2002

Conformal Laplacian and Conical Singularities

Boris Botvinnik
University of Oregon

Serge Preston
Portland State University, serge@pdx.edu

Conformal Laplacian and Conical Singularities

Boris Botvinnik and Serge Preston

Abstract

We study a behavior of the conformal Laplacian operator $\mathbb{L}_g$ on a manifold with *tame conical singularities*: when each singularity is given as a cone over a product of the standard spheres. We study the spectral properties of the operator $\mathbb{L}_g$ on such manifolds. We describe the asymptotic of a general solution of the equation $\mathbb{L}_g u = Q u^\alpha$ with $1 \le \alpha \le \frac{n+2}{n-2}$ near each singular point. In particular, we derive the asymptotic of a Yamabe metric near such singularity.

1 Introduction

1.1. The goal. The problem we consider in this paper has two essential parts. Firstly, we study the conformal Laplacian operator

$$\mathbb{L}_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$$

on a compact Riemannian manifold (M, g) with isolated singularities of a particular type. Namely, each singular point has a neighborhood which is a cone over the product of spheres $S^p \times S^q$ endowed with the standard metric. Secondly, we derive asymptotics for the positive solutions of the semilinear elliptic equation

$$\mathbb{L}_g u = -\Delta_g u + \frac{n-2}{4(n-1)} R_g u = Q u^\alpha \qquad (1)$$

near each singular point. Here $1 \le \alpha \le \frac{n+2}{n-2} = \alpha^*$, and $\dim M = n = p + q + 1 \ge 3$. We call the equation (1) the *Yamabe equation*. Indeed, for $\alpha = \alpha^*$ it corresponds to the *Yamabe problem*.

1.2. Motivation and some prospectives. Presenting this, somewhat technical paper, we would like to address and discuss natural questions which motivate our interest to study the conformal geometry on manifolds with the cone-type singularities over the product of spheres.

First we recall the classical setting for the Yamabe problem. Let N be a compact smooth manifold, $\dim N = n \geq 3$, and $\mathcal{R}\mathrm{iem}(N)$ the space of Riemannian metrics on N. We denote by R_g the scalar curvature and by $d\sigma_g$ the volume form for each Riemannian metric $g \in \mathcal{R}\mathrm{iem}(N)$. Then the (normalized) Einstein-Hilbert functional $I : \mathcal{R}\mathrm{iem}(N) \to \mathbf{R}$ is defined as

$$I : g \mapsto \frac{\int_N R_g d\sigma_g}{\mathrm{Vol}_g(N)^{\frac{n-2}{n}}}.$$

Let $\mathcal{C}(N)$ be the space of conformal classes on of Riemannian metrics on N, and $C \in \mathcal{C}(N)$. The classical Yamabe problem is to find a metric $\breve{g}$ in a given conformal class C such that the Einstein-Hilbert functional attains its minimum on C: $I(\breve{g}) = \inf_{g \in C} I(g)$. This minimizing metric $\breve{g}$ is called a *Yamabe metric*, and the conformal invariant $Y_C(N) := I(\breve{g})$ the *Yamabe constant*. It is a celebrated result in conformal geometry that the Yamabe problem has an affirmative solution for closed manifolds, see [31], [30], [3], [26].

The *Yamabe invariant of N* is defined as $Y(N) = \sup_{C \in \mathcal{C}(N)} Y_C(N)$. It is well known that the Yamabe invariant $Y(N)$ is a diffeomorphism invariant of N. Furthemore, the Yamabe invariant completely determines the existence of a metric of positive scalar curvature: $Y(N) > 0$ if and only if the manifold N admits a metric of positive scalar curvature. On the other hand, there are just few examples of manifolds with special properties (in the dimensions at least than four) for which the value of the Yamabe invariant is actually known. The fundamental problem here is to compute the Yamabe invariant $Y(N)$ in terms of other known topological invariants of N. In particular, it is important to understand a behavior of the Yamabe constant/invariant under such topological operations as connected sum of manifolds and, more generally, a surgery.

O. Kobayashi [14] has proven the following estimate for the Yamabe invariant of a connected sum. Let N_1, N_2 be two compact closed manifolds of $\dim N_1 = \dim N_2 = n \geq 3$ and $N_{1,2} = N_1 \# N_2$. Then

$$Y(N_{1,2}) \geq \begin{cases} -\left(|Y(N_1)|^{n/2} + |Y(N_2)|^{n/2}\right)^{2/n} & \text{if } Y(N_1), Y(N_2) \leq 0, \\ \\ \min\{Y(N_1), Y(N_2)\} & \text{otherwise.} \end{cases} \tag{2}$$

Petean and Yun [23] have proven the same formula for the case when $N_{1,2} = N_1 \cup_V N_2$ is a union of two manifolds along a submanifold V of codimension

at least three. This result allowed Petean to prove that $Y(N) \geq 0$ for any simply connected manifold of dimension at least five, [24] (see [9] for the case of non-simply connected manifolds).

A difficult case here is to study what is happening with the Yamabe invariant under surgery if $Y(N) > 0$. In more detail, let $S^p \subset N$ be an embedded sphere with trivial normal bundle. Denote by N' the manifold obtained from N by a surgery along the sphere S^p. The integer $n-p$ is called a *codimension* of the surgery. Assume that $Y(N) \leq 0$ and $n - p \geq 3$. Then one can use (2) to show that $Y(N') \geq Y(N)$, see [23] and also [1, Corollary 4.10]. In the case when $Y(N) > 0$ the relationship between the invariants $Y(N)$ and $Y(N')$ is not clear. Consider the trace W of the above surgery, i.e. $W = N \times I \cup D^{p+1} \times D^{q+1}$, $q = n - p - 1$, where $S^p \times D^{q+1} \subset N \times \{1\}$ is identified with $S^p \times D^{q+1} \subset \partial(D^{p+1} \times D^{q+1})$. In particular, the boundary of W is a disjoint union of N and N'. According to the elementary Morse theory, one can choose a Morse function $f : W \to [-1, 1]$ with a single nondegenerate critical point p such that $N = f^{-1}(-1)$, $N' = f^{-1}(1)$ and $f(p) = 0$. Let $N_\tau = f^{-1}(\tau)$, then the manifold N_τ is diffeomorphic to N if $\tau < 0$, and to N' if $\tau > 0$. The manifold N_0 has an isolated singularity, the vertex of the cone $C(S^p \times S^q)$ with an appropriate metric, see Fig. 1.1.
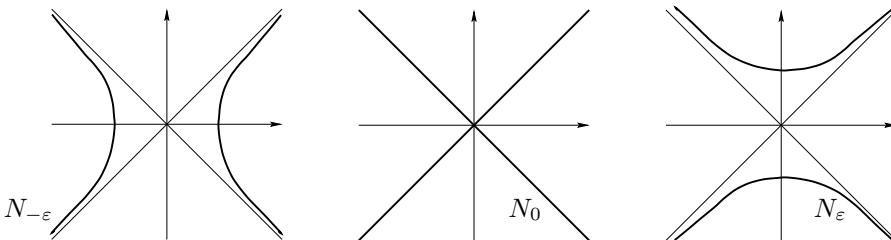

$N_{-\varepsilon}$ N_0 N_ε

Fig. 1.1.

