

Portland State University

**PDXScholar**

---

Electrical and Computer Engineering Faculty  
Publications and Presentations

Electrical and Computer Engineering

---

12-1-1976

# Beam modes in complex lenslike media and resonators

Lee W. Casperson

*Portland State University*

Follow this and additional works at: [https://pdxscholar.library.pdx.edu/ece\\_fac](https://pdxscholar.library.pdx.edu/ece_fac)



Part of the [Electrical and Computer Engineering Commons](#)

**Let us know how access to this document benefits you.**

---

## Citation Details

Lee W. Casperson, "Beam modes in complex lenslike media and resonators," *J. Opt. Soc. Am.* 66, 1373-1379 (1976).

This Article is brought to you for free and open access. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Publications and Presentations by an authorized administrator of PDXScholar. Please contact us if we can make this document more accessible: [pdxscholar@pdx.edu](mailto:pdxscholar@pdx.edu).

## Beam modes in complex lenslike media and resonators

Lee W. Casperson

*School of Engineering and Applied Science, University of California, Los Angeles, California 90024*

(Received 30 December 1975; revision received 5 April 1976)

General sets of higher-order beam modes are derived for light propagation in media having spatial variations of the gain or loss. The resulting expressions are also valid for propagation through conventional optical elements and graded transmission filters. The four basic mode sets obtained include off-axis Hermite-Gaussian and Laguerre-Gaussian modes of both real and complex argument. A procedure is developed for finding the resonant modes of laser oscillators containing arbitrary complex lens elements, and the mode stability properties of lasers can be interpreted physically by means of these formulas.

One of the most basic problems in laser theory is the determination of the propagation modes of electromagnetic radiation in various types of optical media. The simplest limit of propagation in free space has been well understood for many years, and the modes in that case can be most easily described in terms of Laguerre-Gaussian<sup>1</sup> or Hermite-Gaussian<sup>2</sup> functions.

However, for media with spatial variations of the gain or loss most previous investigations have emphasized the spot size, phase-front curvature, and displacement of only the fundamental mode.<sup>3,4</sup> Some higher-order effects have been obtained as superpositions of the waveguide eigenmodes,<sup>5,6</sup> and a complex Hermite-Gaussian mode set has been derived by a Green's-func-

tion method.<sup>7,8</sup> It has been shown that the beam parameters may exhibit damped or growing spatial oscillations depending on whether the gain maximum or minimum occurs in the vicinity of the beam center, and if no gain variations are present the oscillations are undamped.<sup>4-6</sup>

One purpose of the present work is to derive directly from the wave equation several sets of three-dimensional higher-order beam modes that are valid in media having general spatial variations of the gain and index of refraction. The only restriction on the analysis is that constant, linear, and quadratic functions must be sufficient to characterize the transverse gain and index of refraction variations in the vicinity of the beam. With these results it becomes possible for the first time to trace all of the higher-order beam modes through arbitrary systems of lens elements including lenses, complex lenslike media, spherical mirrors, and Gaussian transmission filters. One can also readily calculate the higher-order modes of laser resonators which include complex lens elements. An understanding of such modes is helpful for studies of mode stability in laser oscillators and waveguides.

Media with spatial gain or loss variations occur very commonly in practice. Typical laser amplifiers often exhibit a radial profile of the gain, and such variations are also inevitable when saturation occurs. Gain focusing and gain deflection of laser beams have been demonstrated experimentally in xenon and CO<sub>2</sub> laser systems.<sup>4,9</sup> At the same time that such gain effects are occurring, the index of refraction may also have a radial profile due to thermal gradients, free electrons, or dispersion associated with the optical transitions. Also, complex transmission filters often provide an excellent model for aperture effects in laser systems.<sup>10</sup>

In Sec. I the wave equation is reduced to a set of ordinary differential equations governing the basic Gaussian beam parameters together with a partial differential equation governing all of the possible polynomial factors. The polynomial equation is solved for the four most important polynomial mode sets in Sec. II. These include the Hermite-Gaussian and Laguerre-Gaussian functions of complex and real arguments. It turns out that the polynomial functions of complex argument correspond to beam modes in which the phase fronts may be nonspherical and the intensity distribution may lack well-defined node lines. In the special case that transverse variations of the gain or loss are negligible much simpler alternate sets of off-axis modes can be found involving polynomial functions of real arguments. Then the intensity distribution can be readily obtained and the phase fronts of the beam modes are spherical. An important application of these results is in light guiding structures of the SELFOC variety where the index of refraction exhibits quadratic radial variations. The formalism developed here can be used to trace the propagation of spatially distinct higher-order beam modes in SELFOC devices.

