K'}(\pi)$ over Y and, therefore the commutative diagram of fiber product of vector bundles

$$\begin{array}{ccc} J^1_\nu(\pi) & \xrightarrow{w_K} & J^1_{K\ \nu}(\pi) \\ w_{K'}\Big\downarrow & & \pi^{1\ 0}_K\Big\downarrow \\ J^1_{K'\ \nu}(\pi) & \xrightarrow{\pi^{1\ 0}_{K'}} & Y \end{array} \tag{4.3}$$

Splitting $T(X) = K \oplus K'$ of the almost product structure defines dual splitting of the cotangent bundle:

$$T^*(X) = K^* \oplus K'^*,$$

where $K^* \equiv K'^{\perp},\ K'^* \equiv K^{\perp}$ are annulators of the complemental subbundles. As a result, the isomorphism (4.2) splits

$$I_\nu : J^1_y(\pi) \simeq V_y \otimes T^*_{\pi(y)}(X) = V_y \otimes K^*_{\pi(y)} \oplus V_y \otimes K'^*_{\pi(y)}. \tag{4.4}$$

In particular, this defines the affine subbundle

$$Z_{K0} = \{(x, y, z) \in Z |\ I_\nu(z) \in V_y \otimes K^*_{\pi(y)}\},$$

corresponding to the vector subbundle $V(\pi) \otimes K^*$ in the decomposition (4.4).

Example 3. In this example $X = T \times M \equiv R^{(1+3)}$ is the Galilean space-time, i.e. the 4-dim space-time with the block-diagonal Euclidian metric $H = dt^2 + h$ and the action of Galilean group

$$G = T^4 \times O(3) \times V^3,$$

where T^4 is the group of 4-dim translations in X, $O(3)$ is the orthogonal group of euclidian metric h in physical space M and V^3 is the group of inertia frame transformations.

Example 4. In this example we take $\pi : R^3 \to R^2$ with coordinates (t, x) on the base$X$ (time + one space variable), $K =< \partial_x >$ is the subbundle of derivative along space direction. Consider the constitutive relation defined on the partial 1-jet bundle $J^1_K(\pi)$ leading to the Poincare-Cartan form $\Theta_C = -\sigma \wedge dy = ydy \wedge dx + (\frac{y^2}{2} - \delta z_x)dy \wedge dt + 0dy \wedge dt \wedge dx$ (conservation law). Then the balance equation defined by this constitutive relation is

$$d\sigma = 0 \Leftrightarrow \partial_t y + \partial_x(\frac{y^2}{2} - \delta y_{,x}) = y_t + yy_x - \delta y_{xx} = 0\ -$$

Burgers equation.

Example 5. For the same bundle π as in the previous example and for the same partial 1-jet bundle take the constitutive relation leading to the Poincare-Cartan form $-\sigma \wedge dy = \Theta_C = z_x dy \wedge dx + cos(y)dy \wedge dt + 0dy \wedge dt \wedge dx$.

Then the corresponding balance (conservation) law takes the form

$$d\sigma = 0 \Leftrightarrow \partial_t z_x + \partial_x cos(y) = y_{,tx} - sin(y) = 0\ -$$

sin-Gordon equation.

Remark 6. If we would like to write down the KdV equation $y_t + 6yy_x + y_{xxx} = 0$ in the form similar to one of the last two examples, we would need to use the Poincare-Cartan form

$$\Theta = ydy \wedge dx + (z_{xx} - 3y^2)dy \wedge dt,$$

so that we would need to use the *partial 2-jet bundle* as the domain of corresponding constitutive relation, or to increase dimension of the state space U of fields y^i by adding first derivative z_x in their list.

Definition 2. *Let J^1_p is a partial 1-jet bundle of the bundle π. Denote by $J^2_p(\pi)$ the subbundle*

$$J^2_p(\pi) = J^2(\pi) \cap J^1(J^1_p(\pi)) \subset J^2(\pi).$$

Sections of this bundle are 1-jets of sections of the bundle $J^1(J^1_p(\pi))$ modulo the mixed derivative equality whenever one is applicable (see [19]).

4.3. **Space-time splitting case - bundles $J^1_S(\pi)$.** When the partial jet spaces $J^1_K(\pi)$ with different K mixes to produce more complex partial jet bundle we get a more complicated factor of the full 1-jet bundle. As an example consider the situation where the fiber U of the configurational bundle Y splits into the subspaces of fields that enters the constitutive relations with only time, only space and space-time derivatives (5.1):

$$U = U_0 \oplus U_t \oplus U_x \oplus U_{tx}, \tag{4.5}$$

corresponding to the splitting S of the set of indices:

$$S = S_0 \cup S_t \cup S_x \cup S_{xt} : \quad [0, m] = [0, m_0] \cup [m_0 + 1, m_1] \cup [m_1 + 1, m_2] \cup [m_2 + 1, m]. \tag{4.6}$$

Here U_0 includes the fields $y^i, i \in S_0$ whose first derivatives do not enter the CR, U_t includes the fields $y^i, i \in S_t$ whose time derivative enters the CR but their spacial gradient does not, U_x is formed by the fields $y^i, i \in S_x$ whose spacial gradient but not the time derivative enter the CR, finally, U_{tx} is formed by the fields $y^i, i \in S_{tx}$ all derivative of which enter the CR. We assume that all the fields are tensor or tensor density fields in the space X. This splitting is, therefore, $Diff(T) \times Diff(B)$-invariant.

Using the decomposition (5.1) together with the splitting $T(X) = T(R_t) \oplus T(B)$ (see Sec.2) we can introduce the following

Definition 3. *Let S be a diagram of a splitting the fiber of the bundle π as the sum of subbundles (4.5). We define the* ***partial 1-jet bundle*** *$J^1_S(\pi)$ starting with the equivalence relation for two local sections $s_1, s_2 : X \to Y$ defined in a neighborhood of a point $x \in X$:*

$$s_1 \sim_{S,x} s_2 \Leftrightarrow s^i_{1,t}(x) = s^i_{2,t}(x),\ i \in S_t;\ s^i_{1,x^A}(x) = s^i_{2,x^A}(x),\ i \in S_x, A = 1, \dots, n;$$
$$s^i_{1,x^\mu}(x) = s^i_{2,x^\mu}(x),\ i \in S_{tx}, \mu = 0, \dots, n \tag{4.7}$$

and following the steps of definition of $J^1_K(\pi)$.

Y	U_0	U_t	U_x	U_{tx}
$J^1_S(\pi)$	0	$u^i_{,t}$	$u^i_{,x}$	$u^i_{,t}, u^i_{,x}$
$J^2_S(\pi)$	$u^i_{,t}, u^i_{,x}$	$u^i_{,x}, u^i_{,tt}, u^i_{tx}$	$u^i_{,t}, u^i_{,tx}, u^i_{xx}$	$u^i_{,tt}, u^i_{,tx}, u^i_{xx}$

TABLE 1. 1- and 2-partial jet spaces $J^i_S(\pi)$.

Bundle $J^1_S(\pi)$, as its fiber the space of the first derivatives of sections $s : X \to Y$ of the following type

$$u^i_{,t},\ i \in S_t = [m_0{+}1, m_1];\ u^i_{,x^\mu},\ i \in S_x = [m_1{+}1, m_2];\ u^i_{,t}, u^i_{,x^\mu},\ i \in S_{tx} = [m_2{+}1, m].$$

Statements in the next Proposition follows directly from definitions and we omit their proof.

Proposition 3. (1) *The bundle $J^1_S(\pi)$ is defined correctly with respect to the diffeomorphisms from $Diff(R_t) \times Diff(B)$ of the base manifold $X = R_t \times B^3$ containing arbitrary diffeomorphisms of B^3 and the independent time diffeomorphisms of R_t.*

(2) *Correspondingly to the decomposition (4.5) we have the decomposition of the bundle $Y \to X$ as the fiber product of vector bundles over X*

$$Y = Y_0 \underset{X}{\times} Y_t \underset{X}{\times} Y_x \underset{X}{\times} Y_{xt}, \tag{4.8}$$

where $\pi_0 : Y_0 \to X$ has U_0 as its fiber, $\pi_t : Y_t \to X$ has U_t as its fiber, $\pi_x : Y_x \to X$ has U_x as its fiber, $\pi_{tx} : Y_{tx} \to X$ has U_{tx} as its fiber.

(3) *For the partial 1-jet bundle $J^1_S(\pi)$ we have the following decomposition into the fiber product of* ***affine bundles***

$$J^1_S(\pi) = 0(Y_0) \underset{Y}{\times} J^1_{\langle \partial_t \rangle}(Y_t) \underset{Y}{\times} J^1_{\langle \partial_{x^A} \rangle}(Y_x) \underset{Y}{\times} J^1(Y_{tx}) \tag{4.9}$$

over X.

(4) *(Functoriality) Let the lower square of the diagram*

$$\begin{array}{ccc} J^1_S(\pi_1) & \xrightarrow{j^1(f)_S} & J^1_S(\pi_2) \\ {\scriptstyle \pi^1_1}\downarrow & & {\scriptstyle \pi^2_1}\downarrow \\ Y_1 & \xrightarrow{f} & Y_2 \\ {\scriptstyle \pi_1}\downarrow & & {\scriptstyle \pi_2}\downarrow \\ X & = & X \end{array}$$

represents a morphism of the bundles $f : Y_1 \to Y_2$ over the manifold B such that $f(U_{1\ K_i}) \subset U_{2\ K_i}$ for the subbundles K_i be the subbundles or the splitting S, i.e. $K_i =< 0 >, < \partial_t >, < \partial_{x^A} >, T(X)$. Then there exists the morphism of the bundles $j^1(f)_S : J^1_S(\pi_1) \to J^1_S(\pi_2)$ such that the diagram above is commutative.

In Table 1 there are listed the derivatives of sections corresponding to a decomposition S of the state space U (here $J^2_S(\pi) = J^2(\pi) \cap J^1(J^1_S(\pi))$, see above)

Remark 7. Notice here that the case $J^1_S(\pi)$ includes, as its special cases, RET case, case $Z_p = J^1_K(\pi)$ and the case of the full 1-jet bundle $J^1(\pi)$. Thus, probably, this situation is the most general case of a partial 1-jet bundle over π important in applications (that includes only 1-jet bundles but not higher order bundles).

Introduce the notion of geometric automorphisms of the bundle π that can be lifted to the bundle $J^1_S(\pi)$.

Definition 4. (1) *An automorphism $\phi \in Aut(\pi)$ of the bundle π is called a S-admissible if it is the automorphism of the fiber product bundle decomposition (4.4),i.e. there are automorphisms: ϕ_0 of the bundle $Y_0 \to X$, ϕ_t of the bundle $Y_t \to X$ etc. such that*

$$\phi = \phi_0 \underset{Y}{\times} \phi_t \underset{Y}{\times} \phi_x \underset{Y}{\times} \phi_{tx}.$$

Lie group of S-admissible automorphisms of the bundle π will be denoted by $Aut_S(\pi)$.

(2) *A projectable vector field $\xi \in \mathcal{X}(Y)$ is called S-admissible if transformations of its local flow ϕ_t are S-admissible. Denote by $\mathcal{X}_S(\pi)$ the Lie algebra of all S-admissible projectable vector fields in Y.*

Following simple Lemma describes the structure of S-admissible vector fields.

Lemma 1. *A vector field $\xi \in \mathcal{X}(\pi)$ belongs to $\mathcal{X}_S(\pi)$ if and only if it has the form*

$$\xi = \xi^\mu(x,t)\partial_{x^\mu} + \sum_{i_0 \in S_0} \xi^{i_0}(x, y^{j_0},\ j^0 \in S_0)\partial_{y^{j_0}} + \sum_{i_1 \in S_t} \xi^{i_1}(x, y^{j_1},\ j^1 \in S_t)\partial_{y^{j_1}} + \\ + \sum_{i_2 \in S_x} \xi^{i_2}(x, y^{j_2},\ j^2 \in S_x)\partial_{y^{j_2}} + \sum_{i_3 \in S_{tx}} \xi^{i_3}(x, y^{j_3},\ j^3 \in S_{tx})\partial_{y^{j_3}}. \quad (4.10)$$

Finally, we have the following description of the 1-jet bundle of the bundle $J^1_S(\pi)$. Notice that fibers of this bundle contains values of derivatives of first and second order of some fields y^i.

Proposition 4. *Affine bundle $J^1(J^1_S(\pi))$ over $J^1_S(\pi)$ has its fiber modeled on the vector space*

$$(T^*(X) \otimes U_0) \oplus (T^*_{x,xt,tt}(X) \otimes U_t) \oplus (T^*_{t,xt,xx}(X) \otimes U_x) \oplus (T^*_{t,x,tx,xx}(X) \otimes U_{tx}),$$

where the types of partial derivatives of fields y^i from different components U_* of the field space U included into the second partial jet bundle are marked.

Remark 8. Natural dual affine bundle to the bundle $Z_p = J^1_p(\pi)$ is the subbundle of the bundle Z^*. To see this we recall the natural affine projection $J^1(\pi) \to J^1_p(\pi)$ introduced above. Correspondingly we get the induced monomorphism of dual affine bundles

$$J^1_p(\pi)^* \to J^1(\pi)^* = Z^*.$$

Yet below we will be mostly interested by the mappings from $J^1_p(\pi)$ to the whole space Z^* and $\tilde{Z}$.

5. Balance Equations with the domain in $J^1_p(\pi)$

In this section we define balance equations with the domain being an open subset of a partial 1-jet bundle and the balance systems - basic notion of this work.

Definition 5. (1) *A balance equation (law) with a domain $D \subset J^1_p(\pi)$ is an $n+(n+1)$ semibasic form B defined in the domain D:*

$$B = F^\mu \eta_\mu + \Pi\eta,\ \ F^\mu, \Pi \in C^\infty(D).$$

(2) *A section $s \in \Gamma(\pi)$ is a solution of the balance equation B if*

$$\tilde{d}j^1_p(s)^*(B) = dj^1_p(s)^*(F^\mu \eta_\mu) - j^1_p(s)^* \Pi\eta = 0.$$

Here $\tilde{d}$ is the Iglesias differential, see Appendix II.

(3) *A balance law B is called trivial if any section of the bundle π is its solution.*

Balance laws with a domain $D \subset J^1_p(\pi)$ form a vector space - subspace $\mathcal{BL}(D) \subset \Lambda^*_{sb}(D)$ of the subalgebra of semi-basic forms $\Lambda^*_{sb}(D)$ of the exterior algebra $\Lambda^*(D)$ in the domain D.

Lemma 2. *A balance law $B = q^\mu \eta_\mu + \Pi\eta$ is trivial if and only if $\Pi = \sum_\mu d_\mu q^\mu$, where $d_\mu = \partial_{x^\mu} + z^i_\mu \partial_{y^i}$ is the total derivative by x^μ.*

Proof. Standard. □

Remark 9. Functions q^μ should be such that the jet variables z^i_μ be admissible variables of $J^1_p(\pi)$ provided that $\partial_{y^i} q^\mu \neq 0$. This places a limitations on the type of functions q^μ. Namely,

(1) For the RET case $q^\mu = q^\mu(x)$ can not depend on y.
(2) For the full case there are no restriction to the dependence of $q^\mu(x, y)$ on x, y.
(3) For $Z_K = J^1_K(\pi)$ with $K = T(B)$, one should have $\partial_{y^i} q^0 = 0$ for all i.
(4) For $Z_K = J^1_K(\pi)$ with $K = T(R_t)$ one should have $\partial_{y^i} q^A = 0$ for all $A = 1, \dots, n$.

Definition 6. *Two balance equations (laws) $B_k = F^\mu_k \eta_\mu + \Pi_k \eta$ are called Div-equivalent if $B_2 - B_1 = q^\mu \eta_\mu + (\sum_\mu d_\mu q^\mu)\eta$ for some functions $q^\mu \in C^\infty(Z_p)$.*

It is clear that the balance equations which are Div-equivalent has the same space of solutions $s \in \Gamma(\pi)$.

Definition 7. *A balance system defined in a domain $D \subset J^1_p(\pi)$ is a subspace in the space $\mathcal{BL}(D)$ of the balance laws defined in D.*

Remark 10. As defined, the notion of balance system is very broad. To be more practically useful, one has to deal with the systems large enough to specify all the components $s^i(x)$ of the basic fields y^i and small enough to be determined. Second condition is usually achieved by requiring that the number of equation in a balance system is equal to the number m of basic fields. First condition requires the fulfillment of some regularity conditions (see Section 14 below) that may depend on the problem studied with the balance system.

6. Partial Cartan distribution.

The 1-jet space $Z = J^1(\pi)$ is endowed with the canonical *Cartan distribution* Ca locally (in the adapted coordinates) defined by the 1-forms

$$\omega^i = dy^i - z^i_\mu dx^\mu.$$

Cartan distribution is the direct sum of two distributions:

$$Ca_z = D_z \oplus V(\pi_{10}), \tag{6.1}$$

where $V(\pi_{10}) =< \partial_{z^i_\mu} >$ is the vertical subbundle of the tangent bundle $T(Z)$ with respect to the projection $\pi_{10} : Z \to Y$ and

$$D_z =< d_\mu = \partial_{x^\mu} + \sum_i z^i_\mu \partial_{y^i} >$$

is the subbundle generated by the (truncated) total derivatives by x^μ. Distribution D is defined correctly with respect to the automorphisms of the bundle π but it is not integrable.

Distribution D allows to define the contact lift of vector fields from X to Z:

$$\xi = \sum_\mu \xi^\mu(x)\partial_{x^\mu} \ \to \hat{\xi} = \sum_\mu \xi^\mu(x) D_\mu.$$

In a case where the subbundle K is not integrable we will have to use non-holonomic frames in X and the corresponding coframes.

Let (x^μ, y^i) is a local adopted coordinate chart in the bundle π. Let $\xi_\mu = \xi^\lambda_\mu \partial_\lambda$ be a (local) nonholonomic frame of the tangent bundle $T(X)$. Denote by ψ^μ its dual coframe: $< \psi^\mu, \xi_\nu >= \delta^\mu_\nu$. Using this definition it is easy to see that

$$\hat{\psi}^\sigma = (\xi^\alpha_\beta)^{-1}{}_\mu{}^\sigma dx^\mu.$$

Let $\hat{\psi}^\mu$ be the pullback of the 1-form ψ^μ to $Z = J^1(\pi)$ by the projection π^1 : $Z = J^1(\pi) \to X$.

Following simple Lemma gives the representation of the Cartan distribution in Z in terms of such a non-holonomic frame.

Lemma 3. *Let (x^μ, y^i) is a local adopted coordinate chart in the bundle π. Let ξ_μ be a (local) nonholonomic frame of the tangent bundle $T(X)$. Denote by ψ^μ its dual coframe ($< \psi^\mu, \xi_\nu >= \delta^\mu_\nu$). Let $\hat{\psi}^\mu$ be the pullback of the 1-form ψ^μ to $Z = J^1(\pi)$ by the projection π^1. Introduce the (local) coordinates in the fibers of the bundle $Z = J^1(\pi) \to Y$ by*

$$\widetilde{z}^i_\mu(j^1(s)(x)) = (\xi_\mu \cdot s^i)(x)$$

for all sections $s : X \to Y$. Then

(1) *We have $\tilde{z}^i_\mu = \xi^\lambda_\mu z^i_\mu$.*
(2) *Cartan distribution Ca in Z is defined by the forms*

$$\widetilde{\omega}^i = dy^i - \sum_\mu \tilde{z}^i_\mu \hat{\psi}^\mu,$$

(3) *Cartan distribution is generated by the vector fields*

$$Ca(z) =< \widehat{\xi}_\mu = \xi_\mu + \sum_i \tilde{z}^i_\mu \partial_{y^i},\ \partial_{z^i_\mu},\ \mu = 1, \ldots, n+1 > .$$

Proof. We know that Cartan distribution is generated by the vertical vector fields $\partial_{z^i_\mu}$ trivially annulated by the forms $\widetilde{\omega}^i$ and by the linearly independent vector fields $\widehat{\xi_\mu} = \xi^\lambda_\mu \widehat{\partial_\mu}$. Thus it is sufficient to check that $\widetilde{\omega}^i(\widehat{\xi_\mu}) = 0$ for all i, μ. We have

$$\widetilde{\omega}^i(\widehat{\xi_\mu}) = \langle dy^i - \sum_\mu \tilde{z}^i_\mu \hat{\psi}^\mu, \xi^\lambda_\mu d_\lambda \rangle = \langle dy^i - \sum_\sigma \tilde{z}^i_\sigma \hat{\psi}^\sigma, \xi^\lambda_\mu (\partial_{x^\lambda} + \sum_i z^i_\lambda \partial_{y^i})\rangle =$$
$$= \xi^\lambda_\mu \left(z^i_\lambda - \tilde{z}^i_\sigma \hat{\psi}^\sigma(\partial_\lambda)\right) = \xi^\lambda_\mu \left(z^i_\lambda - \tilde{z}^i_\sigma (\xi^\alpha_\beta)^{-1}{}_\lambda{}^\sigma \right) = \xi^\lambda_\mu z^i_\lambda - \tilde{z}^i_\sigma \delta^\sigma_\lambda = \xi^\lambda_\mu z^i_\lambda - \tilde{z}^i_\lambda = 0. \tag{6.2}$$

To prove the third statement we notice that

$$\widehat{\xi_\mu} = \xi^\lambda_\mu (\partial_\lambda + z^i_\lambda \partial_{y^i}) = \xi_\mu + \xi^\lambda_\mu z^i_\lambda \partial_{y^i} = \xi_\mu + \tilde{z}^i_\mu \partial_{y^i}.$$

□

Recall the following

Definition 8. *An exterior form ν^k on the 1-jet space $Z = J^1(\pi)$ is called* ***contact*** *if for all sections $s \in \Gamma(\pi)$, $j^1(s)^*\nu = 0$.*

Contact forms on Z form the ideal $C\Lambda^*(Z)$ of the exterior algebra $\Lambda^*(Z)$. Forms ω^i defined above in the case of a holonomic frame or forms $\tilde{\omega}^i$ in a case of a non-holonomic frame generate the ideal $C\Lambda^*(Z)$.

In the 2-jet bundle $J^2(\pi)$ with local coordinates $x^\mu, y^i, z^i_\mu, z^i_{\mu\nu}$ similar Cartan distribution is defined generated by the ideal of contact form with the generators

$$\omega^i = dy^i - z^i_\mu dx^m u, \omega^i_\mu = dz^i_\mu - z^i_{\mu\nu} dx^\nu. \tag{6.3}$$

In a contrast to the full 1-jet bundle in the maximal (RET) case the fiber of $J^1_p(\pi)$ is one point and has no local geometrical structure. We will show that in the intermediate case a partial 1-jet bundles $J^1_K(\pi), J^1_S(\pi)$ have the "partial Cartan distribution", corresponding to the structure of the fibers of $J^1_p(\pi) \to Y$. This distribution although depending not just on the subbundle K but on the complemental distribution K' as well (i.e. on the whole almost product structure $T(X) = K \oplus K'$, [23]) plays an important role for the partial jet bundles similar to that of the conventional Cartan distribution.

6.1. **Case of $J^1_K(\pi)$, K - integrable.** We start with the case of a decomposition $T(X) = K \oplus K'$ of the tangent bundle of the base X and the corresponding decomposition $T^*(X) = K^* \oplus K'^*$ of the cotangent bundle into the direct sum of two *integrable* subbundles. Locally, one can choose a coordinate chart $x^\mu =< x^\nu; x^\sigma >$ such that (with respect to the index splitting $\mu =< \nu, \sigma >$)

$$K =< \partial_{x^\nu} >;\ K' =< \partial_{x^\sigma} > .$$

Almost product structure allows to split both subdistributions of the decomposition (5.1) as the sums of K- and K'-subdistributions

$$\begin{cases} V(\pi_1) = V(\pi_1)_K \oplus V(\pi_1)_{K'},\ V(\pi_1)_K(z) =< \partial_{z^i_\nu} >, V(\pi_1)_{K'}(z) =< \partial_{z^i_\sigma} >, \\ D(z) = D_K(z) \oplus D_{K'}(z),\ D_K(z) =< D_\nu(z) >,\ D_{K'}(z) =< D_\sigma(z) > . \end{cases} \tag{6.4}$$

This decomposition is invariant under the automorphisms ϕ of the bundle π whose projection $\bar{\phi}$ to X *preserves the almost product structure* $T(X) = K \oplus K'$.

Now we define the 1-forms on the partial 1-jet bundle $J^1_K(\pi)$:

$$\omega^i_K = dy^i - \sum_\nu z^i_\nu dx^\nu$$

(summation by ν only!) in the domain of the chart x^μ. These 1-forms are defined correctly *with respect to the diffeomorphisms of X preserving the decomposition* $T(X) = K \oplus K'$ (i.e. leaving both distributions of this decomposition invariant). It is easy to check that a section q of the bundle $J^1_K(\pi)$ is the (partial) 1-jet of a section $s : X \to Y$ if and only if $q^*(\omega^j)|_K = 0$ for all $i = 1, \dots, m$.

Thus, we have

Proposition 5. *Let $T(X) = K \oplus K'$ be a decomposition of the tangent bundle of the base X into the direct sum of integrable subbundles.*

(1) *The one forms ω^i defined in the local coordinate chart $x^\mu =< x^\nu; x^\sigma >$ integrating the subbundles K, K' by*

$$\omega^i = dy^i - \sum_\nu z^i_\nu dx^\nu \tag{6.5}$$

generate a distribution on the partial jet space $J^1_K(\pi)$ of codimension m invariant under the diffeomorphisms of X preserving the decomposition $T(X) = K \oplus K'$.

(2) *A section q of the bundle $J^1_K(\pi)$ is the (partial) 1-jet of a section $s : X \to Y$ if and only if $q^*(\omega^j)|_K = 0$ for all $i = 1, \dots, m$.*

(3) *Partial Cartan distribution is the linear span of the vector fields*

$$\partial_{x^\nu} + z^i_\nu \partial_{y^i},\ \partial_{x^\sigma},\ \partial_{z^i_\nu}.$$

(4) *Let χ be a connection in the bundle π. Define the affine subbundle Z_{K0}, (depending on the connection ν and the integrable almost product structure $T(X) = K \oplus K'$) by the equations $z^i_\sigma = 0$. Then the intersection of Cartan distribution Ca of $T(Z)$ with the tangent to the subbundle Z_{K0} is the linear span of the tangent vectors*

$$Ca \cap T(Z_{K0}) =< D_\nu, \partial_{x^\sigma}, \partial_{z^i_\nu}. >$$

(5) *Restriction to Z_{K0} of the projection $Z \to J^1_K(\pi)$ defined the isomorphism of affine bundles $Z_{K0} \to J^1_K(\pi)$ mapping the distribution $Ca \cap T(Z_{K0})$ isomorphically onto the partial Cartan distribution CA_K in $J^1_K(\pi)$.*

6.2. **Case $J^1_K(\pi)$, K - general.** Let now K be a **general** vector subbundle of $T(X)$ and let $T(X) = K \oplus K'$ is the almost product structure containing K as one of the subbundles. Choose a local basis of distribution K (respectively of K') consisting of the vector fields $\xi_\nu, \nu = 1, \dots, k$ (respectively $\xi_\sigma, \sigma = 1, \dots, n+1-k$). Vector fields $\xi_\mu, \mu = 1, \dots, n+1$ form a local frame. Introduce the dual coframe ψ^μ of this frame by requiring that $(< \psi^\mu, \xi_\alpha >= \delta^\mu_\alpha)$.

Frame ξ_μ defines in a domain $W \subset X$ determines the zero curvature connection (absolute parallelism) τ in W. This connection has, in general, a non-zero torsion $T^\mu_{\alpha\beta}$ that can also be defined in terms of the commutators of vector field of the frame or, what is equivalent, in terms of the differentials of the coframe 1-forms:

$$[\xi_\alpha, \xi_\beta] = T^\mu_{\alpha\beta}\xi_\mu;\ d\psi^\mu = T^\mu_{\alpha\beta}\psi^\alpha \wedge \psi^\beta, \tag{6.6}$$

with the tensor $T^\mu_{\alpha\beta}$ being the torsion $T$of the connection τ.

Take the pullbacks $\hat{\psi}_\nu$ of the 1-forms ψ_ν to the bundle $J^1_K(\pi)$. A fiber $J^1_{K\ x,y}$ of the bundle $J^1_K(\pi)$ will be endowed with the defined above (local) coordinates $\tilde{z}^i_\nu = (\xi_\nu \cdot s^i)(x)$ where $s(x)$ is a (local) section of π such that $s(x) = y$ (Notice that these coordinates are defined by the distribution K **only** but not on the complemental distribution K').

Consider the set of 1-forms on $J^1_K(\pi)$

$$\widetilde{\omega}^j = dy^j - \sum_\nu z^j_\nu \hat{\psi}_\nu. \tag{6.7}$$

Let $\sigma = (x^\mu; s^i(x), s^i_\nu(x)) : X \to J^1_K(\pi)$ be a local section of $J^1_K(\pi)$ such that $\sigma^*\omega^j = 0$ for all j. This condition is equivalent to the fulfillment of the conditions

$$s^i_\nu(x) = \xi_\nu s^i(x), j = 1, \dots, k,$$

in other words to the integrability of the section σ. Action of a diffeomorphism of X preserving the almost product structure $T(X) = K \oplus K'$ transforms vector fields of the frame ξ_ν of K, these of the subbundle K', dual coframe and the vertical coordinates $\tilde{z}^i_\nu(x)$ of the section by the action of Jacoby matrix in a coherent way ensuring the correctness of the following definition

Definition 9. *Let K be a general (locally trivial) subbundle of $T(X)$ and let $T(X) = K \oplus K'$ is the almost product structure containing K as one of the subbundles. Let $\xi_\nu, \nu = 1, \dots, k$ be a local frame of distribution K, $\xi_\sigma, \nu = k+1, \dots, n+1$ is the local frame of K', let ψ^μ be a dual coframe of the local frame ξ_μ. Let $\hat{\psi}_\nu$ be the pullback of the 1-forms ψ_ν to the bundle $J^1_K(Y)$. Then we define the (partial) Cartan distribution $Ca_{K\oplus K'}$ on the bundle $J^1_K(\pi)$ as the one determined by the 1-forms*

$$\widetilde{\omega}^j = dy^j - \sum_{\nu=1}^{k} \tilde{z}^j_\nu \widehat{\psi}_\nu. \tag{6.8}$$

Proposition 6. (1) *Distribution Ca_K has the property that a section $\sigma : X \to Z_K = J^1_K(\pi)$ is the K-partial 1-jet of a section $s : X \to Y$ iff $\sigma^*\widetilde{\omega}^j|_K = 0$ for all j.*

(2) *Distribution Ca_K is invariant under the flow lifts of diffeomorphisms of X preserving the AP structure $T(X) = K \oplus K'$.*

(3) *Cartan distribution is generated by the (locally defined) vector fields*

$$\widehat{\xi}_\nu = \xi_\nu + \sum_i \tilde{z}^i_\nu \partial_{y^i},\ \xi_\sigma, \partial_{z^i_\nu}.$$

(4) *Let ν be a connection in the bundle π. Affine subbundle Z_{K0}, defined by the connection ν and the almost product structure $T(X) = K \oplus K'$ (see ()) by the equations $\tilde{z}^i_\sigma = 0$ and the intersection of Cartan distribution Ca of $T(Z)$ with the tangent to the subbundle Z_{K0} is the linear span of the tangent vectors*

$$Ca \cap T(Z_{K0}) =< \widehat{\xi}_\nu, \xi_\sigma, \partial_{\tilde{z}^i_\nu}. >$$

(5) *Restriction to Z_{K0} (see Prop. 5 above) of the projection $Z \to J^1_K(\pi)$ defined the isomorphism of affine bundles $Z_{K0} \to J^1_K(\pi)$ mapping the distribution $Ca \cap T(Z_{K0})$ isomorphically onto the partial Cartan distribution Ca_K in $J^1_K(\pi)$.*

Proof. To prove the second statement of the proposition we write the basic contact forms on the full 1-jet bundles as

$$\widetilde{\omega}^i = \partial_{y^i} - \sum_\mu \tilde{z}^i_\mu \hat{\psi}^\mu.$$

The Cartan distribution Ca is generated, in this basis, by the vector fields $\xi_\nu + \tilde{z}^i_\nu \partial_{y^i}, \xi_\sigma + \tilde{z}^i_\sigma \partial_{y^i},\ \partial_{\tilde{z}^i_\nu}, \partial_{\tilde{z}^i_\sigma}$. Using the connection ν for identifying the affine 1-jet bundles with the corresponding vector bundles (see (5.5)) we see that the projection $J^1(\pi) \to J^1_K(\pi)$ has, in the chosen adopted coordinates nonholonomic in the jet fibers, the form $(x, y, \tilde{z}^i_\nu, \tilde{z}^i_\sigma) \to (x, y, \tilde{z}^i_\nu, 0)$. Under this projection the vector fields $\partial_{z^i_\sigma}$ go to zero while others projects to the vector fields $\xi_\nu + \tilde{z}^i_\nu \partial_{y^i},\ \xi_\sigma,\ \partial_{\tilde{z}^i_\nu},\ 0$ respectively (if we assume that $\tilde{z}^i_\sigma$ goes to zero. These vector fields are horizontal with respect to the partial contact structure on the partial 1-jet space $J^1_K(\pi)$. □

Let only subbundle $K \subset T(X)$ is given and we complete it to the AP-structure in two different ways:

$$T(X) = K \oplus K_1 = K \oplus K_2.$$

Let η_ν (respectively $\eta_{\sigma\ 1}, \eta_{\sigma\ 2}$ be a local basis of distribution K (respectively, of $K_1,\ K_2$). Mapping $\beta : T(X) \to T(X)$ given by

$$\eta_\nu \to \eta_\nu,\ \eta_{\sigma\ 1} \to \eta_{\sigma\ 2}$$

defines the pure gauge automorphism of the tangent bundle $T(X)$ preserving subbundle K and exchanging subbundles K_1 and K_2. Dual mapping β^* defines the automorphism of $T^*(X)$ sending $\psi^{\sigma\ 1} \to \psi^{\sigma\ 2}$ but sending $\psi^{\nu\ 1}$ into other 1-forms $\psi^{\nu\ 2}$. Coordinates $\tilde{z}^i_\nu$ in the fiber of the partial 1-jet bundle $J^1_K(\pi)$ defined by the condition $\tilde{z}^i_{\nu\ 1}(j^1_K(s)) = \eta_\nu \cdot s^i =< \eta_\nu, ds^i >$ are mapped under the isomorphism β to the *same* coordinates $\tilde{z}^i_{\nu\ 2}(j^1_K(s)) = \eta_\nu \cdot s^i =< \eta_\nu, ds^i >$ and the pullbacked formes $\hat{\psi}^{\nu\ 1}$ in $J^1_K(\pi)$ are mapped to the forms $\hat{\psi}^{\nu\ 2}$ - pullbacks of the forms $\psi^{\nu\ 2}$. Thus, the generating forms $\widetilde{\omega}^i_{K1} = dy^i - \tilde{z}^i_\nu \hat{\psi}^{\nu\ 1}$ of the first partial Cartan distribution are mapped to the generating forms $\widetilde{\omega}^i_{K2} = dy^i - \tilde{z}^i_\nu \hat{\psi}^{\nu\ 2}$ of the second Cartan distribution. In terms of vector fields we have the mapping $(\hat{\xi}_\nu, \xi_{\sigma\ 1}, \partial_{z^i_\nu}) \to N_\nu, \xi_{\sigma\ 2}, \partial_{z^i_\nu})$ of the first PCS to the second PCS. This mapping is the (pure) gauge isomorphism of the first structure to the second. It is clear that if the AP-structures $K \oplus K_j$, $J = 1, 2$ are integrable with a local integrating charts $(x^{nu}, x^{\sigma\ j})$ then the geometrical mapping of the change of coordinates between these charts $(x^{nu}, x^{\sigma\ 1}) \to (x^{nu}, x^{\sigma\ 2})$ determines the isomorphism of partial Cartan structures defined above for general case. Thus we have proved the following

Proposition 7. *Let a subbundle $K \subset T(X)$ is given. Let*

$$T(X) = K \oplus K_1 = K \oplus K_2$$

are two ways to complete K to an AP-structure. Then the partial Cartan structures $CA_{K\ j},\ j = 1, 2$ in $J^1_K(\pi)$ defined by these two AP-structures are isomorphic. Isomorphism between these structures leaves invariant the sub-distribution of the (partial) Cartan distribution $< \widehat{\xi}_\nu, \partial_{z^i_\nu} >$. If the AP-structures $K \oplus K_j$ are integrable, isomorphism between corresponding partial Cartan distributions is generated by the (geometrical) coordinate change of adopted (local) charts.