Hence one can think about a surgery on a manifold N as a deformation N_τ, $-1 \leq \tau \leq 1$ through a "singular point" N_0. Furthermore, an appropriate metric on the trace W gives a curve (N_τ, C_τ) of conformal manifolds. We conjecture that the function $\tau \mapsto Y_{C_\tau}(N_\tau)$ is a continuous function. Then under an appropriate choice of conformal classes C_τ, one can approach the following:

Conjecture 1.1 *Let N be a closed compact manifold of $\dim N = n \geq 5$, and N' be obtained from N by a surgery of codimension at least three. Then $Y(N') \leq Y(N)$.*

If Conjecture 1.1 is indeed true, then the Yamabe invariant would be a

cobordism invariant. In particular, it would mean that if N is a simply-connected *Spin*-manifold cobordant to zero, then $Y(N) = Y(S^n)$.

From this viewpoint, it is important to understand the behavior of the conformal Laplacian under surgery, in particular our goal here is to *extend* the conformal geometry to the category of manifolds with cone-type singularities over a product of spheres $S^p \times S^q$. This is the main goal of this paper.

On the other hand, we think that the results of this paper concerning the asymptotic of a solution of the Yamabe equation (1) are of independent value. The asymptotics for singular solutions of the Yamabe equation were studied thoroughly in the case when the manifold in question is the standard sphere S^n punctured at k points, see [19], [15]. In particular, this is related to a gluing construction of metrics of constant scalar curvature under connected sum operation, see [20]. We believe that the results of our paper could be used to glue metrics of constant scalar curvature under surgery.

We have one more application in mind. The paper of the first author and Akutagawa [1] gives an affirmative solution of the Yamabe problem for *cylindrical manifolds*. In particular, in the case of the *positive cylindrical Yamabe constant* the asymptotic of a Yamabe metric is *almost conical* near a cylindrical end, see [1]. In that case, it is easy to see that a manifold with cylindrical ends is equivalent to a manifold with the cone-type singularities. In particular, the Yamabe problem has a solution for manifolds with cone-type singularities over the product $S^p \times S^q$ we study here. As an application, we find an explicit asymptotic for a Yamabe metric in the specific case related to surgery, see Corollary 7.4.

1.3. Conformal Laplacian and the Yamabe equation. Let M be a compact closed manifold of dimension at least three. For a given Riemannian metric g there is the *energy functional*

$$E_g(\varphi) = \int_M \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) d\sigma_g. \qquad (3)$$

Let $1 \leq \alpha \leq \frac{n+2}{n-2} = \alpha^*$. We consider the functional

$$Q_\alpha(\varphi) = \frac{E_g(\varphi)}{\left(\int_M |\varphi|^{\alpha+1} d\sigma_g \right)^{\frac{2}{\alpha+2}}}, \qquad \varphi \in H^1_2(M), \quad \varphi > 0$$

with the Euler-Lagrange equation

$$-\Delta_g u_\alpha + \frac{n-2}{4(n-1)} R_g u_\alpha = Q_\alpha(M) u_\alpha^\alpha. \tag{4}$$

Here $H_2^1(M)$ is the Sobolev space of functions from $L_2(M)$ with their first derivatives (as distributions) also in $L_2(M)$. The Sobolev embedding theorems imply that for each $\alpha \in [1, \alpha^*)$ there exists a smooth function $u_\alpha > 0$ with

$$\int_M u_\alpha^{\alpha+1} d\sigma_g = 1, \quad \text{satisfying} \quad Q_\alpha(u_\alpha) = \min \left\{ Q_\alpha(\varphi) \mid \varphi \in H_1(M) \right\}.$$

We denote $Q_\alpha(M) = Q_\alpha(u_\alpha)$. The sign of the constants $Q_\alpha(M)$ is the same for all $1 \le \alpha \le \alpha^*$ and it depends only on the conformal class of the metric g, see [3]. The existence of a minimizing function in the case $\alpha = \alpha^*$ is the celebrated Yamabe problem, see [31], [30], [2] and [26]. The solution of the Yamabe problem had to overcome the fundamental analytic difficulty concerning Sobolev inequalities for the *critical exponent* α^*. The constant $Q_{\alpha^*}(M)$ depends only on the conformal class $[g]$ and is known as the *Yamabe constant* $Y_{[g]}(M)$ of the conformal class $[g]$. On the other hand, the constant $Q_1(M)$ coincides with the first eigenvalue $\mu_1(\mathbb{L}_g)$ of the conformal Laplacian $\mathbb{L}_g$.

1.4. Manifolds with tame conical singularity. Let M_0 be a compact smooth manifold, $\dim M_0 = n \ge 3$ with the boundary

$$\partial M_0 = S^p \times S^q \quad (\text{where } p + q = n - 1).$$

Let $C(S^p \times S^q)$ be a cone over $S^p \times S^q$ with the vertex point x_*. We glue together M_0 and the cone $C(S^p \times S^q)$ along the boundary $S^p \times S^q$ to obtain a manifold with a conical singularity $x_* \in M$:

$$M = M_0 \cup_{S^p \times S^q} C(S^p \times S^q).$$

Now we describe a metric on the manifold M. Let $S^k(r)$ be a sphere with the standard metric of radius r. First, we assume that $S^p = S^p(r_p)$ and $S^q = S^q(r_q)$. Let (θ, ψ, ℓ) be the standard coordinate system on the cone $C(S^p \times S^q)$, where θ, ψ are the spherical coordinates on S^p, S^q respectively, and $0 \le \ell \le \varepsilon_1$. In particular, $(\theta, \psi, 0) = x_*$ is the singular point, and $(\theta, \psi, \varepsilon_1)$ give spherical coordinates on $S^p \times S^q = \partial M_0$. Let $\varepsilon_0 < \varepsilon_1$. We

decompose $C(S^p \times S^q) = K \cup B$ where

$$K = \{(\theta, \psi, \ell) \in C(S^p \times S^q) \mid 0 \le \ell \le \varepsilon_0\},$$

$$B = \{(\theta, \psi, \ell) \in C(S^p \times S^q) \mid \varepsilon_0 \le \ell \le \varepsilon_1\}.$$

We denote by $\Lambda = \Lambda(p, q)$ the following constant

$$\Lambda = p(p-1)\frac{2-r_p^2}{r_p^2} + q(q-1)\frac{2-r_q^2}{r_q^2} - 2pq.$$

We endow M with a Riemannian metric g satisfying the following properties:

(1) The scalar curvature function $R_g(x) > 0$ if $x \in M_0$.