The procedure for applying the beam-mode formalism to laser resonators is discussed in Sec. III. The re-

sults apply to resonators containing an arbitrary assortment of lenses, complex lenslike elements, and Gaussian apertures. In a specific example it is shown that the mode stability properties of laser amplifiers and oscillators can be interpreted in terms of mode discrimination within the amplifying medium.

## I. DERIVATION OF BEAM EQUATIONS

For any investigation of light propagation the proper starting point is Maxwell's equations. These equations can be combined to yield coupled-wave equations which govern the various field components of a propagating electromagnetic beam. For the usual case of nearly plane waves the dominant transverse Cartesian field components are governed by the much simpler wave equation<sup>4</sup>

$$\nabla^2 \bar{E} + k^2 \bar{E} = 0, \quad (1)$$

where  $\nabla^2$  is the ordinary scalar Laplacian operator acting on the  $x$  or  $y$  components of the electric field  $\bar{E}$ , and  $k$  is the complex spatially dependent wave number  $k = \omega(\mu\epsilon)^{1/2}$ . The wave number may have an imaginary part due to nonzero conductivity or out-of-phase components of the polarization or magnetization. A similar equation holds for the transverse cylindrical field components except that the  $r$  and  $\phi$  variations are coupled. If needed, the weak  $z$  components of the fields may be found from the transverse components by means of Maxwell's equations.

In many practical situations the gain (or loss) and index of refraction have at most quadratic variations in the vicinity of the propagating beam, and one can write

$$k^2(x, y, z) = k_0(z) [k_0(z) - k_{1x}(z)x - k_{1y}(z)y - k_{2x}(z)x^2 - k_{2y}(z)y^2]. \quad (2)$$

For an  $x$ -polarized wave propagating in the  $z$  direction a useful substitution is

$$E_x(x, y, z) = A(x, y, z) \exp\left(-i \int k_0(z) dz\right), \quad (3)$$

and Eq. (1) reduces to

$$\frac{\partial^2 A}{\partial x^2} + \frac{\partial^2 A}{\partial y^2} - 2ik_0 \frac{\partial A}{\partial z} - i \frac{dk_0}{dz} A - k_0(k_{1x}x + k_{1y}y + k_{2x}x^2 + k_{2y}y^2)A = 0, \quad (4)$$

where  $A$  is assumed to vary so slowly with  $z$  that its second derivative can be neglected.

A useful form for an astigmatic off-axis Gaussian beam is

$$A(x, y, z) = B(x, y, z) \exp -i \left( \frac{Q_x(z)x^2}{2} + \frac{Q_y(z)y^2}{2} + S_x(z)x + S_y(z)y \right). \quad (5)$$

With this substitution Eq. (4) may be separated into the set

$$Q_x^2 + k_0 \frac{dQ_x}{dz} + k_0 k_{2x} = 0, \quad (6)$$

$$Q_y^2 + k_0 \frac{dQ_y}{dz} + k_0 k_{2y} = 0, \quad (7)$$

$$Q_x S_x + k_0 \frac{dS_x}{dz} + \frac{k_0 k_{1x}}{2} = 0, \quad (8)$$

$$Q_y S_y + k_0 \frac{dS_y}{dz} + \frac{k_0 k_{1y}}{2} = 0, \quad (9)$$

$$\frac{\partial^2 B}{\partial x^2} - 2i(S_x + Q_x x) \frac{\partial B}{\partial x} + \frac{\partial^2 B}{\partial y^2} - 2i(S_y + Q_y y) \frac{\partial B}{\partial y} - (S_x^2 + S_y^2) B - i(Q_x + Q_y) B - 2ik_0 \frac{\partial B}{\partial z} - i \frac{dk_0}{dz} B = 0. \quad (10)$$

This separation is accomplished by setting equal to zero the various terms in  $x^2$ ,  $y^2$ ,  $x$ , and  $y$ . The significance of the  $Q$  parameters is contained in the relation

$$Q_x(z) = \frac{k_0}{R_x(z)} - i \frac{2}{w_x^2(z)}, \quad (11)$$

where  $R_x$  and  $w_x$  are, respectively, the radius of curva-

ture of the phase fronts and the  $1/e$  amplitude spot size in the  $x$  direction. The ratio  $d_{xi} = -S_{xi}/Q_{xi}$  is the displacement in the  $x$  direction of the amplitude center of the beam, and the ratio  $d_{xr} = -S_{xr}/Q_{xr}$  is the displacement in the  $x$  direction of the phase center of the beam. Here the subscripts  $i$  and  $r$  denote, respectively, the imaginary and real parts of the parameters  $Q_x$  and  $S_x$ , and similar relations apply to the functions  $Q_y$  and  $S_y$ .