6.3. **Case of $J^1_S(\pi)$.** consider now the partial 1-jet bundle $J^1_S(\pi)$ corresponding to the decomposition () of the basic field space U and to the fiber product decompositions for Y and for $J^1_S(\pi)$ given in Proposition (). Integrable product structure $T(X) = \langle \partial_t \rangle \oplus \langle \partial_{x^A} \rangle$ allows to define conventional Cartan distribution Ca_{tx} in the 1-jet bundle $J^1(\pi_{tx})$ and partial Cartan structures Ca_t, Ca_x in the partial 1-jet bundles $J^1_{\partial_t}(\pi_t), J^1_{\partial_{x^A}}(\pi_x)$ respectively. Define now the distribution

$$Ca_S = Ca_t \oplus Ca_x \oplus Ca_{xt}. \tag{6.9}$$

Combining the results for all partial bundles (three of them since the bundle Y_0 has zero fibers) we come to the following

Proposition 8. *Let (S, U_i) is the splitting of the fields space U of the form (). Define in the partial 1-jet bundle $J^1_S(\pi)$ the distribution Ca_S as the direct (fibered) sum of the partial Cartan distribution for all four partial 1-jet bundles $J^1_i(\pi)$*

$$Ca_S = Ca_t \oplus Ca_x \oplus Ca_{xt}.$$

Then,

(1) *Distribution Ca_S is generated by the following 1-forms*

$$\omega^i = dy^i - z^i_t dt,\ i \in S_t,\ \omega^i = dy^i - z^i_A dx^A,\ i \in S_x; \omega^i = dy^i - z^i_\mu dx^\mu,\ i \in S_{tx}.$$

(2) *A section $\sigma : X \to J^1_S(\pi)$ is integrable: $\sigma = i^1_p(s)$ for some section $s : X \to Y$ if and only if*

$$\sigma^* \omega^i = 0,\ \textit{for all } i.$$

6.4. **Contact ideal on Z_p.**

Definition 10. *An exterior form ν^k on the 1-jet space $Z = J^1_p(\pi)$ is called* ***contact*** *if for all sections $s \in \Gamma(\pi)$, $j^1_p(s)^* \nu = 0$.*

Contact forms on Z_p form the ideal $I_p(Ca)$ of the exterior algebra $\Lambda^*(Z_p)$. In local coordinates (x^μ, y^i) denote by P the set of pairs of indices (μ, i) such that coordinate z^i_μ is defined in $Z_p = J^1_p(\pi)$.

Forms $\omega^i = dy^i - \sum_{(\mu,i)\in P} z^i_\mu dx^m u$ defined above generate the ideal $I(Ca)$.

In the 2-jet bundle $J^1(J^1_p(\pi))$ with local coordinates $x^\mu, y^i; z^i_\mu, (\mu, i) \in P; z^i_{\mu nu}, (\mu, i) \in P$ similar partial Cartan distribution is defined, generated by the ideal of contact form with the generators

$$\omega^i = dy^i - \sum_{(\mu,i)\in P} z^i_\mu dx^m u; \omega^i_\mu = dz^i_\mu - z^i_{\mu nu} dx^\nu, \sum_{(\mu,i)\in P}. \tag{6.10}$$

7. Lift of vector fields to $Z_p, \tilde{Z}, W$.

7.1. **Transformations of X.** In the space-time manifolds $X = T \times \bar{B}$ of a concrete physical systems or in some natural bundles over X (tangent bundle, frame bundle, etc.) that are the place for the field variables there are usually defined and undergone the study different groups of transformations reflecting the covariance and invariance properties of the geometrical structures of this theory or even their dynamical behavior. Examples of such groups are:

(1) Diffeomorphism group of X,
(2) Automorphism group of the space-time bundle $\pi_{BX} : X \rightarrow B$,
(3) Group of diffeomorhpismes of the manifold with the boundary $\bar{B}$,
(4) A Lie group G of material symmetries of material manifold B (in a case where B is a material manifold) acting on the frame bundle $F(B)$,
(5) Group of Galilean Transformations acting in the Newtonian space-time $X = T \times E^3$,
(6) Subgroup V of the last group of the transition to the frame moving with constant velocity,
(7) Poincare group acting in the space-time of special relativity (R^{1+3}, η),
(8) A gauge group G^X corresponding to a Lie subgroup $G \subset GL(n, R)$ and acting on the tangent or frame bundle of X,
(9) Affine group acting on Euclidian space $E^3 = (R^3, h)$.

7.2. **Transformation groups in Y.** If a bundle Y is a natural bundle ([11]) or if $Y = Y_1 \oplus Y_2$, where bundle Y_1 is natural, then the action of a group G on X is naturally lifted to the action in Y (respectively in Y_1 and then in Y by trivial extension) in such a way that the projection $\pi : Y \rightarrow X$ becomes a G-morphism.

If Y is not a natural bundle, one can use an Ehresmann connection $\nu : Y \rightarrow J^1(Y)$ in the bundle $\pi : Y \rightarrow X$ to lift vector fields of infinitesimal action of G (and, possibly, the action of the group G itself) to the ν-horizontal vector fields in Y: $\xi \rightarrow \hat{\xi}$. Vector fields $\hat{\xi}$ are projectable vector fields in Y. Such a lift will be the morphism of Lie algebras (i.e. $[\hat{\xi}, \hat{\eta}] = \widehat{[\xi, \eta]}$) *provided the curvature of connection ν vanishes.* Denote by $Aut(\pi)$ the group of automorphisms of the bundle π - diffeomorphisms $\phi \in Diff(Y)$ projecting to the diffeomorphisms $\bar{\phi}$ of X and by $\mathcal{X}(\pi)$ the Lie algebra of $Aut(\pi)$ formed by the projectable vector fields in Y.

Consider the situation where G is a subgroup of the group $\mathcal{X}(\pi)$ of automorphisms of the bundle π - a group of diffeomorphisms of Y preserving fibers of the bundle π. Transformations $g \in G$ project to the diffeomorphisms g_0 of X forming the subgroup $G_0 \subset Diff(X)$. Epimorphism $G \rightarrow G_0,\ g \rightarrow g_0$ of groups has a normal subgroup N of G as its kernel:

$$1 \rightarrow N \rightarrow G \rightarrow G_0 \rightarrow 1$$

is the corresponding exact sequence. N is the intersection of G with the group $G\mathcal{X}(\pi)$ of pure gauge automorphisms - automorphisms of bundle π acting in fibers and, therefore, generating identity diffeomorphism of the base (see [19]).

Remark 11. It is possible that the transformations from G_0 can be naturally lifted to the automorphisms of π: $g \rightarrow \hat{g}$. This happens for instance if the fields y^i are **tensor fields or tensor densities fields** on X. Lifts of elements $h \in G_0$ form a subgroup $\hat{G}_0 \subset \mathcal{X}(\pi)$. Nothing guarantees that $\hat{G}_0 \subset G$ but if this happens, then one get the semidirect product decomposition $G = \hat{G}_0 \times N$ with G acting by automorphisms of N.

One may consider projection $g \to g_0$ as the action of G on X. This action naturally lifts to the action of G by automorphisms of the bundles of exterior forms $\Lambda^k(X) \to X$. Then action of G on Y and $\Lambda^{n+(n+1)}X$ *leaves invariant its subbundle* $\Lambda^{n+(n+1)}_r Y$ over Y because the action of G by automorphisms of the bundle π send fibers of $\pi : Y \to X$ into fibers and therefore leaves the vertical subbundle $V(Y) \subset T(Y)$ invariant. We formulate this result as the following

Lemma 4. *Let $G \subset Aut(\pi)$ be a Lie group of automorphisms of the bundle π.*

(1) *The projection G_0 of the group G to X lifts to the natural bundle of exterior algebras $\Lambda^*(X)$ such that the pullback of the forms*
$$\pi^* : \Lambda^*(X) \to \Lambda^*(Y)$$
is equivariant with respect to the projection $g \to g_0$.

(2) *Subbundles $\Lambda^k_r(Y)$ are invariant under the lifted action of G.*

Let now action of G on X by $g \to g_0$ *preserves the subbundle* $K \subset T(X)$. Then one can naturally define the action of G on the partial 1-jet bundle $J^1_K(\pi)$ in such a way that the projections $J^1_K(\pi) \to Y \to X$ become G-morphisms (see below).

Similarly, if an action of the group G leaves the splitting (5.1) invariant and its projection G_0 leaves invariant the space-time decomposition $T(X) = T(\mathbb{R}_t) \oplus T(B)$ of the tangent bundle, one may lift its action to $J^1_S(\pi)$.

If an action of G can be lifted to the bundle $J^1_p(\pi)$ and to $\Lambda^{n+(n+1)}_2 Y$, by taking the fiber product of these actions we may lift the action of G to the space $W_{0p} = J^1_p \times \Lambda^{n+(n+1)}_2 Y$. As a result we may pose a question to study the lifted action of the group $\hat{G}$ on the constitutive relations $\mathcal{C}$, lifted CR $\hat{\mathcal{C}}$, Cartan-Poincare forms Θ_C and the balance system generated by $\hat{C}$ (see below, Sec.).

Let us look in more details at these prolongations of transformations (in global as well as in infinitesimal variants).

7.3. Lift of vector fields and transformations to $J^1_p(\pi)$.

Definition 11. *Denote by $Aut(\pi^1_p)$ the automorphism group of the double bundle $\pi^1_p : J^1_p(\pi) \to Y \to X$ i.e. diffeomorphisms of $J^1_p(\pi)$ projecting to Y and X. Introduce the corresponding Lie algebra $\mathcal{X}(\pi^1_p)$ of vector fields ξ.*

Recall that for $K = T(X)$, $J^1_p(\pi) = J^1(\pi)$.

Vector fields $\xi \in \mathcal{X}(\pi^1)$ have, in adapted local coordinates (x^μ, u^i, z^i_μ), the form

$$\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x^\nu, y^j)\partial_{y^i} + \xi^i_\mu(x^\nu, y^j, z^j_\nu)\partial_{z^i_\mu}. \tag{7.1}$$

7.3.1. *Case $Z_p = Z = J^1(\pi)$.* Recall ([19, 48]) that there exists the natural lift $\xi \to \xi^1$ of an arbitrary vector field $\xi \in \mathcal{X}(\pi)$ to the vector field ξ^1 in $Z = J^1(\pi)$ defined by the conditions described in the following

Proposition 9. *(see* [19, 48]*).*

(1) *For any vector field $\xi \in \mathcal{X}(Y)$ there is a unique vector field $\xi^1 \in \mathcal{X}(Z)$* ***(1-jet prolongation of*** *ξ) defined by the conditions:*

(a) *Vector field* $\xi^1 \in \mathcal{X}(Z)$ *is projectable to* Y *and*

$$\pi_{10*}(\xi^1) = \xi,$$

(b) *Local flow of the vector field* ξ^1 *preserves the Cartan distribution Co (such a vector field is called an infinitesimal contact transformation).*

(2) *The lift* ξ^1 *of a vector field* $\xi = \xi^\mu(x,y)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i}$ *has in local adapted coordinates the form*

$$\xi^1 = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \left(\frac{d\xi^i}{dx^\mu} - z^i_\nu \frac{d\xi^\nu}{dx^\mu}\right)\partial_{z^i_\mu}, \tag{7.2}$$

where $D_\mu \xi^i = \frac{d\xi^i}{dx^\mu} = \frac{\partial \xi^i}{\partial x^\mu} + z^j_\mu \frac{\partial \xi^i}{\partial y^j}$ *is the total derivative of the function* ξ^i *and similarly for* ξ^μ.

(3) *The mapping* $\xi \to \xi^1$ *is the homomorphism of Lie algebras:*

$$[\xi, \eta]^1 = [\xi^1, \eta^1]$$

for all $\xi, \eta \in \mathcal{X}(Y)$.

(4) *For a projectable vector field* $\xi \in \mathcal{X}(\pi)$, $\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i}$ ***1-jet prolongation*** ξ^1 *coincide with the* ***flow prolongation*** *(see below).*

The **flow prolongation (lift)** is defined by the local flow $\phi_t(x,y)$ of the vector field $\xi \in \mathcal{X}(\pi)$. Let $\bar{\phi}_t(x)$ be the flow induced by ϕ_t in X (having the vector field $\xi^\mu(x)\partial_{x^\mu}$ as the generator. Flow ϕ_t acts on sections $y = s(x)$ of the bundle π by the rule: $s \to (\phi s)(x) = \phi_t s(\bar{\phi}_t^{-1}(x))$. Differentiating by t at $t = 0$ we get the generator of action on the 1-jet part $s^i_{,x^\mu}$ in the form (7.2) (see ([19, 48])).

The flow lift of automorphisms from $Aut(\pi)$ and of corresponding vector fields is the homomorphism of groups (Lie algebras)

$$Aut(\pi) \to Aut(\pi^2_0),\ \mathcal{X}(\pi) \to \mathcal{X}(\pi^2_0) \tag{7.3}$$

that locally, with respect to the adopted chart have the (7.2)

7.3.2. *Case of* $Z_p = J^1_K(\pi)$. Let now K be a subbundle of the tangent bundle $T(X)$ and let $T(X) = K \oplus K'$ be an AP-structure containing K. Let $\eta_\nu (= \partial_{x^\nu}$ in an integrable case) be a local basis of K and denote by $\tilde{z}^i_\nu$ the corresponding local coordinates in the fiber of the bundle $\pi_{10} : J^1_K(\pi) \to Y$ (see above).

Definition 12. (1) *Denote by* $\mathcal{X}_K(\pi)$ *the Lie algebra of* π*-projectable vector fields* ξ *in* Y *such that the field* $\bar{\xi}$ *generated by* ξ *in* X preserves the distribution $K \subset T(M)$: $\bar{\phi}_{t*}K = K$ *for the local flow* $\bar{\phi}_t$ *of the vector field* $\bar{\xi}$.

(2) *Denote by* $\mathcal{X}_{K \oplus K'}(\pi)$ *the Lie algebra of* π*-projectable vector fields* ξ *in* Y *such that the field* $\bar{\xi}$ *generated by* ξ *in* X preserves the distributions $K, K' \subset T(M)$: $\bar{\phi}_{t*}K = K$ *for the local flow* $\bar{\phi}_t$ *of the vector field* $\bar{\xi}$ *(and the same for* K'*).*

Lemma 5. *Let the AP-structure* $T(X) = K \oplus K'$ *is integrable and let* (x^ν, x^σ) *be a (local) integrating chart. Then*

(1) *A* π*-projectable vector field* $\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i}$ *belongs to* $\mathcal{X}_K(\pi)$ *if and only if*

$$\bar{\xi} = \xi^\mu(x^\nu, x^\sigma)\partial_{x^\mu} = \xi^\nu(x)\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma},$$

i.e. if the components $\xi^\sigma(x)$ *do not depend on the variables* x^ν.

(2) *A π-projectable vector field $\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i}$ belongs to $\mathcal{X}_{K\oplus K'}(\pi)$ (preserves the almost product structure $T(X) = K \oplus K'$) if and only if*

$$\bar{\xi} = \xi^\mu(x)\partial_{x^\mu} = \xi^\nu(x^{\nu_1})\partial_{x^\nu} + \xi^\sigma(x^\sigma)\partial_{x^\sigma},$$

Proof.

$$[\bar{\xi}, \partial_{x^\nu}] = -(\partial_{x^\nu}\cdot\xi^{\nu_1})\partial_{x^{\nu_1}} - (\partial_{x^\nu}\cdot\xi^\sigma)\partial_{x^\sigma}.$$

This vector field belongs to K if and only if $\partial_{x^\nu}\cdot\xi^\sigma = 0$ for all ν and σ. The second statement is proved in the same way. □

Proposition 10. *Let the AP-structure $T(X) = K\oplus K'$ is integrable and let (x^ν, x^σ) be a (local) integrating chart.*

(1) *For a vector field $\xi = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} \in \mathcal{X}_K(\pi)$ the following properties are equivalent*
   (a) *There exist a vector field $\xi^1 \in \mathcal{X}(J^1_K(\pi))$ such that*
      (i) *Local flow of the vector field ξ^1 preserves the partial Cartan distribution Ca_K.*
      (ii) $\pi_{10\,*}\xi^1 = \xi$.
   (b) *Vector field ξ has, in a local integrating chart (x^ν, x^σ) the form*

$$\xi = \xi^\nu(x^{\nu_1})\partial_{x^\nu} + \xi^\sigma(x^{\sigma_1})\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i}.$$

   In particular the projection $\bar{\xi}$ of the vector field ξ in X preserved the almost product structure $K\oplus K'$.

(2) *In the case where these conditions are fulfilled the vector field ξ^1 is unique and is given by the formula*

$$\xi^1 = \xi^\nu(x^{\nu_1})\partial_{x^\nu} + \xi^\sigma(x^{\sigma_1})\partial_{x^\sigma} + \xi^i(x^\nu, y)\partial_{y^i} + \left(d_\nu\xi^i - z^i_{\nu_1}\frac{\partial\xi^{\nu_1}}{\partial x^\nu}\right)\partial_{z^i_\nu} \qquad (7.4)$$

(3) *Mapping $\xi \to \xi^1$ is the homomorphism of Lie algebras:*

$$[\xi,\eta]^1 = [\xi^1, \eta^1]$$

for all $\xi, \eta \in \mathcal{X}_{K,K'}(\pi)$.

Proof. Let $\hat{\xi} = \xi^\mu(x)\partial_{x^\mu} + \xi^i(x,y)\partial_{y^i} + \lambda^i_\nu\partial_{z^i_\nu}$ be a prolongation to the partial jet bundle Z_K of the vector field ξ. Then, condition of the preservation of the partial Cartan structure is equivalent to the condition that for all the generators $\omega^i_K = dy^i - \sum_\nu z^i_\nu dx^\nu$ of the contact ideal of exterior forms,

$$\mathcal{L}_{\hat{\xi}}\omega^i_K = \sum_j q^i_j\omega^j_K,$$

for some functions $q^i_j \in C^\infty(Z_p)$. We calculate

$$\mathcal{L}_{\hat{\xi}}\omega^i_K = (di_{\hat{\xi}} + i_{\hat{\xi}}d)(dy^i - \sum_\nu z^i_\nu dx^\nu) = d[\xi^i - z^i_\nu\xi^\nu] + i_{\hat{\xi}}(-dz^i_\nu\wedge dx^\nu) =$$

$$= d\xi^i - \xi^\nu dz^i_\nu - z^i_\nu d\xi^\nu - \lambda^i_\nu dx^\nu + \xi^\nu dz^i_\nu = \xi^i_{,x^\mu}dx^\mu + \xi^i_{,y^j}dy^j - z^i_\nu[\xi^\nu_{,x^{\nu_1}}dx^{\nu_1} + \xi^\nu_{,x^\sigma}dx^\sigma] - \lambda^i_\nu dx^\nu =$$

$$= \sum_j q^i_j(dy^j - \sum_\nu z^j_\nu dx^\nu), \quad (7.5)$$

or

$$(\xi^i_{,x^\sigma} - z^i_\nu\xi^\nu_{,x^\sigma})dx^\sigma + \xi^i_{,y^j}dy^j + [\xi^i_{,x^\nu} - \lambda^i_\nu - z^i_{\nu_1}\xi^{\nu_1}_{,x^\nu}]dx^\nu = \sum_j q^i_j(dy^j - \sum_\nu z^j_\nu dx^\nu).$$

This equality is fulfilled if and only if we have

$$\begin{cases} \xi^i_{,x^\sigma} - z^i_\nu \xi^\nu_{,x^\sigma} = 0, \\ q^i_j = \xi^i_{,y^j}, \\ \xi^i_{,x^\nu} - \lambda^i_\nu - z^i_{\nu_1}\xi^{\nu_1}_{,x^\nu} = -q^i_j z^j_\nu. \end{cases}$$

Since neither ξ^i nor ξ^ν depend on z^i_μ first system is equivalent to the requirement that both ξ^i and ξ^ν are independent on x^σ. Then the second condition determines q^i_j and third - $\lambda^i_\nu = \xi^i_{,x^\nu} + \xi^i_{,y^j} z^j_\nu - z^i_{\nu_1}\xi^{\nu_1}_{,x^\nu}$ and the prolongation $\hat{\xi}$ takes the form described in the Proposition. □

7.3.3. *Case of $Z_p = Z_S = J^1_S(\pi)$.* Consider now the case of partial 1-jet bundle $J^1_S(\pi)$ generated by the (x,t)-decomposition S (see section ??). By Proposition 3 the bundle Y is the fiber product of the bundles $Y = Y_0 \underset{X}{\times} Y_t \underset{X}{\times} Y_x \underset{X}{\times} Y_{xt}$ and the partial 1-jet has the form of the the fiber product $J^1_S(\pi) = 0(Y_0)\underset{Y}{\times} J^1_t(Y_t)\underset{Y}{\times} J^1_x(Y_x)\underset{Y}{\times} J^1(Y_{tx})$.

A natural class of automorphisms of the bundle π is the class of S-automorphisms of π (see Definition 4) and corresponding class of S-admissible vector fields in Y $\xi \in \mathcal{X}_S(\pi)$. In simple words these are geometrical or infinitesimal automorphisms of the bundle π that preserve the S-type of fields under transformation.

In this case we have the canonical integrable AP-structure $T(X) = T(B)\oplus <\partial_t>$ with a local chart (x,t). Applying the arguments used for the study of prolongation $\xi \to \xi^1$ to the partial 1-jet bundles $J_K(\pi)$ we get the following analog of previous Proposition:

Proposition 11. (1) *A vector field $\xi \in \mathcal{X}_S(\pi)$ preserves the AP-structure $T(B)\oplus <\partial_t>$ if and only if*

$$\bar{\xi} = \xi^\mu(x,t)\partial_{x^\mu} = \xi^A(x)\partial_{x^A} + \xi^t(t)\partial_t,$$

(2) *For any S-admissible π-projectable vector field $\xi \in \mathcal{X}_S(\pi)$ following statements are equivalent*
   (a) *There is a vector field $\xi^1 \in \mathcal{X}(Z_S)$ such that*
      (i) *Vector field $\xi^1 \in \mathcal{X}(Z_S)$ is π_{10}-projectable and*

$$\pi_{10*}(\xi^1) = \xi,$$

      (ii) *Local flow of the vector field ξ^1 preserves the partial Cartan distribution Co_S at $Z_S(\pi)$.*
   (b) *Vector field ξ has the following form*

$$\xi = \xi^A(x)\partial_{x^A} + \xi^t(t)\partial_t + \sum_{i_0\in S_0}\xi^{i_0}(x,t;y^{j_0},\ j^0\in S_0)\partial_{y^{j_0}} + \sum_{i_1\in S_t}\xi^{i_1}(t;y^{j_1},\ j^1\in S_t)\partial_{y^{j_1}} + \\ + \sum_{i_2\in S_x}\xi^{i_2}(x;y^{j_2},\ j^2\in S_x)\partial_{y^{j_2}} + \sum_{i_3\in S_{tx}}\xi^{i_3}(x,t;y^{j_3},\ j^3\in S_{tx})\partial_{y^{j_3}}. \quad (7.6)$$

   where dependence of vertical components ξ^i of the vector field ξ on the variables x^A, t is specified by the subset S_* containing index i.

(3) *In the case where these conditions are fulfilled the vector field ξ^1 is unique and is given by the formula (recall that $x^0 = t$)*

$$\xi^1 = \xi + \sum_{i_1 \in S_t} \left(\frac{d\xi^{i_1}}{dx^0} - z^{i_1}_{x^0} \frac{\partial \bar{\xi}^0}{\partial x^0} \right) \partial_{z^{i_1}_{x^0}} + \sum_{i_2 \in S_x} \left(\frac{d\xi^{i_2}}{dx^A} - z^{i_2}_{x^B} \frac{\partial \bar{\xi}^B}{\partial x^A} \right) \partial_{z^{i_2}_{x^A}} + \\ + \sum_{i_3 \in S_{tx}} \left(\frac{d\xi^{i_3}}{dx^\mu} - z^{i_3}_{x^\nu} \frac{\partial \bar{\xi}^\nu}{\partial x^\mu} \right) \partial_{z^{i_3}_{x^\mu}}. \quad (7.7)$$

(4) *Mapping* $\xi \to \xi^1$ *is the homomorphism of Lie algebras:*

$$[\xi, \eta]^1 = [\xi^1, \eta^1]$$

for all $\xi, \eta \in \mathcal{X}_K(Y)$.

7.4. **Case of a general AP-structure** $T(X) = K \oplus K'$**.** Consider now a case where AP-structure $T(X) = K \oplus K'$ is not integrable. Denote by ξ_ν (respectively by ξ_σ local frames of distributions K, K' respectively, by ψ^ν, ψ^σ - dual coframe. Introduce the structural equation

$$d\psi^\mu = T^\mu_{\beta\gamma}(x)\psi^\beta \wedge \psi^\gamma \quad (7.8)$$

of the coframe ψ^μ, where T is the tensor defined above.

Denote by $\tilde{z}^i_\nu$ the vertical coordinates in the partial frame bundle $Z_K = J^1_K(\pi)$ defined by the condition $\tilde{z}^i_\nu(j^1_p(s)) = \xi_\nu s^i$ for all sections s of the bundle π.

Recall that the partial Cartan distribution in $J^1_K(\pi)$ is generated by the 1-forms $\widetilde{\omega}^i = dy^i - \sum_{\nu \in K} \tilde{z}^i_\nu \hat{\psi}^\nu$, where $\hat{\psi}^\nu = \pi^1{}^* \psi^\nu$ is the pullback of a coframe 1-form to the partial jet bundle Z_K.

Let now $\xi = \bar{\xi}^\mu \xi_\mu + \xi^i \partial_{y^i}$ be a projectable vector field in Y with the projection $\bar{\xi}$ in X and let

$$\xi^1 = \bar{\xi}^\mu \xi_\mu + \xi^i \partial_{y^i} + \lambda^i_\nu \partial_{\tilde{z}^i_\nu}$$

be some prolongation of vector field ξ to the bundle Z_K.

We would like to find conditions on the field ξ under which there exists its prolongation to $J^1_K(\pi)$ preserving the partial Cartan structure Ca_K.

We calculate:

$$\mathcal{L}_{\xi^1} \widetilde{\omega}^i = \mathcal{L}_{\xi^1}(dy^i - \sum_{\nu \in K} \tilde{z}^i_\nu \hat{\psi}^\nu) = (di_{\xi^1} + i_{\xi^1} d)(dy^i - \sum_{\nu \in K} \tilde{z}^i_\nu \hat{\psi}^\nu) =$$

$$-i_{\xi^1}(d\tilde{z}^i_\nu \wedge \hat{\psi}^\nu + \tilde{z}^i_\nu d\hat{\psi}^\nu) + d(\xi^i - \tilde{z}^i_\nu < \hat{\psi}^\nu, \bar{\xi}) = -(\xi^1 \cdot \tilde{z}^i_\nu)\hat{\psi}^\nu + < \hat{\xi}, \psi^\nu > d\tilde{z}^i_\nu - \tilde{z}^i_\nu i_{\xi^1}(d\hat{\psi}^\nu) +$$

$$+ d\xi^i - \tilde{z}^i_\nu d < \psi^\nu, \bar{\xi} > - < \psi^\nu, \bar{\xi} > d\tilde{z}^i_\nu = d\xi^i - \tilde{z}^i_\nu d < \psi^\nu, \bar{\xi} > -(\xi^1 \cdot \tilde{z}^i_\nu)\hat{\psi}^\nu - \tilde{z}^i_\nu i_{\xi^1}(d\hat{\psi}^\nu). \quad (7.9)$$

In the last term $i_{\xi^1}(d\hat{\psi}^\nu) = i_{\bar{\xi}}(d\psi^\nu)$ since $d\hat{\psi}^\nu = d\hat{\psi}^\nu$.

Flow of the vector field ξ^1 preserves the partial Cartan distribution if and only if

$$\mathcal{L}_{\xi^1} \widetilde{\omega}^i = q^i_j \widetilde{\omega}^j = q^i_j (dy^i - \sum_{\nu \in K} \tilde{z}^i_\nu \hat{\psi}^\nu)$$

for some functions q^i_j on $J^1_K(\pi)$. Using the calculation above we get to the condition

$$\widehat{d\xi^i} - \tilde{z}^i_\nu d \widehat{< \psi^\nu, \bar{\xi}} > - \lambda^i_\nu \hat{\psi}^\nu - \tilde{z}^i_\nu \widehat{i_{\bar{\xi}}(d\psi^\nu)} = q^i_j (dy^i - \sum_{\nu \in K} \widetilde{z}^i_\nu \hat{\psi}^\nu) \quad (7.10)$$

for all i (we have used $\xi^1 \cdot \tilde{z}^i_\nu = \lambda^i_\nu$).

We have in the last formula $< \psi^\nu, \bar{\xi} >= \xi^\nu$ and we will use the relations

$$d\xi^i = (\xi_\mu \cdot \xi^i)\psi^\mu + (\partial_{y^j}\xi^i)dy^j,\ d\xi^\nu = (\xi_\mu \cdot \xi^\nu)\psi^\mu.$$

Using these two formulas together with (7.8) in (7.10) we write it in the form

$$(\xi_\mu \cdot \xi^i)\widehat{\psi^\mu} + (\partial_{y^j}\xi^i)\widehat{dy^j} - \tilde{z}^i_\nu(\xi_\mu \cdot \xi^\nu)\widehat{\psi^\mu} - \lambda^i_\nu\widehat{\psi^\nu} - \tilde{z}^i_\nu i_{\xi^1}(T^\nu_{\beta\gamma}(x)\widehat{\psi^\beta \wedge \psi^\gamma}) = q^i_j\widehat{dy^i} - q^i_j\tilde{z}^i_\nu\hat{\psi}^\nu. \tag{7.11}$$

Comparing coefficients of dy^j we get

$$q^i_j = \partial_{y^j}\xi^i \tag{7.12}$$

and rewrite the rest of (6.12) as follows

$$(\xi_\mu \cdot \xi^i)\widehat{\psi^\mu} - \tilde{z}^i_\nu(\xi_\mu \cdot \xi^\nu)\widehat{\psi^\mu} - \lambda^i_\nu\widehat{\psi^\nu} - \tilde{z}^i_\nu T^\nu_{\beta\gamma}(x)(\xi^\beta\widehat{\psi^\gamma - \xi^\gamma}\psi^\beta) = -(\partial_{y^j}\xi^i)\tilde{z}^i_{\nu_1}\hat{\psi}^{\nu_1}. \tag{7.13}$$

We remind that in this formula ν runs through indices in K while $\mu, \alpha, \gamma, \beta$ through all indices from 0 to n.

Present λ^i_ν in the form

$$\lambda^i_\nu = \bar{\lambda}^i_\nu(x, y) + \tilde{z}^k_{\nu_1}\lambda^{i\nu_1}_{\nu k}(x, y, \tilde{z}),$$

where first term does not depend on the jet coordinates. Such a representation can always done locally. Substitutive this decomposition into (7.13) and extract the terms that does not contain variables $\tilde{z}^i_\nu$ as a factor

$$(\xi_\mu \cdot \xi^i)\psi^\mu - \bar{\lambda}^i_\nu\psi^\nu = 0.$$

This equality is equivalent to two statements

$$\begin{cases} \xi_\sigma \cdot \xi^i = 0 \Leftrightarrow \xi^i = \xi^i(x^\nu, y), \\ \bar{\lambda}^i_\nu = \xi_\nu \cdot \xi^i. \end{cases} \tag{7.14}$$

After excluding terms without z-variables and using the equality

$$T^\nu_{\beta\gamma}(\xi^\beta\widehat{\psi^\gamma - \xi^\gamma}\psi^\beta) = (T^\nu_{\beta\mu}(x(\xi^\beta - T^\nu_{\mu\beta}(x)\xi^\beta)\widehat{\psi^\mu} = 2T^\nu_{\beta\mu}(x)\xi^\beta\widehat{\psi^\mu}$$

valid due to the antisymmetry of $T^\nu_{\alpha\beta}(x)$ by lower indices, the equality (6.14) will take the form

$$-\tilde{z}^i_\nu(\xi_\mu \cdot \xi^\nu)\widehat{\psi^\mu} - \tilde{z}^k_\nu\lambda^{i\nu}_{\nu_1 k}\widehat{\psi^{\nu_1}} - 2\tilde{z}^i_\nu T^\nu_{\beta\mu}(x)\xi^\beta\widehat{\psi^\mu} = -(\partial_{y^j}\xi^i)\tilde{z}^j_{\nu_1}\hat{\psi}^{\nu_1}. \tag{7.15}$$

Equating here coefficients of the 1-forms ψ^μ with $\mu = \sigma$ we get

$$-\tilde{z}^i_\nu(\xi_\sigma \cdot \xi^\nu) - 2\tilde{z}^i_\nu T^\nu_{\beta\sigma}(x)\xi^\beta = 0,$$

or

$$\xi_\sigma \cdot \xi^\nu + 2T^\nu_{\beta\sigma}(x)\xi^\beta = 0. \tag{7.16}$$

These are structural equations for the X-components of the vector field ξ.

Equating coefficients of the form ψ^{ν_1} in (7.15) we finally get

$$-\tilde{z}^i_\nu(\xi_{\nu_1} \cdot \xi^\nu) - \tilde{z}^k_\nu\lambda^{i\nu}_{\nu_1 k}(x, y, \tilde{z}) - 2\tilde{z}^i_\nu T^\nu_{\beta\nu_1}(x)\xi^\beta = -(\partial_{y^k}\xi^i)\tilde{z}^k_{\nu_1}. \tag{7.17}$$

All terms in this formula except the second one on the left side are linear by $\tilde{z}$-variables. Therefore this second term on the left side is also linear by $\tilde{z}$ and it follows from this that the functions $\lambda^{i\nu}_{\nu_1 k}$ depend on x^μ, y^i but not on $\tilde{z}$.

Equating coefficients at $\tilde{z}^k_\nu$ we get

$$\begin{cases} \lambda^{i\nu}_{\nu_1 k}(x,y) = (\partial_{y^k}\xi^i)\delta^\nu_{\nu_1}, \text{ for } k \neq i, \\ (\xi_{\nu_1}\cdot\xi^\nu) + \lambda^{i\nu}_{\nu_1 i}(x,y) + 2T^\nu_{\beta\nu_1}(x)\xi^\beta = (\partial_{y^k}\xi^i)\delta^\nu_{\nu_1}, \text{ for } k = i. \end{cases}$$

From this we get

$$\begin{cases} \lambda^{i\nu}_{\nu_1 k}(x,y) = (\partial_{y^k}\xi^i)\delta^\nu_{\nu_1}, \text{ for } k \neq i, \\ \lambda^{i\nu}_{\nu_1 i}(x,y) = (\partial_{y^k}\xi^i)\delta^\nu_{\nu_1} - (\xi_{\nu_1}\cdot\xi^\nu) - 2T^\nu_{\beta\nu_1}(x)\xi^\beta, \text{ for } k = i. \end{cases} \tag{7.18}$$

Substituting these expressions and (7.14) into the formulas for λ^i_ν (and reversing places of ν and ν_1) we find λ^i_ν in the form

$$\lambda^i_\nu = \xi_\nu\cdot\xi^i + \tilde{z}^i_{\nu_1}\lambda^{i\nu_1}_{\nu i} = \xi_\nu\cdot\xi^i + \sum_k \tilde{z}^i_{\nu_1}(\partial_{y^k}\xi^i)\delta^{\nu_1}_\nu - \tilde{z}^i_{\nu_1}[(\xi_\nu\cdot\xi^{\nu_1}) - 2T^{\nu_1}_{\beta\nu}(x)\xi^\beta] =$$
$$= d_\nu\xi^i - \tilde{z}^i_{\nu_1}[(\xi_\nu\cdot\xi^{\nu_1}) - 2T^{\nu_1}_{\beta\nu}(x)\xi^\beta]. \tag{7.19}$$

Thus, we have proved the following

Theorem 1. *Let $T(X) = K\oplus K'$ be an almost product structure on X. Let $\{\xi_\mu\} = (\xi_\nu, \xi_\sigma)$ be a (local) frame adopted to the AP-structure and let psi^ν, ψ^σ be the dual coframe. Let the structural equations of this coframe be*

$$d\psi^\mu = T^\mu_{\beta\gamma}(x)\psi^\beta\wedge\psi^\gamma.$$

A vector field $\xi = \xi^\mu\xi_\mu + \xi^i\partial_{y^i}$ in Y have a prolongation to a vector field $\hat{\xi} = \xi + \sum_{\nu\in K}\lambda^i_\nu\partial_{\tilde{z}^i_\nu}$ in the partial 1-jet bundle $Z_p = J^1_K(\pi)$ if and only if the condition

$$\xi_\sigma\cdot\xi^\nu + 2T^\nu_{\beta\sigma}(x)\xi^\beta = 0, \text{ for all } \sigma\in K', \nu\in K$$

is fulfilled. In such a case, this prolongation is unique and is given by

$$\xi^1 = \xi + \Big(d_\nu\xi^i - \tilde{z}^i_{\nu_1}[(\xi_\nu\cdot\xi^{\nu_1}) - 2T^{\nu_1}_{\beta\nu}(x)\xi^\beta\Big)\partial_{\tilde{z}^i_\nu}. \tag{7.20}$$

Remark 12. Theorems on the prolongation proved before for $J^1_K(\pi)$ (Proposition 13) for an integrable AP-structure $T(X) = K\oplus K'$ and that for the full 1-jet bundle (Proposition 12) are special cases of the last result.