(2) Let $g_K = g|_K$ be the standard conic metric

$$g_K = d\ell^2 + \frac{\ell^2 r_p^2}{2}d^2\theta + \frac{\ell^2 r_q^2}{2}d^2\psi$$

induced from the Euclidian metric in $C(S^p \times S^q) \subset \mathbf{R}^{p+1} \times \mathbf{R}^{q+1}$, where $S^p = S^p(r_p)$, $S^q = S^q(r_q)$. (In particular, $R_{g_K}(\theta, \psi, \ell) = \frac{\Lambda}{\ell^2}$, see Section 2.)

Definition 1.1 A manifold (M, g) satisfying the above conditions is called a *manifold with tame conical singularity*.

The conformal Laplacian $\mathbb{L}_g = -\Delta_g + \frac{n-2}{4(n-1)}R_g$ is well-defined on the manifold M without the singular point x_*. In particular, we have the Yamabe equation $\mathbb{L}_g u = Q_\alpha u^\alpha$ on $M \setminus \{x^*\}$ for $1 \le \alpha \le \alpha^* = \frac{n+2}{n-2}$.

1.5. Results. We study the following issues:

(A) The spectral properties of the conformal Laplacian $\mathbb{L}_g$ on a manifold with tame conical singularity.

(B) The asymptotic of a general solution of the linear Yamabe equation (i.e. when $\alpha = 1$) near the singular point.

(C) The asymptotic of a general solution of the non-linear Yamabe equation near the singular point for $1 < \alpha \le \alpha^*$.

The paper is organized as follows. We describe in detail the geometry of M and the Yamabe equation near the singular point in Section 2. We define appropriate weighted Sobolev spaces in Section 3. We study the conformal Laplacian $\mathbb{L}_g$ on M in Section 4, in particular, we prove that under some dimensional restrictions the operator $\mathbb{L}_g$ is positive. We analyze the functional I_α on M and prove a weak version of the Yamabe theorem in Section 5. We study the asymptotic of a general solution of the linear and nonlinear Yamabe equations in Sections 6 and 7 respectively. We put together some technical results and calculations in Appendix (Section 8).

1.6. Acknowledgments. The authors are grateful to Kazuo Akutagawa for useful discussions on the Yamabe invariant and to the referee for his helpful and encouraging remarks. We are also grateful to Ms. Melisa Gilbert for the editorial help.

2   The Yamabe equation on the cone

The cone $K \subset \mathbf{R}^{p+1} \times \mathbf{R}^{q+1} = \mathbf{R}^{n+1}$ is given by

$$K = \left\{ (x_1, \ldots, x_{n+1}) \,\middle|\, \tfrac{1}{r_q^2}\left(x_1^2 + \cdots + x_{p+1}^2\right) - \tfrac{1}{r_q^2}\left(x_{p+2}^2 + \cdots + x_{n+1}^2\right) = 0 \right\}. \quad (5)$$

We use the polar coordinates (r, θ) in $\mathbf{R}^{p+1}$, where $\theta \in S^p$, and (ρ, ψ) in $\mathbf{R}^{q+1}$, where $\psi \in S^q$. Then the equation (5) can be written as $\frac{\rho}{r_p} = \frac{r}{r_q}$.
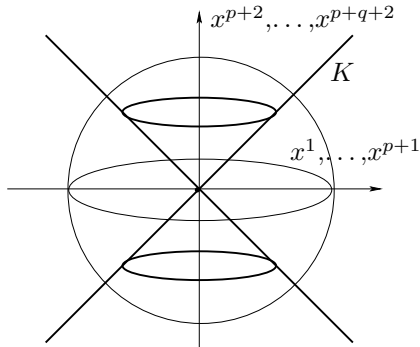

Fig. 2.1.

The coordinate ℓ is the distance from a point on the cone to the vertex x^*. Then $\ell = r\sqrt{2} = \rho\sqrt{2}$. The embedding $K \subset \mathbf{R}^{n+1}$ induces the metric g_K from the standard metric on $\mathbf{R}^{n+1}$. In polar coordinates the metric g_K is given as

$$
\begin{aligned}
g_K &= dr^2 + r^2 d^2\theta + r^2 d^2\psi \\
&= d\ell^2 + \frac{r_p^2 \ell^2}{2} d^2\theta + \frac{r_q^2 \ell^2}{2} d^2\psi.
\end{aligned}
$$

We compute the determinant $|g_K|$:

$$|g_K| = \det(g_{ij}) = 1 \cdot \left(\frac{r_p^2 \ell^2}{2}\right)^p \cdot |g_\theta| \cdot \left(\frac{r_q^2 \ell^2}{2}\right)^q \cdot |g_\psi|, \quad \text{and}$$

$$\sqrt{|g_K|} = \frac{r_p^p r_q^q \ell^{n-1}}{2^{\frac{n-1}{2}}} \cdot \sqrt{|g_\theta|} \cdot \sqrt{|g_\psi|}.$$

Here g_θ and g_ψ are the standard metrics on the spheres $S^p(r_p)$ and $S^q(r_q)$ respectively. In particular, we have the following formula for the volume element:

$$d\sigma_{g_K} = \frac{r_p^p r_q^q \ell^{n-1}}{2^{\frac{n-1}{2}}} d\ell \wedge d\sigma_{g_\theta} \wedge d\sigma_{g_\psi}. \tag{6}$$

The Laplace operator on the cone is then given as

$$\Delta_{g_K} = \frac{1}{\sqrt{|g_K|}} \partial_{x^i} \left(\sqrt{|g_K|}\, g_K^{ij} \partial_{x^j}\right).$$