The next step is to reduce Eq. (10), and one is led to try the changes of variables

$$x' = a_x(z)x + b_x(z), \quad (12)$$

$$y' = a_y(z)y + b_y(z), \quad (13)$$

$$z' = z, \quad (14)$$

where  $a_x(z)$ ,  $b_x(z)$ ,  $a_y(z)$ , and  $b_y(z)$  are as yet unspecified functions of  $z$ . With these substitutions, Eq. (10) becomes

$$a_x^2 \frac{\partial^2 B}{\partial x'^2} - 2 \left( ia_x S_x + i(x' - b_x) Q_x + ik_0 \frac{(x' - b_x)}{a_x} \frac{da_x}{dz'} + ik_0 \frac{db_x}{dz'} \right) \frac{\partial B}{\partial x'} + a_y^2 \frac{\partial^2 B}{\partial y'^2} - 2 \left( ia_y S_y + i(y' - b_y) Q_y + ik_0 \frac{(y' - b_y)}{a_y} \frac{da_y}{dz'} + ik_0 \frac{db_y}{dz'} \right) \frac{\partial B}{\partial y'} - (S_x^2 + S_y^2) B - i(Q_x + Q_y) B - 2ik_0 \frac{\partial B}{\partial z'} - i \frac{dk_0}{dz'} B = 0. \quad (15)$$

Solutions of this equation are derived in Sec. II.

## II. THE POLYNOMIAL-GAUSSIAN MODES

We have obtained so far a general set of off-axis Gaussian modes governed by Eqs. (6)–(9), and the solutions of these equations are known for  $z$ -independent media.<sup>4</sup> Additional amplitude and phase variations can be found as solutions of Eq. (15). In the special case that  $B$  is independent of  $x'$  and  $y'$  the solutions correspond to the fundamental Gaussian mode. If  $B$  is not independent of  $x'$  and  $y'$  Eq. (15) can still be solved exactly, and the most important solutions are derived in

the following paragraphs. These include the off-axis Hermite-Gaussian modes of complex and real argument and the off-axis Laguerre-Gaussian modes of complex and real argument.

### A. Hermite-Gaussian modes of complex argument

The substitution

$$B(x', y', z') = C(x', y', z') \exp[-iP(z')] \quad (16)$$

in Eq. (15) makes possible the arbitrary separation

$$\frac{dP}{dz'} = -\frac{S_x^2 + S_y^2}{2k_0} - i \frac{Q_x + Q_y}{2k_0} - \frac{ma_x^2 + na_y^2}{k_0} - \frac{i}{2k_0} \frac{dk_0}{dz'}, \quad (17)$$

$$a_x^2 \frac{\partial^2 C}{\partial x'^2} - 2 \left( ia_x S_x + i(x' - b_x) Q_x + ik_0 \frac{(x' - b_x)}{a_x} \frac{da_x}{dz'} + ik_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + a_y^2 \frac{\partial^2 C}{\partial y'^2} - 2 \left( ia_y S_y + i(y' - b_y) Q_y + ik_0 \frac{(y' - b_y)}{a_y} \frac{da_y}{dz'} + ik_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} - 2ik_0 \frac{\partial C}{\partial z'} + 2(ma_x^2 + na_y^2) C = 0, \quad (18)$$

where  $P(z')$  is a phase parameter. Equation (18) may be made to resemble the Hermite differential equation if the quantities in brackets are, respectively, set equal to  $a_x^2 x'$  and  $a_y^2 y'$ . Then this equation becomes

$$a_x^2 \left( \frac{\partial^2 C}{\partial x'^2} - 2x' \frac{\partial C}{\partial x'} + 2mC \right) + a_y^2 \left( \frac{\partial^2 C}{\partial y'^2} - 2y' \frac{\partial C}{\partial y'} + 2nC \right) - 2ik_0 \frac{\partial C}{\partial z'} = 0. \quad (19)$$

The corresponding constraints on  $a_x(z')$  and  $b_x(z')$  may be written

$$iQ_x + \frac{ik_0}{a_x} \frac{da_x}{dz'} = a_x^2, \quad (20)$$

$$a_x S_x + ia_x^2 b_x + k_0 \frac{db_x}{dz'} = 0, \quad (21)$$

with similar equations for  $a_y(z')$  and  $b_y(z')$ .

The product function

$$C(x', y', z') = H_m(x') H_n(y') \quad (22)$$

satisfies Eq. (19) provided that  $H_m$  and  $H_n$  are solutions of the Hermite differential equations<sup>11</sup>

$$\frac{d^2 H_m}{dx'^2} - 2x' \frac{dH_m}{dx'} + 2mH_m = 0, \quad (23)$$

$$\frac{d^2 H_n}{dy'^2} - 2y' \frac{dH_n}{dy'} + 2nH_n = 0. \quad (24)$$

Thus the general solution for the propagation of optical beams in complex  $z$ -dependent lenslike media has been expressed in terms of Hermite-Gaussian functions of complex argument. It is only necessary to solve the coupled ordinary differential Eqs. (17), (20), and (21), and for brevity a discussion of such solutions is omitted.