Under the action of an automorphism $\phi\in Aut_p(\pi)$ a vector field $\xi = \xi^\mu\partial_\mu + xi^i\partial_i$ is transformed as follows

$$\phi_*(\xi) = (\bar{\phi}^\mu_{,nu}\xi^\nu)\partial_\mu + (\phi^i_{,\nu}\xi^\nu + \phi^j_{,i}\xi^i)\partial_j.$$

Automorphism ϕ can be flow lifted to the bundle space $J^1_p(\pi)$ as follows: Let $s(x)$ be a local section of the bundle π, then for the action of ϕ to the section s:

$\phi^* s(x) = \phi(s(\bar{\phi}^{-1}(x)))$ we find

$$z^i_\mu(j^1_p(\phi^* s)(x)) = \partial_\mu \phi^i(s(\bar{\phi}^{-1}(x))) = (\phi^i_{,y^j} \circ s)(x)\partial_\lambda s^j(x)\partial_\mu(\bar{\phi}^{-1\ \lambda})(x) + ((\partial_\lambda \phi^i)\circ s)(x)\partial_\mu(\bar{\phi}^{-1\ \lambda})(x) =$$
$$= J(\phi)^i_j \frac{\partial s^j}{\partial x^\lambda} J(\bar{\phi}^{-1})^\lambda_\mu + J(\phi)^i_\lambda J(\bar{\phi}^{-1})^\lambda_\mu, \quad (7.21)$$

in other words z^i_μ transforms by affine transformation

$$z^i(\phi^1(x,y,z)) = J(\phi)^i_j J(\bar{\phi}^{-1})^\lambda_\mu z^j_\lambda + J(\phi)^i_\lambda J(\bar{\phi}^{-1})^\lambda_\mu. \quad (7.22)$$

For the vertical vector fields we have

$$\phi^i_*(\partial_{z^i_\mu}) f = \partial_{z^i_\mu}(f \circ \phi^i(x,y,z)) = \frac{\partial f}{\partial z^j_\nu} J(\phi)^i_j J(\bar{\phi}^{-1})^\lambda_\mu,$$

therefore, for a projectable vector field $\xi = \xi^\mu(x)\partial_\mu + \xi^i(x,y)\partial_i + \xi^i_\mu \partial_{z^i_\mu}$ we have

$$(\phi^1_* \xi) = \left(\xi^i_\mu J(\phi)^i_j J(\bar{\phi}^{-1})^\lambda_\mu\right) \partial_{z^j_\lambda} + \dots \quad (7.23)$$

7.5. **Prolongation of π-automorphisms to the dual bundles $\Lambda^k_r Y$ and $\tilde{Z}$.** Automorphisms of the bundle π (and, correspondingly, projectable vector fields $\xi \in \mathcal{X}(\pi)$) have a natural (flow) prolongation to the projectable diffeomorphisms (and projectable vector fields) of the double bundle $\Lambda^k_r Y \to Y \to X$ of exterior forms on Y (see [19] or [24]). This lift $\xi \to \xi^{1*}$ is defined by the pullback ϕ^*_t of the exterior forms on Y by the local flow ϕ_t of a vector field $\xi \in \mathcal{X}(\pi)$. Since the commutator of a projectable vector field $\xi^\mu(x)\partial_{x^\mu} + \xi^i(x^\nu, y^j)\partial_{y^i}$ and an arbitrary vertical vector field $\eta = \eta^i(x^\nu, y^j)\partial_{y^i}$ *is vertical*, lifted local automorphisms of π (and the corresponding infinitesimal transformations - Lie derivatives with respect to the lifted vector fields) preserve the subbundles $\Lambda^k_r Y \subset \Lambda^k Y$ and, therefore, define the lifts of automorphism transformations (global, local or infinitesimal) to the corresponding automorphisms of the (double) bundles $\Lambda^k_r Y \to Y \to X$.

Another way to lift a general vector field $\xi \in \mathcal{X}(Y)$ is defined by the following construction that was studied in [24] for the case where the metric G is Euclidian (put $\lambda = 0$ in the formulas of following definition).

Definition 13. *(Definition-Proposition,* [24]*.) Let α be a pullback to $\Lambda^{n+1}_2 Y$ of a π-semibasic form $\alpha = \alpha^\nu(x,y)\eta_\nu$ on Y. Let $\xi \in \mathcal{X}(Y)$.*

(1) *Then there exist and is unique a vector field $\xi^{*\alpha}$ on Λ^{n+1}_2 satisfying to the following conditions*
   (a) *Vector field $\xi^{*\alpha}$ is $\pi_{\Lambda^{n+1}_2 Y\ Y}$-projectable and*

$$\pi_{\Lambda^{n+1}_2 Y\ Y\ *}\xi^{*\alpha} = \xi,$$

   (b)

$$\mathcal{L}_{\xi^{*\alpha}} \Theta^{n+1}_2 = -d\alpha.$$

(2) *Vector field $\xi^{*\alpha}$ has the local form*

$$\xi^{*\alpha} = \xi + \xi^{*\alpha\ p}\partial_p + \xi^{*\alpha\ p^\mu_i}\partial_{p^\mu_i},\ \text{where}$$
$$\xi^{*\alpha\ p} = -p\left(\frac{\partial\xi^\mu}{\partial x^\mu} + \xi^\mu\frac{\partial\lambda}{\partial x^\mu}\right) - p^\mu_i\left(\frac{\partial\xi^i}{\partial x^\mu} - \xi^i\frac{\partial\lambda}{\partial x^\mu}\right) - \frac{\partial\alpha^\mu}{\partial x^\mu} - \alpha^\nu\lambda_{,x^\nu};$$
$$\xi^{*\alpha\ p^\mu_i} = -p^\nu_i\left(\frac{\partial\xi^\mu}{\partial x^\nu} - \xi^\mu\lambda_{,x^\nu}\right) - p^\mu_j\frac{\partial\xi^j}{\partial y^i} - p^\mu_i\left(\frac{\partial\xi^\nu}{\partial x^\nu} - \xi^\nu\lambda_{,x^\nu}\right) - p^\nu_i\xi^\mu\lambda_{,x^\nu} - \frac{\partial\alpha^\mu}{\partial y^i}. \tag{7.24}$$

(3) *Let a vector field $\xi \in \mathcal{X}(Y)$ be π-projectable. Then the 0-lift ξ^{*0} of ξ coincide with the flow prolongation ξ^{1*} defined above.*

Remark 13. In the work [29] there was defined the class of "covariant canonical transformations of $\Lambda^{n+1}_2(Y)$" as π-projectable transformations of $\Lambda^{n+1}_2(Y)$ preserving the multisymplectic form Θ^{n+1}_2. Later on we will use transformations from this class to discuss the transformations of constitutive relations.

Let now $\phi \in Aut(\pi)$ be an automorphism of the bundle π. Arguments in the beginning of this subsection shows that the flow lift ϕ^{1*} of ϕ to the bundle $\Lambda^k Y$ leaves its subbundles $\Lambda^k_r Y$ invariant. In particular, ϕ^{1*} acts on the subbundles $\Lambda^{n+1}_2 Y, \Lambda^{n+2}_2 Y$ leaving their subbundles $\Lambda^{n+1}_1 Y, \Lambda^{n+2}_1 Y$ invariant and leaving canonical forms $\Theta^{n+1}_2 Y$ and $\Theta^{n+2}_2 Y$ invariant. Therefore, ϕ^{1*} generates the automorphism $\widetilde{\phi}^*$ of the bundle $\tilde{Z} = \tilde{Z}^{n+1} \oplus \tilde{Z}^{n+2}$ leaving both terms invariant.

Let $H : \tilde{Z}^{n+1} \to \Lambda^{n+1}_2 Y$ to be a section (see Sec.) of the bundle $\Lambda^{n+1}_2 Y \to \tilde{Z}^{n+1}$. Then for the induced form $\Theta_H = H^*\Theta^{n+2}_2 = H(x, y, p^\mu_i)\eta + p^\mu_i dy^i \wedge \eta_\mu$ we have

$$\tilde{\phi}^*\Theta_H(x, y, p^\mu_i) = \tilde{\phi}^* H^*\Theta^{n+2}_2 = (H\circ\tilde{\phi})^*\Theta^{n+2}_2 = H\circ\tilde{\phi}(x, y, p^\mu_i)\eta + p^\mu_i dy^i \wedge \eta_\mu. \tag{7.25}$$

Thus, though the $(0, n+1)$-term of the form Θ_H is changed, its $(1, n)$-term is invariant. For the infinitesimal action of vector field $\tilde{\xi}^*$ - generator of the 1-parametrical group of difeomorphisms ϕ^{1*}_t we get from the previous formula

$$\mathcal{L}_{\tilde{\xi}^*}\Theta_H = (\tilde{\xi}^* \cdot H)\eta.$$

In particular, we will be using this formula for the sections defined by a connection ν in the bundle π with $H = p^\mu_i \Gamma^i_\mu(x)$.

For our study we need to lift a projectable vector field ξ to the bundle $\tilde{Z} = \tilde{Z}^{n+1} \oplus \tilde{Z}^{n+2} = \Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y = \Lambda^{n+1}_2 Y/\Lambda^{n+1}_1 Y \oplus \Lambda^{(n+2)}_2 Y/\Lambda^{(n+2)}_1 Y$. Next result allows to lift ξ to the bundle $\tilde{Z}^{n+2} = \Lambda^{(n+2)}_2 Y$.

Proposition 12. *For any projectable vector field $\xi \in X(\pi)$ there exists unique projectable vector field $\xi^{*(n+2)}$ on the bundle $\Lambda^{(n+2)}_2 Y$ that leaves the canonical form $p_i dy^i \wedge \eta$ invariant. That vector is given by the relation*

$$\xi^{*(n+2)} = \xi + \xi_k\partial_{p_k},\ \xi_k = p_k\left(\xi^\mu\frac{\partial\lambda}{\partial x^\mu} - \frac{\partial\xi^\mu}{\partial x^\mu}\right) - p_j\frac{\partial\xi^j}{\partial y^k}, \tag{7.26}$$

where $\lambda = ln(|G|)$.

Proof. We have, for a vector field of the form $\xi^{*(n+2)} = \xi + \xi_k \partial_{p_k}$

$$\mathcal{L}_{\xi^{*(n+2)}}(p_i dy^i \wedge \eta) = (di_{\xi^{*(n+2)}} + i_{\xi^{*(n+2)}} d)(p_i dy^i \wedge \eta) = d[p_i \xi^i \eta - p_i \xi^\mu dy^i \wedge \eta_\mu] + i_{\xi^{*(n+1)}}(dp_i \wedge dy^i \wedge \eta_\mu) =$$
$$= [d(p_i \xi^i)\eta - d(p_i \xi^\mu) \wedge \eta_\mu) - p_i \xi^\mu dy^i \wedge d\eta_\mu] + [\xi^i dy^i \wedge \eta - \xi^i dp_i \wedge \eta + \xi^\mu dp_i \wedge dy^i \wedge \eta_\mu] =$$
$$= [\xi^i dy^i \wedge \eta - \xi^i dp_i \wedge \eta + \xi^\mu dp_i \wedge dy^i \wedge \eta_\mu] + [\xi^i dp_i \wedge \eta + p_i \frac{\partial \xi^i}{\partial y^j} dy^j \wedge \eta - \xi^\mu dp_i \wedge dy^i \wedge \eta_\mu +$$
$$+ \frac{\partial \xi^\mu}{\partial x^\mu} p_i dy^i \wedge \eta - (p_i \xi^\mu \frac{\partial \lambda}{\partial x^\mu}) dy^i \wedge \eta] = (\frac{\partial \xi^\mu}{\partial x^\mu} p_i + p_i \frac{\partial \xi^i}{\partial y^j} - (p_i \xi^\mu \frac{\partial \lambda}{\partial x^\mu}) + \xi_i) dy^i \wedge \eta. \tag{7.27}$$

Here we have used the relation $d\eta_\mu = \lambda_{,x^\mu}\eta$.

Equating the obtained expression to zero we get the expression for ξ_i as in the Proposition. □

Combining the last result with the prolongation ξ^{*0} from the Definition-Proposition 10 and with the prolongation $\xi^{*(n+2)}$ from the previous Proposition and using factorization by $\Lambda^{(n+1)+(n+2)}_1 Y$ we get the following

Corollary 1. *For any projectable vector field $\xi \in X(\pi)$ there exists unique projectable vector field ξ^{1*} in the space $\tilde{Z} = \Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y$ - prolongation of ξ,* preserving the $(n+1)+(n+2)$ form $p^\mu_i dy^i \wedge \eta_\mu + p_k dy^k \wedge \eta$.

7.6. **Transformations of W_0 and W_1.** Taking the fiber product of the action of $A = \mathcal{X}(\pi^1)$ (global, local or infinitesimal) in $J^1_p(\pi)$ and its action by the (global, local or infinitesimal) automorphisms of the the bundle $\Lambda^k_r Y \to Y \to X$ induced first by the projection to Y and then by the lift to $\Lambda^k_r Y$, described in the last subsection, we define the (global, local or infinitesimal) action of the group $Aut(\pi^1)$ on the bundles $W^{co} = J^1_p(\pi) \underset{Y}{\times} \Lambda^{(n+1)+(n+2)}_2 Y$ and $W_p = J^1_p(\pi) \underset{Y}{\times} (\Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y) = Z_p \times \tilde{Z}$.

Combining this action with the homomorphism $\mathcal{X}(\pi) \to \mathcal{X}(\pi^1)$ induced by the lift prolongation we get the action of $\mathcal{X}(\pi)$ by the projected diffeomorphisms of W^{co} and W_p.

The action of the group A in W^{co} and W_p will allow to define its action on the vector spaces $\tilde{\mathcal{CR}}$ and $\mathcal{CR}$ of covering and usual constitutive relations respectively (see Section 9 below).

8. Prolongation of the connections to Z_p and $\tilde{Z}$.

A (pseudo-Riemannian) metric G in X determines the linear Levi-Civita connection Γ^G in the tangent bundle $T(X)$. By duality an, by tensor and exterior product it defines connection in the cotangent bundle $T^*(X)$, in the bundles of tensors, in the exterior forms bundles $\Lambda^k(X)$ and similarly in other natural bundles over X, see [11]. On the frame bundle $F(X)$ connection Γ^G is defined by the $so(n,R)$-valued 1-form

$$\omega^G(x,f) = \Gamma^i_{kj}(x)dx^k + \tau^i_j(x),$$

where $\tau(x)$ is the $so(n,R)$-valued 1-form of Maurer-Cartan, while on the tangent bundle $T(X)$ it is defined by the equations

$$d\xi^\mu - \Gamma^\mu_{\nu\sigma}(x)\xi^\nu dx^\sigma = 0,$$

where (x^μ, ξ^ν) are the adopted coordinates in the tangent bundle $T(X)$. Thus, the lift to the tangent bundle of the vector field ∂_{x^μ} is

$$\hat{\partial}_{x^\mu}(x,\xi) = \partial_{x^\mu} + \Gamma^\sigma_{\nu\mu}\xi^\nu \partial_{\xi^\sigma}.$$

A (nonlinear, Ehresmann) connection ν ([19, 27]) in the bundle $\pi : Y \to X$ determines and is determined by the section $q_\nu : (x^\mu, y^i) \to (x^\mu, y^i, \Gamma^i_\mu(x,y))$ of the bundle $J^1(\pi) \to Y$ ([19]). We will always assume this connection to be complete. In a local adapted chart (x^μ, y^i) this connection is defined by the equations

$$dy^i - L^i_\mu(x,y)dx^\mu = 0, \tag{8.1}$$

so that the horizontal lift of a basic vector field ∂_{x^μ} is the (projectable) vector field in Y of the form

$$\partial_{x^\mu} + L^i_\mu(x,y)\partial_{y^i}. \tag{8.2}$$

Connection ν defines the connection on the bundle $VY \to Y$ *linear over* Y. Namely, applying the functor of vertical tangent bundle to the section $q_\nu : Y \to J^1(\pi)$ we get a mapping $Vq_\nu : VY \to VJ^1(\pi)$. Let

$$\begin{array}{ccc} VJ^1(\pi) & \xrightarrow{i_Y} & J^1(VY) \\ {\scriptstyle \pi}\downarrow & & {\scriptstyle \pi_{10}}\downarrow \\ VY & \xrightarrow{=} & VY \end{array} \tag{8.3}$$

be the canonical involution ([48]), then the composition

$$\mathcal{V}_Y\nu := i_Y \circ Vq_\nu : VY \to J^1(VY)$$

determines the connection on $VY \to X$ called the *vertical prolongation of the connection* ν. If $Y^i = dy^i$ are coordinates in VY complemental to the adopted coordinates (x^μ, y^i) then the connection $\mathcal{V}_Y\nu$ is defined by the equations (7.1) and

$$dY^i - \frac{\partial L^i_\mu}{\partial y^j} Y^j dx^\mu = 0. \tag{8.4}$$

More then this, connection ν defined canonically (by the flow prolongation of the flows of horizontal vector fields, see [19], Ch.X) the connections on the bundles $T(Y) \to X$, $T^*(Y) \to X$, $\Lambda^k Y \to X$ satisfying to the proper forms of Leibniz relations with respect to the pairing, tensor and exterior products (see [19]) which we denote by the same letter ν.

If we would like this extension to preserve the subbundles Λ^k_r we would have to modify it using the connection Γ^G on the base manifold X in order to extend the horizontal translation to the bundle $Z^* \simeq T(X) \otimes_Y V^*(\pi) \otimes_Y \wedge\Lambda^n(X)$. We denote by $\hat{\Gamma}$ the obtained connection. We have

Proposition 13. (1) *Vertical subbundle $V(Y) \to X$ of the bundle $T(Y) \to X$ is invariant under the ν-parallel translation along any curve $\gamma : (a,b) \to X$.*
(2) *Ehresmann connection ν in the bundle π and linear connection Γ on X (i.e. on the tangent bundle $T(X) \to X$) define canonically the connection $\hat{\Gamma}$ in*

the bundle $Z^* \to X$ whose horizontal lift is defined as follows

$$\partial_{x^\mu} \to \partial_{x^\mu} + \Gamma^i_\mu(\nu)\partial_{y^i} + \left(-\frac{\partial \Gamma^i_\mu(\nu)}{\partial y^j} p^\sigma_i + \Gamma^\gamma_{\mu\gamma} p^\sigma_j\right)\partial_{p^\sigma_j}. \tag{8.5}$$

(3) *Lift of vector fields $\xi \in \mathcal{X}(X)$ with the help of ν to Y and then - by flow lift to Z^* coincide with the $\hat{\Gamma}$-horizontal lift of ξ.*
(4) *Subbundles $\Lambda^k_r Y$ of the bundles $\Lambda^k(Y)$ are invariant under the $\hat{\Gamma}$-parallel translation along any curve $\gamma : (a,b) \to X$.*
(5) *Canonical multisymplectic forms Ω^k_r and $\tilde{\Omega}$ are invariant under the (local) flows of $\hat{\Gamma}$-horizontal vector fields $\hat{\xi},\ \xi \in \mathcal{X}(X)$:*

$$L_{\hat{\xi}}\Omega^k_r = 0.$$

(6) *Pullback $\pi^* : \Lambda^k(X) \to \Lambda^k(Y)$ embeds the exterior k-form bundle of X into the exterior k-form bundle of Y. Subbundles $\pi^*(\Lambda^s X) \subset \Lambda^s Y$ are invariant under the parallel translation.*

Proof. First statement follows from the fact that parallel translation in Y maps fibers of π into fibers. For the proof of second statement see [6]. Third statement follows from comparison of the comparison of expressions of vector fields lifted in two ways. Third statement follows from the first one. Forth and the fifth statements follows from the fact that (local) flows of ν-horizontal vector fields in Y are projectable to X and, therefore, the pullback by these transformations of the forms from $\pi^*(\Lambda^s X)$ is, by duality, realized by the projected flows. □

Using the connection Γ^G one can lift the connection ν in the bundle π to the connection $\hat{\nu}$ in the bundle $\pi^1 : J^1(\pi) \to X$. This lift can be achieved by different ways, (see the proof proved by I.Kolar and others ([19]) that "all the natural operators transforming a general connection on $\pi : Y \to X$ and a linear connection on M (here Γ^G) into a general connection on $J^1(\pi) \to X$ form the one-parametrical family $tP + (1-t)\mathcal{T}^1,\ t \in R$" where $P, \mathcal{T}^1$ are two distinguished connections.

Here we will use only one of these connections, namely the version of connection P for the partial 1-jet bundles defined as follows (see [19], Sec.45.7). Section q_ν determines the identification of the 1-jet bundle $J^1_p(\pi)$ with the associated vector bundle $\mathcal{V}Y \otimes T^*_p(X)$

$$I : J^1(\pi) \simeq VY \otimes T^*_p(X)$$

similar to one for conventional 1-jet bundle. Here $T^*_p(X)$ is the subbundle of $T^*(X)$ dual to the subbundle $K \subset T(X)$ defined by the AP structure $K \oplus K'$. Vertical prolongation $\mathcal{V}\nu$ of the connection ν was defined above. On the other hand, connection ν determines the horizontal lift of vector fields in X preserving the AP structure $K \oplus K'$ to the vector fields in the cotangent bundle $T^*(X)$ whose (local) flow leaves the dual decomposition $T^*(X) = K^\perp \oplus K'^\perp$ invariant. Therefore, vector fields from $X_{K\oplus K'}(X)$ are lifted to the vector fields in $T^*_K(X) \equiv K'^\perp$. As a result we get the lift of the vector fields

$$T_{K\oplus K'}(X) \to \mathcal{X}(VY \otimes T^*_{K\oplus K'}(X)) \to \mathcal{X}(J^1_K(\pi)). \tag{8.6}$$

This lift determines the subbundle $H_{K\oplus K'}(\nu)$ of the bundle $T(J^1_K(\pi))$ complemental to the vertical tangent subbundle of the bundle $\pi^1_K : J^1_K(\pi) \to X$ and, therefore, the connection in the bundle π^1_K. More then this, horizontal distribution $H_{K\oplus K'}(\nu)$ of this connection splits naturally as the sum of two subbundles - horizontal lifts of bundles K and K' respectively:

$$H_{K\oplus K'}(\nu) = K(\nu) \oplus K'(\nu). \tag{8.7}$$

Denote obtained connection by ν^1_K. We get

Proposition 14. *Let $T(X) = K \oplus K'$ be an AP-structure on X. Let ν be a connection on the bundle $\pi : Y \to X$ and Γ^G is the Levi-Civita connection in X. There is the canonically defined connection ν^1_K in the bundle $\pi^1_K : J^1_K(\pi) \to X$ whose horizontal distribution splits*

$$H_{K\oplus K'}(\nu) = K(\nu) \oplus K'(\nu)$$

as the direct sum of subbundles - horizontal lifts of distributions K, K'.

In the case of partial 1-jet space $J^1_S(\pi)$ where we have the natural product structure $< \partial_t > \oplus T(B)$ similar result is valid for a connection ν on the bundle π *provided the parallel translations by the lifts of vector fields from $\mathcal{X}_{<\partial_t>\oplus T(B)}(X)$ leaves the fiber product structure (5.6) of the bundle $\pi : Y \to X$ invariant.* It is easy to prove the following

Proposition 15. *Let ν be a connection on the bundle $\pi : Y \to X$ such that the fiber product structure $Y = Y_0 \underset{X}{\times} Y_t \underset{X}{\times} Y_x \underset{X}{\times} Y_{xt}$ is invariant under the ν-parallel translation. Then there is the prolongation of ν to the connection ν^1_S in the bundle $\pi^1_S : J^1_S(\pi) \to X$ such that the parallel translation with respect to ν^1_S preserves the fiber product structure $J^1_S(\pi) = 0(Y_0) \underset{Y}{\times} J^1_t(Y_t) \underset{Y}{\times} J^1_x(Y_x) \underset{Y}{\times} J^1(Y_{tx})$. Horizontal distribution of this connection splits*

$$H_S(\nu^1_S) =< \partial_t >^1 \oplus T(B)^1$$

as the sum of two distributions - horizontal lifts of sub-distributions $< \partial_t >$, $T(B)$ respectively.

Remark 14. Connection ν^1_S is the fiber product of the connections in the four bundles in the decomposition $J^1_S(\pi) = 0(Y_0) \underset{Y}{\times} J^1_t(Y_t) \underset{Y}{\times} J^1_x(Y_x) \underset{Y}{\times} J^1(Y_{tx})$ over X. First component of this product is the component of connection ν in the bundle $Y_0 \to X$ itself - no prolongation is necessary for this bundle.

In the future will need the following result allowing to lift a connection ν $q_\nu : Y \to J^1(\pi)$ in the bundle π to the vertical part $\hat{\nu} : J^1(\pi) \to J^1(J^1(\pi))$ of the second order connection on the bundle π^1 that defines the connection on the bundle $\pi_{10} : J^1(\pi) \to Y$.

Proposition 16. *(*[13]*, Proposition 2.6.1) Let Γ be a symmetric linear connection on X and $\nu : J \to J^1(\pi)$ be a connection in the bundle π. Then there is an involution $s_\Gamma : J^1(J^1(\pi)) \to J^1(J^1(\pi))$ over $J^1(\pi)$ such that composition*

$$\hat{\nu}_\Gamma = s_\Gamma \circ \Gamma\nu : J^1(\pi) \to J^1(J^1(\pi)), \tag{8.8}$$

given in local coordinates by

$$\hat{\nu}_\Gamma = dx^\lambda \otimes (\partial_{x^\lambda} + L^i_\lambda \partial_{y^i} + [\partial_{x^\lambda} L^i_\mu + y^j_\lambda \partial_{y^j} L^i_\mu + \Gamma^G{}_{\lambda\mu}{}^\nu (y^i_\nu - L^i_\nu)]\partial_{z^i_\mu}) \tag{8.9}$$

Connection $\hat{\nu}$ allows to extend the vertical 1-form $\theta_C \in \Gamma(V(\pi_{10})^*)$ to the 1-form on the whole space $J^1_p(\pi)$ assuming that it annulate the horizontal subspaces $Hor(\hat{\nu}) \subset T(J^1_p(\pi))$.

Chapter III. Constitutive Relations and Balance Systems.

Below we will use the notation $Z_p = J^1_p(\pi)$ that unite the following special cases: RET, where $J^1_p(\pi) = \{0\}$ is the trivial (with the one point fiber) bundle over Y; $Z_K = J^1_K(\pi)$ for a subbundle K in a AP-structure $T(X) = K \oplus K'$, case $Z_S = J^1_S(\pi)$ for a splitting S of the field bundle (see (4.5-6)),; finally $Z = J^1(\pi)$ - full 1-jet bundle. We will be using both longer and shorter notations whichever is more convenient at a point. If we consider a prolongation $\xi \rightarrow \xi^1$ of a vector field $\xi \in \mathcal{X}(Y)$ to the partial 1-jet bundle, *we will always presume that the vector field ξ satisfies to the conditions for the existence of the prolongation ξ^1 preserving the partial Cartan structure (see Section 6).*

9. General Constitutive Relations (CR).

In this section we define general constitutive relations and the Poincare-Cartan forms defined by these relations. We also give examples of several types of constitutive relations.

Consider the following (constitutive) commutative diagram

$$\begin{array}{ccc}
Z_p = J^1_p(\pi) & \xrightarrow{\hat{C}} & \Lambda^{(n+1)+(n+2)}_2(Y) \\
=\big\downarrow & & \chi\big\downarrow \\
Z_p = J^1_p(\pi) & \xrightarrow{C} & \tilde{Z} = \Lambda^{(n+1)+(n+2)}_2(Y)/\Lambda^{(n+1)+(n+2)}_1(Y) \\
\pi^1_0\big\downarrow & & \pi^{(n+1)+(n+2)}\big\downarrow \\
Y & = & Y \\
\pi\big\downarrow & & \pi\big\downarrow \\
X & = & X
\end{array} . \qquad (9.1)$$

Definition 14. (1) *A (general)* ***constitutive relation*** *(CR) C of a field theory with the configurational bundle $\pi : Y \rightarrow X$ and the partial 1-jet space $Z_p = J^1_p(\pi)$ is a smooth morphism of bundles over Y*

$$C : J^1_p(\pi) \rightarrow \tilde{Z} = \Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y.$$

In local coordinates (x^μ, y^i, z^i_μ) on $J^1_p(\pi)$ and (p^μ_i, q_i) on $\tilde{Z}$ a CR-mapping $\mathcal{C}$ has the form

$$C(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)) \qquad (9.2)$$

(2) *A general constitutive relation $\mathcal{C}$ is called* ***regular*** *if the mapping $\mathcal{C}$ is the diffeomorphism of Z_p onto the submanifold of $\tilde{Z}$.*

(3) *A covering constitutive relation $\hat{C}$ of the field theory with the configurational bundle $\pi : Y \rightarrow X$ and the partial 1-jet space $Z_p = J^1_p(\pi)$ is a smooth mapping of bundles*

$$\hat{C} : J^1_p(\pi) \rightarrow \Lambda^{(n+1)+(n+2)}_2 Y.$$

In local coordinates (x^μ, y^i, z^i_μ) *on* $J^1_p(\pi)$ *and* (p, p^μ_i, q_i) *on* $\Lambda^{n+1}_2(Y) \oplus \Lambda^{n+2}_2(Y)$ *a CCR-mapping* $\widehat{\mathcal{C}}$ *has the form*

$$\widehat{\mathcal{C}}(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; p(x^\mu, y^i, z^i_\mu); F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)) \tag{9.3}$$

(4) *A constitutive relation* C *(respectively a covering CR* $\hat{C}$*) is called a conservative relation (CR) (resp. a covering CCR) if* $\Pi_i = 0, i = 1, \dots, m$.
(5) *For a given constitutive relation* $\mathcal{C}$ *denote by* $\mathcal{C}_-$ *the constitutive relation obtained from* C *by the changing sign of the production ((n + 2)) part:*

$$C_-(x^\mu, y^i, z^i_\mu) = (x^\mu, y^i; F^\mu_i(x^\mu, y^i, z^i_\mu); -\Pi_i(x^\mu, y^i, z^i_\mu)) \tag{9.4}$$

Remark 15. Physical case corresponds to the choice $n = 3$.

Remark 16. Definition given here is very broad, including, in particular, a zero mapping. Thus, to get a useful class of constitutive relations one has to put some nondegeneracy conditions to this mapping including but not reducing to the regularity of a CR defined above.

Remark 17. We can also define constitutive relations defined in a domain $U \subset Z_p$ instead of the whole space Z_p. This may be necessary in a situation where some constraints in the from of inequalities on the derivatives of the fields y^i are present.

Example 6. In the maximal (RET) case $U = U_0$, the partial 1-jet space $J^1_p(\pi)$ coincide with Y (its fiber is a point R^0) and the constitutive relation is just the section of the bundle $\tilde{Z} \to Y$. As we will see in the next section it is convenient and natural to consider the CCR $\widetilde{\mathcal{C}}$ for the RET constitutive relation $\mathcal{C}$ defined on the full 1-jet bundle Z of the bundle π (formally we could take it have η-component zero, but it would be a less convenient choice).

We have the following simple

Proposition 17. (1) *Constitutive relations (and the covering constitutive relations) form the* $C^\infty(J^1_p(\pi))$*-module* $\mathcal{CR}$ *and* $\widehat{\mathcal{CR}}$ *respectively.*
(2) *Let* $\hat{C} \in \widehat{\mathcal{CR}}$ *be a CCR, then combining the defining mapping* $\hat{C} : J^1_p(\pi) \to \Lambda^{(n+1)+(n+2)}_2(Y)$ *with the projection by* $\Lambda^{(n+1)+(n+2)}_1(Y)$ *we associate with a CCR* $\hat{C}$ *the constitutive relation* $C \in \mathcal{CR}$.

Using the canonical forms on the bundle $\Lambda^{(n+1)+(n+2)}_2 Y$ we **define** the **Poincare-Cartan form of the covering constitutive relation** $\hat{C}$

$$\Theta_{\hat{C}} = \mathcal{C}^*(\Theta^{n+1}_2 + \Theta^{n+2}_2) = p\eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta. \tag{9.5}$$

Definition 15. *Canonical linear mapping* $: \mathcal{CR} \to \widehat{\mathcal{CR}}$ *(section of the projection above) is defined by the formula*

$$(\tilde{\mathcal{C}})(z) = (x^\mu, y^i; -z^j_\nu F^\nu_j(x^\mu, y^i, z^i_\mu); F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)).$$

CCR $\tilde{C}$ *will be called the* ***lifted CCR of the constitutive relation*** C.

For the lifted CCR $\tilde{C}$ of a CR C defined in the full 1-jet bundle $Z = J^1(\pi)$ we have

$$\Theta_{\tilde{\mathcal{C}}} = S_\eta(F^\mu_i dz^i_\mu) = -z^i_\mu F^\mu_i \eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta. \tag{9.6}$$

Here S_η is the vertical endomorphism (3.1). Notice that we get the same result by applying the vertical endomorphism to any 1-form $\lambda \in \Lambda^1(Z)$ of the form $\lambda = F^\mu_i dz^i_\mu + F_i dy^i + F_\mu dx^\mu$.

Definition 16. *Let ν be an Ehresmann connection in the bundle $\pi : Y \rightarrow X$. Let C be a general constitutive relation. Determine the ν-lift $\tilde{C}_\nu$ of constitutive relation C by section $\delta_\nu : \tilde{Z} \rightarrow \Lambda^{(n+1)+(n+2)}$ (see (3.20), Sec.3)*

$$\tilde{C}_\nu(z) = (x^\mu, y^i; \Gamma^j_\mu F^\mu_j(x^\mu, y^i, z^i_\mu); F^\mu_i(x^\mu, y^i, z^i_\mu); \Pi_i(x^\mu, y^i, z^i_\mu)).$$

Taking the pullback of the canonical form $\tilde{\Theta}_\nu$ on the bundle $\tilde{Z}$ we get the ν-induced **Poincare-Cartan form of the constitutive relation** $\tilde{C}_\nu$

$$\Theta_{\tilde{C}_\nu} = \tilde{C}^*_\nu(\delta^*_\nu \Theta^{n+1}_2 + \Theta^{n+2}_2) = (F^\mu_i \Gamma^i_\mu)\eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta. \tag{9.7}$$

It is the special case of the following construction.

Remark 18. Below (Sec.12) we show that an action of a transformations on the Poincare-Cartan form Θ_C corresponding to a CR C produce an additional term of the type $A\eta$. As a result for the compatibility with the action of transformations one have to consider the classes of Poincare-Catran forms Θ_C - their images in the factor bundle $\Lambda^{(n+1)+(n+2)}_2(J^1_p(\pi))/\Lambda^{(n+1)+(n+2)}_1(J^1_p(\pi))$ rather then simply the forms on $\tilde{Z}$.

Remark 19. Equivalent definition of the general constitutive relations can be given in terms of section of the corresponding bundles:

$$\begin{array}{ccc}
\pi^*_1(\Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y) & \xrightarrow{\pi^*_{(n+1)+(n+2)}} & J^1_p(Y) \\
\pi_{1\ (n+1)+(n+2)} \downarrow & & \pi_1 \downarrow \\
\tilde{Z} = \Lambda^{(n+1)+(n+2)}_2 Y/\Lambda^{(n+1)+(n+2)}_1 Y & \xrightarrow{\pi_{(n+1)+(n+2)}} & Y \\
 & & \pi \downarrow \\
 & & X
\end{array} \tag{9.8}$$

Then we can use the following

Definition 17. *A (general) constitutive relation C of the field theory with the configurational bundle $\pi : Y \rightarrow X$ and the partial 1-jet space $Z_p = J^1_p(\pi)$ is a smooth section $\mathcal{C}$ of the vector bundle $\pi^*_{(n+1)+(n+2)}$ in the diagram above.*

In the local fiber coordinates (x^μ, y^i, z^i_μ) a section of the bundle $\pi^*_{(n+1)+(n+2)}$ has the form

$$\mathcal{C}(x^\mu, y^i, z^i_\mu) = F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta_\mu, \tag{9.9}$$

where F^μ_i, Π_i are functions on the space $J^1_p(Y)$.

Below we list several types of the constitutive relations that are widely used in physics and continuum mechanics.

Example 7. A **Lagrange constitutive relation** defined by a smooth (Lagrangian) function $L \in C^{\infty}(Z_p)$ is given by the mapping:

$$C_L(x^\mu, y^i, z^i_\mu) = (p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = \frac{\partial L}{\partial y^i}). \tag{9.10}$$

Correspondingly, a **covering Lagrange constitutive relation** is defined by a smooth function $L \in C^{\infty}(Z_p)$ giving the mapping $Z_p \rightarrow \Lambda^{n+(n+1)}_2 Y$

$$\hat{C}_L(x^\mu, y^i, z^i_\mu) = (p = -z^i_\mu \frac{\partial L}{\partial z^i_\mu}, p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = \frac{\partial L}{\partial y^i}). \tag{9.11}$$

Notice that the covering Lagrange relation defined here does not coincide with the covering CR defined by the Legendre transformation of the Lagrangian $L\eta$. Relation between these two covering CR will be studied elsewhere.

Example 8. A **semi-lagrangian CR** is defined by a smooth function $L \in C^{\infty}(Z_p)$ and an arbitrary functions $Q_i \in C^{\infty}(Z_p), i = 1, \ldots, m$:

$$C_{L,Q_i}(x^\mu, y^i, z^i_\mu) = (p^\mu_i = \frac{\partial L}{\partial z^i_\mu}; \Pi_i = Q_i(x^\mu, y^i, z^i_\mu)). \tag{9.12}$$

Remark 20. In a case when the domain of C is a partial 1-jet bundle $Z_p = J^1_p(\pi)$ the component F^μ_i of CR C is equal zero if the derivative z^i_μ is absent from the fibers of the partial jet bundle $J^1_p(\pi)$. In the case of RET semi-Lagrangian CR is trivially zero.