The matrix $\left(g_K^{ij}\right)$ is block diagonal:

$$\left(g_K^{ij}\right) = \begin{pmatrix} 1 & & & \\ & \frac{2g_\theta^{ij}}{\ell^2 r_p^2} & & \\ & & \frac{2g_\psi^{ij}}{\ell^2 r_q^2} & \\ & & & \end{pmatrix}.$$

We obtain:

$$\Delta_{g_K} = \frac{2^{\frac{n-1}{2}}}{\ell^{n-1}} \partial_\ell \left(\frac{\ell^{n-1}}{2^{\frac{n-1}{2}}} \partial_\ell\right) + \frac{2}{r_p^2 \ell^2} \Delta_\theta + \frac{2}{r_q^2 \ell^2} \Delta_\psi$$

$$= \frac{\partial^2}{\partial \ell^2} + (n-1)\frac{1}{\ell}\frac{\partial}{\partial \ell} + \frac{2}{r_p^2 \ell^2} \Delta_\theta + \frac{2}{r_q^2 \ell^2} \Delta_\psi.$$

The metric g_K is a particular case of a "double wrapped metric" on the product $I \times S^p \times S^q$. In general, a double wrapped metric is given as

$$d\ell^2 + \varphi_p^2(\ell) d^2\theta + \varphi_q^2(\ell) d^2\psi.$$

In our case

$$\varphi_p(\ell) = \frac{r_p \ell}{\sqrt{2}}, \quad \varphi_q(\ell) = \frac{r_q \ell}{\sqrt{2}}, \quad \dot\varphi_p = \frac{r_p}{\sqrt{2}}, \quad \dot\varphi_q = \frac{r_q}{\sqrt{2}}, \quad \ddot\varphi_p = 0, \quad \ddot\varphi_q = 0.$$

We choose the orthonormal bases for the tangent space to K:

$$F_0 = \partial_\ell, \quad F_i, \ i \in \overline{p}, \quad F_j, \ j \in \overline{q},$$

where $\overline{p} = \{2, \ldots, p\}$, $\overline{q} = \{p+1, \ldots, p+q\}$ are orthonormal basis for the standard metrics on S^p and S^q correspondingly. Then we have the following formulas for the curvature operator $\mathcal{R}$ acting on the bundle of 2-forms (see, for instance the book by P. Petersen [25]):

$$\mathcal{R}(F_0 \wedge F_i) \quad = \quad 0, \quad i \in \overline{p},$$

$$\mathcal{R}(F_0 \wedge F_j) \quad = \quad 0, \quad i \in \overline{q},$$

$$\mathcal{R}(F_{i_1} \wedge F_{i_2}) \quad = \quad \frac{2 - r_p^2}{r_p^2 \ell^2} F_{i_1} \wedge F_{i_2}, \quad i_1, i_2 \in \overline{p},$$

$$\mathcal{R}(F_{j_1} \wedge F_{j_2}) \quad = \quad \frac{2 - r_q^2}{r_q^2 \ell^2} F_{j_1} \wedge F_{j_2}, \quad j_1, j_2 \in \overline{q},$$

$$\mathcal{R}(F_i \wedge F_j) \quad = \quad -\frac{1}{\ell^2} F_i \wedge F_j, \quad i \in \overline{p}, \ j \in \overline{q}.$$

Thus we have:

$$\mathcal{R}ic(F_0) \quad = \quad 0,$$

$$\mathcal{R}ic(F_i) \quad = \quad \left((p-1)\frac{2 - r_p^2}{r_p^2 \ell^2} - q\frac{1}{\ell^2} \right) F_i, \tag{7}$$

$$\mathcal{R}ic(F_j) \quad = \quad \left((q-1)\frac{2 - r_q^2}{r_q^2 \ell^2} - p\frac{1}{\ell^2} \right) F_i.$$

We compute the scalar curvature:

$$
\begin{aligned}
R_{g_K} &= \operatorname{Tr} \mathcal{R}ic = p\left[(p-1)\frac{2-r_q^2}{r_p^2\ell^2} - \frac{q}{\ell^2}\right] + q\left[(q-1)\frac{2-r_q^2}{r_q^2\ell^2} - \frac{p}{\ell^2}\right] = \\
&= \frac{1}{\ell^2}\cdot\left[p(p-1)\frac{2-r_p^2}{r_p^2} + q(q-1)\frac{2-r_q^2}{r_q^2} - 2pq\right] = \frac{\Lambda}{\ell^2}, \quad\text{with}
\end{aligned}
$$

$$
\Lambda = p(p-1)\frac{2-r_p^2}{r_p^2} + q(q-1)\frac{2-r_q^2}{r_q^2} - 2pq.
$$

We refer to the Appendix for some properties of Λ as a function of p, q, r_p, r_q.

Now we can rewrite the Yamabe equation $\mathbb{L}_g u = Q_\alpha u^\alpha$ for the cone K. We obtain:

$$
\frac{\partial^2 u}{\partial\ell^2} + \frac{n-1}{\ell}\frac{\partial u}{\partial\ell} + \frac{2}{r_p^2\ell^2}\Delta_\theta u + \frac{2}{r_q^2\ell^2}\Delta_\psi u - \frac{n-2}{4(n-1)}\frac{\Lambda}{\ell^2}u + Q_\alpha u^\alpha = 0,
$$

where $1 \le \alpha \le \alpha^*$ and with $\displaystyle\int_K u^{\alpha+1} < 1, \quad u_\alpha > 0.$

We notice that $\int_K u^{\alpha+1} < 1$ since $\int_M u^{\alpha+1} = 1$.

3 Sobolev spaces on M

In this section we introduce appropriately weighted Sobolev spaces on a manifold with tame conical singularities and review their basic properties. We denote $d\sigma_g$ the volume form on M corresponding to the metric g. According to the computations above, the form $d\sigma_g$, restricted on the cone part K, is given by

$$
d\sigma_g = \frac{\ell^{n-1}}{2^{\frac{n-2}{2}}}\sqrt{|g_\theta|}\sqrt{|g_\psi|}d\ell \wedge d\sigma_\theta \wedge d\sigma_\psi.
$$

Let $L_2(M)$ be the Hilbert space of functions φ on M. Clearly, the restriction $\varphi|_K = \varphi(\ell,\theta,\psi)$ of a function $\varphi \in L_2(M)$ on the cone K satisfies the property

$$
\int_0^{\varepsilon_0}\int_{S^p}\int_{S^q}|\varphi(\ell,\theta,\psi)|^2\ell^{n-1}d\ell \wedge d\sigma_\theta \wedge d\sigma_\psi < \infty. \tag{8}
$$

The proof of the following lemma is standard.