### B. Hermite-Gaussian modes of real argument

In the previous paragraphs a general formalism has been developed for the propagation of higher-order beam modes in media with spatial variations of the gain and index of refraction. A complicating feature of the

resulting formulas has been the complexity of the arguments of the Hermite polynomial functions. In many practical laser systems and transmission media the spatial variations of the gain do not have a significant effect on the propagation characteristics. When this occurs, a much simpler set of beam modes can be found in which the arguments of the polynomial functions are purely real. The purpose of this section is to derive exactly the general off-axis beam-mode solutions for a lenslike medium having spatial variations of only the refractive index.

Since we are still interested in polynomial-Gaussian solutions, the initial separation of the wave equation is the same as before. The difference is that real polynomial solutions of Eq. (15) must be found rather than complex solutions. Instead of Eqs. (17) and (18) the substitution of Eq. (16) may be used to obtain the separation

$$\frac{dP}{dz'} = -i \frac{Q_{xr} + Q_{yr}}{2k_0} + \frac{(m + \frac{1}{2})Q_{xi} + (n + \frac{1}{2})Q_{yi}}{k_0} - \frac{S_x^2 + S_y^2}{2k_0} - \frac{i}{2k_0} \frac{dk_0}{dz}, \quad (25)$$

$$a_x \frac{\partial^2 C}{\partial x'^2} - 2 \left( ia_x S_x + i(x' - b_x) Q_x + ik_0 \frac{(x' - b_x)}{a_x} \frac{da_x}{dz'} + ik_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + a_y^2 \frac{\partial^2 C}{\partial y'^2} - 2 \left( ia_y S_y + i(y' - b_y) Q_y + ik_0 \frac{(y' - b_y)}{a_y} \frac{da_y}{dz'} + ik_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} - 2ik_0 \frac{\partial C}{\partial z'} - 2mQ_{xi}C - 2nQ_{yi}C = 0, \quad (26)$$

where again the subscripts  $r$  and  $i$  denote, respectively, the real and imaginary parts of the corresponding functions.

The requirement that  $a_x$ ,  $b_x$ ,  $a_y$ ,  $b_y$ , and  $C(x', y', z')$  all be real separates Eq. (26) into the two equations

$$a_x^2 \frac{\partial^2 C}{\partial x'^2} - 2[-a_x S_{xi} - (x' - b_x) Q_{xi}] \frac{\partial C}{\partial x'} + a_y^2 \frac{\partial^2 C}{\partial y'^2} - 2[-a_y S_{yi} - (y' - b_y) Q_{yi}] \frac{\partial C}{\partial y'} - 2mQ_{xi}C - 2nQ_{yi}C = 0, \quad (27)$$

$$\left( a_x S_{xr} + (x' - b_x) Q_{xr} + k_0 \frac{(x' - b_x)}{a_x} \frac{da_x}{dz'} + k_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + \left( a_y S_{yr} + (y' - b_y) Q_{yr} + k_0 \frac{(y' - b_y)}{a_y} \frac{da_y}{dz'} + k_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} + k_0 \frac{\partial C}{\partial z'} = 0. \quad (28)$$

If one imposes the conditions

$$a_x^2 = -Q_{xi}, \quad (29)$$

$$a_x S_{xi} = b_x Q_{xi}, \quad (30)$$

with similar conditions on the  $y'$  variables, Eq. (27) reduces to

$$Q_{xi} \left( \frac{\partial^2 C}{\partial x'^2} - 2 \frac{\partial C}{\partial x'} + 2mC \right) + Q_{yi} \left( \frac{\partial^2 C}{\partial y'^2} - 2 \frac{\partial C}{\partial y'} + 2nC \right) = 0. \quad (31)$$

This equation is satisfied by a Hermite polynomial function like that in Eq. (22). Thus a tentative set of Hermite-Gaussian beam modes has been obtained in which the arguments of the Hermite polynomials are real. It only remains to be shown that these solutions also satisfy Eq. (28).

For the mode functions given in Eq. (22) the derivative  $\partial C / \partial z'$  vanishes. Therefore Eq. (28) will be satisfied if the quantities in brackets and large parentheses are individually set equal to zero. This leads to constraints on  $a_x$  and  $b_x$  in the form

$$Q_{xr} + \frac{k_0}{a_x} \frac{da_x}{dz} = 0, \quad (32)$$

$$S_{xr} + \frac{k_0}{a_x} \frac{db_x}{dz} = 0. \quad (33)$$

But these equations are equivalent to Eqs. (29) and (30). This equivalence can be demonstrated by direct substitutions involving the imaginary parts of Eqs. (6) and (8) together with the condition that  $k_{1x}$  and  $k_{2x}$  are real. Thus the propagation of off-axis light beams can be characterized by real Hermite functions as long as there are no spatial variations of the gain. These results are similar to previously obtained expressions<sup>12</sup> except that here the amplitude center of the beam may be away from the  $z$  axis. In particular, the amplitude center may be shown to propagate according to the paraxial ray equation.<sup>4</sup> It should be emphasized that these mode solutions are not a special case of the modes with complex arguments. Thus a mode from one set can only be expressed as a summation of modes in the other set.