Remark 21. For a semi-Lagrangian CR there exists natural - Lagrangian lift to the CCR:

$$\hat{C}_{L,Q} = (L - z^i_\mu L_{,z^i_{,mu}})\eta + \frac{\partial L}{\partial z^i_\mu} dy^i \wedge \eta_\mu + Q_i dy^i \wedge \eta. \tag{9.13}$$

It will be used below for formulating corresponding Noether Theorem.

A very important example of a semi-Lagrangian CR is the following

Example 9. $L + D$**-system**. Let L be a smooth function $L \in C^{\infty}(Z_p)$. Let the time derivatives z^i_0 of all basic fields belong to Z_p and let $D \in C^{\infty}(Z_p)$ be one more function (dissipative potential). Define the constitutive relation $\mathcal{C}_{L,D}$ that differs from Lagrangian CR C_L by the condition $\Pi_i = L_{,y^i} + D_{,z^i_0}$. Thus, the corresponding Poincare-Cartan form is

$$\Theta_{L,D} = \Theta_L + D_{z^i_{,0}} dy^i \wedge \eta. \tag{9.14}$$

Example 10. Vector-potential CR. Consider a RET case. Let $h = h^\mu(x,y)\eta_\mu$ be a semi-basic n-form on Y. Define a constitutive relation by the formula

$$C_h(x^\mu, y^i) = (p^\mu_i = \frac{\partial h^\mu}{\partial y^i}; \Pi_i = \Pi_i(x,y)). \tag{9.15}$$

This is the case of the *dual formulation in terms of Lagrange-Liu variables* (y^i replaces the λ^i here), see Sections 15 and 17 below.

Example 11. One can combine Semi-Lagrangian and vector-potential examples into the following one. Let $L, Pi_i,\ i = 1, \ldots, m \in C^{\infty}(J^1_p(\pi)$ and let Let $h =$

$h^\mu(x,y)\eta_\mu$ be a semi-basic n-form on Y. Define a constitutive relation by the mapping

$$C_h(x^\mu, y^i, z^i_\mu) = (p^\mu_i = L_{z^i_\mu} + \frac{\partial h^\mu}{\partial y^i}; \Pi_i = \Pi_i(x,y)). \tag{9.16}$$

Example 12. For the 5F-fluid system (see Sec.2 above) with the trivial bundle $Y = X \times R^5 \to X$ and the basic fields (ρ, v^A, ϑ) the constitutive relations define and are defined by its Poincare-Cartan form

$$\Theta_{C_5} = [\rho d\rho \wedge \eta_0 + \rho v^B d\rho \wedge \eta_B] + [(\rho v^A) dv^A \wedge \eta_0 + (\rho v^A v^B - t^{AB}) dv^A \wedge \eta_B] +$$
$$+ [\rho\epsilon d\vartheta \wedge \eta_0 + (\rho\epsilon V^A + q^A) d\vartheta \wedge \eta_B]] + [f_A dv^A \wedge \eta + t^A_B \frac{\partial V^B}{\partial x^A} d\vartheta \wedge \eta + r]. \tag{9.17}$$

In the simplest case of $5F - fluid$ system considered in Sec.2, the *state space* represents the fiber of the bundle $\pi^1 : J^1_S(\pi) \to X$ where $S_0 = \{\rho\}, S_x = \{v^A, \vartheta\}, S_t = S_{tx} = \emptyset$. Thus, only derivatives of velocity components and of temperature by spacial coordinates x^A are present in the 1-jet fiber of the state space. State space S itself is the fiber bundle over the basic fields space U: $\varrho : S \to U$.

10. Balance System $\mathcal{B}_{\mathcal{C}}$ defined by a Constitutive Relation $\mathcal{C}$.

In the Lagrangian Field Theory the Poincare-Cartan form Θ_L appears in the second term of a local variation in the direction of a vector field $\xi \in \mathcal{X}(J^1(\pi))$ obtained by the variational version of the Cartan formula $L_\xi(L\eta) = (i_\xi d + di_\xi)(L\eta)$

$$\delta_\xi(L\eta)(j^1(s)) = j^{1\ *}(s)[i_\xi e(L)(s) + di_\xi \Theta_L(z)] \tag{10.1}$$

see [11]. Here $s \in \Gamma(\pi)$ is a section of bundle π.

The semi-basic $n+1$-form $e(L) \in \Lambda^{n+1}(J^1(\pi))$ is the Euler-Lagrange form and the Euler-Lagrange system of the field theory with the Lagrangian L for a section s has the form:

$$j^{1\ *}(s) i_\xi e(L) = 0,\ \forall \xi. \tag{10.2}$$

In its turn, the form Θ_L is the Poincare-Cartan form (4.3) and the same Euler-lagrange system of equations is obtained by (4.6)

$$j^{1\ *}(s) di_\xi \Theta_L(z) = 0. \tag{10.3}$$

For a general constitutive relation (9.2) and the corresponding Poincare-Cartan form Θ_C we have an analog of variational formula (10.1) given by Cartan formula

$$L_\xi \Theta_C = i_\xi d\Theta_C + di_\xi \Theta_C. \tag{10.4}$$

Thus, we can try to formulate the balance laws corresponding to the CR mapping C by following one of two ways suggested above. We can take the pullback via $j^{1\ *}_p(s)$ of the first or second term in (10.4) and request it to be zero for a set of variation vector fields ξ, large enough to separate the balance equations corresponding to the CR $\mathcal{C}$. Yet, as we see below, both of these cases meet some interesting restrictions. In order to get the balance equations the variations ξ should satisfy some conditions defined by the F^μ_i-part of the constitutive relation C. More specifically we should have m linearly independent vector fields ξ in order to extract all m balance equations from the invariant formulation of the type (10.2) or (10.3). Locally this is always possible but still leads to some restrictions to the type of variations. We will see that there is a way around this difficulty if one uses in CR version of the formula (10.2) the *reduced horizontal differential* $\hat{d}$ (comp. [21] or Appendix IV) instead of the conventional De-Rham differential d. On the other hand studying these restrictions we will determine the special place of the *semi-Lagrangian constitutive relations* in between the general CR - these are CR defined on the full 1-jet bundle $J^1(\pi)$ for which there are no limitations for them on the nature of variations vector fields.

10.1. **Poincare-Cartan formulation of a balance system.** We assume that a connection $\nu : Y \to J^1(\pi)$ is fixed. We start with the Poincare-Cartan way of obtaining the balance equations and for this we take an arbitrary vector field $\xi \in \mathcal{X}(Z_p)$ locally having form

$$\xi = \xi^\mu \partial_{x^\mu} + \xi^i \partial_{y^i} + \xi^i_\mu \partial_{z^i_\mu}, \tag{10.5}$$

and plug it into the (ν-dependent) Poincare-Cartan form Θ_{C_ν}

$$i_\xi \Theta_{C_\nu} = (F^\mu_i \Gamma^i_\mu)\xi^\lambda \eta_\lambda + F^\mu_i \xi^i \eta_\mu - F^\mu_i \xi^\nu dy^i \wedge \eta_{\mu\nu} - \Pi_i \xi^\mu dy^i \wedge \eta_\mu + (\Pi_i \xi^i)\eta. \tag{10.6}$$

Take the vector field ξ vertical, i.e. assume that $\xi^\mu = 0, \mu = 1, \dots, n+1$. Then *any* addition of a term of the form $h(z)\eta$ choused by an adopted change of coordinates or by another choice of a connection ν will be eliminated and the result, $i_\xi \Theta_C$ is defined canonically.

Now we apply the I-differential $\tilde{d}$

$$\tilde{d} i_\xi \Theta_C = d(F^\mu_i \xi^i) \wedge \eta_\mu + (F^\mu_i \xi^i (\partial_{x^\mu} \lambda_G)\eta - d(F^\mu_i \xi^\nu) dy^i \wedge \eta_{\mu\nu} + \sum_{\mu<\nu} F^\mu_i \xi^\nu dy^i \wedge ((\partial_{x^\nu}\lambda_G)\eta_\mu - (\partial_{x^\mu}\lambda_G)\eta_\nu) - (\Pi_i \xi^i)\eta + \Pi_i \xi^\mu dy^i \wedge \eta_\mu. \quad (10.7)$$

Thus, requesting the vector field ξ to be **vertical** (i.e. putting $\xi^\mu = 0$) we get

$$\tilde{d} i_\xi \Theta_C = d(F^\mu_i \xi^i) \wedge \eta_\mu + (F^\mu_i \xi^i \partial_{x^\mu} \lambda_G)\eta - (\Pi_i \xi^i)\eta = \xi^i (F^\mu_{i,x^\mu}\eta + F^\mu_{i,y^j} dy^j \wedge \eta_\mu + F^\mu_{i,z^j_\nu} dz^j_\nu \wedge \eta_\mu) + (F^\mu_i \xi^i \partial_{x^\mu}\lambda_G)\eta - \Pi_i \xi^i \eta + F^\mu_i d\xi^i \wedge \eta_\mu. \quad (10.8)$$

Applying now the pullback by the 1-jet $j^1_p(s)$ of a section $s \in \Gamma(\pi)$ and using

$$j^{1*}_p(s)[F^\mu_{i,y^j} dy^j \wedge \eta_\mu + F^\mu_{i,z^j_\nu} dz^j_\nu \wedge \eta_\mu] = F^\mu_{i,y^j} s^j_{,\mu}\eta + F^\mu_{i,z^j_\nu} s^j_{\nu\mu}\eta,$$

we get

$$j^{1\,*}_p(s)\tilde{d} i_\xi \Theta_C = \xi^i [(F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i (\partial_{x^\mu}\lambda_G) - \Pi_i \circ j^1_p(s)]\eta + + F^\mu_i \circ j^1_p(s) \left(\xi^i_{,x^\mu} + \xi^i_{,y^j} s^j_{,x^\mu} + \xi^i_{,z^j_\nu} s^j_{,x^\nu x^\mu} \right)\eta \quad (10.9)$$

In the right side in parentheses stays the pullback by $j^1(s)$ of the total derivative $d_\mu \xi^i = \xi^i_{,x^\mu} + z^i_\mu \xi^i_{,y^j} + z^i_{\mu\nu}\xi^i_{,z^j_\nu}$ of a component ξ^i of vector field ξ^i along section s by x^μ contracted with the form F^μ_i: $j^{1*}(s)(F^\mu_i d_\mu \xi^i)\eta$.

In order to extract the balance equations from the invariant form of variational principle obtained by equating to zero the obtained expression we have to require the second term in (10.9) to be zero. Thus, vector field ξ should satisfy to the additional condition that is explicitly formulated in the next definition.

Definition 18. (1) *For a constitutive relation C (or, more precise, for current form $F = F^\mu_i dy^i \wedge \eta_\mu \in \Lambda^n_2 Z/\Lambda^2_1 Z$) denote by $\mathcal{X}(\mathcal{C})$ the sheaf associated to the* pre-sheaf of vector fields *over Y that for an open set $U \subset Y$ consists of vertical vector fields $\xi \in \Gamma(U, V(\pi))$ whose flow prolongation $\xi^1 = \xi^i \partial_{y^i} + (d_\mu \xi^i)\partial_{z^i_\mu}$ satisfies to the condition ($\xi^{1\ i} = \xi^i$)*

$$FDiv(\xi) = F^\mu_i d_\mu \xi^i = 0 \quad (10.10)$$

*in U. Vector fields - sections of the sheaf $\mathcal{X}(\mathcal{C})$ will be called $\mathcal{C}$-**admissible**.*

(2) *A constitutive relation $\mathcal{C}$ with the current form F is called **locally separable** if each point $y \in Y$ has a neighborhood U_y such that there are m vector fields ξ_k in the space of sections $\Gamma(U_y, \mathcal{X}(\mathcal{C}))$ linearly independent at each point $y_1 \in U_y$.*

(3) *A constitutive relation C is called **separable in an open subset** $W \subset Y$ if there are m vector fields $\Gamma(W, \mathcal{X}(\mathcal{C}))$ linearly independent at each point of W.*

Having introduced these notions we can now formulate the Poincare-Cartan version of the variational principle for the balance system with the domain $J^1_p(\pi)$ and the constitutive relation C.

Definition 19. *Let $\mathcal{C}$ be a constitutive relation with the domain $J^1_p(\pi)$. We say that a section $s : U_s \rightarrow Y$ of the bundle $\pi : Y \rightarrow X$, U_s being an open subset in X, satisfies to the* ***balance system (system of balance laws) defined by CR*** *$\mathcal{C}$ if* ***for* all $\mathcal{C}$-admissible *vector fields*** *$\xi \in \mathcal{X}(C)|_{\pi_1^{-1}(U_s)}$ (i.e. over U_s)*

$$\tilde{d}(j^1_p(s)^*(i_\xi \Theta_C)) = j^1_p(s)^*(\tilde{d}(i_\xi \Theta_C)) = 0. \quad (10.11)$$

Here $\tilde{d}$ is the Iglesias differential of a (n+(n+1))-form on X (see Appendix II). In simple terms

$$\tilde{d}(\omega^n + \omega^{n+1}) = d\omega^n - \omega^{n+1}.$$

Let $z \in Z_p$ and let $V \subset X$ be a neighborhood of the projection $x = \pi_1(z) \in X$ over which the bundle Y is trivial $Y|_V \approx V \times U$ and $W \subset U$ be an open set such that $V \times W$ is the neighborhood of the point $y = \pi_{10}(z)$. We may assume that V and W are domains of adopted chart (x^μ, y^i). Vector fields $\xi_j = \partial_{y^j} \in V(\pi)|_{V\times W}$ have the property that in this local chart $\frac{d\xi^j}{dx^\mu} = 0$ in $\pi_{10}^{-1}(V \times W)$. Therefore these m vector fields in $X(V \times W)$ are C-admissible for any constitutive relation C with the domain in $J^1_p(\pi)$ for all four choices of partial 1-jet bundles (see Proposition 10b and Theorem 1 for the case of $J^1_K(\pi)$, Proposition 11b for the case of $J^1_S(\pi)$). As a result we get

Proposition 18. *Any constitutive relation C is locally separable.*

Globally defined C-admissible vector fields have an important meaning for the balance system $\mathcal{B}_\mathcal{C}$ (see next section). In the next section the case of a special situation will be described, for some type of the bundles π a natural class of m linearly independent globally defined vertical vector fields $\xi \in \mathcal{X}(Y)$ that are admissible for all CR $\mathcal{C}$. Now we formulate the main result of this section in the Poincare-Cartan formulation.

Theorem 2. *If a constitutive relation $\mathcal{C}$ is locally separable, then the following statements for a section $s \in \Gamma(\pi)(D_s)$, $D_s \subset X$ are equivalent:*

(1)
$$\tilde{d}j^{1\ *}_p(s) i_\xi \Theta_\mathcal{C} \equiv j^{1\ *}_p(s)\tilde{d} i_\xi \Theta_\mathcal{C} = 0, \ \textit{for all}\ \xi \in \mathcal{X}(\mathcal{C})|_{D_s}, \quad (10.12)$$

(2) *Section s is the solution of the following system of balance laws -* ***balance system:***

$$(F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\mu}\lambda_G) = \Pi_i(j^1_p(s)),\ i = 1, \dots, m. \quad (10.13)$$

Proof. It is clear that (1) follows from (2). If (1) is true, choose m (local) linearly independent vector fields $\xi_k \in \mathcal{X}(C)$ in a neighborhood of a point $x \in X$. Then, for such (locally defined) vector fields ξ the last term in (10.9) is zero and we get the system of linear equations

$$\xi^i_k(s(x))((F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\mu}\lambda_G) - \Pi_i \circ j^1_p(s)) = 0$$

at each point y in a neighborhood of the point x for i unknowns $(F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i(\partial_{x^\mu}\lambda_G) - \Pi_i \circ j^1_p(s)$ in the parenthesis. By the condition, matrix $\xi^i_k(s(x))$ of this system is nondegenerate, so system has only zero solution. Being true in a neighborhood of any point $x \in X$, (2) is true in all X. □

Below the balance system system (10.13) will be refereed to as the ★.

Example 13. For a Lagrangian constitutive relation (See Example 3) the balance system (10.13) takes the form

$$(\frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s))_{,x^\mu} + \frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s)(\partial_{x^\mu}\lambda_G) = \frac{\partial L}{\partial y^i}(j^1_p(s)), \tag{10.14}$$

or

$$\frac{\partial}{\partial x^\mu}(\frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s)) + \frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s)(\partial_{x^\mu}\lambda_G) - \frac{\partial L}{\partial y^i}(j^1_p(s)) = 0,$$

i.e. is the system of Euler-Lagrange equations for the Lagrangian form $L\eta$. Here $\lambda_G = ln(|G|)$.

Example 14. $L + D$**-system**. For a $L + D$-system where the Poincare-Cartan form is

$$\Theta_{L,D} = \Theta_L + D_{z^i_0} dy^i \wedge \eta,$$

the corresponding balance system has the form

$$\mathcal{E}_L(s)_i = \frac{\partial}{\partial x^\mu}(\frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s)) + \frac{\partial L}{\partial z^i_\mu} \circ j^1_p(s)(\partial_{x^\mu}\lambda_G) - \frac{\partial L}{\partial y^i}(j^1_p(s)) = \frac{\partial D}{\partial z^i_0} \circ j^1(s). \tag{10.15}$$

This system has the form of Euler-Lagrange equations with the dissipative Rayleigh potential D (see ([30])).

Example 15. System of conservation laws. If we take $\Pi_i = 0$ in the constitutive relation C then the balance system ★ takes the form of the system of conservation laws

$$(F^\mu_i \circ j^1_p(s))_{,\mu} = 0. \tag{10.16}$$

10.2. **Euler-Lagrange formulation of the balance system.** Now we will see what happens if we apply the standard order $i_\xi d$ of operations that is used in the Lagrangian Theory (10.2) to the *lifted Poincare-Cartan form* $\Theta_{\tilde{C}_-}$ of the constitutive relation $\mathcal{C}$ (see ()), or, more generally, to an arbitrary covering constitutive relation of the form

$$\Theta_{\widehat{\mathcal{C}_-}} = p\eta + F^\mu_i dy^i \wedge \eta_\mu - \Pi_i dy^i \wedge \eta.$$

We reversed the sign of the source term in the Poincare-Cartan form to compensate for the different order of operation of contraction and applying the differential.

Doing these calculations we will be repeatedly using the relation between the contact forms of partial contact structures on Z_p and the differentials of basic variables (see Sec.4):

$$dy^i = \omega^i + \sum_{(\mu,i)\in P} z^i_\mu dx^\mu,\ dz^i_\mu = \omega^i_\mu + \sum_{(\mu,i)\in P} z^i_{\mu\sigma} dx^\sigma,$$

and the total derivative d_ν on Z_p (see Appendix IV), or, more exactly, on the $J^1(Z_p) = J^2_p(\pi)$

$$d_\nu f = \partial_{x^\nu} f + \sum_{(\nu,i)\in P} z^i_\nu \partial_{y^i} f + \sum_{(\sigma,i)\in P} z^i_{\nu\sigma} \partial_{z^i_\sigma} f.$$

For a given section $s \in \Gamma_V(\pi),\ V \subset X$ we request the fulfilment of the equation

$$j^1(s)^*(i_\xi \tilde{d}\Theta_{\hat{C}_-}) = 0 \tag{10.17}$$

for large enough family of (locally defined) vector fields $\xi \in \mathcal{X}(Z_p)$ (not necessary vertical with respect to the projection $\pi^1 : Z_p \to X$) guarantying the sections s to be a solution of the *balance system* of m independent balance equations. Remark that vector fields ξ^1 for $\xi \in \mathcal{V}(\pi)$ in the first subsection above are a special case of considered vector fields (for $\xi \in \mathcal{V}(\pi)$, $\xi^1 = \xi^j \partial_{y^j} + d_\mu \xi^j \partial_{z^j_\mu}$).

Thus, we take the $(n+1)+(n+2)$-form $\Theta_{\hat{C}_-}$ of the form (9.5) and apply first the Iglesias differential $\tilde{d}$ and then i_ξ for a vector field $\xi = \xi^\nu \partial_\nu + \xi^j \partial_{y^j} + \xi^i_\mu \partial_{z^i_\mu}$.

We will denote by Con an arbitrary contact forms that appears in calculations. Assuming the summation by repeated indices agreement we recall that only z^i_μ or derivatives by these variables with $(\mu, i) \in P$ are present on the formulas.

We get, using that $d\eta_\mu = \lambda_{G,\mu}\eta$,

$$\begin{gathered} i_\xi \tilde{d}\Theta_{\hat{C}_-} = i_\xi[d(p\eta + F^\mu_i dy^i \wedge \eta_\mu) + \Pi_i dy^i \wedge \eta] = \\ i_\xi\left[dp \wedge \eta + dF^\mu_i \wedge dy^i \wedge \eta_\mu - F^\mu_i dy^i \wedge d\eta_\mu + \Pi_i dy^i \wedge \eta\right] = \\ (i_\xi dp) \wedge \eta - dp \wedge i_{\bar{\xi}}\eta + (\xi \cdot F^\mu_i) dy^i \wedge \eta_\mu - \xi^i dF^\mu_i \wedge \eta_\mu + dF^\mu_i \wedge dy^i \wedge i_{\bar{\xi}}\eta_\mu - \\ - F^\mu_i \lambda_{G,\mu}(\xi^i \eta - dy^i \wedge i_{\bar{\xi}}\eta) + \Pi_i \xi^i \eta - \Pi_i dy^i \wedge i_{\bar{\xi}}\eta = \\ = (\xi \cdot p)\eta + (\xi \cdot F^\mu_i)(\omega^i + z^i_\nu dx^\nu) \wedge \eta_\mu - \xi^i dF^\mu_i \wedge \eta_\mu - \xi^i F^\mu_i \lambda_{G,\mu}\eta + \xi^i \Pi_i \eta - \\ - dp \wedge i_{\bar{\xi}}\eta + dF^\mu_i \wedge (\omega^i + z^i_\nu dx^\nu) \wedge i_{\bar{\xi}}\eta_\mu + \lambda_{G,\mu} F^\mu_i dy^i \wedge i_{\bar{\xi}}\eta - \Pi_i(\omega^i + z^i_\nu dx^\nu) \wedge i_{\bar{\xi}}\eta] = \end{gathered} \tag{10.18}$$

Now we are using the fact that $dF^\mu_i = (d_\nu F^\mu_i) dx^\nu + F^\mu_{i,y^j}\omega^j + F^\mu_{i,z^j_\nu}\omega^j_\nu$ and, similarly, $dp = d_\nu dx^\nu + Con$ and continue

$$\begin{gathered} = (\xi \cdot p)\eta + (\xi \cdot F^\mu_i) z^i_\nu dx^\nu \wedge \eta_\mu - \xi^i d_\nu F^\mu_i dx^\nu \wedge \eta_\mu - \xi^i F^\mu_i \lambda_{G,\mu}\eta + \xi^i \Pi_i \eta + \\ + [-d_\nu p dx^\nu - \Pi_i z^i_\nu dx^\nu + \lambda_{G,\mu} F^\mu_i z^i_\nu dx^\nu] \wedge i_{\bar{\xi}}\eta + (d_\nu F^\mu_i) dx^\nu \wedge z^i_\lambda dx^\lambda \wedge i_{\bar{\xi}}\eta_\mu + Con = \\ [\xi \cdot p + (\xi \cdot F^\mu_i) z^i_\mu - \xi^i d_\mu F^\mu_i - \xi^i F^\mu_i \lambda_{G,\mu} + \xi^i \Pi_i]\eta + [-d_\nu p - \Pi_i z^i_\nu + \lambda_{G,\mu} F^\mu_i z^i_\nu] dx^\nu \wedge \xi^\sigma \eta_\sigma + \\ + (d_\nu F^\mu_i) dx^\nu \wedge z^i_\lambda dx^\lambda \wedge \xi^\sigma \eta_{\mu\sigma} + Con = . \end{gathered} \tag{10.19}$$

Now we will use the formula (16.5, Appendix I) for $dx^\lambda \wedge \eta_{\mu\sigma}$ from which it will follow that if $\nu = \mu, \lambda = \sigma$, then $dx^\nu \wedge dx^\lambda \wedge \eta_{\mu\sigma} = \eta$ and when $\nu = \sigma, \lambda = \mu$, then $dx^\nu \wedge dx^\lambda \wedge \eta_{\mu\sigma} = -\eta$. Using this in the last term in the previous formula we get

$$(d_\nu F^\mu_i) dx^\nu \wedge z^i_\lambda dx^\lambda \wedge \xi^\sigma \eta_{\mu\sigma} = [z^i_\sigma d_\mu F^\mu_i \xi^\sigma - z^i_\mu \xi^\sigma d_\sigma F^\mu_i]\eta. \tag{10.20}$$

Using this result we continue

$$= [\xi\cdot p + (\xi\cdot F^\mu_i)z^i_\mu - \xi^i d_\mu F^\mu_i - \xi^i F^\mu_i \lambda_{G,\mu} + \xi^i\Pi_i + [-d_\nu p - \Pi_i z^i_\nu + \lambda_{G,\mu}F^\mu_i z^i_\nu]\xi^\nu +$$
$$+ z^i_\sigma d_\mu F^\mu_i \xi^\sigma - z^i_\mu \xi^\sigma d_\sigma F^\mu_i]\eta + Con =$$
$$= \{\xi\cdot(p + z^i_\mu F^\mu_i) - \xi^i_\mu F^\mu_i - \xi^\nu d_\nu(p + z^i_\mu F^\mu_i) + \xi^\sigma z^i_{\mu\sigma}F^\mu_i + \xi^\sigma z^i_\sigma[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i] -$$
$$- \xi^i[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i]\}\eta + Con =$$
$$= \{(\xi - \xi^\nu d_\nu)(p + z^i_\mu F^\mu_i) - \xi^i_\mu F^\mu_i + \xi^\sigma z^i_{\mu\sigma}F^\mu_i + (\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i]\}\eta + Con \tag{10.21}$$

We notice now that for a function $f \in C^\infty(Z_p)$, lifted to the partial 2-jet bundle $J^2_p(\pi)$, $\xi\cdot f - \xi^\nu d_\nu f = (\xi^i - \xi^\nu z^i_\nu)\partial_{y^i} f + (\xi^i_\sigma - \xi^\nu z^i_{\sigma\nu})\partial_{z^i_\sigma} f$. Starting from this moment we assume that all the forms are lifted to $J^2_p(\pi)$, vector fields are flow prolonged there (we will see that our considerations do not depend on this prolongation). Applying this for $f = p + z^i_\mu F^\mu_i$ we see that the expression in figure brackets is equal to

$$(\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i] + (\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)F^\mu_i + (\xi^i - \xi^\nu z^i_\nu)\partial_{y^i}(p + z^j_\mu F^\mu_j) + (\xi^i_\sigma - \xi^\nu z^i_{\sigma\nu})\partial_{z^i_\sigma}(p + z^j_\nu F^\nu_j) =$$
$$= (\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i - \partial_{y^i}(p + z^j_\nu F^\nu_j)] + (\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)[F^\mu_i - \partial_{z^i_\mu}(p + z^j_\nu F^\nu_j)]. \tag{10.22}$$

Thus, we get, finally

$$i_\xi \tilde{d}\Theta_{\hat{C}_-} =$$
$$= \{(\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i - \partial_{y^i}(p + z^j_\nu F^\nu_j)] + (\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)[F^\mu_i - \partial_{z^i_\sigma}(p + z^j_\nu F^\nu_j)]\}\eta + Con =$$
$$\{-\omega^i(\xi)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i - \partial_{y^i}(p + z^j_\nu F^\nu_j)] - \omega^i_\mu(\xi)[F^\mu_i - \partial_{z^i_\mu}(p + z^j_\nu F^\nu_j)]\}\eta + Con. \tag{10.23}$$

Here

$$\begin{cases}\omega^i = dy^i - \sum_{(\mu,i)\in P} z^i_\mu dx^\mu,\\ \omega^i_\mu = dz^i_\mu - \sum_{(\mu,i)\in P} z^i_{\mu\sigma}dx^\sigma\end{cases}$$

are generating (partial) contact forms on the bundle $J^1(J_p(\pi))$.

Remark 22. Notice that the quantities $Q_i = \omega^i(\xi) = \xi^i - z^i_\mu \xi^\mu$ form the *characteristic of the vector field* $\xi = \xi^\mu\partial_\mu + \xi^i\partial_{y^i}$, see [41], Ch.2.

These arguments proves the following

Proposition 19. *Let $\hat{C}$ be a CCR defined in a domain of the partial 1-jet bundle Z_p. Then, for any $\xi \in \mathcal{X}(Z_p)$,*

$$i_\xi \tilde{d}\Theta_{\hat{C}_-} = -\omega^1_{\hat{C}}(\xi) - \omega^2_{\hat{C}}(\xi), \tag{10.24}$$

where

$$\begin{cases}\omega^1_{\hat{C}}(\xi) = \omega^i(\xi)[d_\mu F^\mu_i + \lambda_{G,\mu}F^\mu_i - \Pi_i - \partial_{y^i}(p + z^j_\nu F^\nu_j)],\\ \omega^2_{\hat{C}}(\xi) = \sum_{(\mu,i)\in P}(F^\mu_i - \partial_{z^i_\mu}(p + z^j_\nu F^\nu_j))\omega^i_\mu(\xi)\end{cases} \tag{10.25}$$

will be called respectively as the ***first (*****Euler-Lagrange*****) and the second contact forms of the covering constitutive relation** C.

Remark 23. Calculation leading to the last Proposition is valid in a case of a covering constitutive relation depending on the derivatives of higher order. Such a CCR, which we write, for (formal) simplicity as defined on the infinite jet bundle of the bundle π (see Appendix IV or [11, 21, 41]:

$$\hat{C} : J^{\infty}(\pi) \rightarrow \Lambda^{(n+1)+(n+2)}(Y)$$

but depending on derivatives of order $\leqq N$ defines in the same way the Poincare-Cartan form $\Theta_{\hat{C}}$ and we can formulate the balance system in the same way, postulating the fulfillment of the equation $j^{\infty}(s)^*(i_\xi \tilde{d}\Theta_{\hat{C}_-}) = 0$ for variations $\xi \in \mathcal{X}(J^{infty}(\pi))$ in the number sufficient for separating the balance laws. Using the corresponding properties of higher order contact form $\omega^i_\Lambda = dz^i_\Lambda - \sum_\lambda z^i_{\Lambda+\lambda} dx^\lambda$, Λ being a multi-index and total derivatives (see Appendix IV), for instance,

$$dz^i_\Lambda = \omega^i_\Lambda + Con,\ (\xi - \xi^\sigma d_\sigma)\cdot f = \omega^i(\xi) f_{,y^i} + \sum_\Lambda \omega^i_\Lambda(\xi) f_{,z^i_\Lambda},$$

we get the result similar to (10.24):

$$i_\xi \tilde{d}\Theta_{\hat{C}_-} = -\omega^1_{\hat{C}}(\xi) - \omega^2_{\hat{C}}(\xi) + \sum_{\Lambda||\Lambda|>1} \omega^i_\Lambda(\xi)\partial_{z^i_\Lambda}(p + z^i_\mu F^\mu_i). \tag{10.26}$$

It follows from this that for the CCR $\tilde{C}$ of a constitutive relation C of higher order where $p + z^i_\mu F^\mu_i = 0$ no new limitations for the admissible variations ξ beyond those considered here will appear. More detailed study of constitutive relations of higher order will be done in the continuation of this work.

Remark 24. Equality (10.23) contains the jet variables of the second order $z^i_{\mu\sigma}$. Yet, **only** ω^i_μ with $(\mu, i) \in P$ **are present in the formula** (10.23)! For instance in the RET case *all these terms are absent* from (10.23).

In order that the equation resulting from taking the pullback by $j^1(s)$ would not depend on the variables not in $J^1_p(\pi)$ we require that all the coefficients of these variables would be zero. This leads to the strong conditions to the admissible variations ξ.

Proposition 20. *Let $\hat{C}$ be a covering constitutive relation. The following properties 1),2) are equivalent*

(1) *Second contact form of CCR $\hat{C}$ is identically zero:*

$$\omega_{\hat{C}} = F^\mu_i - \partial_{z^i_\mu}(p + z^j_\nu F^\nu_j)\omega^i_\mu = 0, \tag{10.27}$$

(2) *Locally*

$$F^\mu_i = \partial_{z^i_\mu} L,\ L \in C^\infty(Z_p),$$

i.e. CR $\mathcal{C}$ is (locally) semi-Lagrangian.

(3) *If properties 1),2) are fulfilled, then*

$$p = L - z^i_\mu \partial_{z^i_\mu} L + l(x, y)$$

with an arbitrary function $l(x, y)$.

Proof. Rewrite the first equality as

$$\partial_{z^i_\sigma} p = -z^i_\mu \partial_{z^i_\sigma} F^\mu_i. \tag{10.28}$$

Right sides of these equalities satisfy to the mixed derivative test

$$\partial_{z^j_\lambda}(-z^k_\mu \partial_{z^i_\sigma} F^\mu_k) = \partial_{z^i_\sigma}(-z^k_\mu \partial_{z^j_\lambda} F^\mu_k),$$

or $\partial_{z^j_\lambda} F^\sigma_i = \partial_{z^i_\sigma} F^\lambda_j$. From this equality valid for all couples of indices $(j,\lambda),(i,\sigma)$ the second statement follows.

To prove the opposite - reverse the arguments.

□

Theorem 3. *Let $\mathcal{C}_{L,\Pi}$ be a semi-Lagrangian CR: $F^\mu_i = L_{,z^i_\mu}$. Let $\hat{C}_{L,\Pi}$ be a CCR covering $\mathcal{C}$ and such that $p = L - z^i_\mu \partial_{z^i_\mu} L + l(x,y)$ with some function $l(x,y)$. Then,*

1. *Equality 10.23 takes the form*
$$i_\xi \tilde{d}\Theta_{\hat{C}_{L,\Pi,-}} = -\omega^i(\xi)[d_\mu L_{,z^i_\mu} + \lambda_{G,\mu} L_{,z^i_\mu} - \Pi_i - \partial_{y^i}(L + l(x,y))] + Con.$$
2. *Following statements are equivalent*
   (a) *For a section $s \in \gamma(\pi)$ and for all $\xi \in \mathcal{X}(Z_p)$*
$$j^1_p(s)^* i_\xi \tilde{d}\Theta_{\hat{C}_{L,\Pi\ -}} = 0.$$
   (b) *Section s is the solution of the system of balance equations*

$$(L_{,z^i_\mu} \circ j^1_p(s))_{,\mu} + (\lambda_{G,\mu} \circ s) L_{,z^i_\mu} \circ j^1_p(s) = \Pi_i \circ j^1_p(s) + \partial_{y^i}(L + l(x,y)). \quad (10.29)$$

Proof. Proof of the first statement follows from the equation (10.23) if we substitute $F^\mu_i = \partial_{z^i_\mu} L$.

Equivalence of the statements in second part follows from the fact that when we will apply the pullback by the 1-jet section $j^1_p(s)$ of the bundle $\pi^1 : Z_p \to X$ to the equality in the first statement of Theorem, contact form vanishes and from the fact that *locally* there always exist m linearly independent vector fields ξ_k such that the $m \times m$-matrix $\omega^i(\imath_k)$ is invertible. □

Corollary 2. *Let $C_{L,\Pi}$ be a semi-Lagrangian CR: $F^\mu_i = L_{z^i_\mu}$. Then,*

1. *For $\Pi_i = l(x,y) = 0$ the balance system (10.29) takes the form of the system of Euler-Lagrange Equations (10.14).*
2. *For $\Pi_i = \Pi^1_i - \partial_{y^i}(L + l(x,y))$, the balance system (10.29) takes the form of the balance system ★ with the source term $\Pi^1_i dy^i \wedge \eta$.*

Consider now the case of a RET constitutive relation $C : Y \to \tilde{Z}$ and a CCR $\hat{C}$: covering C (i.e. having the same (1,n)- and (n+2)- components but an arbitrary (0,n+1)-component $p(x,y)$, with the Poincare-Cartan form

$$\Theta_{\hat{C}} = p\eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta.$$

Following the arguments leading to the basic relation (10.23), namely continuing with the first equality in (10.21) we get, due to the independence of $\hat{C}$ on the jet variables

$$\begin{aligned} i_\xi \tilde{d}\Theta_{\hat{C}_-} &= (\xi^i - z^i_\nu \xi^\nu)\Pi_i + (-\xi^i + z^i_\nu \xi^\nu)(d_\mu F^\mu_i + F^\mu_i \lambda_{G,\mu}) + (\xi \cdot p - \xi^\nu d_\nu p) + z^i_\mu(\xi \cdot F^\mu_i - \xi^\sigma d_\sigma F^\mu_i) + Con = \\ &= (z^i_\nu \xi^\nu - \xi^i)(d_\mu F^\mu_i + F^\mu_i \lambda_{G,\mu} - \Pi_i) + \omega^i(\xi)\partial_{y^i}(p + z^j_\mu F^\mu_j) + Con = \\ &\quad - \omega^i(\xi)\left[d_\mu F^\mu_i + F^\mu_i \lambda_{G,\mu} - \Pi_i - \partial_{y^i}(p + z^i_\mu F^\mu_i)\right] + Con. \quad (10.30) \end{aligned}$$

Here we have used the fact $\xi \cdot p - \xi^\nu d_\nu p = \xi^i p_{,y^i} + \xi^\sigma p_{,x^\sigma} - \xi^\nu (p_{,x^\nu} + z^i_\nu p_{,y^i}) = (\xi^i - z^i_\nu \xi^\nu)\partial_{y^i} p = \omega^i(\xi)\partial_{y^i} p$ and, similarly, $z^i_\mu(\xi \cdot F^\mu_i - \xi^\sigma d_\sigma F^\mu_i) = z^i_\mu \omega^j(\xi)\partial_{y^j} F^\mu_i$ (we remind that in the RET case all functions depend on (x, y) only).