Lemma 3.1 *Let φ be a function on M with $\int_{M \setminus K} |\varphi|^2 d\sigma_g < \infty$, and with the asymptotic behavior $\varphi|_K = \varphi(\ell, \theta, \psi) = \ell^s(1 + O(1))$ as $\ell \to 0$. Then the following statements are equivalent:*

(1) *The integral* (8) *converges.*

(2) *The function $\varphi \in L_2(M)$.*

(3) $s > -\frac{n}{2}$.

Let $\chi \in C^\infty(M \setminus x_*)$ be a positive weight function satisfying

$$\chi(x) = \begin{cases} 1 & \text{if } x \in M_0, \\ \frac{1}{\ell} & \text{if } x \in K \end{cases} \quad \text{and} \quad 0 < \chi(x) \leq 1 \text{ for all } x \in M \setminus x_*.$$

It is easy to construct such a function. Then a *weighted Sobolev space* $H_2^k(M) = H_2^k(M, g)$ is defined as a space of locally integrable functions $\varphi \in L_1^{loc}(M, g)$ with finite norm

$$\|\varphi\|_{H_k^2(M)}^2 = \int_M \left(\sum_{i=0}^{k} \chi^{2(k-i)} \sum_{|\mu|=i} |D^\mu \varphi|^2 \right) d\sigma_g. \tag{9}$$

The derivatives here are understood as distributions (i.e. $D^\mu \varphi \in L_1^{loc}(M, g)$, $|\mu| \leq k$). We denote by $C_*^\infty(M)$ the space of smooth functions on M that are equal to zero in some neighborhood of the singular point x_*. We notice that although the norm of a function φ in Sobolev spaces defined here depends on the choice of the weight χ, the property of φ belonging to the corresponding Sobolev space does not.

Remark. The spaces $H_2^k(M, g)$ defined above coincide with the spaces $W_2^k(M, 1, \rho^{2k})$ defined by H. Triebel [29, Section 3.24, 3.26], where $\rho(x) = \chi(x)$.

In particular, according to [29, Section 3.2.3], the spaces $H_2^k(M, g)$ are complete Hilbert spaces with the scalar product

$$\langle \varphi, \psi \rangle_{H_k^2(M)} = \int_M \left(\sum_{i=0}^{k} \chi^{2(k-i)} \sum_{|\mu|=i} D^\mu \varphi D_\mu \psi \right) d\sigma_g.$$

The spaces $H_2^k(M,g)$ are closely related to the functional spaces $V_{2,0}^k(M,g)$ and $W_{2,\beta}^k(M)$ defined in the book [16]. The space $V_{2,0}^k(M,g)$ is a closure of the space $C_*^\infty(M)$ in the norm (8), [16, Section 6.1.1], and the space $W_{2,\beta}^k(M,g)$ is defined as a space of locally integrable functions $\varphi \in L_{loc}^1(M, d\sigma_g)$ with the following finite norm:

$$\|\varphi\|_{W_{2,\beta}^\ell(M)}^2 = \left(\int_M r^{2\beta} \sum_{|\mu| \le \ell} |D^\mu \varphi|^2 d\sigma_g \right)^{1/2},$$

where the weight function r is related to our weight function as $\chi \sim r^{-1}$ (see [16, Section 7.1.2]). Clearly we have the following embeddings

$$V_{2,0}^k(M,g) \subset H_k^2(M,g) \subset W_{2,0}^k(M,g). \tag{10}$$

These embeddings are continuous; moreover, if $\beta > k - n/2$, the spaces $V_{2,\beta}^k(M,g)$ and $W_{2,\beta}^k(M,g)$ coincide and their norms are equivalent (see [16, Theorem 7.1.1]). In our case, when $\beta = 0$ (when $k < n/2$) three spaces in (10) coincide. This implies the following result.

Lemma 3.2 *Let $n = \dim M \ge 5$. Then the space $C_*^\infty(M)$ is dense in the spaces $H_2^1(M,g)$ and $H_2^2(M,g)$.*

Remark. Lemma 3.2 is the first point when the analysis implies the dimensional restriction $n \ge 5$. We do not know whether some modification of Lemma 3.2 holds in dimensions $n = 3, 4$.

Now we recall the embedding theorem, following the exposition of H. Triebel [29], for the spaces $H_2^k(M,g) = W_2^k(\Omega, \rho^0, \rho^2)$ (where $\rho = \chi$).

Theorem 3.3 *Let M be a manifold with a tame conical singularity as above, $n \ge 5$. Let $2 \le p \le 1 + \alpha^*$, $\alpha^* = \frac{n+2}{n-2}$. Then there exists a continuous linear embedding*

$$i_p : H_2^k(M,g) \subset L_p(M)$$

for $k \ge 1$. Furthermore, the embedding operator i_p is compact for $2 \le p < 1 + \alpha^*$.

Proof. This result follows by combining the embedding theorems of Triebel (see the book [29, Theorems 3.5.1, and 3.8.3]) for the domains in $\mathbf{R}^n$ with conical points on the boundary with the classical embedding theorems for Sobolev spaces on compact manifolds (see, for example, [3, Theorems 2.10, 2.33]). $\square$

There is an important special case here. Clearly, we have the embeddings:

$$H_2^2(M,g) \subset H_2^1(M,g) \subset L_2(M).$$

Theorem 3.3 implies that the embedding $H_1^1(M,g) \subset L_2(M)$ is compact. We will need the following lemma which follows from Theorem 3.3.

Lemma 3.4 *Let $n \geq 5$, then in the space $H_2^1(M,g)$ the norms*

$$\|\nabla\varphi\|_2^2 + \|\chi\varphi\|_2^2 \quad and \quad \|\nabla\varphi\|_2^2 + \|\varphi\|_2^2$$

are equivalent, in particular, for some $C > 0$

$$\|\chi\varphi\|_2^2 \leq C \left(\|\nabla\varphi\|_2^2 + \|\varphi\|_2^2 \right).$$

The following lemma allows us to compare norms in $L_2(M_0)$ and $L_2(M)$.