### C. Laguerre-Gaussian modes of complex argument

The Hermite-Gaussian modes found previously are a complete set which can be used for following the propagation of optical beams in complex or real media.

However, many laser systems of practical interest have a basic cylindrical symmetry, and the Cartesian Hermite-Gaussian functions are an awkward choice for the governing mode sets. In particular it is not uncommon for the output modes of practical laser oscillators to have a basic Laguerre-Gaussian form. The purpose

of this section is to develop a new mode set governing the propagation of off-axis Laguerre-Gaussian modes of complex argument in complex lenslike media.

It is helpful to assume at the outset  $Q_x = Q_y = Q$  and  $a_x = a_y = a$ . Then Eqs. (17) and (18) may be replaced by

$$\frac{dP}{dz'} = -\frac{S_x^2 + S_y^2}{2k_0} - \frac{iQ}{k_0} - a^2 \frac{2p+l}{k_0} - \frac{i}{2k_0} \frac{dk_0}{dz'}, \quad (34)$$

$$a^2 \frac{\partial^2 C}{\partial x'^2} - 2 \left( iaS_x + i(x' - b_x)Q + ik_0 \frac{(x' - b_x)}{a} \frac{da}{dz'} + ik_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + a^2 \frac{\partial^2 C}{\partial y'^2} - 2 \left( iaS_y + i(y' - b_y)Q + ik_0 \frac{(y' - b_y)}{a} \frac{da}{dz'} + ik_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} - 2ik_0 \frac{\partial C}{\partial z'} + 2a^2(2p+l)C = 0. \quad (35)$$

Equation (35) can be simplified by setting the quantities in large parentheses, respectively, equal to  $a^2 x'$  and  $a^2 y'$  and by requiring that  $C$  be independent of  $z'$ . The result is

$$\frac{\partial^2 C}{\partial x'^2} + \frac{\partial^2 C}{\partial y'^2} - 2x' \frac{\partial C}{\partial x'} - 2y' \frac{\partial C}{\partial y'} + 2(2p+l)C = 0. \quad (36)$$

The constraining equations are the same as Eqs. (20) and (21) if the subscript  $x$  (or  $y$ ) is everywhere removed from the functions  $Q$  and  $a$ .

Equation (36) can be expressed in cylindrical coordinates by means of the substitutions

$$x' = r' \cos \phi', \quad y' = r' \sin \phi', \quad (37)$$

and one obtains

$$\frac{\partial^2 C}{\partial r'^2} + \left( \frac{1}{r'} - 2r' \right) \frac{\partial C}{\partial r'} + \frac{1}{r'^2} \frac{\partial^2 C}{\partial \phi'^2} + 2(2l+l)C = 0. \quad (38)$$

It is evident from the derivation that both of the variables  $r'$  and  $\phi'$  may be complex. If  $C$  varies sinusoidally according to

$$C(r', \phi') = D(r') \begin{cases} \cos l\phi' \\ \sin l\phi' \end{cases}, \quad (39)$$

then Eq. (38) becomes

$$\frac{dP}{dz'} = -i \frac{Q_r}{k_0} + \frac{2p+l+1}{k_0} Q_i - \frac{S_x^2 + S_y^2}{2k_0} - \frac{i}{2k_0} \frac{dk_0}{dz'} \quad (43)$$

$$a^2 \frac{\partial^2 C}{\partial x'^2} - 2 \left( iaS_x + i(x' - b_x)Q + ik_0 \frac{(x' - b_x)}{a} \frac{da}{dz'} + ik_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + a^2 \frac{\partial^2 C}{\partial y'^2} - 2 \left( iaS_y + i(y' - b_y)Q + ik_0 \frac{(y' - b_y)}{a} \frac{da}{dz'} + ik_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} - 2ik_0 \frac{\partial C}{\partial z'} - 2(2p+l)Q_i C = 0 \quad (44)$$