If we take $\hat{C} = \tilde{C}$ to be lifted CCR then the term $\partial_{y^i}(p + z^i_\mu F^\mu_i)$ vanishes and, using the Proposition 18 above we finish the proof of the following

Theorem 4. *Let C be a constitutive relation of the RET type and $\tilde{C}$ - corresponding lifted CCR. Then the following statements are equivalent*

(1) *For a given section $s \in \Gamma(\pi)$ and for all $\xi \in \mathcal{X}(Y)$*

$$s^*(i_\xi \tilde{d}\Theta_{\hat{C}_-}) = 0 \tag{10.31}$$

(2) *Section s is the solution of the balance system* ★.

Consider now the *general case* but take $p + z^j_\nu F^\nu_j = 0$ i.e. consider $\hat{C} = \tilde{C}$ to be the *lifted CCR* of a CR $\mathcal{C}$ (see previous section). Then equality (10.23-24) takes the form

$$i_\xi \tilde{d}\Theta_{\tilde{C}_-} = \{(\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,\mu} F^\mu_i - \Pi_i] + (\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)F^\mu_i\}\eta + Con \tag{10.32}$$

Taking the pullback of equality (10.23) by the 1-jet $j^1(s)$ of a section $s \in \Gamma(\pi)$ we get

$$j^1(s)^* i_\xi \tilde{d}\Theta_{\tilde{C}_-} = j^1(s)^*((\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)F^\mu_i)\eta + j^1(s)^*(\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,x^\mu} F^\mu_i - \Pi_i]\eta. \tag{10.33}$$

Equating this to zero we see that if we want the ξ-weighted balance equation

$$j^1(s)^*(\xi^\sigma z^i_\sigma - \xi^i)[d_\mu F^\mu_i + \lambda_{G,x^\mu} F^\mu_i - \Pi_i] = 0$$

to be true for a section s *under some conditions independent on section s* (i.e. independent on the $z^i_{\mu\sigma}$!) we have to require that for the vector field ξ used for the "variation"

$$\omega^2_{\hat{C}}(\xi) = (\xi^\sigma z^i_{\mu\sigma} - \xi^i_\mu)F^\mu_i = 0.$$

Since F^μ_i and ξ do not depend on is $z^i_{\mu\sigma}$ vector field ξ should be such that $\xi^\sigma = 0$ for σ such that *for some i* $(\sigma, i) \in P$ (this is trivially true if ξ is π^1-*vertical*).

Then the ξ-weighted balance equation for such vector field ξ will be fulfilled if and only if $j^1(s)^*(\xi^i_\mu F^\mu_i) = 0$ and, for s-independent condition we have to require $\xi^i_\mu F^\mu_i = 0$. This brings us to the extension of the condition $FDiv(\xi) = 0$ to the space of all π^1-vertical vector fields (see above)

Definition 20. (1) *A vector field $\xi \in X(Z_p)$ is called P-**vertical** if $\xi^\sigma = 0$ for σ such that* for some i $(\sigma, i) \in P$.

(2) *A π_{10}-projectable P-vertical vector field $\xi \in X(U), U \subset Z_p$ is called $\mathcal{C}$-**admissible** if $\sum_{(\mu,i)\in P} \xi^i_\mu F^\mu_i = 0$ in U.*

Denote by $\mathcal{X}(\mathcal{C})$ the sheaf generated by the pre-sheaf of the C-admissible vector fields in Z_p.

(3) *A constitutive relation $\mathcal{C}$ is called **separable in** U if the space of projections to Y of the space of C-admissible vector fields in U has, at each point $\pi_{10}(U)$ dimension m.*

Remark 25. Flow prolongation ξ^1 of the C-admissible π-vertical vector fields $\xi \in \mathcal{X}(Y)$ is a special case of C admissible vector field $\xi \in \mathcal{X}(Z_p)$. Notice, though, that while the condition $F^\mu_i \xi^i_\mu$ is *linear algebraic* for a general vector field $\xi \in \mathcal{X}(Z_p)$, it is differential for the lifts ξ^1 of vector fields $\xi \in \mathcal{X}(Y)$ (see next section for examples of specific forms of these relations).

Example 16. Let $Z_p = J^1_{<\partial_x>}(\pi)$ be a partial 1-jet bundle defined by the distribution $< \partial_x >$. In other words we assume that a constitutive relation C depends on the spacial but not on the time variables of the fields y^i. Then a vector field ξ is P-vertical if $\xi^{x^A} = 0,\ A = 1, 2, \dots, n$ while component ξ^0 corresponding time derivative ∂_t may be arbitrary.

Thus, previous arguments proves the following

Theorem 5. *If a constitutive relation $\mathcal{C}$ is locally separable, then the following statements for a section $s \in \Gamma(\pi)(U),\ U \subset X$ are equivalent:*

(1)
$$j^1(s)^*(i_\xi \tilde{d}\Theta_{\tilde{C}_-}) = 0 \text{ for all } \xi \in \mathcal{X}(\mathcal{C}|_U). \tag{10.34}$$

(2) *Section s is the solution of the following system of balance laws -* ***balance system****:*

$$(F^\mu_i \circ j^1_p(s))_{,x^\mu} + F^\mu_i \circ j^1_p(s)(\partial_{x^\mu}\lambda_G) = \Pi_i(j^1_p(s)),\ i = 1, \dots, m. \tag{$\bigstar$}$$

10.3. **Reduced horizontal differential formulation of the balance system.** Recall (see [21, 13] or Appendix IV) that the reduced horizontal differential $\hat{d}$ acts from $J^k(\pi)$ to $J^{k+1}(\pi)$ for all k by the formulas (20.9-10).

Now, let us postulate the balance system corresponding to the CR $\mathcal{C}_-$ in the form

$$j^1_p{}^*(s) i_{\xi^1} \tilde{\hat{d}}\Theta_{\mathcal{C}_-} = 0, \tag{10.35}$$

for *all* variations $\xi \in \mathcal{X}(Z_p)$.

Notice that the additional term in $\Theta^{n+1}_{C_-}$ of the form $h(z)\eta$ produced by an arbitrary transformation by an adopted transformation $\phi \in Aut(\pi)$ *will be eliminated* by applying the reduced horizontal differential $\hat{d}$ (see formula (20.9-10), Appendix IV) from which it follows that $\hat{d}(q\eta) = 0$), so that this equation is independent on a choice of representation of the Poincare-Cartan form $\widetilde{\Theta}_C$.

We have, *for any vector field $\xi \in \mathcal{X}(Z_p)$*

$$\begin{aligned} i_\xi \tilde{\hat{d}}\Theta_{\mathcal{C}_-} &= i_\xi[\hat{d}(F^\mu_i dy^i \wedge \eta_\mu) + \Pi_i dy^i \wedge \eta] = i_\xi[-(d_\mu F^\mu_i) dy^i \wedge \eta - F^\mu_i dy^i \wedge \hat{d}\eta_\mu) + \Pi_i dy^i \wedge \eta] = \\ &= i_\xi[(-d_\mu F^\mu_i - F^\mu_i \lambda_{G,x^\mu} + \Pi_i) dy^i \wedge \eta] = \xi^i[-d_\mu F^\mu_i - F^\mu_i \lambda_{G,x^\mu} + \Pi_i]\eta - \\ &\quad - \xi^\mu[-d_\mu F^\mu_i - F^\mu_i \lambda_{G,x^\mu} + \Pi_i] dy^i \wedge \eta_\mu, \end{aligned} \tag{10.36}$$

since $\hat{d}\eta_\mu = \lambda_{G,x^\mu}\eta$. Now we take the pullback by the section $j^1_p(s)$ and get

$$i_\xi \tilde{\hat{d}}\Theta_{\mathcal{C}_-} = [-d_\mu F^\mu_i - F^\mu_i \lambda_{G,x^\mu} + \Pi_i](\xi^i - \xi^\mu s^i_{,\mu})\eta \tag{10.37}$$

Requiring (10.35) to be fulfilled **for all** vector fields $\xi \in \mathcal{X}(Z_p)$ we guarantee the possibility to have, for a given section s an m vector fields ξ such that for their component $\xi^i\partial_{y^i} + \xi^\mu\partial_\mu$ the differences $(\xi^i - \xi^\mu s^i_{,\mu})$ are linearly independent in a neighborhood of any point $x \in X$ (we actually can choose these vector fields to be 1-jet prolongations of vector fields $\xi \in \mathcal{X}_\pi(Y)$). Therefore, the condition (10.35) will be fulfilled for all vector fields $\xi \in \mathcal{X}(Z_p)$ if and only if the balance system of equations ★

$$j^1_p{}^*(s)[d_\mu F^\mu_i + F^\mu_i \lambda_{G,x^\mu}] = \Pi_i(j^1_p(s)),\ i = 1\ldots, m$$

is satisfied by the section s. Thus, we get

Theorem 6. *Let $\mathcal{C}$ be a constitutive relation. For a section $s \in \Gamma(\pi)$ the following statements are equivalent*

(1) *For all vector fields $\xi \in \mathcal{X}(Z_p)$,*

$$j^1_p{}^*(s) i_{\xi^1} \tilde{\tilde{d}} \Theta_{\mathcal{C}_-} = 0.$$

(2) *Section s is the solution of the balance system* ★

$$j^1_p{}^*(s)[d_\mu F^\mu_i - F^\mu_i \lambda_{G,x^\mu}] = \Pi_i(j^1_p(s)),\ i = 1\ldots, m. \qquad (\star)$$

Remark 26. If we would like to use the conventional horizontal differential d_H instead of reduced one in the formulated above we would still remove the term of the form $q(z)\eta$ of the Poincare-Cartan form of CR $\mathcal{C}$, but in the calculation above we would get, for a *vertical vector field* ξ on Y an extra term

$$i_{\xi^1}[F^\mu_i d_H dy^i \wedge \eta_\mu] = i_{\xi^1}[F^\mu_i dx^\lambda \wedge dz^i_\lambda \wedge \eta_\mu] = -i_{\xi^1}[F^\mu_i dz^i_\mu \wedge \eta] = F^\mu_i \xi^1_\mu{}^i \eta$$

and, after taking the pullback by a section $s : X \to Y$ we would get an extra term $(F^\mu_i d_\mu \xi^i)\eta$. This brings us back to the requirement that vector field ξ of variation is C-admissible.

11. $\mathcal{C}$-ADMISSIBLE VECTOR FIELDS.

In this section we will start studying the vector space $\mathcal{X}(\mathcal{C})$ of C-admissible vertical vector fields $\xi = \xi^i \partial_{y^i} \in V(\pi)$, i.e. vector fields satisfying to the condition

$$\omega^2_C(\widehat{\xi}) = FDiv(\xi) = F^\mu_i d_\mu \xi^i = 0.$$

Here $\widehat{\xi}$ is the (arbitrary) lift of vector field $\xi \in \mathcal{X}(\pi)$. Let $\phi \in Aut_p(\pi)$ be an automorphism of the bundle π and let $\xi = \xi^\mu \partial_\mu + \xi^i \partial_{y^i} \in \mathcal{X}(\pi)$ be any projectable vector field in Y. Then we have

$$\phi_*(\xi) = (\bar{\phi}^\mu_{,\nu} \xi^\nu)\partial_\mu + (\phi^i_{,\nu}\xi^\nu + \phi^j_{,y^i}\xi^i)\partial_{y^j}. \tag{11.1}$$

Let $\xi = \xi^i \partial_{y^i} \in \mathcal{X}(\mathcal{C})$ and let $\phi \in Aut_p(\pi)$ be as above.

Lemma 6. *For a transformed constitutive relation $C^\phi = \widetilde{\phi}^* \circ C \circ \phi^{1\ -1}$ we have $Fdiv(C^\phi) = \phi^* Fdiv(C)$.*

Proof. By the (12.5-6), (where we substitute ϕ^{-1} for ϕ!) for the transformed CR C^ϕ the (1,n)-component of its Poincare-Cartan form $\Theta_{C,\nu}$ is transformed as follows

$$F^\mu_i dy^i \wedge \eta_\mu \rightarrow detJ(\bar{\phi}^{-1})J(\bar{\phi})^\mu_\nu (F^\nu_j \circ (\phi^{-1})^1)(\phi^{-1})^j_{,y^i} dy^i \wedge \eta_\nu.$$

On the other hand, by () the π_{10}-vertical component of a π^1 vertical vector field ξ transforms under the flow lifted transformation of Z_p as $\xi^i_\mu \partial_{z^i_\mu} \rightarrow \left(\xi^i_\mu J(\phi)^i_j J(\bar{\phi}^{-1})^\lambda_\mu\right)\partial_{z^j_\lambda}$. Combining these two laws of transformation we see that

$$F(C^\phi)^\mu_i (\phi_* \xi)^i_\mu = detJ(\bar{\phi}^{-1})\left(F(C)^\mu_i \xi^i_\mu\right),$$

and, therefore, these two quantities equals zero simultaneously. □

Considering change of local admissible variables and corresponding local automorphism in the intersection of the domains of local charts we see that

Corollary 3. *Condition $F^\mu_i \xi^i_\mu = 0$ defining the class of π^1-vertical vector fields $\mathcal{X}(\mathcal{C})$ is independent on the local adopted chart (x^μ, y^i).*

A natural question that leads directly to the "entropy condition" for a balance system (9.13) (see Part II of this work for more details)

$$(F^\mu_i \circ j^1_p(s))_{,x^\mu} = \Pi_i(j^1_p(s)),\ i = 1, \ldots, m, \tag{$\bigstar$}$$

defined by a CR $\mathfrak{C}$ is - are there, except of the linear combinations of balance equations in the system $\star$, balance laws for the bundle $Y^1_p(\pi)$ that follows from the balance system $\star$ in the following sense:

Definition 21. *Fix a CR $\mathcal{C}$ and consider the corresponding balance system ($\star$). We call a balance law*

$$(K^\mu \circ j^1_p(s))_{,\mu} = Q \circ j^1_p(s). \tag{11.2}$$

of the same type (i.e. with the coefficients defining on $J^1_p(\pi)$) given by a (n+1)+(n+2)-form $K^\mu \eta_\mu + Q\eta$ on the space $Y^1_p(\pi)$ ***generated by the CR*** *$\mathcal{C}$ (or the* ***secondary balance laws*** *for the system ($\bigstar$)) if any solution $s : X \rightarrow Y$ of the balance system ($\bigstar$) is at the same time solution of the balance law (11.2).*

All the balance laws that follows from the balance system (including the balance laws in the system (★) themselves and their linear combinations) form the vector space $\mathcal{BL}_{\mathcal{C}}$. The simple class of secondary balance laws beyond the linear combinations of the balance laws of the system (★) is determined by the following

Proposition 21. *Let a vertical vector field $\xi = \xi^i \partial_{y^i} \in V(\pi)$ belongs to the $\mathcal{X}(\mathcal{C})$, i.e. the condition $FDiv(\xi) = 0$ is fulfilled. Then the balance law*

$$j^{1}_{p}{}^{*}(s)d(\xi^i F^{\mu}_{i}\eta_\mu) = \xi^i \Pi_i \eta \Leftrightarrow ((\xi^i F^{\mu}_{i}) \circ j^1(s))_{,x^\mu} = (\xi^i \Pi_i) \circ j^1(s)$$

belongs to the space $\mathcal{BL}_{\mathcal{C}}$.

Proof. Follows from $d(j^{1}_{p}{}^{*}(s)\xi^i F^{\mu}_{i}\eta_\mu) = \xi^i d(j^{1}_{p}{}^{*}(s)F^{\mu}_{i}\eta_\mu) + j^{1}_{p}{}^{*}(s)FDiv(\xi)\eta$. □

Remark 27. Vector fields $\xi = \xi^i \partial_{y^i}$ with *constant* components ξ^i in a local coordinate system are obviously C-admissible. To such a vector field ξ there correspond, by the Proposition the (secondary) balance law that is, of course, the linear combination of the original balance laws with constant coefficients. More geometrically, one may consider the abelian m-dim subalgebras of the Lie algebra $V(\pi)$ of vertical vector fields on the bundle π. Vector fields of such a subalgebra (generating, by Frobenius theorem the local charts) gives the necessary number of vertical vector fields satisfying to the C-admissibility condition.

On the contrary, variable C-admissible vector fields ξ generate some nontrivial secondary balance laws. In the part II of this work we will study such secondary laws more detailed.

As it is well known, the vector space of divergent-free vector field (divergence being defined by a pseudo-Riemannian metric or, more fundamental, by a volume form - exterior form of maximal degree nonzero at every point of the manifold M) is closed under the bracket of vector fields and, therefore form the Lie subalgebra of Lie algebra $\mathcal{X}(Y)$. Asking the same question about the vector space $\mathcal{X}(C)$ we get, in general, the negative answer.

To illustrate the notion of an $\mathcal{C}$-admissible vector field we consider two simple examples of CR with 1 space variable

Example 17. Consider a case of the full 1-jet bundle $J^1(\pi)$ and of one field $y(t,x)$ being the function of time t and one space variable x. There is only one balance law

$$\partial_t F^0 + \partial_x F^1 = \Pi,$$

and the corresponding balance relation has the Poincare-Cartan form $\Theta_C = F^0 dy \wedge dx + F^1 dy \wedge dt + \Pi dy \wedge dt \wedge dx$. Here F^μ are, in general, functions of all variables $(t, x; y; z_t, z_x)$. As a result, the condition for a vertical vector field $\xi = \xi(t, x; y)\partial_y$ to be C-admissible takes the form

$$F^0 d_t \xi + F^1 d_x \xi = 0,$$

or

$$(F^0 \partial_t + F^1 \partial_x)\xi + (F^0 z_t + F^1 z_x)\partial_y \xi = 0.$$

Let F^μ are independent of the jet variables - $F^\mu = F^\mu(t, x, y)$. Then from the equation above it follows that $\partial_y \xi = 0$, i.e. $\xi = \xi(t, x)$ and $(F^0 \partial_t + F^1 \partial_x)\xi = 0$. This last equation tells that the function $\xi(t, x)$ is constant along the trajectories of the vector field $F^0 \partial_t + F^1 \partial_x$ on the plane. Locally, in a neighborhood of points where this vector field is nonsingular, it tells that the function ξ is an arbitrary function of a transverse variable. Consider an example where $F^0 = y, F^1 = c - const.$ Then the condition reduces to the equation

$$(y\partial_t + c\partial_x)\xi = 0,$$

so that $\xi = f(x - \frac{c}{y}t)$ with an arbitrary differentiable function of one variable.

If $\xi_i = f_i(x - \frac{c}{y}t)\partial_y$ are two such C-admissible vector fields, then their commutator

$$[\xi_1, \xi_2] = (-\frac{ct}{y^2})(f_1 f_2' - f_2 f_1')\partial_y$$

is not a C-admissible vector field because the coefficient of ∂_y does not have the required form.

Example 18. As a second example we consider a balance system with 2-dim space-time X (with coordinates $x^0 = t, x^1 = x$) and two fields y^1, y^2. For such a system the condition $FDiv(\xi) = 0$ takes the form

$$F^0_i d_{x^0}\xi^i + F^1_i d_{x^1}\xi^i = F^0_i \partial_t \xi^i + F^1_i \partial_x \xi^i + [z^k_0 F^i_0 + z^k_1 F^1_i]\partial_{y^k}\xi^i = 0.$$

Restrict to the RET case where $F^\mu_i = F^\mu_i(x, y)$ do not depend on the jet variables z^i_μ. Then the condition above splits into three conditions

$$\begin{cases} F^0_i \partial_{y^k}\xi^i = 0, \\ F^1_i \partial_{y^k}\xi^i = 0, \\ (F^0_i \partial_t + F^1_i \partial_x)\xi^i = 0. \end{cases} \tag{11.3}$$

First two equations show that the (nonzero) covectors $F^\mu_i dy^i,\ \mu = 0, 1$ are annulated by the linear transformation with the matrix $J(\xi) = \partial_{y^k}\xi^i$. This is (generically) possible in two cases:

(1) $J(\xi) \equiv 0$. This means that $\xi^i = \xi^i(x)$ do not depend on the fiber coordinates y^i. This being true, the third condition takes the form similar to that in the previous example

$$(F^0_i(x, y)\partial_t + F^1_i(x, y)\partial_x)\xi^i(x) = 0, \forall y.$$

Let, for instance, $F^0_i = y^i$ be the density of the field y^i. Then the last condition takes the form

$$(y^i \partial_t + F^1_i(x, y)\partial_x)\xi^i(x) = 0, \forall y.$$

Decomposing the flux term F^1_i by y^i into Taylor series (or differentiating it by y^i) we get the evolutional equation

$$\partial_t \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} + \left(F^1_{i,y^j}(t, x, y = 0)\right)\partial_x \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix} = 0 \tag{11.4}$$

and the family of ordinary differential equations $Q^\alpha_i \partial_x \xi^i = 0,\ \alpha = (\alpha_1, \alpha_2)$ for the components ξ^i of vector field ξ number and type of which is determined by the character of dependence of $F^1_i(x, y)$ on y.

If, for instance $F^1_i = F^1_{ij}(x)y^j$ are linear by y, no additional conditions are present and the evolutional system (11.4) is locally solvable for a good enough initial condition $\xi^i(t, x)|_{t=0} = \xi^i(x)$. This gives us a family of vector fields $\xi \in \mathcal{X}(\mathcal{C})$ depending on two functions of one variable as in the previous example.

(2) Case $rk(J(\xi)) = 1$. In this case $F^\mu_i, \mu = 0, 1$ belongs to the kernel of the 2×2 matrix $J(\xi)$ and are, therefore, proportional:

$$F^1_i = \lambda(x, y) F^0_i, \tag{11.5}$$

with some function $\lambda(x, y)$. Third equation takes the form

$$F^0_i(\partial_t + \lambda\partial_x)\xi^i = 0.$$

If, for instance $F^0_i = y^i$, last equation takes the form

$$(\partial_t + \lambda\partial_x)(y^i\xi^i) = 0.$$

Starting with a CR $\mathcal{C}$ of the form considered in this example that satisfies the relation (11.5) we find the function λ, then the system of equations (11.3) for the components ξ^i of the vector field $\xi(x, y)$ takes the form

$$\begin{cases} y^i\partial_{y^k}\xi^i = 0, \; i = 1, 2, \\ (\partial_t + \lambda\partial_x)(y^i\xi^i) = 0. \end{cases} \tag{11.6}$$

First two equations can be rewritten in the form

$$\partial_{y^k}(y^1\xi^i + y^2\xi^2) - \xi^k = 0.$$

Introduce the function $q(x, y) = y^1\xi^i + y^2\xi^2$. Then first two equations give us $\xi^i = \partial_{y^i}q$. substituting ξ^i in such a form into the first two equations we get for the function $q(x, y)$ two equations

$$\begin{cases} y^1 q_{,11} + y^2 q_{,12} = (y^1\partial_{y^1})q_{,y^1} = 0, \\ y^1 q_{,21} + y^2 q_{,22} = (y^1\partial_{y^1})q_{,y^2} = 0. \end{cases}$$

In terms of polar coordinates ρ, θ in the plane (y^1, y^2) these two conditions means that $\xi^i = \xi^i(\theta)$ are independent on the radial variable ρ and depend on the angular variables $\theta = tan^{-1}(\frac{y^2}{y^1})$ and t, x.

This is equivalent to the statement that $q = y^1\tilde{q}(x, \frac{y^2}{y^1})$.

Substituting this to the third equation we get it in the form

$$(\partial_t + \lambda(x, y)\partial_x)\tilde{q} = 0.$$

Any solution of this first order wave type equation with the parameter $\frac{y^2}{y^1}$ determines the vector field $\xi \in \mathcal{X}(\mathcal{C})$.

There exists a geometrical situation where there is a natural class of m linearly independent at each point globally defined vector fields ξ admissible for all the constitutive relations.

Proposition 22. *Let $\pi : Y \to X$ be a trivial principal bundle of a connected abelian n-dimensional Lie group A. Then the (globally defined) fundamental vector fields on Y (generated by the right action of A on Y) satisfy to the condition $\omega^2_{\mathcal{C}}(\widehat{\xi}) = 0$.*

Proof. Trivial. □

Corollary 4. *Let $\pi : Y \to X$ be a trivial principal bundle of a connected abelian m-dimensional Lie group A (in particular, a* trivial vector bundle *over X). Let $\mathcal{C}$ be an arbitrary constitutive relation. Then the equivalence statement of Theorem 2 is valid for any constitutive relation $\mathcal{C}$.*

Remark 28. One might expect that in a case of a non-trivial bundle $\pi : Y \to X$ there may be less then n linearly independent *global* vector fields in $\mathcal{X}(F)$. It would be interesting to study a topological meaning of such a phenomena.

For a **projectable vertical vector field** we can formulate this condition (and more generally, condition (7.7)) in a covariant way. Recall that $F^\mu_i dy^i \wedge \eta_\mu$ is the section of the bundle $\pi^{1*}_0(V^*(\pi) \otimes \Lambda^{n-1}(X))$ over Z_p. Last bundle have, as its $\Lambda^n(X)$-dual, the bundle $\pi^{1*}_0(V(\pi) \otimes \Lambda^1(X))$ over Z_p and we may consider the associating of $d_\mu \xi^i$ to the vector field ξ as the mapping of **total differential**

$$D : \xi = \xi^i(x, y)\frac{\partial}{\partial y^i} \to d_\mu \xi^i dx^\mu \otimes \frac{\partial}{\partial y^i} : V(\pi) \to \pi^{1*}_0(V(\pi) \otimes \Lambda^1(X)). \tag{11.7}$$

Then condition (9.10) takes the form

$$FDiv(\xi) = \langle F^\mu_i dy^i \wedge \eta_\mu, D\xi\rangle = 0,$$

where notation $FDiv(\xi)$ for the expression on the right was introduced in the previous section.

Example 19. Consider the case of a vector fields of the type

$$\xi = \xi^i(x^\mu, y^i, z^i_\mu)\frac{\partial}{\partial y^i}.$$

For such a vector field and an arbitrary section $s \in \Gamma(\pi)$

$$j^{1*}(s)\langle F^\mu_i dy^i \wedge \eta_\mu, D\xi\rangle = [F^\mu_i \circ j^1(s)(\xi^i_{,x^\mu} + \xi^i_{y^j} s^j_{,x^\mu}) + F^\mu_i \circ j^1(s)\xi^i_{,z^j_\nu} s^j_{x^\nu x^\mu}]\eta. \tag{11.8}$$

Since second derivative components of 2-jets of sections s at a point (x, y, z) *can be arbitrary*, condition of C-admissibility for these vector fields ξ splits into two conditions:

$$\begin{cases} F^\mu_i(\xi^i_{,x^\mu} + \xi^i_{,y^j} z^j_\mu) = F^\mu_i(\frac{\partial}{\partial x^\mu} + z^j_\mu \frac{\partial}{\partial y^j})\xi^i & = 0, \\ F^{(\mu}_i \xi^i_{,z^j_{\nu)}} & = 0, \text{ for all } \mu, \nu, j, \end{cases} \tag{11.9}$$

where in the second condition the symmetrization by $\mu\nu$ is done.

One can rewrite second condition in the form

$$F^\mu_i \xi^i_{,z^j_\nu} = \omega^{\mu\nu}_j,$$

for an arbitrary family of skew-symmetrical tensors $\omega^{\mu\nu}_j$ on Z_p. For a projectable vector field ξ the second equation is trivially satisfied.

11.1. **$\mathcal{C}$-admissible vertical vector fields: 5F-fluid system.** Here we consider the condition of C-admissibility for a vector field $\xi \in \mathcal{X}(Y)$ in the case of the 5F-fluid balance system C_5, see (2.7) in the Newtonian space-time with Euclidian metric $(E^h, h = dt^2 + \sum_A dx^{A\ 2})$ and the global coordinates (t, x^A). In order to simplify calculations we choose basic fields to be $(\rho; v^A, A = 1, 2, 3; \vartheta)$ and will use the internal energy ϵ balance instead of the full energy density e balance. We assume that the constitutive relations for 5-fields system, i.e. functions ϵ, t^A_B, q^A may depend on the spacial gradients of dynamical variables $\nabla v, \nabla\vartheta$.

We have for C_5:

$$\Theta_{C_5} = [\rho d\rho \wedge \eta_0 + \rho v^B d\rho \wedge \eta_B] + [(\rho v^A) dv^A \wedge \eta_0 + (\rho v^A v^B - t^{AB}) dv^A \wedge \eta_B] +$$
$$+ [\rho\epsilon d\vartheta \wedge \eta_0 + (\rho\epsilon V^A + q^A) d\vartheta \wedge \eta_B]] + [f_A dv^A \wedge \eta + (f_A v^A + t^A_B \frac{\partial V^B}{\partial x^A} + r) d\vartheta \wedge \eta]. \tag{11.10}$$

Since we assume that the constitutive relations C_5 are independent on the derivatives of ρ, for a general vector field $\hat{\xi} \in \mathcal{X}(Z_p)$, condition of C-admissibility $F^\mu_i \xi^i_\mu = 0$ has the form

$$(z^i_{A\mu}\xi^\mu - \xi^i_A) F^A_i = 0; A = 1,2,3, \mu = 0,1,2,3\ i = v^B, \vartheta.$$

Due to the independence of the 2-jet variables $z^i_{A\mu}$ and to the fact that none of the flow components F^μ_i is zero, we get: $\xi^\mu = 0,\ \mu = 0,1,2,3$. Thus, C_5-admissible vector field ξ is π-vertical and condition of C_5-admissibility takes the form

$$\sum_{(A,i)\in P} \xi^i_A F^A_i = (\rho v^A v^B - t^{AB}) \xi^{v^B}_A + (\rho\epsilon v^A + q^A) \xi^\vartheta_A = 0. \tag{11.11}$$

Any π^1-vertical vector field $\xi \in \mathcal{X}(Z_p)$ that satisfy to this linear algebraic condition is C_5-admissible.

Consider now the case where the vector field above is the flow prolongation ξ^1 of a π-vertical vector field $\xi = \xi^\rho \partial_\rho + \xi^{v^A} \partial_{v^A} + \xi^\vartheta \partial_\vartheta$, i.e. use

$$\xi^1 = \xi + d_\mu \xi^i \partial_{z^i_\mu} = \xi + d_\mu \xi^\rho \partial_{z^\rho_\mu} + d_\mu \xi^{v^A} \partial_{z^{v^A}_\mu} + d_\mu \xi^\vartheta \partial_{z^\vartheta_\mu}.$$

Condition of admissibility $\sum_{(A,i)\in P} F^A_i d_A \xi^i = 0$ (summation goes over the space derivatives since no time derivatives of dynamical fields enters the constitutive relations) is obtained by substituting $\xi^i_A = d_A \xi^i$ into the algebraic equation (11.11).

$$\rho v^A v^B d_A \xi^{v^B} + \rho\epsilon v^A d_A \xi^\vartheta = t^{AB} d_A \xi^{v^B} - q^A d_A \xi^\vartheta. \tag{11.12}$$

This equation splits corresponding to the order of the terms in the total derivatives $d_A = \partial_{x^A} + z^i_A \partial_{y^i}$ and we get:

$$\begin{cases} \text{Coefficient of } z^i_A: \ \rho v^A v^B \xi^{v^B}_{,y^i} + \rho\epsilon v^A \xi^\vartheta_{,y^i} = t^{AB} \xi^{v^B}_{,y^i} - q^A \xi^\vartheta_{,y^i},\ i = \rho, v^C, \vartheta; A = 1,2,3, \\ \text{No } z^i_A: \quad \rho v^A v^B \xi^{v^B}_{,A} + \rho\epsilon v^A \xi^\vartheta_{,A} = t^{AB} \xi^{v^B}_{,A} - q^A \xi^\vartheta_{,A}. \end{cases} \tag{11.13}$$

or, in more details,

$$\begin{cases} \rho v^A v^B \xi^{v^B}_{,\rho} + \rho\epsilon v^A \xi^\vartheta_{,\rho} = t^{AB} \xi^{v^B}_{,\rho} - q^A \xi^\vartheta_{,\rho}, \\ \rho v^A v^B \xi^{v^B}_{,v^C} + \rho\epsilon v^A \xi^\vartheta_{,v^C} = t^{AB} \xi^{v^B}_{,v^C} - q^A \xi^\vartheta_{,v^C}, \\ \rho v^A v^B \xi^{v^B}_{,\vartheta} + \rho\epsilon v^A \xi^\vartheta_{,\vartheta} = t^{AB} \xi^{v^B}_{,\vartheta} - q^A \xi^\vartheta_{,\vartheta}, \\ \rho v^A v^B \xi^{v^B}_{,A} + \rho\epsilon v^A \xi^\vartheta_{,A} = t^{AB} \xi^{v^B}_{,A} - q^A \xi^\vartheta_{,A}. \end{cases} \tag{11.14}$$

Looking at the first three (families of) equations in this system we see that the left sides of these equations do not depend on the jet variables. Therefore right sides of these equations do not depend on jet- variables $\nabla v, \nabla\vartheta$ too.

For the second equation this gives, since t^A_B, q^A do not depend on v^C that

$$t^{AB} \xi^{v^B}_{,v^C} - q^A \xi^\vartheta_{,v^C} = (t^{AB} \xi^{v^B} - q^A \xi^\vartheta)_{,v^C} = \lambda^A_{v^C}(\rho, v, \vartheta)$$

with some functions $\lambda^A_{v^C}(\rho, v, \vartheta)$ of named variables.

Since function $(t^{AB}\xi^{v^B} - q^A\xi^\vartheta)$ in the left side does not depend on C and due to the mixed derivative test $\partial_{v^C_1}\lambda^A_{v^C_2} = \partial_{v^C_2}\lambda^A_{v^C_1}$, we have $\lambda^A_{v^C} = \partial_{v^C}h^A(\rho, v, \vartheta)$ with some functions $h^A(\rho, v, \vartheta)$. From this it follows that

$$(t^{AB}\xi^{v^B} - q^A\xi^\vartheta - h^A)_{,v^C} = 0 \Rightarrow t^{AB}\xi^{v^B} - q^A\xi^\vartheta = h^A(\rho, v, \vartheta) + h^A_1((\rho, \vartheta, \nabla v, \nabla\vartheta)). \tag{11.15}$$

Thus, second equation takes the form

$$\rho v^A v^B \xi^{v^B}_{,v^C} + \rho\epsilon v^A \xi^\vartheta_{,v^C} = h^A_{,v^C}(\rho, v, \theta). \tag{11.16}$$

Take here derivative by $\nabla\theta$. We get

$$\rho\frac{\partial\epsilon}{\partial\nabla\theta}v^A\xi^\vartheta_{,v^C} = 0.$$

Thus, either $\frac{\partial\epsilon}{\partial\nabla\theta} = 0$ or $\xi^\vartheta_{,v^C} = 0$. Taking derivative by ∇v we get that either $\frac{\partial\epsilon}{\partial\nabla\theta} = \frac{\partial\epsilon}{\partial\nabla v} = 0$ or $\xi^\vartheta_{,v^C} = 0$.

Consider first the second alternative - $\xi^\vartheta_{,v^C} = 0$. substituting this into the equation (11.16) we get

$$t^A_B\xi^{v^B}_{,v^C} = h^A_{,v^C} \tag{11.17}$$

for all A, C. Thus, either t^A_B does not depend on the gradient variables or $\xi^{v^B}_{,v^C}$ is degenerate and the part of t^A_B that depend on gradients belongs to the kernel of $\xi^{v^B}_{,v^C}$.

Consider, *generically*, a case where $\xi^{v^B}_{,v^C} = 0 \rightarrow h^A_{,v^C} = 0$ as well.

Thus, ξ^ϑ, ξ^{v^B} do not depend on v^C. Taking derivatives by v^A, v^B in the first equation we get $\xi^\vartheta_{,\rho} = 0$ (ϵ does not depend on v^C), then, using this and differentiating by v^A we get $\xi^{v^B}_{,\rho} = 0$ as well. Repeating this procedure with the third equation we will see that ξ^ϑ, ξ^{v^B} do not depend on ϑ as well. Thus,

$$\xi^{v^B}_{,v^C} = 0 \rightarrow \xi^\vartheta, \xi^{v^B} - const.$$

Another choice would be to have $\epsilon = \epsilon(\rho, \theta)$ (which is realized, for instance in the case of Navier-Stokes fluid ([31]).