Lemma 3.5 *Let $n \geq 5$. Then for any constant $a > 0$ there exists a constant $C > 0$ such that*

$$\|\nabla u\|_{L_2(M)} + a\|u\|_{L_2(M_0)} \geq C\|u\|_{L_2(M)}$$

for any function $u \in H_2^1(M)$.

Proof. Notice, first, that for a function u as in the formulation of Lemma 3.5, we have $u|_{M_0} \in L_2(M_0)$. Assume that for some $a > 0$ there is no such constant $C > 0$. Then there exists a sequence $u_n \in H_2^1(M)$, such that

$$\|u_n\|_{L_2(M)} = 1, \quad and \quad \|\nabla u_n\|_{L_2(M)} \to 0, \quad \|u_n\|_{L_2(M_0)} \to 0$$

as $n \to \infty$. Recall that the embedding $H_2^1(M) \subset L_2(M)$ is compact; thus, passing to a subsequence, if necessary, we get $u_n \to u_\infty$ in $L_2(M)$ for some $u_\infty \in L_2(M)$. Clearly we have $\|u_\infty\|_{L_2(M)} = 1$, but $u_\infty|_{M_0} = 0$ almost everywhere since $u_n \to 0$ in $L_2(M_0)$ as $n \to \infty$. Now we choose a function $\varphi \in C_0^\infty(M)$ (in particular, φ has compact support). We have

$$|\langle \nabla u_n, \nabla\varphi\rangle_{L_2(M)}| \leq \|\nabla u_n\|_{L_2(M)} \cdot \|\nabla\varphi\|_{L_2(M)}.$$

As $n \to \infty$ the right-hand side goes to zero, thus the left-hand side too. However

$$\langle \nabla u_n, \nabla\varphi\rangle_{L_2(M)} = \langle u_n, \Delta\varphi\rangle_{L_2(M)} \longrightarrow \langle u_\infty, \Delta\varphi\rangle_{L_2(M)}$$

(using that $\mathbf{Supp}(\varphi)$ is compact). Thus, $\langle u_\infty, \Delta\varphi\rangle_{L_2(M)} = 0$ for all $\varphi \in C_0^\infty(M)$, i.e. $u_\infty \in L_2(M)$ is a weak solution of the equation $\Delta_g u = 0$. Since the Laplacian Δ_g is an elliptic operator, u_∞ is analytic in M. Since $u_\infty|_{M_0} \equiv 0$, then $u_\infty \equiv 0$ on M. This contradicts to $\|u_\infty\|_{L_2(M)} = 1$. $\square$

Let $H_{2,*}^k(M, g)$ be the corresponding Sobolev space of functions on M with some open neighborhood of the singular point x_* removed.

Lemma 3.6 *Let $f = f(l, \theta, \psi) \in H_{2,*}^k(M)$ be a measurable function, so that $f \in H_2^k(M_*)$, and f has asymptotic behavior $f \sim l^q$ as $l \longrightarrow 0$ near the point x_*. Then, $f \in H_2^k(M)$ if and only if $k < q + \frac{n}{2}$.*

Proof. We have that the integral in the formula (9) near $\ell = 0$ has the form

$$\int_0^\varepsilon \ell^{2(q-k)} \ell^{n-1} d\ell.$$

The integral in (9) (taken over $M \backslash U$ where U is some open neighborhood of x_*) is finite by the condition on f. It is easy to see that the Sobolev norm of the function f over all M in (9) is finite if and only if $2(q-k)+n-1 > -1$, or, if $q - k + \frac{n}{2} > 0$. This gives the condition $k < q + \frac{n}{2}$. $\qquad \square$

Later it will be convenient for us to refer to a class of functions with particular asymptotic near the singular point x_*.

Definition 3.1 *Let $1 \le \alpha \le \alpha^*$. A locally integrable function $f \in L_{1,*}(M)$ is called an α-basic function if $f \sim \ell^{-\frac{2}{\alpha-1}}$ as $\ell \to 0$, namely,*

$$c \cdot \ell^{-\frac{2}{\alpha-1}} \le |f(\ell, \theta, \psi)| \le C \cdot \ell^{-\frac{2}{\alpha-1}}$$

for some positive constants c, C.

In particular, if $\alpha = \alpha^* = \frac{n+2}{n-2}$, an α-basic function means that $f \sim \ell^{-\frac{n-2}{2}}$ near x_*.

It is easy to see that Lemma 3.6 implies the following result:

Proposition 3.7 *Let $1 \le \alpha \le \alpha^*$. Then an α-basic function f belongs to $L_2(M)$ if and only if $\frac{n+4}{n} < \alpha \le \alpha^*$. Furthermore, f does not belong to $H_2^1(M)$ if $\alpha \in [1, \alpha^*)$.*

Proof. Let f be an α-basic function. Then $f \in H_2^k(M)$ if and only if $k < \frac{n}{2} - \frac{2}{\alpha-1}$. Then $k = 0$ gives $\alpha - 1 > \frac{4}{n}$ or $\alpha > \frac{n+4}{n}$, and $k = 1$ gives $\frac{2}{\alpha-1} < \frac{n}{2} - 1 = \frac{n-2}{2}$, or $\alpha - 1 > \frac{4}{n-2}$. But this is exactly opposite to the main condition for the parameter α: $\alpha \le \alpha^*$. $\qquad \square$

4 Spectral properties of the conformal Laplacian

4.1. General remarks. In this section we study the conformal Laplacian

$$\mathbb{L}_g = -\Delta_g + c(x), \quad \text{with} \quad c(x) = \frac{n-2}{4(n-1)} R_g(x) \qquad (11)$$

on a manifold M with tame conical singularity.

The classical Laplace operator Δ_g has been studied thoroughly on compact manifolds and on manifolds with certain singularities. In particular, the spectral properties of the Laplacian and its heat kernel are well-understood. The conformal Laplacian is studied predominantly through its relation to the conformal geometry in general, and the Yamabe problem in particular, although there has been important work done on its general properties as well (for example, work by T. Parker, S. Rosenberg [22], by T. Branson [6], see also the references given in the book by P. Gilkey [11]).

On the other hand, operators of the type (11) were studied much as *Schrödinger operators*. For example, the book by S. Mizohata [21, Chapter 8]) studies the Schrödinger operator $-\Delta + c(x)$ on $\mathbf{R}^3$ where the function $c(x)$ has a singularity at the origin of order $|x|^{-\frac{3}{2}+\varepsilon}$ or weaker. Under these conditions, the operator $-\Delta + c(x)$ is bounded from below and has a unique self-adjoint extension (*Friedrichs' extension*). Additional asymptotic conditions (when $|x| \to \infty$) on $c(x)$ ensure that the spectrum of $-\Delta + c(x)$ is discrete and of finite multiplicity.