The requirement that  $a$ ,  $b_x$ ,  $b_y$ , and  $C(x', y', z')$  all be real separates Eq. (44) into the two equations

$$a^2 \frac{\partial^2 C}{\partial x'^2} - 2 \left[ -aS_{xi} - (x' - b_x)Q_i \right] \frac{\partial C}{\partial x'} + a^2 \frac{\partial^2 C}{\partial y'^2} - 2 \left[ -aS_{yi} - (y' - b_y)Q_i \right] \frac{\partial C}{\partial y'} - 2(2p+l)Q_i C = 0, \quad (45)$$

$$\left( aS_{xr} + (x' - b_x)Q_r + k_0 \frac{(x' - b_x)}{a} \frac{da}{dz'} + k_0 \frac{db_x}{dz'} \right) \frac{\partial C}{\partial x'} + \left( aS_{yr} + (y' - b_y)Q_r + k_0 \frac{(y' - b_y)}{a} \frac{da}{dz'} + k_0 \frac{db_y}{dz'} \right) \frac{\partial C}{\partial y'} + k_0 \frac{\partial C}{\partial z'} = 0. \quad (46)$$

$$\frac{d^2 D}{dr'^2} + \left( \frac{1}{r'} - 2r' \right) \frac{dD}{dr'} - \frac{l^2}{r'^2} D + 2(2p+l)D = 0. \quad (40)$$

The change of variables  $\rho = r'^2$  leads to

$$\rho \frac{d^2 D}{d\rho^2} + (1-\rho) \frac{dD}{d\rho} - \frac{l^2 D}{4\rho} + \frac{2p+l}{2} D = 0, \quad (41)$$

and the substitution  $D(\rho) = \rho^{l/2} L(\rho)$  yields the Laguerre differential equation<sup>11</sup>

$$\rho \frac{d^2 L}{d\rho^2} + (1+l-\rho) \frac{dL}{d\rho} + pl = 0. \quad (42)$$

When the preceding substitutions are collected, one has a general set of off-axis Laguerre-Gaussian modes of complex argument.

#### D. Laguerre-Gaussian modes of real argument

It was shown previously that in media having no transverse variations of the gain it is possible to obtain a Hermite-Gaussian mode set in which the arguments of the Hermite polynomials are real. The purpose of this section is to show that it is also possible to express the beam propagation in such media in terms of Laguerre-Gaussian functions of real argument. Instead of Eqs. (34) and (35) we start with the separation

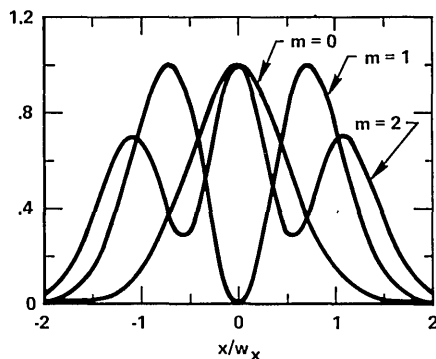


FIG. 1. Intensity distribution of low-order beam modes for an amplifier with a radial gain profile. The modes are normalized to a maximum intensity of unity.

If one imposes the conditions given in Eqs. (29) and (30) with the subscripts  $x$  and  $y$  removed from  $a$  and  $Q$ , then Eq. (45) reduces to the same form as Eq. (36). Therefore the solution techniques of Sec. IIC all apply, and the resulting modes are again described in terms of off-axis Laguerre-Gaussian functions. The difference here is that the arguments of the Laguerre polynomials are now real. As in Sec. IIB these modes may readily be shown to be consistent with Eq. (46).

### III. RESONATOR MODES

The beam equations developed in Secs. I and II govern the propagation of Gaussian beams through very general types of lenslike media. If the media are  $z$  independent the basic  $Q$  and  $S$  parameters evolve according to simple matrix formulas.<sup>13</sup> These formulas also apply to other lens elements including lenses, mirrors, and Gaussian transmission filters, and a table of matrices has been given.<sup>10</sup> It now becomes a straightforward matter to apply this formalism to the determination of the modes of resonators containing an arbitrary assortment of lens elements. The general resonance condition is that the electromagnetic field must repeat itself after one round trip through the resonator. In the present context this means that all of the parameters  $Q_x$ ,  $S_x$ ,  $a_x$ ,  $b_x$ , etc., must repeat, and the phase shift must be an integral multiple of  $2\pi$ . The application of these conditions yields the values of the beam parameters themselves. This procedure can be applied directly to obtain the modes of resonators with Gaussian mirror reflectivity profiles, and the resulting Hermite-Gaussian modes agree with those obtained previously using integral equation<sup>14</sup> or eikonal equation techniques.<sup>15</sup>