In such a case we return to the relation (11.16)

$$t^A_B\xi^{v^B}_{,v^C} - q^A\xi^\vartheta_{,v^C} = h^A_{,v^C}.$$

Taking here derivative by ∇v and *assuming generically that* $\xi^{v^B}_{,v^C}$ *nondegenerate* we get

$$\frac{t^A_{B,\nabla v}}{q^A_{,\nabla v}} = (\xi^{v^B}_{,v^C})^{-1} \cdot \xi^\vartheta_{,v^C} = \lambda_B(\rho, \theta)$$

does not depend on v^C or on any gradients. Similarly we get

$$\frac{t^A_{B,\nabla\vartheta}}{q^A_{,\nabla\vartheta}} = (\xi^{v^B}_{,v^C})^{-1} \cdot \xi^\vartheta_{,v^C} = \lambda_B(\rho, \theta).$$

Then $q^A_{,\nabla v}\lambda_B = t^A_{B,\nabla v}, q^A_{,\nabla \vartheta}\lambda_B = t^A_{B,\nabla \vartheta}$ and, integrating by $\nabla v, \nabla \vartheta$ we get:

$$\begin{cases} q^A\lambda_B(\rho,\vartheta) = t^A_B + \mu^A_B(\rho,\vartheta), \\ \xi^\theta_{,v^C} = \xi^{v^B}_{,v^C}\lambda_B(\rho,\vartheta). \end{cases}$$

Here $\lambda_B(\rho,\vartheta), \mu^A_B(\rho,\vartheta)$ are *constitutional functions*, depending on the constitutive relations C_5 but not on the vector field ξ.

From the second equation we get

$$\xi^\vartheta = \xi^{v^B}\lambda_B(\rho,\vartheta) + \zeta(\rho,\vartheta).$$

Using expressions for t^A_B and ξ^ϑ from these two equations we get

$$t^A_B\xi^{v^B} - q^A\xi^\vartheta = -q^A\zeta(\rho,\vartheta) - \mu^A_B\xi^{v^B}_{,v^C}.$$

Taking here derivative by v^C and substituting into the second equation of system (11.14) we get

$$\rho v^A(v^B\xi^{v^B}_{,v^C} + \epsilon\xi^\vartheta_{,v^C}) = -\mu^A_B\xi^{v^B}_{,v^C}.$$

substituting here $\xi^\vartheta_{,v^C} = \xi^{v^B}_{,v^C}\lambda_B$ we get

$$[\rho v^A v^B + \rho v^A\epsilon\lambda_B(\rho,\vartheta) + \mu^A_B(\rho,\vartheta)]\xi^{v^B}_{,v^C} = 0. \tag{11.18}$$

Generically, to be true in an open set of basic fields space U we have to have $\xi^{v^B}_{,v^C} = 0$. Then $\xi^\vartheta_{,v^C} = 0$ and from this it follows as above that $\xi^{v^B}, \xi^\vartheta - const.$

Proposition 23. *Assume that in the 5F-fluid system constitutive fields (i.e. t^A_B, q^A, ϵ) do not depend on the velocity v^A and on the $\nabla\rho$. Then,* ***generically****, i.e. without special conditions for the constitutive relations, components ξ^{v^C}, ξ^ϑ of a vector field $\xi \in V(Y) \cap \mathcal{X}(C)$ are constant and an admissible π-vertical vector field $\xi \in V(Y)$ has the form*

$$\xi = \xi^\rho(t,x,\rho,v,\vartheta)\partial_\rho + \xi^{v^B}(t)\partial_{v^B} + \xi^\vartheta\partial_\vartheta.$$

.

More detailed study of the geometrical properties of the 5F-fluid balance system including the consideration of special, non-generic cases of the constitutive relations will be done in other paper.

12. Action of geometrical transformations on the constitutive relations.

In this section we study the action on the constitutive relations of the natural prolongations of the projectable transformations $\phi \in Aut_p(\pi)$ of Y studied in the Section 5.

12.1. **Action of $\phi \in Aut_\pi(Y)$ on the covering constitutive relations.** Let $\hat{C} : J^1_p(\pi) \to \Lambda^{n+1}Y \oplus \Lambda^{n+2}Y$ be a covering constitutive relation. Decompose corresponding Poincare-Cartan form as follows:

$$\Theta_{\hat{C}} = \Theta^{0,n+1}_{\hat{C}} + \Theta^{1,n}_{\hat{C}} + \Theta^{n+2}_{\hat{C}} = p\eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta. \tag{12.1}$$

An automorphism $\phi \in Aut_p(\pi)$ of the bundle π can be lifted to the contact automorphism ϕ^1 of the partial 1-jet bundle $Z_p = J^1_p(\pi)$ over Y and X (see Sec. 5). It can also be lifted to the bundle automorphism $\phi^{1*} = (\phi^{*(n+1)}, \phi^{*(n+2)})$ of the bundle $\Lambda^{n+1}_2 Y \oplus \Lambda^{n+2}_2 Y$ preserving its subbundle $\Lambda^{n+1}_2 Y \oplus \Lambda^{n+2}_2 Y$. Automorphism ϕ^{1*} leaves canonical form(s) $\Theta^{n+1}_2 \oplus \Theta^{n+2}_2$ invariant (Sec.5).

More generally, let $\psi \in Aut(\pi_1)$ belongs to the group of automorphisms of the double bundle $\pi^1 : J^1_p(\pi) \to Y \to X$. Transformation $\psi \in Aut(\pi^1)$ generate the automorphism $\phi \in Aut(\pi)$. In its turn, transformation ϕ extends to the automorphism $\phi^1 \in Aut_p(\pi_1)$ (see Section 6). This allows to present

$$\psi = \phi^1 \circ \psi_{gau},$$

where the automorphism ψ_{gau} projects to the identity diffeomorphism of Y and, thus, represents pure gauge transformation of $J^1_p(\pi)$. Correspondingly, the action of ψ on the Poincare-Cartan form of a CCR $\hat{C}$ is the composition

$$\psi^* \Theta_{\hat{C}} = \psi^*_{gau} \phi^{1\ *} \Theta_{\hat{C}}.$$

Calculate this action explicitly.

Lifted automorphism (ϕ^1, ϕ^{1*}) transforms the constitutive relation $\hat{C}$ into the constitutive relation

$$\hat{C}^\phi = \phi^{1*} \circ \hat{C} \circ \phi^{1\ -1}. \tag{12.2}$$

For the Poincare-Cartan form of $\hat{C}^\phi$ we have (using for the transformed CCR $\hat{C}^\phi$ the fact that ϕ^{1*} preserves the multisymplectic forms Θ^{n+1}_2 and Θ^{n+2}_2)

$$\begin{aligned} \Theta_{\hat{C}^\phi} &= (\hat{C}^\phi)(\Theta^{n+1}_2 \oplus \Theta^{n+2}_2) = (\phi^{1*} \circ \hat{C} \circ \phi^{1\ -1})^* (\Theta^{n+1}_2 \oplus \Theta^{n+2}_2) = \\ &= \phi^{1\ -1\ *} \circ \hat{C}^* \circ (\phi^{1*})^* (\Theta^{n+1}_2 \oplus \Theta^{n+2}_2) = \phi^{1\ -1\ *} \circ \hat{C}^* (\Theta^{n+1}_2 \oplus \Theta^{n+2}_2) = \phi^{1\ -1\ *} \Theta_{\hat{C}}. \end{aligned} \tag{12.3}$$

On the other hand

$$\begin{aligned} \phi^{1*-1} \Theta_{\hat{C}} &= \phi^{1*-1}[p\eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta] = (p \circ \phi^{1\ -1}) \cdot \bar{\phi}^{*-1}\eta + \\ &+ (F^\mu_i \circ \phi^{1\ -1}) d(\phi^{-1\ i})(x,y) \wedge \bar{\phi}^{*-1}\eta_\mu + \Pi_i \circ \phi^{1\ -1} d(\phi^{-1\ i}) \wedge \bar{\phi}^{-1\ *}\eta. \end{aligned} \tag{12.4}$$

here we have used the fact that dy^i and η, η_μ are pullbacked form the spaces Y and X respectively and that pullback by π_{10} or by π^1 commutes with the pullback by ϕ^1 and its projections to Y and X respectively.

To shorten the notations we will make the next calculation for $\widehat{\phi} \in Aut_p(\pi^1)$ - automorphsim of the double bundle $Z_p \rightarrow Y \rightarrow X$ rather then ϕ^{-1} (lift $\widehat{\phi} = \psi^1$ of a automorphism $\psi \in Aut_p(\pi)$ is a special case of this more general case). Automorphism $\widehat{\phi}$ induces automorphism $\phi \in Aut_p(\pi)$ and automorphism of splitting structure (if $J^1_P(\pi)$ is the proper partial 1-jet bundle) in X. We notice that $\bar{\phi}^*\eta = detJ(\bar{\phi})\eta$ where $detJ(\bar{\phi})$ is the Jacobian of the (local) diffeomorphism $\bar{\phi}$ defined by the volume form η. On the other hand

$$\bar{\phi}^*\eta_\mu = \bar{\phi}^* i_{\partial_{x^\mu}}\eta = i_{\bar{\phi}^{-1}_*\partial_{x^\mu}}\bar{\phi}^*\eta = detJ(\bar{\phi})J(\bar{\phi}^{-1})^\nu_\mu\eta_\nu$$

since $\bar{\phi}^{-1}_*\partial_{x^\mu} = J(\bar{\phi}^{-1})^\nu_\mu\partial_{x^\nu}$.

Altogether

$$\begin{aligned}\widehat{\phi}^*\Theta_{\hat{C}} &= p\circ\widehat{\phi}\cdot detJ(\bar{\phi})\eta + (F^\mu_i\circ\widehat{\phi})(\phi^i_{,x^\sigma}dx^\sigma + \phi^i_{,y^j}dy^j)\wedge[detJ(\bar{\phi})J(\bar{\phi}^{-1})^\nu_\mu\eta_\nu]+\\ &+\Pi_i\circ\widehat{\phi}(\phi^i_{,x^\sigma}dx^\sigma + \phi^i_{,y^j}dy^j)\wedge detJ(\bar{\phi})\eta = [detJ(\bar{\phi})J(\bar{\phi}^{-1})^\nu_\mu(F^\mu_i\circ\widehat{\phi})\phi^i_{,y^j}]dy^j\wedge\eta_\nu+\\ &+[detJ(\bar{\phi})(\Pi_i\circ\widehat{\phi})\phi^i_{,y^j}]dy^j\wedge\eta + [p\circ\widehat{\phi}\cdot detJ(\bar{\phi}) + (F^\mu_i\circ\widehat{\phi})(\phi^i_{,x^\nu}[detJ(\bar{\phi})J(\bar{\phi}^{-1})^\nu_\mu]\eta.\end{aligned} \tag{12.5}$$

Splitting the terms we can write last result as follows.

$$\begin{cases}[\widehat{\phi}^*\Theta_{\hat{C}}] = {}^{\widehat{\phi}}p\eta + {}^{\widehat{\phi}}F^\mu_i dy^i\wedge\eta_\mu + {}^{\widehat{\phi}}\Pi_i dy^i\wedge\eta, \text{ where}\\ {}^{\widehat{\phi}}p = [p\circ\widehat{\phi} + (F^\mu_i\circ\widehat{\phi})(\phi^i_{,x^\nu}J(\bar{\phi}^{-1})^\nu_\mu]\cdot detJ(\bar{\phi}),\\ {}^{\widehat{\phi}}F^\mu_i = detJ(\bar{\phi})J(\bar{\phi}^{-1})^\mu_\nu(F^\nu_j\circ\widehat{\phi})\phi^j_{,y^i},\\ {}^{\widehat{\phi}}\Pi_i = detJ(\bar{\phi})(\Pi_j\circ\widehat{\phi})\phi^j_{,y^i}.\end{cases} \tag{12.6}$$

To use these formulas for calculating $C^{\widehat{\phi}}$ and $\Theta_{C^{\hat{\phi}}}$ one should replace $\widehat{\phi}$ in (12.3) by inverse mapping $\widehat{\phi}^{-1}$.

From the last result it follows that $\Theta^{1,n}_{\hat{C}}$ and $\Theta^{n+2}_{\hat{C}}$ transforms *tensorially* under the action of $\widehat{\phi}$ while the component $\Theta^{0,n+1}_{\hat{C}}$ transforms affine.

Remark 29. If we take $\widehat{\mathcal{C}}$ to be ν-lifted CCR $\hat{C}_\nu = q_\nu\circ\mathcal{C}$ (see Sec.) for an arbitrary constitutive relation $\mathcal{C}: Z_p\rightarrow\tilde{Z}$ and $\Theta_{\mathcal{C},\nu}$ - corresponding ν-lifted Poincare-Cartan form of $\mathcal{C}$. for a one-parameter group of automorphisms $\widehat{\phi}_t$ of Z_p we define one-parameter group of automorphisms ψ_t of $\Lambda^{(n+1)+(n+2)}_2$ by projecting $\hat{\phi}_t$ to the one-parameter group of automorphisms ϕ_t of Y and then lifting it to $\Lambda^{(n+1)+(n+2)}_2$ using results of section 5. We have

$$\begin{aligned}\widehat{\phi}^*_{-t}\Theta_{\hat{C},\nu} &= \widehat{\phi}^*_{-t}(\hat{C}^*\tilde{\Theta}_\nu) = \widehat{\phi}^*_{-t}C^*\psi^*_t\psi^*_{-t}\tilde{\Theta}_\nu = (\phi^{1*}_{-t}C^*\psi^*_t)[\tilde{\Theta}_\nu + (F^\mu_{Pi}\Gamma^i_\mu)\circ\psi_t\eta] =\\ &= (\psi_t\circ C\circ\widehat{\phi}_{-t})^*[\tilde{\Theta}_\nu + (F^\mu_{Pi}\Gamma^i_\mu)\circ\psi_t\eta] = C^{\phi_t\,*}[\tilde{\Theta}_\nu + ((F^\mu_{Pi}\Gamma^i_\mu)\circ\psi_t)\eta] = \Theta_{C^{\hat{\phi}_t},\nu} + C^{\widehat{\phi}_t\,*}((F^\mu_{Pi}\Gamma^i_\mu)\circ\psi_t)\eta,\end{aligned} \tag{12.7}$$

where we have used the fact that ψ_t acts on the form $\Theta_{C,\nu}$ leaving its $dy^i\wedge\eta_\mu$ part invariant. Here $C^{\widehat{\phi}_t} = \psi_t\circ C\circ\widehat{\phi}_{-t}$.

If we take the expression obtained above mod Λ^{n+1}_1 the last term vanished and we get the formula for transformation of the $(1,*)$-part of the Poincare-Cartan form $\Theta_{\mathcal{C},\nu}$ of a CR $\mathcal{C}$ *independent on a choice of connection* ν! (another argument would be that the last term vanishes if we contract it with a vertical vector field ξ):

$$\phi^{1*}\Theta^{(1,*)}_{C,\nu} = [detJ(\bar{\phi})J(\bar{\phi}^{-1})^{\nu}_{\mu}(F^{\mu}_{i}\circ\phi^{1})\phi^{i}_{,y^j}]dy^j\wedge\eta_{\nu} + [detJ(\bar{\phi})(\Pi_i\circ\phi^1)\phi^{i}_{,y^j}]dy^j\wedge\eta. \tag{12.8}$$

In terms of separate balance laws

$$\sigma_i = F^{\mu}_{i}\eta_{\mu} + \Pi_i\eta, \tag{12.9}$$

using the pullback of the basic forms η, η_ν by $\bar{\phi}$

$$\begin{cases}\eta^{\bar{\phi}} = detJ(\bar{\phi})\eta;\\ \eta^{\bar{\phi}}_{\mu} = detJ(\bar{\phi})J(\bar{\phi}^{-1})^{\nu}_{\mu}\eta_{\nu}\end{cases}$$

we have for the transformed balance laws

$$\sigma^{\phi}_{i} = ((F^{\nu}_{j}\circ\phi^{1})\phi^{j}_{,y^i})\eta^{\bar{\phi}}_{\nu} + ((\Pi_j\circ\phi^1)\phi^{j}_{,y^i})\eta^{\bar{\phi}}, \tag{12.10}$$

Transformation $C\rightarrow\mathcal{C}^{\phi}$ acts, in a natural way, on the sheaf of solutions $Sol(\mathcal{C})$ transforming it to the sheaf of solutions of the balance system $\mathcal{B}^{\phi}_{\mathcal{C}}$:

$$Sol(C)\rightleftarrows Sol(C^{\phi}). \tag{12.11}$$

A pure gauge automorphisms ψ_{gau} in the decomposition $\psi = \phi^1\circ\psi_{gau}$ acts simply by

$$\psi^{*}_{gau}\Theta_C = F^{\mu}_{i}\circ\psi_{gau}dy^i\wedge\eta_{\mu} + \Pi_i\circ\psi_{gau}dy^i\wedge\eta. \tag{12.12}$$

and the individual balance laws σ_i after transformation take the form

$$\sigma^{\psi_{gau}}_{i} = F^{\mu}_{i}\circ\psi_{gau}\eta_{\mu} + \Pi_i\circ\psi_{gau}\eta. \tag{12.13}$$

Let now $\xi\in\mathcal{X}_p(\pi)$ be an *infinitesimal automorphism* (vector field) of the bundle π, i.e. a projectable vector field in Y satisfying to the conditions of Section 7 for lifting to the partial 1-jet bundle $J^1_p(\pi)$:

$$\xi = \xi^{\mu}(x)\partial_{x^\mu} + \xi^{i}(x,y)\partial_{y^i}.$$

Let ξ^1 be its prolongation to the projectable contact vector field in $J^1_p(\pi)$ (see Sec.7). Thus, we have

$$\xi^1 = \xi^{\mu}(x)\partial_{x^\mu} + \xi^{i}(x,y)\partial_{y^i} + \left(d_{\mu}\xi^i - z^{i}_{\nu}\frac{\partial\xi^{\nu}}{\partial x^{\mu}}\right)\partial_{z^i_\mu},$$

where summation in the last term is taken over the z^i_μ that are present in the partial 1-jet bundle. In the RET case we do not need to introduce any prolongation.

Let $\widetilde{\xi}^*$ be the prolongation of ξ to the projectable vector field in $\tilde{Z} = \Lambda^{(n+1)+(n+2)}_{2}/\Lambda^{(n+1)+(n+2)}_{1}$ preserving canonical multisymplectic forms $\Theta^{n+1}_{2}, \Theta^{n+2}_{2}$ (See Sec.7):

$$\widetilde{\xi}^* = \xi^{\mu}(x)\partial_{x^\mu} + \xi^{i}(x,y)\partial_{y^i} + \left(-p^{\nu}_{i}\left(\frac{\partial\xi^{\mu}}{\partial x^{\nu}} - \xi^{\mu}\lambda_{,x^\nu}\right) - p^{\mu}_{j}\frac{\partial\xi^{j}}{\partial u^{i}} - p^{\mu}_{i}\left(\frac{\partial\xi^{\nu}}{\partial x^{\nu}} - \xi^{\nu}\lambda_{G,x^\nu}\right) - p^{\nu}_{i}\xi^{\mu}\lambda_{G,x^\nu}\right)\partial_{p^\mu_i} + \\ + \left(p_k\left(\xi^{\mu}\frac{\partial\lambda_G}{\partial x^{\mu}} - \frac{\partial\xi^{\mu}}{\partial x^{\mu}}\right) - p_j\frac{\partial\xi^j}{\partial u^k}\right)\partial_{p_k}. \tag{12.14}$$

Let now ϕ^1_t be a local flow in $J^1_p(\pi)$ of the vector field ξ^1 and ψ_t be a local flow in $\tilde{Z}$ of the vector field $\widetilde{\xi^*}$.

Taking in the expression for the transformed mapping $\psi_t \circ C \circ \phi^1_{-t}$ derivative by t at $t = 0$ we get the generalized Lie derivative of mapping C with respect to the vector fields $(\xi^1, \widetilde{\xi^*})$ (see [19], Chapter 11) - the vector field over the mapping $C : Z_p \to \tilde{Z}$:

$$\mathcal{L}_{(\xi^1, \widetilde{\xi^*})}\mathcal{C} = C_*(\xi^1) - \widetilde{\xi^*} \circ \mathcal{C}. \tag{12.15}$$

In local adapted coordinates we have

$$L_{(\xi^1,\xi^{1*})}\mathcal{C} = [\left(\xi^\nu \partial_{x^\nu} + \xi^j \partial_{u^j} + \left(d_\sigma \xi^j - z^j_\nu \frac{\partial \xi^\nu}{\partial x^\sigma}\right)\partial_{z^j_\sigma}\right) F^\mu_i +$$
$$+ \left(\left(F^\nu_i \left(\frac{\partial \xi^\mu}{\partial x^\nu} - \xi^\mu \lambda_{G,x^\nu}\right) + F^\mu_j \frac{\partial \xi^j}{\partial u^i} + F^\mu_i \left(\frac{\partial \xi^\nu}{\partial x^\nu} - \xi^\nu \lambda_{G,x^\nu}\right) + F^\nu_i \xi^\mu \lambda_{G,x^\nu}\right)\right)]\partial_{p^\mu_i} +$$
$$+ \left[\left(\xi^\nu(x)\partial_{x^\nu} + \xi^i(x,y)\partial_{u^i} + \left(d_\mu \xi^i - z^i_\nu \frac{\partial \xi^\nu}{\partial x^\mu}\right)\partial_{z^i_\mu}\right)\Pi_k - \left(\Pi_k(\xi^\mu \frac{\partial \lambda_G}{\partial x^\mu} - \frac{\partial \xi^\mu}{\partial x^\mu}) - \Pi_j \frac{\partial \xi^j}{\partial u^k}\right)\right]\partial_{p_k} \tag{12.16}$$

In the case of partial 1-jet bundles we assume restrictions to the automorphisms and vector fields that were introduced in Section 7. For instance in a case of $J^1_S(\pi)$ we assume that the automorphisms of π preserve the structure of fiber product (4.8).

Definition 22. (1) *A diffeomorphism Φ of $W_p = Z_p \times \tilde{Z}$ is called a **generalized symmetry transformation** of constitutive relation $\mathcal{C}$ if $\Phi(\Gamma_{\mathcal{C}}) = \Gamma_{\mathcal{C}}$ for the graph $\Gamma_{\mathcal{C}}$ of the mapping $\mathcal{C}$. A generalized symmetry Φ of $\mathcal{C}$ is called a **trivial symmetry** of $\mathcal{C}$ if restriction of Φ to $\Gamma_{\mathcal{C}}$ is identity.*

(2) *A couple of diffeomorphisms $\phi^1 \in Diff(Z_p)$, $\psi \in Diff(\tilde{Z})$ is said to generate the **symmetry transformation** of $\mathcal{C}$ if the diffeomorphism $\Psi = \psi \times \phi^{-1}$ of W_p is the generalized symmetry of $\mathcal{C}$. This is equivalent to the condition*

$$\psi \circ \mathcal{C}(z) = \mathcal{C} \circ \phi(z) \text{ for all } z \in Z_p.$$

A symmetry is, of course, a special case of a generalized symmetry.

(3) *An automorphism $\phi \in Aut_p(\pi)$ is called a **geometrical symmetry transformation** of a constitutive relation C if the diffeomorphism $\Psi = \tilde{\phi}^* \times \phi^{1\ -1}$ of W_p is the symmetry of C, i.e. if $\mathcal{C}^{\phi_t} = \mathcal{C}$.*

(4) *An automorphism $\phi \in Aut_p(\pi)$ is called a **geometrical symmetry transformation** of a covering constitutive relation $\widehat{\mathcal{C}}$ if the diffeomorphism $\Psi = \tilde{\phi}^* \times \phi^{1\ -1}$ of W_p is the symmetry of $\widehat{\mathcal{C}}$, i.e. if $\widehat{\mathcal{C}}^{\phi_t} = \widehat{\mathcal{C}}$.*

(5) *Let $\xi \in \mathcal{X}_p(\pi)$ be a projectable vector field. We say that ξ is a **geometrical infinitesimal symmetry** of the constitutive relation $\mathcal{C}$ if $L_{(\xi^1, \widetilde{\xi^*})}\mathcal{C} = 0$.*

Properties presented in the next Proposition follows directly from the given definitions. Last statement follows from (12.2-3)

Proposition 24. (1) *A vector field $\xi \in \mathcal{X}_p(\pi)$ is an infinitesimal symmetry of $\mathcal{C}$ if (and only if) the (local) phase flow diffeomorphisms $\Phi^\xi_t = \psi_t \times \phi^1_{-t}$ of*

W_p defined by the prolongation of ξ map $\Gamma_{\mathcal{C}}$ into itself:

$$\Phi^{\xi}_t(\Gamma_{\mathcal{C}}) = \Gamma_{\mathcal{C}},$$

i.e. if the (local) phase flow ϕ_t of vector field ξ is the geometrical symmetry of $\mathcal{C}$.

(2) *Generalized symmetries of $\mathcal{C}$ form the group $GSym(\mathcal{C}) \subset Diff(W_p)$.*
(3) *Trivial symmetries of $\mathcal{C}$ form the normal subgroup $TSym(\mathcal{C})$ of $GSym(\mathcal{C})$.*
(4) *Geometrical symmetries Φ form the subgroup $Sym(\mathcal{C}) \subset Aut_p(\pi)$.*
(5) *Infinitesimal symmetries of $\mathcal{C}$ form Lie algebra $\mathfrak{g}(\mathcal{C}) \subset \mathcal{X}_p(\pi)$ with the bracket of vector fields in Y as the Lie algebra operation.*
(6) *A vector field $X \in \mathcal{X}(W_p)$ is the generator of the 1-parametrical group of generalized symmetries of $\mathcal{C}$ if and only if it is tangent to the graph $\Gamma_{\mathcal{C}}$.*
(7) *If $\xi \in \mathcal{X}_p(Y)$ is the infinitesimal geometrical symmetry, then for the local phase flow ϕ_t of ξ*

$$\Theta_{\hat{C}^{\phi_t}} = \phi^{1\ -1\ *}\Theta_{\hat{C}} = \Theta_{\hat{C}}.$$

Condition that generalized Lie bracket (12.17) is zero has the form of a system of differential equation of the first order for the components of the constitutive relation $\mathcal{C}$:

$$\begin{cases} \xi^1 \cdot F^\mu_i + \left(\left(F^\nu_i\left(\frac{\partial \xi^\mu}{\partial x^\nu} - \xi^\mu \lambda_{G,x^\nu}\right) + F^\mu_j \frac{\partial \xi^j}{\partial y^i} + F^\mu_i\left(\frac{\partial \xi^\nu}{\partial x^\nu} - \xi^\nu \lambda_{G,x^\nu}\right) + F^\nu_i \xi^\mu \lambda_{G,x^\nu}\right)\right) = 0, \\ \xi^1 \cdot \Pi_k - \left(\Pi_k(\xi^\mu \frac{\partial \lambda_G}{\partial x^\mu} - \frac{\partial \xi^\mu}{\partial x^\mu}) - \Pi_j \frac{\partial \xi^j}{\partial y^k}\right) = 0 \end{cases} \tag{12.17}$$

Vector field ξ^1 in these equations for a fixed μ represents the acts on the components of the vector function (with values in the space dual to the vertical tangent vector of the bundle π, i.e. in $V(\pi^*)$ lifted to the space Z_p.

For the vertical vector fields $\xi = \xi^i \partial_{y^i}$ the system (11.5) takes the form

$$\begin{cases} \left(\xi^j \partial_{y^j} + d_\nu \xi^j \partial_{z^j_\nu}\right) F^\mu_i + F^\mu_j \frac{\partial \xi^j}{\partial y^i} = \left(\delta^j_i \xi^1 + \frac{\partial \xi^j}{\partial y^i}\right) F^\mu_j = 0, \\ \left(\xi^j \partial_{y^j} + d_\nu \xi^j \partial_{z^j_\nu}\right) \Pi_k + \Pi_j \frac{\partial \xi^j}{\partial y^k} = \left(\delta^j_i \xi^1 + \frac{\partial \xi^j}{\partial y^i}\right) \Pi_j = 0 \end{cases} \tag{12.18}$$

We can rewrite last system as the system of conditions to the vertical vector field $\xi = \xi^i \partial_{y^i}$:

$$\begin{cases} \left(F^\mu_{i,z^j_\sigma} d_\sigma + F^\mu_j \frac{\partial}{\partial y^i}\right) \xi^j + F^\mu_{i,y^j} \xi^j = 0,\ \mu = 1, \dots, n; i = 1, \dots, m, \\ \left(\Pi_{k,z^j_\sigma} d_\sigma + \Pi_j \frac{\partial}{\partial y^k}\right) \xi^j + \Pi_{k,y^j} \xi^j = 0,\ k = 1, \dots, m. \end{cases} \tag{12.19}$$

Recall that here $D_\sigma \xi^i = \frac{\partial \xi^i}{\partial x^\sigma} + z^j_\sigma \frac{\partial \xi^i}{\partial y^j}$.

Let $\phi \in Aut_p(\pi)$ be a geometrical symmetry of a CR $\mathcal{C}$. For a solution $s \in \Gamma(\pi)$ of the balance system (10.12-13), i.e. for a section $s \in \Gamma(\pi)$ such that

$$j^{1\ *}_p(s) i_\xi \tilde{d} \Theta_{\widehat{\mathcal{C}_-}} = 0$$

for all $\xi \in \mathcal{X}(\mathcal{C})$ we have

$$(j^1_p(\phi^* s))^* i_\xi \tilde{d}\Theta_{\widehat{\mathcal{C}_-}} = [\phi^{1\ *}(j^1_p(s))]^* i_\xi \tilde{d}\Theta_{\widehat{\mathcal{C}_-}} = (j^1_p(s))^* \circ \phi^{1\ *} i_\xi \tilde{d}\Theta_{\widehat{\mathcal{C}_-}} = \\ = (j^1_p(s))^* i_{\phi^1_* \xi} \phi^{1\ *} \tilde{d}\Theta_{\widehat{\mathcal{C}_-}} = (j^1_p(s))^* i_{\phi^1_* \xi} \tilde{d}\phi^{1\ *}\Theta_{\widehat{\mathcal{C}_-}} = (j^1_p(s))^* i_{\phi^1_* \xi} \tilde{d}\Theta_{\widehat{\mathcal{C}_-}}. \quad (12.20)$$

Here we have used the symmetry condition in the form presented in Proposition 24, 7). Last expression is equal zero if (vertical) vector field $\phi^1_* \xi \in \mathcal{X}(\mathcal{C})$. But, by Lemma 6, Sec.10 $\phi^1_* \mathcal{X}(\mathcal{C}) = \mathcal{X}(\mathcal{C}^\phi)$. Since for a geometrical symmetry transformation $\mathcal{C}^\phi = \mathcal{C}$ we have proved the following

Theorem 7. *Let $\phi \in Aut_p(\pi)$ be a symmetry of the CR $\mathcal{C}$. Then the mapping $s \to \phi^* s$ maps the set $Sol(C)$ of solutions of the balance system (8.12-13) into itself.*

In the second part of the work we will study action of transformations on the balance systems in more details, including covariance transformations, equivalence relations etc.

12.2. **Homogeneous constitutive relations.** If the state space of a theory contains enough fields to make the constitutive relations free from the explicit dependence on $(t, x) \in X$ (general relativity or theory of uniform materials are two examples), then the corresponding balance system simplifies and while studying it one does not need to introduce assumptions on the character of the space-time dependence of the balance system. Definition given below is an invariant way to distinguish a class of such CR.

Any local chart x^μ in X defined the local (translational) action of R^n in X associating with the basic vectors e_μ the vector field ∂_{x^μ}. Vice versa, any n-dimensional commutative subalgebra $\mathfrak{h}$ of the Lie algebra of vector fields $\mathcal{X}(U)$, U being an open connected subset of X, defines the locally transitive action of R^n in U and, therefore, a local chart in a neighborhood of any point in U.

Definition 23. (1) *Let ν be a connection in the bundle π satisfying to the conditions of Propositions 14 or 15 with "partial" meaning $K \oplus K'$ or S respectively. We will call a constitutive relation $\mathcal{C}$ **ν-homogeneous** if any point $z \in Z_p$ there exists a local chart in a neighborhood $U_x,\ x = \pi^1(z)$ such that the Poincare-Cartan form $\Omega_{\mathcal{C}}$ of the CR $\mathcal{C}$ is invariant under the local flows $\phi^{\xi^1}_t$ of the lifts $\hat{\xi}^1$ of ν-horizontal vector fields $\hat{\xi}, \xi \in \mathfrak{h}$ in the neighborhood of $y = \pi_{10}(z)$:*

$$\mathcal{L}_{\hat{\xi}^1} \Omega_{\mathcal{C}} = 0 \ mod\ q\eta.$$

(2) *A constitutive relation C is called a homogeneous if there is a connection ν on the bundle π such that C is ν-homogeneous.*

Proposition 25. *Let ν be a connection in the bundle π. Then the following properties of a constitutive relation C are equivalent:*

(1) *C is ν-homogeneous,*

(2) *For all* $\xi \in \mathfrak{h}$*, the* ν*-horizontal lift* $\hat{\xi}$ *is the infinitesimal symmetry of the constitutive mapping* $\mathcal{C}$ *in sense of Definition 22.*
(3) *The graph* $\Gamma_C \subset Z_p \times \tilde{Z}$ *of mapping* $\mathcal{C}$ *is invariant under the flow generated by (flow) lifts of* ν*-horizontal vector fields* $\hat{\xi}$*,* $\xi \in \mathfrak{h}$*.*

Proof. Trivially follows from the Definition 23 and Proposition 24. □

Remark 30. In a case where connection ν is flat, the association $\xi \rightarrow \hat{\xi}^1$ is the Lie algebra endomorphism $\mathfrak{h} \rightarrow Aut(\pi^1) \subset \mathcal{X}(Z_p)$.

Remark 31. It would be interesting to study the influence of the curvature of connection ν on the properties of ν-homogeneous constitutive relations.

13. Noether Theorem.

Noether Theorem of the Lagrangian Field Theory associates the conservation law with one-parameter groups of symmetries (or with the corresponding vector fields) the *conservation laws* that is valid for any solution of the Euler-Lagrange equations (on shell). Conserved currents are defined in terms of the (multi)-momentum mapping that in the case of a multisymplectic field theory was constructed in [29].

In the situation considered in this work we might expect a similar result to be true at least for the semi-Lagrangian constitutive relation $C_{L,\Pi}$ or RET case (see Sec.9). On the other hand, in the general case, with serious restrictions to the admissible variations ξ one can hardly expect the Noether Theorem type results. In this section we study possible formulations of the (first) Noether Theorem in for semi-Lagrangian, RET and general constitutive relation C. We follow the works [25, 29, **?**] in the presentation of Noether Theorem of multisymplectic field theory.

We consider separately cases of semi-Lagrangian constitutive relation C and corresponding Lagrangian lift to the CCR $\hat{C}$ (see Sec.9) and the general case. In the first case results are parallel to the Lagrangian case, in the second one they are much more limited.

Let a Lie group $G \subset Sym(C) \subset Aut_p(\pi)$ be a subgroup of the geometrical symmetry group of a constitutive relation $\mathcal{C}$. Let $\mathfrak{g}$ be the Lie algebra of the group G and $\mathfrak{g}^*$ be its dual space. Lie algebra g acts on Y by projectable infinitesimal transformations, i.e there exists homomorphism of Lie algebras $\mathfrak{g} \rightarrow X_p(\pi)$. For an element $\xi \in \mathfrak{g}$ we denote by the same letter the corresponding vector field in Y, by ξ^1 - the lifted vector field in $J^1_p(\pi)$ preserving Cartan distribution, by ξ^{1*} - the vector field in $\Lambda^{(n+1)+(n+2)}_2 Y$ leaving invariant the canonical multisymplectic form:

$$L_{\xi^{1*}}(\Theta^{n+1}_2 + \Theta^{n+2}_2) = 0.$$

In a more general fashion consider the Lie subalgebra $\mathfrak{g} \subset \mathcal{X}_{\pi^1}(Z_p)$ of the Lie algebra of projectable (to Y and to X) vector fields $\hat{\xi}$ in Z_p which consists of the *infinitesimal symmetries* of the CCR $\hat{C}$. In other words we assume that the projection $\xi \in \mathcal{X}(\pi)$ of $\hat{\xi}$ in Y is defined and being lifted to the vector field ξ^* in the bundle $\Lambda^{(n+1)+(n+2)}$ preserving canonical form(s) $\Theta^{n+1}_2 + \Theta^{n+2}_2$ is such that $\mathcal{L}_{(\hat{\xi},\xi^*)}C = 0$ (see Sec.12)). Then as is proved in Sec.12 in terms of Poincare-Cartan form $\Theta_{\hat{C}}$ this condition takes the form

$$L_{\xi^1}(\Theta^{n+1}_{\hat{C}} + \Theta^{n+2}_{\hat{C}}) = 0, \tag{13.1}$$

obtained by differentiating condition 7) in the Proposition 23. This splits into two conditions - independent preservation of forms $\Theta^{n+1}_{\hat{C}}$ and $\Theta^{n+2}_{\hat{C}}$. First condition is the natural generalization of the invariance condition of the Lagrangian field Theory ([25]).

Former situation ($\xi \in \mathcal{X}_p(\pi)$) is the special case of the later one where $\hat{\xi} = \xi^1$.