4.2. Basic results. We start with the following standard property of $\mathbb{L}_g$ (this follows from the basic results on the Sobolev spaces).

Proposition 4.1 *The conformal Laplacian $\mathbb{L}_g = -\Delta_g + \frac{n-2}{4(n-1)} R_g$ is densely defined in $\mathbb{L}_2(M)$ with the domain $H_2^2(M, g)$. Furthermore, the operator $\mathbb{L}_g$ is symmetrical and continuous with respect to the norm in $H_2^2(M, g)$.*

Proof. It is obvious that this operator is symmetric with the domain $C_*^\infty(M)$. We look at the norm

$$|\mathbb{L}_g \varphi|^2 = \left(-\Delta_g \varphi + \frac{n-2}{4(n-1)} R_g \varphi \right)^2 \leq 2 \left((\Delta_g \varphi)^2 + \left(\frac{n-2}{4(n-1)} \right)^2 R_g^2 \varphi^2 \right).$$

Integration over M gives:

$$\|\mathbb{L}_g\varphi\|_{L_2(M)}^2 \leq 2\left(\int_M (\Delta_g\varphi)^2\, d\sigma_g + C\int_M R_g^2\varphi^2 d\sigma_g\right).$$

Notice now that $|R_g(x)| \leq C_1\chi(x)$ for some constant C_1 (Indeed, on $M\setminus K$ the scalar curvature is bounded from above while on the cone K it is equal to $\Lambda\chi(x)$.) Thus using the definition of the norm (9) we obtain

$$\|\mathbb{L}_g\varphi\|_{L_2(M)}^2 \leq 2\left(\int_M (\Delta_g\varphi)^2\, d\sigma_g + C_1\int_M \chi^2\varphi^2 d\sigma_g\right) \leq C_g\|\varphi\|_{H_2^2(M,g)}^2.$$

This proves Proposition 4.1. $\qquad\square$

Theorem 4.2 *Let (M,g) be a manifold with tame conical singularity of $\dim M = n \geq 3$. Then the quadratic form $\langle \mathbb{L}_g u, u\rangle$ defined on $H_2^1(M,g)$ is such that*

$$D\|u\|_{H_2^1(M,g)}^2 \leq \langle \mathbb{L}_g u, u\rangle \leq C\|u\|_{H_2^1(M,g)}^2$$

for some constants $C > 0$ and $D > -\infty$ (i.e. the form $\langle \mathbb{L}_g u, u\rangle$ is bounded from below). Furthermore,

(1) *if $\Lambda > 0$, then*

$$C_1\|u\|_{H_2^1(M,g)}^2 \leq \langle \mathbb{L}_g u, u\rangle \leq C_2\|u\|_{H_2^1(M,g)}^2 \tag{12}$$

   for some constants $C_1 > 0$, $C_2 > 0$;

(2) *if $\Lambda \leq 0$ then for each r_p, r_q there is only a finite number of nonpositive eigenvalues (each of finite multiplicity) of the operator $\mathbb{L}_g$.*

Corollary 4.3 *Let $\Lambda > 0$. Then for any $u \in H_2^1(M,g)$ the operator $\mathbb{L}_g$ satisfies*

$$\|\mathbb{L}_g(u)\|_{L_2(M)} \geq C\|u\|_{L_2(M)}$$

for some constant $C > 0$.

Corollary 4.4 *There exists the Friedrichs' self-adjoint extension $\widetilde{\mathbb{L}}_g$ of the operator $\mathbb{L}_g$. The extension $\widetilde{\mathbb{L}}_g$ is semi-bounded with the same bounding constant. The range of $\widetilde{\mathbb{L}}_g$ coincides with $L_2(M)$.*

Proof. This follows directly from 4.2 and the Neumann Theorem (see [7, Theorem 17]). $\qquad\square$

Corollary 4.5 *The conformal Laplacian $\mathbb{L}_g$ is essentially self-adjoint, and its self-adjoint extension is unique.*

Proof. It follows from the fact that $\mathbb{L}_g$ is strictly positive and symmetric, see [7, Theorems 28, 29]. $\square$

Remark. We have that the range $R(\widetilde{\mathbb{L}}_g) = L_2(M)$. Thus, the range $R(\mathbb{L}_g)$ is dense in $L_2(M)$. Thus, essential self-adjointness of $\mathbb{L}_g$ follows also from [21, Lemma 8.14]. A direct proof of the density of $R(\mathbb{L}_g)$ in $L_2(M)$ may be given following the proof (presented in [21, Chapter 8, Section 13]) for the Schrödinger operator $-\Delta + c(x)$. The latter one utilizes the asymptotic estimates of the Green function for $\mathbb{L}_g$ similar to one obtained by V. Maz'ya, S. Nasarow, B. Plamenewski (see [18]).

Theorem 4.6 *Let $\Lambda > 0$. Then the self-adjoint extension $\widetilde{\mathbb{L}}_g$ of the conformal Laplacian $\mathbb{L}_g$ has discrete positive spectrum of finite multiplicity.*

Proof. We use compactness of the embedding $\mathcal{D}(\widetilde{\mathbb{L}}_g) = H_2^2(M, g) \subset L_2(M)$ and Rollich Theorem (see, say, [21, Theorem 3.3.]) to prove that the inverse operator $\widetilde{\mathbb{L}}_g^{-1}$ is compact and has discrete spectrum $\{\mu_n\}$ ($\lambda_n^{-1} = \mu_n \to 0$ as $n \to \infty$) of finite multiplicity. Also $\langle \mathbb{L}_g u, u \rangle \geq C \|u\|_{L_2(M)}$ gives that $\sigma(\widetilde{\mathbb{L}}_g) \subset \mathbf{R}_+$. $\square$

Remark. One can also prove that $\mathcal{D}(\widetilde{\mathbb{L}}_g) = H_2^2(M, g)$ using arguments similar to those used by H. Triebel [29, Theorem 6.4.1] for the operator $-\Delta + \chi^2$.

4.3. Proof of Theorem 4.2. The upper estimate follows from the symmetry of the form $\langle \mathbb{L}_g u, u \rangle$ and the definition of the Sobolev spaces.