Our principal example here concerns the influence of the laser medium on the stability of the amplifier and oscillator modes. Attention is first restricted to the special case of steady-state on-axis beam propagation ( $S_x = S_y = b_x = b_y = 0$ ) in a symmetric medium ( $k_{1x} = k_{1y} = 0$ ). This limit is sufficient to illustrate the mode discrimination properties of lasers in which the amplifier has radial gain variations. From Eq. (6) the steady-state beam parameter is

$$Q_x = \pm (-k_0/k_{2x})^{1/2}. \quad (47)$$

If  $k$  is separated into its real and imaginary parts ac-

cording to  $k = \beta + i\alpha$ , Eq. (47) becomes

$$Q_x = \pm [-\beta_0(\beta_{2x} + i\alpha_{2x})]^{1/2} = \beta_0/R_x - i2/w_x^2, \quad (48)$$

provided that the gain per wavelength is small ( $\beta_0 \gg \alpha_0$ ). Equation (11) has been employed in expressing the right-hand side of Eq. (48). The arbitrary sign may always be chosen in such a way that a confined beam results ( $w_x^2 > 0$ ).<sup>13</sup> For example, if there is a gain maximum at the axis of the laser ( $\alpha_{2x} > 0$ ) and no refraction profile ( $\beta_{2x} = 0$ ), Eq. (48) reduces to

$$Q_x = (1 - i)(\beta_0\alpha_{2x}/2)^{1/2} = \beta_0/R_x - i2/w_x^2, \quad (49)$$

which is an explicit expression for the steady-state spot size and phase-front curvature. From Eq. (20) one obtains

$$a_x^2 = iQ_x, \quad (50)$$

and with Eq. (17) the phase is governed by

$$\frac{dP}{dz} = -i \frac{(m + \frac{1}{2})Q_x + (n + \frac{1}{2})Q_y}{\beta_0}. \quad (51)$$

The  $x$  dependence of some low-order Hermite-Gaussian mode intensity profiles are plotted in Fig. 1 using the relationship  $I \propto EE^*$  together with the beam parameters of Eqs. (49) and (50). For the  $m=0$  and  $m=1$  modes the intensity profiles are similar to those of conventional Hermite-Gaussian beams. With the higher-order modes, however, the intensity distribution is different, and some of the usual node lines may be absent. This situation is illustrated with the  $m=2$  mode shown in the figure. The phase-front curvature of the higher-order modes may also be more complicated than the familiar spherical shape.

The axial phase variations are especially significant in connection with mode discrimination. From Eqs. (49) and (51) the phase is

$$P(z) = -(1+i)[(m + \frac{1}{2})(\alpha_{2x}/2\beta_0)^{1/2} + (n + \frac{1}{2})(\alpha_{2y}/2\beta_0)^{1/2}]z. \quad (52)$$

Since the field amplitude includes the factor  $\exp(-iP)$ , it follows from Eq. (52) that the higher-order modes are amplified more weakly due to the gain profile than is the fundamental mode ( $m=n=0$ ). Thus a laser amplifier having a gain maximum at the axis tends to discriminate against higher-order modes.

On the other hand, if the gain minimum is at the laser axis ( $\alpha_{2x} < 0$ ,  $\alpha_{2y} < 0$ ) Eq. (49) must be replaced by

$$Q_x = -(1+i)(-\beta_0\alpha_{2x}/2)^{1/2} = \beta_0/R_x - i2/w_x^2. \quad (53)$$

The corresponding expression for the phase is

$$P(z) = -(1-i)[(m + \frac{1}{2})(-\alpha_{2x}/2\beta_0)^{1/2} + (n + \frac{1}{2})(-\alpha_{2y}/2\beta_0)^{1/2}]z. \quad (54)$$

The amplifier now discriminates against the fundamental mode. While this is a straightforward analytic demonstration of mode selection in laser amplifiers having radial gain variations, the basic conclusions can also be inferred from previous investigations of mode stability in laser amplifiers and oscillators.<sup>13,16</sup> In mode stability analyses it has been shown that a positive

value of  $\alpha_{2x}$  implies that perturbations of the fundamental mode will damp out while a negative value of  $\alpha_{2x}$  implies that perturbations will grow. If the perturbations are expanded in terms of higher-order beam modes, the stability conclusions are consistent with the mode discrimination properties found here.

The importance of the mode discrimination effects can best be illustrated by means of a practical example. The 3.51  $\mu\text{m}$  xenon laser typically has an amplitude gain coefficient of about  $\alpha_0 = 10 \text{ m}^{-1}$ , which corresponds to an intensity gain of 87 dB/m. If the gain decreases quadratically to zero at the discharge wall at a radius of  $2^{1/2} \text{ mm}$ , it follows from the relation  $\alpha = \alpha_0 - \alpha_{2x}x^2/2 - \alpha_{2y}y^2/2$  that the quadratic gain coefficients are  $\alpha_{2x} = \alpha_{2y} = \alpha_2 = 10^7 \text{ m}^{-3}$ . From Eqs. (3) and (52) the intensity of the steady-state beam grows according to

$$I_{mn} = I_0 \exp\left\{ \left[ 2\alpha_0 - (m + \frac{1}{2})(\lambda\alpha_{2x}/\pi)^{1/2} - (n + \frac{1}{2})(\lambda\alpha_{2y}/\pi)^{1/2} \right] z \right\} \\ = I_0 \exp\left\{ [2\alpha_0 - (m+n+1)(\lambda\alpha_2/\pi)^{1/2}] z \right\}. \quad (55)$$