Definition 24. *Let $\hat{C}$ be a covering constitutive relation in Z_p.*

(1) *A vector field $\xi \in X(Y)$ is called a* ***variational symmetry*** *if the Lie derivative*

$$\mathcal{L}_{\xi^1}\Theta^{n+1}_{\hat{C}} \in \mathcal{I}(Ca)$$

(belongs to the differential ideal of (partial) contact structure of Z_p, see Sec.6) and also ξ^1 is tangent to the boundary subbundle B and verifies $\mathcal{L}_{\xi^1}\Pi_C = 0$.

(2) *A vector field $\xi \in X(Y)$ is called a* ***Noether (divergence) symmetry*** *if there is a n-form $\alpha \in \Lambda^n(Y)$ whose pullback α to Z_p is exact on B: $\alpha_B = d\beta$ and such that*

$$\mathcal{L}_{\xi^1}\Theta^{n+1}_{\hat{C}} - d\alpha \in \mathcal{I}(Ca),$$

and vector field ξ^1 is tangent to B and verifies $\mathcal{L}_{\xi^1}\Pi_C = 0$.

(3) *A vector field $\xi_Z \in \mathcal{X}(Z_p)$ is called a* ***Cartan symmetry*** *of C if*
   (a) *Flow of ξ_Z preserves the differential ideal $\mathcal{I}(Ca)$: $\mathcal{L}_{\xi_Z}\theta \in \mathcal{I}(Ca)$ for all $\theta \in \mathcal{I}(Ca)$,*
   (b) *There exists a n-form α on Z_p that is exact on B: $\alpha_B = d\beta$ and such that*

$$\mathcal{L}_{\xi_Z}\Theta^{n+1}_{\hat{C}} - d\alpha \in \mathcal{I}(Ca),$$

   (c) *Vector field ξ_Z is tangent to B and verifies $\mathcal{L}_{\xi_Z}\Pi_C = 0$.*

Every variational symmetry is Noether symmetry as well. If ξ is a Noether symmetry, then its flow prolongation is a Cartan symmetry. Vice versa, a π_{10}-projectable Cartan symmetry is the flow prolongation of its projection which is the Noether symmetry. In the next proposition proof of which is the same as in [25] some properties of symmetries of these three types are collected.

Proposition 26. *Let $\hat{C}$ be a covering constitutive relation defined at Z_p.*

(1) *Variational symmetries form the Lie subalgebra $\mathfrak{vg}_{\widehat{C}}$ of $\mathcal{X}(Y)$.*
(2) *Noether symmetries form the Lie subalgebra $\mathfrak{ng}_{\widehat{C}}$ of $\mathcal{X}(Y)$.*
(3) *Cartan symmetries form the Lie subalgebra $\mathfrak{cg}_{\widehat{C}}$ of $\mathcal{X}(Z_p)$.*
(4) *For the prolongations of the first two types of vector fields we have the following sequence of embeddings of Lie subalgebras of $\mathcal{X}(Z_p)$:*

$$\mathfrak{vg}^1_{\widehat{C}} \hookrightarrow \mathfrak{ng}^1_{\widehat{C}} \hookrightarrow \mathfrak{cg}_{\widehat{C}}$$

(5) *A geometrical infinitesimal symmetry $\xi \in X_p(\pi)$ of CCR $\hat{C}$ is the variational symmetry of the CCR $\hat{C}$.*
(6) *An infinitesimal symmetry $\hat{\xi} \in \mathcal{X}_{\pi^1}(Z_p)$ is the Cartan symmetry of the CCR $\hat{C}$.*

Now we define the canonical multimomentum mapping (MM) following [29].

Definition 25. *The multimomentum mapping $J : \Lambda^{n+1}_2 Y \to \Lambda^n(X) \otimes \mathfrak{g}^*$ is defined as*

$$J(z^*)(\xi) = i_{\xi^{1*}}\Theta^{n+1}_2(z^*),\ \textit{for all } z^* \in \Lambda^{n+1}_2,\ \xi \in \mathfrak{g}.$$

Lemma 7. *([29, 25]) For the MM-mapping J and an arbitrary $\hat{\xi} \in \mathcal{X}(Z_p)$*

$$i_{\widehat{\xi}^*} d\Theta^{n+1}_2 = -dJ(z^*)(\widehat{\xi}).$$

Proof. We have $0 = L_{\widehat{\xi}^*}\Theta^{n+1}_2 = i_{\widehat{\xi}^*} d\Theta^{n+1}_2 + d i_{\widehat{\xi}^*}\Theta^{n+1}_2 = i_{\widehat{\xi}^*} d\Theta^{n+1}_2 + dJ(z^*)(\widehat{\xi})$. □

The MM mapping $J^{\hat{C}}$ for an arbitrary covering constitutive relation $\hat{C}$ is defined here in the same way as it was defined in [29] for the Legendre transformation corresponding to a Lagrangian $L\eta$.

Definition 26. *A multimomentum mapping of a covering constitutive relation $\hat{C}$ is the mapping $J^{\hat{C}} : Z_p = J^1_p(\pi) \to \Lambda^n(X) \otimes \mathfrak{g}^*$*

$$J^{\hat{C}}(z)(\hat{\xi}) = \hat{C}^*_z J([\hat{C}^{n+1}(z))(\hat{\xi}) = i_{\hat{\xi}} \Theta^{n+1}_{\hat{C}}(z),$$

where $\hat{\mathcal{C}}^{n+1}$ is the (n+1)-component of the constitutive mapping $\hat{\mathcal{C}}$.

Remark 32. Notice that $J^{\hat{C}}(z)$ depends only on the current component of the constitutive mapping and, therefore, $J^{\hat{C}}(z) = J^{\hat{C}_-}(z)$.

Lemma 8. *If the mapping $\hat{\mathcal{C}}$ is regular (i.e. if it is the diffeomorphism onto its image), then*

$$i_{\hat{\xi}} d\Theta^{n+1}_{\hat{\mathcal{C}}} = -dJ^{\hat{\mathcal{C}}}(z)(\hat{\xi}), \quad \textit{for all } z \in J^1_p(\pi),\ \xi \in \mathfrak{g}.$$

Proof. Follows from the previous Lemma by using the $\mathcal{G}$-equivariance of the constitutive relation C giving $\hat{\mathcal{C}}_* \hat{\xi} = \xi^* \circ \hat{\mathcal{C}}$ (recall that ξ is the projection of $\hat{\xi}$ to Y). More specifically, we have

$$\begin{aligned}(dJ^{\hat{C}}(z)(\hat{\xi}) &= (d\hat{C}^* J)(z)(\hat{\xi}) = \hat{C}^*(dJ)(\hat{\xi}) = dJ(\hat{C}(z))(\hat{C}_*(\hat{\xi})) = dJ(\hat{C}(z))(\xi^*(\hat{C}(z))) = \\ &= \hat{C}^* i_{\xi^*(\hat{C}(z))} d\Theta^{n+1}_2 = \hat{C}^* i_{\hat{C}_{z*}(\hat{\xi}(z))} d\Theta^{n+1}_2 = i_{\hat{\xi}} \hat{C}^*_z d\Theta^{n+1}_2 = i_{\hat{\xi}} d\hat{C}^*_z \Theta^{n+1}_2 = i_{\hat{\xi}} d\Theta^{n+1}_{\hat{C}}\end{aligned} \tag{13.2}$$

□

Theorem 8. *(Noether Theorem) Let $\hat{C}$ be a semi-Lagrangian covering constitutive relation with*

$$\Theta_{\hat{C}} = (L - z^i_\mu L_{z^i_\mu})\eta + L_{z^i_\mu} dy^i \wedge \eta_\mu + (\Pi_i - L_{,y^i}) dy^i \wedge \eta,$$

and let $\xi \in X_p(Y)$ be a variational symmetry of $\hat{C}$. Then for all solutions $s \in \Gamma(\pi)$ of the balance system ★ the following balance equation is true

$$d[(j^1(s))^* J^{\hat{\mathcal{C}}}(z)(\xi)] = (\omega^i(\xi)\Pi_i) \circ j^1(s))^* \eta. \tag{13.3}$$

Proof. We have, by the Cartan formula for the Lie derivative and using Theorem 3

$$0 = (j^1(s))^* \mathcal{L}_{\xi^1} \Theta^{n+1}_{\hat{C}} = (j^1(s))^* \left(d i_{\xi^1} \Theta^{n+1}_{\hat{C}} + i_{\xi^1} d\Theta^{n+1}_{\hat{C}} \right) = d(j^1(s))^* i_{\xi^1} \Theta^{n+1}_{\hat{C}} + (\omega^i \Pi_i) \circ j^1(s))^* \eta,$$

since s is a solution of the balance system (10.29) with Π_i replaced by $\Pi_i - L_{,y^i}$. □

In the same way the following statement is proved

Theorem 9. *(Noether Theorem) Let $\hat{C}$ be a semi-Lagrangian covering constitutive relation with*

$$\Theta_{\hat{C}} = (L - z^i_\mu L_{z^i_\mu})\eta + L_{z^i_\mu} dy^i \wedge \eta_\mu + (\Pi_i - L_{,y^i}) dy^i \wedge \eta,$$

and let $\xi_Z \in X(Z_p)$ be a Cartan symmetry of $\hat{C}$ (in particularly, $\xi_Z = \xi^1$ for $\xi \in X_p(Y)$ to be a Noether symmetry of $\hat{C}$). Then for all solutions $s \in \Gamma(\pi)$ of the balance system ★ the following balance equation is true

$$d[(j^1(s))^* i_{\xi_Z} \Theta^{n+1}_{\hat{C}} - \alpha] = (\omega^i(\xi_Z)\Pi_i) \circ j^1(s))^* \eta. \tag{13.4}$$

Corollary 5. *If, in addition to the conditions of the Theorem 8 the balance system $\mathcal{B}_{\mathcal{C}}$ is the conservation system (i.e. if $\Pi_i = 0,\ i = 1,\dots,m$), then for all $\xi \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ of the balance system $\mathcal{B}_{\mathcal{C}}$ the* Noether conservation law *holds:*

$$d[(j^1(s))^* i_{\xi_Z}\Theta^{n+1}_{\hat{C}} - \alpha] = 0, \tag{13.5}$$

Remark 33. Associating to each $\xi \in \mathfrak{g}$ the corresponding balance law (13.3) defined the linear mapping

$$\mathfrak{g} \to \mathfrak{BL}_{\hat{\mathcal{C}}}$$

to the space of secondary balance laws of the system B_C (see Sec.11).

Let now condition (13.1) is fulfilled i.e. G is symmetry of *both flux* ***and*** *source terms* of the constitutive relation C. Then

$$\begin{aligned} di_{\hat{\xi}}\Theta^{n+2}_{\hat{C}} &= -i_{\hat{\xi}}d\Theta^{n+2}_{\hat{C}} = -i_{\hat{\xi}}(d\Pi_i \wedge dy^i \wedge \eta) = \\ &= [-(\hat{\xi}\cdot\Pi_i)dy^i \wedge \eta - \xi^i d\Pi_i \wedge \eta + \xi^\mu d\Pi_i \wedge dy^i \wedge \eta_\mu] = \\ &= -(\hat{\xi}\cdot\Pi_i)(\omega^i + z^i_\nu dx^\nu)\wedge\eta - \xi^i(\Pi_{i,x^\sigma}dx^\sigma + \Pi_{i,y^j}(\omega^j + z^j_\sigma dx^\sigma) + \Pi_{i,z^j_\sigma}(\omega^j_\sigma - z^j_{\sigma\lambda}dx^\lambda)\wedge\eta + \\ &+ \xi^\mu d\Pi_i \wedge (\omega^i + z^i_\nu dx^\nu)\wedge\eta_\mu] = \mathrm{Con} + \xi^\mu d\Pi_i \wedge z^i_\nu dx^\nu \wedge \eta_\mu = \\ &= \mathrm{Con} + \xi^\mu[\Pi_{i,x^\sigma}dx^\sigma + \Pi_{i,y^j}(\omega^j + z^j_\sigma dx^\sigma) + \Pi_{i,z^j_\sigma}(\omega^j_\sigma - z^j_{\sigma\lambda}dx^\lambda)] \wedge z^i_\mu\eta = \\ &= \mathrm{Con}. \end{aligned} \tag{13.6}$$

Here Con means a contact form. During this calculation we repeatedly used the equality $dx^\nu \wedge \eta = 0$. Notice that the same statement follows directly from the fact that the pullback by $j^1(s)$ of the $n+1$-form is necessary closed. Applying now the pullback by $j^1(s)$ we get the following

Proposition 27. *Let, in addition to the conditions of Theorem 7, G is the symmetry of the source part of the constitutive relation, i.e. (13.1) is true. Then*

$$di_{\hat{\xi}}\Theta^{n+2}_{\hat{C}} = Cont \Rightarrow dj^1(s)^* i_{\hat{\xi}}\Theta^{n+2}_{\hat{C}} = 0$$

for all sections s. Therefore, locally (and in a top. trivial domain, globally)

$$j^1(s)^* i_{\hat{\xi}}\Theta^{n+2}_{\hat{C}} = d\Phi_{\hat{C}}(s,\hat{\xi},z)$$

for some (n+1) form $\Phi_{\hat{C}}$ ($\mathfrak{g}$-potential of the source $\hat{C}^{n+2}$) linearly depending on the vector field ξ.

Remark 34. notice that in the last Proposition, as in the Noether Theorems above $i_{\hat{\xi}}\Theta^{n+2}_{\hat{C}} = (\Pi_i\xi^i)\eta$.

Corollary 6. *If G is the Lie group of symmetries of a regular constitutive relation C then (locally) in the conditions of Theorem 9,*

$$d[J^{\hat{\mathcal{C}}}(j^1(s)(x)(\xi) - \Phi_{\mathcal{C}}(s,\xi,z)] = 0$$

for all solutions $s \in \Gamma(\pi)$ of the balance system $\mathcal{B}_{\mathcal{C}}$.

For the RET constitutive relations we get the results similar to those for semi-Lagrangian case valid for the *lifted* covering constitutive relations (comp. Sec.10, Thm.4):

Theorem 10. *(Noether Theorem) Let $\tilde{C}$ be a lifted covering constitutive relation of the RET type with*

$$\Theta_{\tilde{C}} = -z^i_\mu F^\mu_i \eta + F^\mu_i dy^i \wedge \eta_\mu + \Pi_i dy^i \wedge \eta,$$

and let $\xi \in X(Y)$ be a variational symmetry of $\tilde{C}$. Then for all solutions $s \in \Gamma(\pi)$ of the balance system ★ the following balance equation is true

$$d[(j^1(s))^* J^{\tilde{C}}(z)(\xi)] = (\omega^i(\xi)\Pi_i) \circ j^1(s))^* \eta. \tag{13.7}$$

Now we formulate the Noether Theorem for the balance system $\mathcal{B}_\mathcal{C}$ corresponding to a general regular constitutive relation. This result is limited since the infinitesimal symmetry vector fields ξ should be C-admissible.

Theorem 11. *Let $\mathcal{C}$ be a* regular *constitutive relation defined on a partial 1-jet bundle $J^1_p(\pi)$ and $\tilde{C}$ - its lifted (covering) constitutive relation. Let a Lie group $G \subset Sym(C) \subset Aut(\pi)$ be a symmetry group of the flux part Θ^{n+1}_C of the constitutive relation $\mathcal{C}$ such that its Lie algebra $\mathfrak{g}$ consists of C-admissible vector fields $\mathfrak{g} \subset \mathcal{X}(\mathcal{C})$ on Z_p. Then for all $\hat{\xi} \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ of the balance system $\mathcal{B}_\mathcal{C}$,*

$$d[J^{\tilde{C}}(j^1(s)(x)(\widehat{\xi})] = j^1(s)^* i_\xi \Theta^{n+2}_{\tilde{C}} = j^1(s)^*[\omega^i(\widehat{\xi})\Pi_i]\eta, \tag{13.8}$$

where $\omega^i = dy^i - \sum_{\mu,(\mu,i)\in P} z^i_\mu dx^\mu$ are the basic Cartan forms in $J^1_p(\pi)$.

Proof. We have

$$dJ^{\tilde{C}}(j^1(s))(\hat{\xi}) = j^1(s)^* dJ^{\tilde{C}_-}(\hat{\xi}) = -j^1(s)^* i_{\hat{\xi}} d\Theta^{n+1}_{\tilde{C}_-} =$$
$$= -j^1(s)^* i_{\hat{\xi}} \Theta^{n+2}_{\tilde{C}_-} = j^1(s)^*[\omega^i(\xi^1)\Pi_i]\eta. \tag{13.9}$$

where ω is some contact form (we have used the result of Lemma 12 Appendix IV). Here we have used formulation (10.34) (see Theorem 5) of the balance system. In the last equality we have used the relation $\omega^i(\hat{\xi}) = \xi^i$. As a result we get

$$dJ^C(j^1(s))(\xi) = j^1(s)^* i_{\xi^1} \Theta^{n+2}_C = j^1(s)^*[\omega^i(\xi^1)\Pi_i]\eta.$$

□

Corollary 7. *If, in addition to the conditions of the last Theorem the balance system B_C is the conservation system (i.e. if $\Pi_i = 0,\ i = 1, \dots, m$), then for all $\xi \in \mathfrak{g}$ and for all solutions $s \in \Gamma(\pi)$ of the balance system B_C the Noether conservation law holds:*

$$d[J^C(j^1(s)(x)(\xi)] = 0, \tag{13.10}$$

13.1. **Energy-Momentum Balance Law.** Let ν be a connection in the bundle $\pi : Y \to X$ with the form $dy^i - \Gamma^i_\mu dx^\mu$. Consider a ν-homogeneous constitutive law $\mathcal{C}$ and the corresponding balance system $\mathcal{B}_\mathcal{C}$. Let (x^μ, y^i) be a local adopted chart in the bundle π.

Let ∂_{x^μ} be a basic vector field in X and let $\xi_\mu = \partial_{x^\mu} + \Gamma^i_\mu \partial_{y^i}$ be its horizontal lift in Y. Flow lift of the vector field ξ_μ is

$$\hat{\xi}_\mu = \partial_{x^\mu} + \Gamma^i_\mu \partial_{y^i} + d_\nu \Gamma^i_\mu \partial_{z^i_\nu}.$$

Now we assume that $\hat{\xi}_\mu \in \mathcal{X}(\mathcal{C})$. Remind that this is true for all μ and all connections ν in semi-Lagrangian case and in the RET case. In the general case that requires fulfillment of two conditions: $\hat{\xi}_\mu$ is P-vertical and

$$\omega_C(\hat{\xi}_\mu) = F^\nu_i d_\nu \Gamma^i_\mu = 0. \tag{13.11}$$

Calculate now

$$i_{\hat{\xi}_\mu}\Theta^{n+1}_C = F^\nu_i \Gamma^i_\mu \eta_\nu - F^\nu_i dy^i \wedge \eta_{\mu\nu},$$

and $\omega^i(\hat{\xi}_\mu) = \Gamma^i_\mu - z^i_\mu$. Therefore, the balance law (13.3,13.7) takes, for the vector field $\hat{\xi}_\mu$, the form

$$dj^1_p(s)^*[F^\nu_i \Gamma^i_\mu \eta_\nu - F^\nu_i dy^i \wedge \eta_{\mu\nu}] = j^1_p(s)^*(\Pi_i(\Gamma^i_\mu - z^i_\mu))\eta. \tag{13.12}$$

Introducing 1-jet of section s into the form in brackets and omitting the form η we get the energy-momentum balance law in the form (comp. [13], Chapter 3)

$$d[(F^\nu_i \Gamma^i_\mu) \circ j^1_p(s) - \delta^\nu_\mu F^\sigma_i s^i_\sigma + F^\nu_i s^i_{,\mu}]_{,x^\nu} = \Pi_i(s)(\Gamma^i_\mu(s) - s^i_{,\mu})). \tag{13.13}$$

Energy-momentum Tensor for the constitutive relation C has, thus, the form

$$T^\nu_\mu = F^\nu_i \Gamma^i_\mu - \delta^\nu_\mu F^\sigma_i z^i_\sigma + F^\nu_i z^i_\mu. \tag{13.14}$$

13.2. **Case of pure gauge symmetry transformation.** Let $\xi = \xi^i \partial_{y^i}$ be a vertical (pure gauge) symmetry transformation of a constitutive relation $\mathcal{C}$. Then, the flow lift of vector field ξ to Z_p is $\xi^1 = \xi + d_\mu \xi^i \partial_{z^i_\mu}$.

We calculate

$$i_{\xi^1}\Theta^{n+1}_C = \xi^i F^\mu_i \eta_\mu, \omega^i(\xi^1) = \xi^i.$$

Therefore, the Noether balance equation corresponding to the vector field ξ has the form

$$dj^1_p(s)^*(\xi^i F^\mu_i \eta_\mu) = j^1_p(s)^*(\xi^i \Pi_i \eta). \tag{13.15}$$

Substituting s explicitly we get this equation in the form

$$[(\xi^i F^\mu_i)(j^1_p(s))]_{,x^\mu} = (\xi^i \Pi_i)(j^1_p(s)) \; - \tag{13.16}$$

- the secondary balance law defined by the $\mathcal{C}$-admissible vector field ξ (see Sec. 11) and part II of this work.

14. Evolutional balance systems.

Here we consider the case where $Z_p = Z_S = J^1_S(\pi)$. A general balance system (9.13) does not necessary produce an evolutional dynamical system for all the state fields y^i, the extreme case of such a situation be when no time derivatives enters the constitutive relation and $F^0_i = 0$ for all i. To specify type of balance systems that produce a dynamical system for all the fields y^i we have to put some restrictions on the CR $\mathcal{C}$.

We simplify our consideration here by assuming here that the CR C satisfies to the condition:

Time derivatives y^i_t of the fields u^i may enter the constitutive relation C only through the term F^0_i of $\mathcal{C}$.

Then, the balance system ★ can be schematically written in the form

$$\sum_{j\in S} \frac{\partial F^0_j}{\partial y^k} y^k_{,t} + \sum_{S_t\cup S_{tx}} \frac{\partial F^0_j}{\partial y^k_t} y^k_{,tt} = G_j(y, y_x, y_{xx}), \tag{14.1}$$

where terms containing time derivatives are gathered on the left.

Example 20. Consider a ν-horizontal constitutive relation $\mathcal{C}$ such that $U_t = U_{tx} = 0$ (so that no time derivatives of the fields y^i enters $\mathcal{C}$), If in such a case the condition of *regularity* is fulfilled

$$det\left(\frac{\partial F^0_i}{\partial y^j}\right) \neq 0, \tag{14.2}$$

guarantees that the system can be written in the normal form

$$y^i_t = H^i(y, y_x),$$

and, in an analytical case, the Cauchy problem for this system is is locally solvable.

Restrict to the case where the bundle $Y \to X$ is the *vector bundle.* Introduce the pullback of the bundle $\pi : Y \to X$ to Z_p:

$$\begin{array}{ccc} \pi^{1\ *}(Y) & \longrightarrow & Y \\ \Big\downarrow & & \pi\Big\downarrow \\ Z_p & \xrightarrow{\pi^1} & X \end{array}. \tag{14.3}$$

Let the $m \times m$ matrix $A_2(z) = \frac{\partial F^0_j}{\partial y^k_t}(z)$ has, in a neighborhood of a point z a constant rank $s_2 \leqq |S_t \cup S_{tx}|$. Then, the kernel $K_2(z)$ of the matrix $\frac{\partial F^0_j}{\partial y^k_t}$ defines, at each point z the vector subspace $K_2(z) \subset \pi^*_{1\ z}(Y_x)$. If the rank of matrix $A_2(z)$ is constant, we get (locally) the subbundle K_2 of the pullback bundle $\pi^{1\ *}(Y)$ of the fields y^i whose *second time derivatives* y^i_{tt} do not enter the balance system (14.1).

In the same way, at each point there is defined the rank s_1 of the matrix $A_1(z) = \frac{\partial F^0_j}{\partial y^k}(z)$ and the subspace $K_1(z)$ - kernel of the linear mapping defined by the matrix $A_1(z)$. If that rank is locally constant, then the vector subbundle K_1 of the bundle $\pi^*_{1\ z}(Y)$ is defined in the same way as K_2 - subbundle of vector fields whose *first time derivatives* y^j_t do not enter the balance system at the point z.

Define the intersection $K(z) = K_1(z) \cap K_2(z)$. Generically, if the rank of this intersection $k_e(z)$ is (locally) constant one get the subbundle $K \subset \pi_{1\ z}^*(Y)$ of the fields *whose time derivative do not enter the balance system* (9.13).

We have for the defined sub-bundles the inclusion $K \subset K_1$. Choose a compliment K^1 to the subbundle K of the bundle K_1 (if a Riemannian metric is defined on the fibers U_x of the vector bundle $\pi : Y \to X$ then it is natural to take $K^1 = K^\perp$).

In the same way, choose a vector subbundle K^2 complemental to K_2 in the bundle $\pi^{1\ *}(Y)$ (orthogonal if a Riemannian metric is defined on the fibers U_x of the bundle $\pi : Y \to X$).

Thus, we get the decomposition

$$\pi_z^{1\ *}(Y) = K_z \oplus K_z^1 \oplus K_z^2 \tag{14.4}$$

of the pullback of the state bundle Y into the sum of subbundles with the corresponding fields that

1. $y^i \in K^2$ enters the $\mathcal{C}_0^{n+1}$ (i.e. $dy^i \wedge \eta_0$-term of CR) with their time derivative $y^i_{,t}$,
2. $y^i \in K^2$ enters the $\mathcal{C}_0^{n+1}$ (i.e. $dy^i \wedge \eta_0$-term of CR) but their time derivative $y^i_{,t}$ does not enter $\mathcal{C}$,
3. Neither $y^i \in K^2$ no its time derivative $y^i_{,t}$ enter the $\mathcal{C}$.

Remark 35. If the bundle $Y \to X$ is not a vector bundle, similar decomposition exists for the vertical tangent $V(\pi)$ of the bundle Y and can be used instead.

Decomposition (14.4) allows to split the system of balance laws (locally, if the ranks of matrices $A_i, i = 1, 2$ and dimension of intersection $K_1 \cap K_2$ are locally constant) into the three subsystems - hyperbolic for the fields in K^2, parabolic - for the fields in K^1 and stationary - for the fields in K.

Theorem 12. *Let $\pi : Y \to X$ is the vector bundle and let the CR $\mathcal{C}$ satisfies to the condition:* ***time derivatives y^i_t of the fields y^i may enter the constitutive relation C only through the terms F^0_i of*** *the CR $\mathcal{C}$. Assume that the the ranks of matrices $A_i, i = 1, 2$ and dimension of intersection $K_1 \cap K_2$ of the subbundles $K_1, K_2 \subset \pi^{1\ *}(Y)$ defined above are constant throughout the Z_S. Then the pullback of the bundle Y to the partial 1-jet bundle Z_S splits into the sum of three vector subbundles*

$$\pi_{1\ z}^*(Y) = K \oplus K^1 \oplus K^2 \tag{14.5}$$

and the balance system (14.1) splits into the hyperbolic, parabolic and stationary subsystems

$$\begin{cases} y^i_{tt} &= P_i(x, y^j, z^j_\mu),\ \ y^i \in K^2, \\ y^i_t &= P_i(x, y^j, z^j_\mu, \mu \neq 0; y^k_t, y^k \in K^2),\ \ y^i \in K^1, \\ 0 &= P_i(x, y, z^i_\mu, \mu \neq 0, y^k_t, y^k \in K^2 \cap K^1. \end{cases} \tag{14.6}$$

At a point $z \in Z_p$ let $(h(z), p(z), e(z))$ be corresponding dimensions of subbundles $K^2,\ K^1,\ K$. we call the triple of numbers $(h(z), p(z), e(z))$ the index of a system (14.1) at a point z and numbers in this index - hyperbolic, parabolic and stationary dimensions at the point z. It is clear that sum of these dimensions is equal to m: $h(z) + p(z) + e(z) = m$.

15. RET balance systems. Lagrange-Liu dual formulation.

In this section we suggest a bundle picture of the Rational Extended Thermodynamics in terms of dual variables. We will be using terminology from Sec.3. Recall that for the conventional RET case where $F^0_i = y^i$ ([32] or Sec.3 above) whenever the entropy density $h^0(x,y)$ is convex by vertical variables y^i, the change of variables $\{y^i\} \to \lambda^i = \frac{\partial h^0}{\partial y^i}$ is globally defined diffeomorphsim $\wp_x : U_x \to \Lambda_x$ of the fibers U_x onto the space Λ of variables λ^i. This allows to introduce the dual bundle $\pi_{Y^*X} : Y^* \to X$ with the fiber Λ with the corresponding isomorphism of bundles

$$\begin{array}{ccc} Y & \xrightarrow{\ \wp\ } & Y^* \\ \pi_{Y^*X}\big\downarrow & & \pi_{YX}\big\downarrow \\ X & = & X \end{array} \tag{15.1}$$

Since in this section we are repeatedly using notation Λ for the space of dual variables, it will be convenient to change the notation for for the space (or bundle) of exterior k-forms from the Λ^k to Ω^k.

Taking the pullback of the bundle of $n+(n+1)$-forms on X via the projection π_{XY^*} or, what is the same, forming the fiber product of the bundle π_{Y^*X} with the (n+(n+1))-bundle $\pi_{n+(n+1)}$ (see Sec.2) we get the following commutative square

$$\begin{array}{ccccc} \Lambda \times \Omega^{n+(n+1)} & \longrightarrow & Y^* \underset{X}{\times} \Omega^{n+(n+1)} & \longrightarrow & \Omega^{n+(n+1)}(X) \\ \pi_{\Omega\Lambda}\big\downarrow & & \pi_{\Omega Y^*}\big\downarrow & & \pi_{\Omega X}\big\downarrow \\ \Lambda & \longrightarrow & Y^* & \xrightarrow{\pi_{Y^*X}} & X \end{array}, \tag{15.2}$$

where the left column represent a typical fiber of the middle column bundle over a point $x \in X$.

A point of a fiber of the bundle $\pi_{\Omega\Lambda}$ can be presented as

$$(\lambda, \sum_\mu q^\mu \eta_\mu + p\eta),$$

where $q^\mu(x,\lambda), p(x,\lambda)$ are functions defined on the space Y^*.

Introduce the 1-jet bundle $J^1(\pi_{Y^*\Omega})$ of the bundle $\pi_{\Omega Y^*}$. A point of the fiber of this 1-jet bundle $\pi_{Y^*\Omega} : J^1(\pi_{Y^*\Omega}) \to \Lambda(= Y^*_x) \times (\Omega^{n+(n+1)}_x)$ (over a fixed base point $x \in X$) can be presented as

$$q^\mu_i \eta_\mu \wedge d\lambda^i + p_i \eta \wedge d\lambda^i, \tag{15.3}$$

were we have used the standard isomorphism $J^1(E \to U) \simeq E \bigotimes T^*(U)$ of bundles over U induced by a connection in the bundle $E \to U$. In this case we are using a connection induced in the central column of the bundle (15.2) by the connection Γ^G in the bundle of n+(n+1)-forms over X.

Organize the spaces introduced above into the following bundle picture, where on the right are the local coordinates in the fibers of the bundles

$$
\begin{array}{ccccc}
J^1(\Lambda \times \Omega^{n+(n+1)}) & \xrightarrow{\hookrightarrow} & J^1(Y^* \underset{X}{\times} \Omega^{n+(n+1)}) & (\lambda^i; q^\mu, q^\mu_i; p, p_i) & \\
\pi_1^* \big\downarrow & & \pi_1^* \big\downarrow & & \\
(\Lambda \times \Omega^{n+(n+1)}) & \xrightarrow{\hookrightarrow} & Y^* \underset{X}{\times} \Omega^{n+(n+1)} & (\lambda^i; q^\mu, p) & \\
\pi_{\Lambda\Omega} \big\downarrow & & \pi_{\Lambda\Omega} \big\downarrow & & (15.4) \\
\Lambda & \xrightarrow{\hookrightarrow} & Y^* & (\lambda^i) & \\
\pi \big\downarrow & & \pi_{Y^* X} \big\downarrow & & \\
. & \xrightarrow{\hookrightarrow} & X & &
\end{array}
$$

A choice of a section $\hat{h}$ of bundle $\pi_{\Lambda\Omega}$ determines the dual entropy density $\hat{h}^0(\lambda)$, its flow $\hat{h}^\nu(\lambda)$ and the entropy production $\Sigma(\lambda)$ as the function of dual variables λ_i.

A choice of a section $\mathfrak{c} = (q^\mu_i(\lambda), p_i(\lambda))$ of the 1-jet bundle $\pi^*_2 = \pi_{\Lambda\Omega} \circ \pi^*_1$ determines, in addition to the previous quantities, the quantities q^μ_i and p_i as functions of dual variables λ^i.

If we identify

$$\tilde{F}^\mu_i \equiv q^\mu_i(\lambda),\ \tilde{\Pi}_i \equiv p_i(\lambda), \tag{15.5}$$

we see that a choice of a section $\mathfrak{c}$ of the jet bundle π^*_2 **is equivalent to the choice of *all the constitutive relations* of the theory simultaneously**.

Recall that a section $\mathfrak{c}$ of the bundle π^*_2 is called **holonomic** if it is a 1-jet of a section $\hat{h}$ of the bundle $\pi_{Y^*\times\Omega^n}$:

$$\mathfrak{c}(\lambda) = j^1(\hat{h})(\lambda).$$

Now we notice that **if the Ω^n-component $\mathfrak{c}^n$ of the section $\mathfrak{c}$ is holonomic, fields $\tilde{F}^\mu_i(\lambda), \hat{h}^\nu(\lambda)$ satisfy to the relations**

$$d_y\hat{h}^\mu = \tilde{F}^\mu_i d\lambda^i \Leftrightarrow \tilde{F}^\mu_i = \frac{\partial \hat{h}^\mu}{\partial \lambda^i} \tag{15.6}$$

and vice versa.

To see this we recall (see, for instance [21, 19]) that the 1-jet space $J^1(\Lambda \underset{X}{\times} \Omega^n)$ is endowed with the canonical contact structure defined by the forms

$$\theta^\mu = dq^\mu - q^\mu_i d\lambda^i.$$

Necessary and sufficient conditions for a section $\mathfrak{c} = (\hat{h}^\mu(\lambda), q^\mu_i(\lambda))$ to be holonomic is the fulfillment of relations

$$\mathfrak{c}^*(\theta^\mu) = d\hat{h}^\mu - q^\mu_i d\lambda^i = 0$$

for all $\mu = 0, 1, \ldots, n$ which is the other form of relations (15.6) with the identification (15.5) above.

Assume now that the dual space Λ of variables λ^i is the vector space and consider now the Liouville vector field ζ in the (vector) space Λ

$$\zeta = \lambda^i \frac{\partial}{\partial \lambda^i}.$$

We **require additionally that the section $\mathfrak{c}$ satisfies to the (residual entropy)** condition

$$i_\zeta(\Pi_i(\lambda)\eta \wedge d\lambda^i) = \Pi_i\lambda^i = \Sigma(\lambda) \geqq 0 \tag{15.7}$$

In such a way we ensure the fulfilment of condition (2.9) *including the positivity of entropy production* Σ (see 2.10).

As a result we have proved the following

Proposition 28. *The following statements are equivalent*

(1) *Constitutive relations defined by the section* $\mathfrak{c}$ *of the bundle* π_2^* *satisfy to the entropy principle.*
(2) Ω^n*-component of section* $\mathfrak{c}$ ***is holonomic*** *and* $\Omega^(n+1)$*-component* $\Pi_i d\lambda^i \wedge \eta$ *of section* $\mathfrak{c}$ *satisfies to the* ***positivity condition***

$$i_\zeta[\Pi_i(\lambda)d\lambda^i \wedge \eta] = \Pi_i\lambda^i\eta = \Sigma(\lambda)\eta \geqq 0. \tag{15.8}$$

In the last inequality we use the nonnegativity defined by the mass form $dM = \rho\eta$.

Example 21. Let a function $\Psi(\lambda)$ be given such that the radial monotonicity condition

$$\zeta \cdot \Psi \geqq 0 \tag{15.9}$$

is fulfilled. This condition is equivalent to the geometrical requirement that the sublevel domains $\Psi^{-1}(-\infty, c)$ of the function Ψ are "star-shaped" domains with respect to the origin.

Consider a production vector Π_i of the form

$$\Pi_i = \frac{\partial\Psi}{\partial\lambda^i} \Leftrightarrow \Pi_i d\lambda^i \wedge \eta = d\Psi \wedge \eta$$

with the function $\Psi(\lambda)$. Then the positivity condition $\lambda^i\Pi_i \geqq 0$ is fulfilled due to the condition (15.9).

Now we would like to present the balance system in terms of dual fields $\lambda^i(x)$ instead of the original fields $y^i(x)$ in the way similar to the Euler-Lagrange Equations in the multisymplectic Poincare-Cartan formalism (see above):

$$\partial_{x^\mu}[(j^{1\ *}(\lambda)\hat{F}_i^\mu)(\lambda(x))] = (j^{1\ *}(\lambda)\Pi_i))(\lambda(x)). \tag{15.10}$$

To do this we start with a section

$$\mathfrak{c} = j^1(\hat{h}) - \Pi = (\lambda^i; \hat{h}^\mu(\lambda)\eta_\mu + \Sigma(\lambda)\eta; \frac{\partial\hat{h}^\mu}{\partial\lambda^i}d\lambda^i \wedge \eta_\mu - \Pi_i(\lambda)d\lambda^i \wedge \eta) \tag{15.11}$$

of the 1-jet bundle π_2^* satisfying to the conditions of the Proposition 28 above.