With the previously given numbers this is

$$I_{mn} = I_0 \exp\left\{ [20 - 3.34(m+n+1)] z \right\}. \quad (56)$$

Thus the fundamental mode  $I_{00}$  has a gain of about  $17 \text{ m}^{-1}$  while the  $I_{11}$  mode has a much smaller gain of about  $10 \text{ m}^{-1}$ .

On the other hand if the gain profile were reversed so that there is zero gain at the tube axis ( $\alpha_0 = 0$ ) and maximum gain at the tube walls ( $\alpha_2 = -10^7 \text{ m}^{-3}$ ), then Eq. (54) implies that the intensity is governed by

$$I_{mn} = I_0 \exp\left\{ [2\alpha_0 + (m+n+1)(\lambda\alpha_2/\pi)^{1/2}] z \right\} \\ = I_0 \exp\left\{ [3.34(m+n+1)] z \right\}. \quad (57)$$

The fundamental mode  $I_{00}$  has a gain of about  $3 \text{ m}^{-1}$  while the  $I_{11}$  mode has a much larger gain of about  $10 \text{ m}^{-1}$ . This kind of negative gain profile occurs in many types of practical lasers due to gain saturation, heating at the laser axis, or various types of wall interactions. When such effects are present the higher-order modes will always dominate over the fundamental mode. This situation can usually be reversed by intentionally introducing stabilizing apertures in the system.<sup>17</sup>

#### IV. CONCLUSION

A general formalism has been developed for tracing the evolution of high-order off-axis Gaussian modes in media having spatial variations of the gain (or loss) and index of refraction. The resulting modes are described in terms of Hermite-Gaussian or Laguerre-Gaussian functions in which the arguments of the polynomials are complex. With these results it becomes possible to employ familiar Gaussian beam techniques in analyzing higher-order mode propagation through arbitrary sequences of lens elements and lenslike media. Requiring that the beam parameters repeat after one round trip leads directly to the resonator modes of laser oscillators. The mode stability characteristics are interpreted physically in terms of the selection properties of laser media with radial gain variations. In media with no radial variations of the gain or loss, simpler alternate mode sets are possible in which the arguments of the polynomial functions are real.

<sup>1</sup>G. Goubau and F. Schwing, IRE Trans. Antennas Propag. AP-9, 248 (1961).

<sup>2</sup>G. D. Boyd and J. P. Gordon, Bell Syst. Tech. J. 40, 489 (1961).

<sup>3</sup>H. Kogelnik, Appl. Opt. 4, 1562 (1965).

<sup>4</sup>L. W. Casperson, Appl. Opt. 12, 2434 (1973), and references therein.

<sup>5</sup>Y. Suematsu, T. Shimizu, and T. Kitano, Electron. Commun. Jpn. 53-B, 77 (1970).

<sup>6</sup>D. Marcuse, *Light Transmission Optics* (Van Nostrand Reinhold, New York, 1972), p. 275.

<sup>7</sup>J. A. Arnaud, J. Opt. Soc. Am. 61, 751 (1971).

<sup>8</sup>J. A. Arnaud, *Progress in Optics* (North-Holland, Amsterdam, 1973), Vol. 11, p. 247.

<sup>9</sup>L. W. Casperson and S. J. Sheldrake, Opt. Commun. 12, 349 (1974).

<sup>10</sup>L. W. Casperson and S. D. Lunnam, Appl. Opt. 14, 1193 (1975).

<sup>11</sup>Handbook of Mathematical Functions, Natl. Bur. Stand. Appl. Math. Ser. 55 (U. S. Government Printing Office, Washington, D. C., 1964), Chap. 22.

<sup>12</sup>H. Kogelnik and T. Li, Appl. Opt. 5, 1550 (1966).

<sup>13</sup>L. W. Casperson, IEEE J. Quantum Electron. QE-10, 629 (1974).

<sup>14</sup>N. G. Vakimov, Radio Eng. Electron. Phys. 10, 1439 (1965).

<sup>15</sup>J. A. Arnaud, Proc. IEEE 62, 1561 (1974).

<sup>16</sup>U. Ganiel and Y. Silberberg, Appl. Opt. 14, 306 (1975).

<sup>17</sup>U. Ganiel, A. Hardy, and Y. Silberberg, Opt. Commun. 14, 290 (1975).