Taking the differential $\tilde{d}$ of the vertical part of section $\mathfrak{c}$ - the (n+1)+(n+2) form $\mathfrak{c}_v = \frac{\partial\hat{h}^\mu}{\partial\lambda^i}d\lambda^i \wedge \eta_\mu - \Pi_i(\lambda)d\lambda^i \wedge \eta$ we get

$$d(\frac{\partial\hat{h}^\mu}{\partial\lambda^i})\wedge d\lambda^i\wedge\eta_\mu + \Pi_i(\lambda)d\lambda^i\wedge\eta = -\partial_{x^\mu}\left(\frac{\partial\hat{h}^\mu}{\partial\lambda^i}\right)d\lambda^i\wedge\eta - \frac{\partial\hat{h}^\mu}{\partial\lambda^i}\lambda_{G,x^\mu}d\lambda^i\wedge\eta + \Pi_i(\lambda)d\lambda^i\wedge\eta.$$

Now we take the interior derivative of this form in the direction of an arbitrary vertical vector field $\xi \in T(\Lambda)$ (corresponding, in Poincare-Cartan formalism, to the vertical variation of a section $(\hat{h},\Sigma)$ in the direction of ξ) and get

$$i_\xi \mathfrak{c}_v = -\partial_{x^\mu}\left(\frac{\partial \hat{h}^\mu}{\partial \lambda^i}\right)\xi^i\wedge\eta - \frac{\partial \hat{h}^\mu}{\partial \lambda^i}\lambda_{G,x^\mu}\xi^i\wedge\eta + \Pi_i(\lambda)\xi^i\wedge\eta$$

Taking now the pullback of this n+(n+1) form with respect to a section $\lambda=\lambda(x)$ of the bundle $\pi_{Y^*X}: Y^* = X\times\Lambda\rightarrow X$ we get

$$\begin{aligned}\lambda^*(i_\xi \mathfrak{c}_v) &= i_{\xi\circ\lambda(x)}\lambda^* d\mathfrak{c}_v = i_{\xi\circ\lambda(x)}\lambda^* d\lambda^*\mathfrak{c}_v = \\ &= \xi^i(x,\lambda(x))[-(D_\mu\left(\frac{\partial \hat{h}^\mu}{\partial \lambda^i}\right)(x,\lambda(x)) - \frac{\partial \hat{h}^\mu}{\partial \lambda^i}\lambda_{G,x^\mu} + \Pi_i(x,\lambda(x))]\eta. \quad (15.12)\end{aligned}$$

Equating this expression to zero and requiring that the last equation would be fulfilled for a section $\lambda(x)$ **for arbitrary (vertical) vector field ξ in the space** Λ we see that the condition $\lambda^*(i_\xi\mathfrak{c}_v)=0$ is equivalent to the fulfillment of the balance system of equations (15.10)

$$\partial_{x^\mu}\left(\frac{\partial \hat{h}^\mu}{\partial \lambda^i}(\lambda(x))\right) = \Pi_i(\lambda(x))$$

which is, with the identification $F^\mu_i = \frac{\partial \hat{h}^\mu}{\partial \lambda^i}$ equivalent to the dual system of balance equations (15.10). Thus we have proved the following statement

Theorem 13. *Let*

$$S = j^1(\hat{h}) - \Pi = (\lambda^i;\hat{h}^\mu(\lambda)\eta_\mu + \Sigma(\lambda)\eta; \frac{\partial \hat{h}^\mu}{\partial \lambda^i}d\lambda^i\wedge\eta_\mu - \Pi_i(\lambda)d\lambda^i\wedge\eta)$$

be a (constitutive) section of the 1-jet bundle π^*_2 satisfying to the conditions of the Proposition 28 above. Then the following statements about a section $\lambda(x)$ of the bundle $\pi_{Y^*X}: Y^* = X\times\Lambda\rightarrow X$ are equivalent:

(1) *For any vertical vector field ξ in the space Λ*
$$\lambda^*(i_\xi\mathfrak{c}_v) = 0$$
(2) *With the identification $\hat{F}^\mu_i(\lambda) = \frac{\partial \hat{h}^\mu}{\partial \lambda^i}$, the system of dual fields $\lambda=\lambda(t,x^\nu)$ satisfy to the balance system (15.10), to the entropy principle and to the second law of thermodynamics.*

16. Conclusion.

Basic structures of a multisymplectic theory of systems of balance laws (balance systems) was developed in this paper. Constitutive relations of balance systems appears in this scheme as a generalized Legendre transformations C between the (partial) 1-jet bundles of the configurational bundle $\pi: Y\rightarrow X$ and the dual bundle of the semi-basic exterior (n+1)+(n+2)-forms on Y. Action of geometrical (gauge) transformations on the constitutive laws C and on the corresponding Poincare-Cartan forms is studied. Noether Theorem is proved for the symmetry groups of a constitutive law C and the energy-momentum balance law for a ν-homogenous balance laws is considered. Entropy principle if formulated for a general balance systems is formulated and restrictions it put on the constitutive laws are studied. These considerations are applied to the Rational Extended Thermodynamics (RET) to construct the dual geometrical picture of RET, present the balance system of

RET in an invariant form and to interpret the entropy principle as the holonomicy of the current component of the constitutive relations.

In the second part of this work we will study the partial jet bundles of higher order compatible with the covariance groups of a balance system (see [28, 54, 51]) and extend the scheme presented here to this situation. Action of the groups of point transformations and the gauge groups on the phase and dual jet-bundles of a field theory in producing, rearranging and ordering the systems of balance laws ("*balance systems*") of mixed tensorial structure and of different differential order will be studied in the framework of the present scheme. More detailed study of the structure of secondary balance laws of a balance system is the other direction of the future work. Applications to the continuum mechanics (uniform materials, nonlinear visco-elasticity and the electrodynamics of continua) will be considered.

Another direction of future work would be to extend the constructed scheme to the case of the base manifolds with the boundary $(X, \partial X)$. Even in the case of a homogeneous Thermodynamics the mathematical (geometrical) description of interaction of a thermodynamical system with the environment presents a challenge (see, for instance, the works [36, 38, 39]).

In the conclusion I would like to express my deep gratitude to Ernst Binz whose interest and discussions during my short visit to Mannheim in September 2006 were extremely helpful to me and to Professor W. Muschik for the discussions stimulating my interest to the problems of field thermodynamics and the entropy principle.

17. Appendix I. Properties of forms η_μ.

Here we collect some properties of the forms η_μ that are repeatedly used in the text.

We have

$$\eta_\mu = i_{\partial_{x^\mu}}\eta = (-1)^\mu \sqrt{|G|} dx^0 \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n \tag{17.1}$$

and $dx^\mu \wedge \eta_\mu = \eta$.

The differential $d\eta_\mu$ has the form

$$d\eta_\mu = (-1)^\mu \sqrt{|G|}_{,x^\mu} dx^\mu \wedge dx^0 \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n = (\partial_{x^\mu}\lambda_G)\eta, \tag{17.2}$$

where $\lambda_G = ln(\sqrt{|G|})$.

Introduce the (n-1)-forms

$$\eta_{\mu\nu} = i_{\partial_{x^\nu}} i_{\partial_{x^\mu}} \eta.$$

Then we have

$$\eta_{\mu\nu} = \begin{cases} i_{\partial_{x^\nu}}\eta_\mu = (-1)^{\mu+\nu}\sqrt{|G|} dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n, & \text{if } \nu < \mu; \\ i_{\partial_{x^\nu}}\eta_\mu = (-1)^{\mu+\nu-1}\sqrt{|G|} dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n, & \text{if } \nu > \mu. \end{cases}, \tag{17.3}$$

and, in particular, for all μ, ν,

$$\eta_{\mu\nu} = -\eta_{\nu\mu}. \tag{17.4}$$

We also have

$$dx^\sigma \wedge \eta_{\mu\nu} = \begin{cases} \eta_\mu & \text{if } \sigma = \nu, \\ -\eta_\nu & \text{if } \sigma = \mu, \\ 0, & \text{otherwise,} \end{cases} \tag{17.5}$$

for all μ, ν. To see this we first check it explicitly for $\nu < \mu$ and then, for $\nu > \mu$ we use $dx^\sigma \wedge \eta_{\mu\nu} = -dx^\sigma \wedge \eta_{\nu\mu}$ and use the proved result.

For the differentials of these forms we calculate for the case $\mu < \nu$

$$d\eta_{\mu\nu} = d[(-1)^{\mu+\nu}\sqrt{|G|}dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n] =$$
$$(-1)^{\mu+\nu}\sqrt{|G|}[\partial_{x^\nu}\lambda_G)dx^\nu + \partial_{x^\mu}\lambda_G)dx^\mu] \wedge dx^0 \wedge x^{\nu-1} \wedge dx^{\nu+1} \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n =$$
$$= (-1)^\mu \partial_{x^\nu}\lambda_G)\sqrt{|G|}dx^0 \wedge \ldots \wedge x^{\mu-1} \wedge dx^{\mu+1} \ldots \wedge dx^n + (-1)^{\nu+1}\partial_{x^\mu}\lambda_G)\sqrt{|G|}dx^0 \wedge \ldots \wedge x^{\nu-1} \wedge dx^{\nu+1} \ldots \wedge dx^n =$$
$$= ((\partial_{x^\nu}\lambda_G)\eta_\mu - (\partial_{x^\mu}\lambda_G)\eta_\nu). \tag{17.6}$$

and then notice that using (20.5) we get the same result for the case $\nu > \mu$.

18. Appendix II. Formalism of Rational Extended Thermodynamics (RET).

Here we describe, in a short form the basic structure of the Rational Extended Thermodynamics developed by I.Muller and T.Ruggeri, [31, 32]. For the complete presentation of the formalism of Rational Extended Thermodynamics we refer to the monograph [32], Chapter 3. Here we introduce only necessary material in the form suited for our purposes. To be more consistent to the standard notations in the book [32] we will use in this section the notations u^i for the basic fields instead of y^i. Constructions of this section are mostly specializations of those of Section 2.

18.1. **Space-time base.** A state of material body will be described by the collection of the time-dependent fields $\{u^i, i = 1, \ldots, m\}$ defined in a domain $B \subset E^3$ of the physical euclidian) space (E, h) with the boundary ∂B. We assume that the Pseudo-Riemannian metric G is defined in X. An example of such a metric is the Euclidian metric $g = dt^2 + h$ or Lorentz metric. We introduce (global) coordinates $x^\mu, \mu = 1, 2, 3$ in B and the time $t = x^0$. Altogether fields u^i are defined in the n-dim physical space-time $X = \mathbb{R}_t \times \bar{B}$.

Denote by η the volume n-form $\eta = \sqrt{|G|}dt \wedge dx^1 \wedge dx^2 \wedge dx^n$ corresponding to the metric G.

18.2. **State (configurational) bundle.** Basic fields of a continuum thermodynamical theory u^i (except of the entropy that will be included later) take values in the space $U \subset \mathbb{R}^m$ which we will call the **basic state space** of the system.

Following the framework of a classical field theory (see [1, 11]) we organize these fields in the bundle

$$\pi_U : Y \to X,\ X = \mathbb{R}_t \times \bar{B},\ Y = X \times U$$

with the base X being the cylinder $\mathbb{R} \times \bar{B}$ in the Newtonian space-time and the fiber U.

To formulate balance equations in terms of exterior forms we will use the spaces of (3+4)- exterior forms in $X = \mathbb{R}^4$ introduced in Section 1. This space has as its basis elements $\eta_\mu, \mu = 0,1,2,3; \eta$ and is the space of smooth) sections of the bundle of exterior forms of orders 3 and 4 $\Lambda^{3+4}(X) = \Lambda^3(X) \oplus \Lambda^4(X)$ over X.

Taking the pullback of the bundle $\Lambda^{3+4}(X) \to X$ to Y (or, what is the same, construct the fiber product of π_{YX} and $\pi_{\Lambda X}$ we get the following commutative diagram

$$\begin{array}{ccccc} U \times \Lambda^{3+4} & \longrightarrow & Y \underset{X}{\times} \Lambda^{3+4} & \longrightarrow & \Lambda^{3+4}(X) \\ \pi_{\Lambda U}\downarrow & & \pi_{\Lambda Y}\downarrow & & \pi_{\Lambda X}\downarrow \\ U & \longrightarrow & Y & \xrightarrow{\pi_{YX}} & X \end{array} . \tag{18.1}$$

Left column of this diagram represents a typical fiber of bundle $\pi_{\Omega Y}$ over a point $x \in X$. Notice also that the sections of the bundle $\pi_{YX} \circ \pi_{\Omega Y} : Y \underset{X}{\times} \Omega^{3+4} \to X$ are the "semibasic" (3+4) exterior forms on the space Y of the bundle π_{YX}, see [23], Sec.4.2.

18.3. **Balance Equations.** Fields u^i are to be determined as solutions of the field equations having the form of **balance equations** for the currents F^μ_i, where $F^0_i = u^i$

$$F^\mu_{i,\mu} = u^i_{,t} + F^\nu_{i,x^\nu} = \Pi_i,\ i = 1, \dots, n. \tag{18.2}$$

Here $\Pi_i(u,x)$ is called the **production** of the component u^i and $\sum_{\nu=1}^3 F^\nu_i(u,x)\frac{\partial}{\partial x^\nu}$ - the **flow** of the component u^i. These quantities are assumed to be function of the fields u^i and, possibly, of the point $x^\mu \in X$. Usually in RET one restricts the attention to the case where there F^μ_i, Π_i do not depend explicitly on the space-time point x^μ.

Remark 36. In the rational Extended Thermodynamics one consider a case where balance equations are written *for all the basic fields in the state space and only for them* and where flows F^μ_i and productions Π_i depend on the fields u^i *but not on their gradients or time derivatives.*

To close system of equations (18.2) for u^i one has to choose the flows and production forms as functions of u^i - to choose the **constitutive equations** of the body. Such a choice should be done for each balance equation. As we will see below, utilizing of the entropy condition allows to reduce this process to the choice of **entropy flow** 3-form and to the choice of production 4-forms subject to the positivity condition.

18.4. **Entropy condition.** Entropy $h^0(u)$ is assumed to be a function of the basic state variables u^i. It satisfies to the balance law

$$d(h^\mu \eta_\mu) = \Sigma, \tag{18.3}$$

with the **positive** production 4-form

$$\Sigma = \sigma(u)\eta, \sigma \geqq 0, \tag{18.4}$$

and the flow 3-form $H(u) = \sum_{\nu=1}^3 h^\nu \eta_\nu$.

Remark 37. To clarify the geometrical meaning of positivity of an exterior 4-form recall that for each material there is defined the **mass form** $dM = \rho\eta$. Using this form we **define a given 4-form** $f\eta$ **to be nonnegative (positive) if** $f/\rho \geqq 0$ (> 0).

Entropy principle requires that any solution of the balance equations (18.2) would also satisfy to the equation (18.3) and that the production σ of entropy (in the system) should be non-negative.

In addition to this a requirement of **convexity**

$$\frac{\partial^2 h^0}{\partial u \partial u} \sim \text{negative definite} \tag{18.5}$$

has to be fulfilled.

Remark 38. * The last condition shows that the symmetrical bilinear form

$$g_{ij}(u) = -\frac{\partial^2 h^0}{\partial u \partial u} \tag{18.6}$$

can be considered as a **degenerate Riemannian metric** in the state space U. This is the Ruppeiner thermodynamical metric ([31]). It would be interesting to interpret the curvature of this metric in the context of RET.

Requirement of the fulfillment of the entropy balance equation (18.3) for all solutions of balance equations (18.2) for the fields u^i leads to strong limitations on the form of constitutive equations. Namely, this condition is equivalent to the following two statements: There exists a functions $\lambda^i(u)$ (Lagrange multipliers) on the space U such that for all values of variables u^i

$$\frac{\partial h^\mu}{\partial u^i} = \lambda_j \cdot \frac{\partial F^\mu_j}{\partial u^i} \Leftrightarrow dh^\mu = \lambda_j \cdot dF^\mu_j, \tag{18.7}$$

and

$$\Sigma = \lambda^i \Pi_i \geqq 0. \tag{18.8}$$

First of the equation (18.7) defines the Lagrange-Liu multipliers

$$\frac{\partial h^0}{\partial u^i} = \lambda^i. \tag{18.9}$$

Differentiating by u^j we get

$$Hess(h^0) = \frac{\partial^2 h^0}{\partial u^i \partial u^j} = \frac{\partial \lambda^i}{\partial u^j}$$

from which it follows that if the entropy density h^0 is a strongly convex function of its arguments u^i, then the change of variables $u \to \lambda$ is **globally invertible**. Thus, we get the diffeomorphic mapping

$$\wp : U \to \Lambda \tag{18.10}$$

from the state space U onto the space $\Lambda \subset R^n$ of values of variables $\lambda = \{\lambda^i\}$.

18.5. **Dual formulation.** As a result one may present all the quantities as the functions of dual variables λ_i:

$$\tilde{F}^\mu_i = \tilde{F}^\mu_i(\lambda), \tilde{\Pi}_i \equiv \tilde{\Pi}_i(\lambda); \tilde{h}^\mu = \tilde{h}^\mu(\lambda). \tag{18.11}$$

Combining balance equations (18.2) with this change of variables we rewrite these equations in the form

$$\frac{\partial \tilde{F}^\mu_i}{\partial \lambda^j} \cdot \frac{\partial \lambda^j}{\partial x^\mu} = \tilde{\Pi}_i(\lambda) \Leftrightarrow \frac{\partial^2 \hat{h}^\mu}{\partial \lambda^i \partial \lambda_j} \frac{\partial \lambda_j}{\partial x^\mu} = \tilde{\Pi}_i(\lambda),\ i = 1, \ldots, n, \tag{18.12}$$

where the **four-vector potential** (or 3-form)

$$\hat{h}^\mu = \lambda^i \cdot \tilde{F}_i^\mu - \tilde{h}^\mu(\lambda) \tag{18.13}$$

was introduced. In terms of $\hat{h}$ the relation (18.7) takes the form

$$d\hat{h}^\mu(\lambda) = \hat{F}_i^\mu d\lambda^i, \tag{18.14}$$

summation is assumed by repeating indices.

In terms of 3-forms

$$\hat{h} = \lambda^i \cdot \tilde{F}_i^\mu \eta_\mu - \tilde{h}^\mu(\lambda)\eta_\mu = \lambda_i \tilde{F}_i - \tilde{h}. \tag{18.15}$$

From the relation (18.14) it follows that

$$\tilde{F}_i^\mu = \frac{\partial \hat{h}^\mu}{\partial \lambda^i} \Rightarrow \tilde{h}^\mu(\lambda) = -\hat{h}^\mu + \lambda^i \cdot \frac{\partial \hat{h}^\mu}{\partial \lambda^i}. \tag{18.16}$$

As a result, $4n+4$ constitutive functions $\tilde{F}_i^\mu$ and $\tilde{h}^\mu(\lambda)$ can, in terms of λ variables be derived from the 4 functions $\hat{h}^\mu$ - coefficients of 3-form $\hat{h}$.

Remark 39. As long as we are not dealing with variables λ^i as fields in space and time (functions of x^μ) the presentation of $\hat{h}$ as a four-vector potential or as a 3-form in the 4D space-time is pure formal. We use this representation as the starting point for construction of double bundles of the geometrical form of RET (see Section 15).

After presenting currents $\tilde{F}_i^\mu$ in the form (18.16) what is left of the requirements of entropy principle (provided the condition of convexity of h^0 is fulfilled) is the **residual inequality**

$$\Sigma(\lambda) = \lambda^i \Pi_i \geqq 0. \tag{18.17}$$

Two statements containing here determine the entropy production Σ in terms of the production 4-forms Π_i and **require** positivity of Σ.

Reversing the arguments leading to the statements (18.15) and (18.16) one proves the following basic result of RET leading to the dual formulation of balance equations (18.2) and the entropy principle (18.3)

Theorem 14. [32] *The following statements are equivalent under the condition of the convexity of entropy density $h^0(u^i)$ as the function of fields u^i:*

(1) *Entropy principle is fulfilled for the balance equations (3.2) and the entropy balance equation (3.3) for given constitutive functions $F(u), \Pi(u), h(u), \Sigma(u)$.*
(2) *Constitutive fields $F(u), h(u), \Sigma(u)$ are obtained by the relations (18.12),(18.15), (18.16) from the four-potential $\hat{h}(\lambda)$ (formal 3-form) and the production 4-forms $\Pi_i(\lambda)$ for which the residual inequality*

$$\lambda_i \Pi_i \geqq 0$$

is fulfilled.

19. Appendix III. Iglesias Differential.

Differential $\tilde{d}$ is a special case of operators introduced by D. Iglesias and used in [16].

$$\begin{cases} \tilde{d} : \Omega^{k+(k+1)} = \Omega^k(X) \oplus \Omega^{k+1}(X) \to \Omega^{(k+1)+(k+2)} = \Omega^{k+1}(X) \oplus \Omega^{k+2}(X) : \\ \tilde{d}(\alpha^k + \beta^{k+1}) = ((-d\alpha + \beta) + d\beta). \end{cases} \tag{19.1}$$

Lemma 9. $\tilde{d}\circ\tilde{d}=0$.

Proof. We have

$$\tilde{d}\tilde{d}(\alpha^k+\beta^{k+1})=\tilde{d}((-d\alpha+\beta)+d\beta)=[-d(-d\alpha+\beta)+d\beta]+d(d\beta)=-d\beta+d\beta+0.$$

□

The complex

$$0\rightarrow\Omega^1(X)\oplus\Omega^0(X)\rightarrow\ldots\rightarrow\Omega^k(X)\oplus\Omega^{k-1}(X)\rightarrow\ldots\Omega^n(X)\oplus\Omega^{n-1}(X)\rightarrow 0\oplus\Omega^n(X)\rightarrow \tag{19.2}$$

is generated by de Rham complex of a manifold X and corresponds to the couples of forms $\alpha^k+\beta^{k+1}$. This complex can be considered as dual to the complex of chains generated by couples $(C^{k+1},\partial C^k)$ of submanifolds $C^{k+1}\subset X^n$ of dimension k with the boundary ∂C^k: Duality is defined by integration

$$<\alpha^k+\beta^{k+1},(C^{k+1},\partial C^k)>=\int_C\beta+\int_{\partial C}\alpha.$$

We have, obviously,

$$<\tilde{d}(\alpha^k+\beta^{k+1}),(C^{k+1},\partial C^k)>=0$$

for all $(C,\partial C)$ iff $\tilde{d}(\alpha^k+\beta^{k+1})=0$.

20. Appendix IV. Reduced horizontal differential.

here we recall the properties of horizontal differential d_H and introduce an augmented horizontal differential $\hat{d}$ that is used in Sec.10.

Recall [21, 13] that the r-jet bundles $J^r(\pi)$ of a bundle $\pi:Y\rightarrow X$ form the inverse system

$$X\xleftarrow{\pi}Y\xleftarrow{\pi^1_0}J^1(\pi)\xleftarrow{\pi^2_1}\xleftarrow{\cdots}\xleftarrow{\pi^r_{r-1}}J^r(\pi)\xleftarrow{\cdots} \tag{20.1}$$

whose inverse limit $J^\infty(\pi)$ is the infinite order jet bundle of the bundle π.

Adapted local coordinates (x^μ,y^i) in Y determine the local coordinates (x^μ,y^i,y^i_Λ), where multi-index $\Lambda=(\lambda_k,\lambda_{k-1},\ldots,\lambda_1)$ is a collection of natural numbers modulo permutations. We denote by $\partial_\Lambda=\partial_{\lambda_k}\circ\partial_{\lambda_{k-1}}\circ\ldots\partial_{\lambda_1}$ the composition of derivations.

Corresponding to the inverse system (20.1) we have the inverse system of projectable vector fields X_r on the r-jet bundles $\pi^r_0:J^r(\pi)\rightarrow X$.

Dually, there is the direct system

$$\Lambda^*(X)\xrightarrow{\pi^*}\Lambda^*(Y)\xrightarrow{\pi^{1*}_0}\Lambda^*(J^1(\pi))\xrightarrow{\cdots}\xrightarrow{\pi^{r\ *}_{r-1}}\Lambda^*(J^r(\pi)))\ldots \tag{20.2}$$

induced by the pullback of the forms from the lower order jet bundles to the higher order jet bundles. Limit of this direct system is the exterior Z-graded algebra called the bundle $\mathfrak{D}^*_\infty=\Lambda^*(J^\infty(\pi))$ of exterior forms on $J^\infty(\pi)$.

Bundle of algebras $\mathfrak{D}^*_\infty$ is locally generated by the basic forms dx^μ and the contact forms

$$\theta^i_\Lambda=dy^i_\Lambda=y^i_{\Lambda+\lambda}dx^\lambda,\ 0\geqq|\Lambda|.$$

As a result, the vector subspace $\mathfrak{D}^s_\infty=\Lambda^s(J^\infty(\pi))$ of exterior s-forms has the canonical decomposition

$$\mathfrak{D}^s_\infty=D^{0,s}_\infty\oplus\mathfrak{D}^{1,s-1}_\infty\oplus\ldots\oplus D^{s,0}_\infty.$$

elements of $\mathfrak{D}^{k,s-k}_\infty$ are called k-contact forms. Denote by $h_k : \mathfrak{D}^s_\infty \to \mathfrak{D}^{k,s-k}_\infty,\ k \leqq s$ the k-contact projection. Especially important is the horizontal projection $h_0 : \mathfrak{D}^s_\infty \to \mathfrak{D}^{0,s}_\infty$ given by

$$dx^\mu \to dx^\mu,\ dy^i_\Lambda \to y^i_{\lambda+\Lambda} dx^\lambda. \tag{20.3}$$

Accordingly, the exterior differential on $\mathfrak{D}^*_\infty$ is decomposed into the sum

$$d = d_h + d_v \tag{20.4}$$

of horizontal differential d_H and vertical differential d_v so that when

$$\begin{cases} d : D^{k,s-k}_\infty \to D^{k+1,s-k}_\infty \oplus D^{k,s-k+1}_\infty, \\ d_H : D^{k,s-k}_\infty \to D^{k,s-k+1}_\infty, \\ d_v : D^{k,s-k}_\infty \to D^{k+1,s-k}_\infty. \end{cases} \tag{20.5}$$

We have homology properties

$$d^2_H = d^2_V = d_V d_H + d_H d_V = 0$$

and the relation

$$h_0 \circ d = d_H \circ h_0.$$

Introduce the *total derivative* d_μ - lift of partial derivation ∂_μ to the by the rules to the vector field in $J^\infty(\pi)$ in the sense of [21, 41]:

$$d_\mu f(x,y,z) = \frac{\partial f}{\partial x^\mu} + z^i_\mu \frac{\partial f}{\partial y^i} + \sum_{i,\{\mu_1\ldots\mu^k\mu\}} z^i_{\mu_1\ldots\mu_k\mu} \frac{\partial f}{\partial z^i_{\mu_1\ldots\mu_k}}.$$

It acts on the exterior forms by the rules

$$\begin{cases} d_\mu(\nu \wedge \sigma) = d_\mu \nu \wedge \sigma + \nu \wedge d_\mu \sigma, \\ d_\mu d\sigma = dd_\mu \sigma,\ \sigma, \nu \in \mathfrak{D}^*_\infty. \end{cases} \tag{20.6}$$

Then the horizontal differential is locally given by expression

$$d_H \omega = dx^\mu \wedge d_\mu(\omega), \omega in \mathfrak{D}^*_\infty. \tag{20.7}$$

From these properties the following relations follows

$$\begin{cases} d_H f = d_\lambda f dx^\lambda, f \in \mathfrak{D}^0_\infty, \\ d_\lambda(dx^\mu) = 0,\ d_H(dx^\mu) = 0, \\ d_\lambda(dz^i_\Lambda) = dz^i_{\Lambda+\lambda}),\ d_H(dz^i_\Lambda) = dx^\lambda \wedge dz^i_{\Lambda+\lambda}, \\ d_\lambda(\theta^i_\Lambda) = \theta^i_{\lambda+\Lambda},\ d_H(\theta^i_\Lambda) = dx^\lambda \wedge \theta^i_{\Lambda+\lambda}. \end{cases} \tag{20.8}$$

Directly from the definition of total derivative the following properties follows

Lemma 10. *Acting on the functions from* $C^\infty(J^\infty(\pi))$,

(1) $[d_\mu, \partial_{x^\nu}] = 0$,
(2) $[d_\mu, \partial_{y^i}] = 0$.

Working with the partial 1-jet bundles $J^1_p(\pi)$ (see Sec. 4-7) we have to use the reduced version of the total derivative. We keep the same notation for this derivative silently assuming that when working on a special kind of partial 1-jet bundle we use the appropriate version of d_μ. Thus, on $J^i_p(\pi)$ with the model vector bundle having as its fiber over $y \in Y$ the factor-space of $T^*(X) \otimes V(\pi)$: for instance for $J^1_K(\pi)$ the fiber has the form $T^*_K(X)\otimes V(\pi)$ of the cotangent bundle $T^*(X)$, fiber coincide with $T^*(X)\otimes V(\pi)$ in the case of the full 1-jet bundle and reduces to zero in the RET case. In local coordinates (x^μ, y^i) denote by P the set of pairs of indices (μ, i) such that coordinate z^i_μ is defined in $J^1_p(\pi)$, or, what is the same, such that $dx^\mu\otimes\partial_{y^i}$ generate the nonzero element of the fiber of vector model for Z_p. Thus, we define,

$$d_\mu f = \partial_{x^\mu} f + \sum_{(\mu,i)\in P} z^i_\mu \partial_{y^i} f + \sum_{(\mu,i)\in P} z^i_{\mu\sigma}\partial_{z^i_\mu} f. \tag{20.9}$$

It is easy to see that total derivative defined in this way preserves the properties (20.6) and the properties listed in Lemma 10.

In addition to the horizontal differential d_H we will be using an "reduced horizontal differential" $\hat{d}$. We define operator $\hat{d}$ by the properties

$$\begin{cases} \hat{d}f = d_\mu f dx^\mu, \\ \hat{d}(dx^\nu) = \hat{d}(dy^i) = \hat{d}(dz^i_\Lambda) = 0, \\ \hat{d}(\omega_1\wedge\omega_2 = (\hat{d}\omega_1)\wedge\omega_2 + \omega_1\wedge(\hat{d}\omega_2). \end{cases} \tag{20.10}$$

In other words we define $\hat{d}$ first on the semi-basic subalgebra of algebra $\mathfrak{D}^*_\infty$ and then extend the differential $\hat{d}$ to the whole algebra by requiring that

$$\hat{d}dy^i = 0, \hat{d}dz^i_{\mu_1\ldots\mu_k} = 0.$$

Lemma 11. *Operator $\hat{d}:\Lambda^k(J^\infty(\pi))\to\Lambda^{k+1}(J^\infty(\pi))$ preserves the subcomplex $\Lambda^*_Y(J^\infty(\pi))$ of π^1_0-semibasic forms (with the generators dy^i, dx^μ) and maps the subspaces of the forms annulated by r π-vertical arguments into itself*

$$\hat{d}:\Lambda^k_r(J^\infty(\pi))\to\Lambda^{k+1}_r(J^\infty(\pi)).$$

Since $d_\lambda d_\mu = d_\mu d_\lambda$ if applied to the functions, it is easy to check that $\hat{d}\circ\hat{d} = 0$, so $\hat{d}$ is the differential operator: $\hat{d}^2 = 0$.

Remark 40. Notice that operator $\hat{d}$ does not commute with the usual differential d, for instance $\hat{d}dy^i = 0$ but $d\hat{d}y^i = d(z^i_\mu dx^\mu)\neq 0$.

Lemma 12. *Let $C\Lambda(J^\infty(\pi))$ be the ideal in $\Lambda(J^\infty(\pi))$ of the contact forms (forms annulating the Cartan distribution), then for any form $\nu\in\Lambda(J^\infty(\pi))$ we have*

$$(\hat{d}-d)\nu\in C\Lambda(J^\infty(\pi)).$$

Proof. Both operators d and $\hat{d}$ are derivations of the exterior algebra, therefore it is sufficient to prove the statement for generators of this algebra $f(x,u,z), dx^\mu, dy^i, dz^i_{\bar\mu},\ \bar\mu = \mu_1,\ldots,\mu_k$. For differentials the result is obvious - both operators annulate them. For the functions we have

$$\hat{d}f = f_{,x^\mu}\hat{d}x^\mu + f_{,y^i}\hat{d}y^i + \sum_{\bar\mu} f_{,z^i_{\bar\mu}}\hat{d}z^i_{\bar\mu} = f_{,x^\mu}dx^\mu + f_{,y^i}z^i_\mu dx^\mu + \sum_{\bar\mu} f_{,z^i_{\bar\mu}} z^i_{\bar\mu\nu}dx^\nu.$$

Subtracting from this expression the similar (but simpler) expression for df we get

$$(\hat{d}-d)f = f_{,y^i}(z^i_\mu dx^\mu - dy^i) + \sum_{\bar\mu} f_{,z^i_{\bar\mu}}(z^i_{\bar\mu\nu}dx^\nu - dz^i_{\bar\mu}) = -f_{,y^i}\omega^i + \sum_{\bar\mu} f_{,z^i_{\bar\mu}}\omega^i_{\bar\mu},$$

that finishes the proof. □

Proposition 29. *Let ϕ be an automorphism of the bundle π and ϕ^{∞} - its contact (=flow) prolongation to the $J^{\infty}(\pi)$. Then*

$$\hat{d}\phi^{\infty\ *}\omega \equiv \phi^{\infty\ *}\hat{d}\omega \ mod\ C\Lambda^*$$

for all π^1_0-semibasic forms ω on $J^{\infty}(\pi)$. Here $C\Lambda^*(J^{\infty}(\pi))$ is the ideal in generated by the Cartan forms (forms annulating the Cartan distribution.

Proof. Mapping ϕ^{∞}_* of tangent spaces leaves the Cartan distribution invariant, therefore the pullback $\phi^{\infty\ *}$ of the forms leaves the Contact ideal $C\Lambda^*(J^{\infty}(\pi))$ invariant. Therefore, for all forms ν by the previous Lemma

$$\phi^{\infty\ *}(\hat{d}-d)\nu \in C\Lambda^*(J^{\infty}(\pi)).$$

On the other hand by the same Lemma $(\hat{d}-d)\phi^{\infty\ *}\nu \in C\Lambda^*(J^{\infty}(\pi))$ as well. This last inclusion can be written

$$\hat{d}\phi^{\infty\ *}\nu - \phi^{\infty\ *}d\nu \in C\Lambda^*(J^{\infty}(\pi))$$

since the pullback commutes with the de Rham differential. Subtracting obtained inclusions we get the result stated in the Proposition.

Both $\phi^{\infty\ *}$ and $\hat{d}$ are linear and respect the wedge product in the corresponding sense. Therefore, one can check the statement for the generators $f,\ dx^{\mu}, du^i$ only.

For dx^{μ} total differential reduces to the usual de Rham differential on X and ϕ^{∞} acts by the projection $\bar{\phi}: X \to X$. Thus, the statement reduces for the usual property of d.

For dy^i we have

$$\hat{d}\phi^{\infty\ *}dy^i = \hat{d}d\phi^i = \hat{d}[\phi^i_{,x^{\nu}}dx^{\nu} + \phi^i_{,y^j}dy^j] = d_{\mu}\phi^i_{,x^{\nu}}dx^{\mu}\wedge dx^{\nu} + d_{\mu}\phi^i_{,y^j}dx^{\mu}\wedge du^j =$$

$$= (\partial_{x^{\mu}}\partial_{x^{\nu}}\phi^i + z^j_{\mu}\partial_{y^j}\partial_{x^{\nu}}\phi^i)dx^{\mu}\wedge dx^{\nu} + (\phi^i_{x^{\nu}u^j} + z^k_{\mu}\phi^i_{y^jy^k})dx^{\mu}\wedge dy^j =$$

$$= \partial_{x^{\mu}}\partial_{x^{\nu}}\phi^i dx^{\mu}\wedge dx^{\nu} + (z^j_{\mu}dx^{\mu} - dy^j)\phi^i_{y^jx^{\nu}}\wedge dx^{\nu} + z^k_{x^{\mu}}dx^{\mu}\wedge\phi^i_{u^jy^k}dy^j =$$

$$= \partial_{x^{\mu}}\partial_{x^{\nu}}\phi^i dx^{\mu}\wedge dx^{\nu} - \omega^j\wedge\phi^i_{y^jx^{\nu}}dx^{\nu} - (dy^k - z^k_{x^{\mu}}dx^{\mu})\wedge\phi^i_{y^jy^k}dy^j + \wedge\phi^i_{y^jy^k}dy^k\wedge dy^j =$$

$$= -\omega^j\wedge\phi^i_{y^jx^{\nu}}dx^{\nu} - \omega^k\wedge\phi^i_{y^jy^k}dy^j = -\omega^j\wedge(\phi^i_{y^jx^{\nu}}dx^{\nu} + \phi^i_{y^jy^k}dy^k),$$

here we canceled two terms with second derivatives of ϕ^i due to the antisymmetry of wedge products of basic forms. Since $\phi^{\infty\ *}\hat{d}dy^i = 0$, statement is proved for dy^i. □

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