For the proof of the lower estimate, consider first an arbitrary smooth function $u \in C_*^\infty(M)$. We have

$$
\begin{aligned}
\langle \mathbb{L}_g u, u \rangle &= \int_M \left(-u \cdot \Delta_g u + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g \\
&= \int_M \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g.
\end{aligned}
\tag{13}
$$

Here we use $u \in C_*^\infty(M)$, so u is zero in some neighborhood of x_*. Indeed, let $u \equiv 0$ in $\{|\ell|, \nu\} \subset K \subset M$ for some ν. Then $M \setminus \{|\ell| < \frac{\nu}{2}\}$ is a

manifold with the boundary $\{|\ell| = \frac{\nu}{2}\}$, and u is zero in a neigborhood of this boundary.

We decompose M as follows:

$$M = (M_0 \cup B_\sigma) \cup ((B \setminus B_\sigma) \cup K), \quad \text{with}$$

$$B_\sigma = \{x = (r, \ell) \in C(S^p \times S^q) \mid \sigma - r\ell|\ell| < \sigma\},$$

so that the scalar curvature $R_g \geq R_0 > 0$ on $M_0 \cup B_\sigma$, while on $(B \setminus B_\sigma) \cup K$ the metric g may have nonpositive scalar curvature R_g (satisfying the above "tame" conditions). We consider two cases: $\Lambda > 0$ and $\Lambda \leq 0$.

Case $\Lambda > 0$. We have assumed (see above) that in this case $R_g > 0$ everywhere on M, and $R_g \sim \chi^2$ in the neighborhood of x_*. In particular, scalar curvature is positive on the (compact) belt $B \setminus B_\sigma$. Thus, on $M_0 \cup (B \setminus B_\sigma)$ the scalar curvature is bounded from below by a positive constant. Using this we get the estimate

$$C_1 \chi^2 \leq R_g \leq C_2 \chi^2$$

everywhere on M (for some positive constants $C_1, C_2 > 0$). We multiply this inequality by u^2, then we integrate over M, and (13) implies

$$C_1 \|u\|^2_{H^1_2(M)} \leq \langle \mathbb{L}_g u, u \rangle \leq C_2 \|u\|^2_{H^1_2(M)}$$

for some positive constants $C_1, C_2 > 0$ as required.

Case $\Lambda \leq 0$. We use the above decomposition and (13) to write

$$\langle \mathbb{L}_g u, u \rangle = \int_{M_0 \cup B_\sigma} \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g$$

$$+ \int_{(B \setminus B_\sigma) \cup K} \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g = A + B.$$

We estimate the term A:

$$A = \int_{M_0 \cup B_\sigma} \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g \geq C \int_{M_0 \cup B_\sigma} \left(|\nabla_g u|^2 + \chi^2 u^2 \right) d\sigma_g$$

since $R_g \geq R_0 > 0$ on $M_0 \cup B_\sigma$.

For the integral B we have:

$$B = \int_K \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g$$

$$+ \int_{B \setminus B_\sigma} \left(|\nabla_g u|^2 + \frac{n-2}{4(n-1)} R_g u^2 \right) d\sigma_g = \mathbf{I} + \mathbf{J}$$

We start with the study of the integrands in the integral $\mathbf{J}$. In the compact closure $\overline{B \setminus B_\sigma}$ we have the norms of the gradients of a function u with respect to the metrics g and g_K:

$$\|\nabla_g u\| = g^{ij}(x)\xi_i \xi_j, \quad \|\nabla_{g_K} u\| = g_K^{ij}(x)\xi_i \xi_j, \quad \xi_i = \partial_i u.$$

We notice that the ratio

$$\frac{\|\nabla_g u\|_g \sqrt{|g|}}{\|\nabla_{g_K} u\|_{g_K} \sqrt{|g_K|}} = \frac{g_K^{ij}(x)\xi_i \xi_j \sqrt{|g|}}{g_K^{ij}(x)\xi_i \xi_j \sqrt{|g_K|}}$$

is bounded from below by a positive constant c on the compact set $\overline{B \setminus B_\sigma}$. One can take c to be a minimum of the ratio taken over the compact subspace

$$(x, \xi) \in \left\{ T(\overline{B \setminus B_\sigma}) \mid |\xi| = 1 \right\}$$

of the tangent bundle $T(\overline{B \setminus B_\sigma})$. Thus, on the compact set $\overline{B \setminus B_\sigma}$ we have the bound

$$\|\nabla_g u\|_g \sqrt{|g|} \geq c \|\nabla_{g_K} u\|_{g_K} \sqrt{|g_K|} \tag{14}$$

for all u. At the same time on this compact set $\overline{B \setminus B_\sigma}$ we have

$$\frac{\ell^2 |R_g| \sqrt{|g|}}{\sqrt{|g_K|}} \leq c_1$$

for some positive constant c_1 (since the functions u are continuous and positive on this compact set). Thus, on this compact belt we have

$$R_g \sqrt{|g|} \geq c_2 \frac{\Lambda}{\ell^2} \sqrt{|g_K|}, \quad \text{with some} \quad c_2 = \frac{c_1}{|\Lambda|} > 0. \tag{15}$$

Taking $c_3 = \min(c, c_2)$ and combining the estimates (14), (15) we get

$$\mathbf{J} \geq c_3 \int_{B \setminus B_\sigma} \left[\|\nabla_{g_K} u\|^2 + \frac{\Lambda}{\ell^2} u^2 \right] \sqrt{|g_K|} \, dx.$$

Thus,

$$B = \mathbf{I} + \mathbf{J} \geq c_4 \int_{K \cap (B \setminus B_\sigma)} \left[\|\nabla_{g_K} u\|^2 + \frac{\Lambda}{\ell^2} u^2] \sqrt{|g_K|} \right] dx,$$

here $c_4 = \min(1, c_3) > 0$. Thus, we have estimated the integral B from below by an integral over part of the cone, say $\ell \leq \varepsilon_4$, containing only *standard conic metric*.

Thus, we consider the quadratic form

$$\langle \mathbb{L}_{\text{cone}} u, u \rangle = \int_{[0,\varepsilon] \times S^p_{r_p} \times S^q_{r_q}} \left[|\nabla_\ell u|^2 - u \cdot (\Delta_\theta + \Delta_\psi) u + \frac{n-2}{4(n-1)} \frac{\Lambda}{\ell^2} u^2 \right] \ell^{n-1} d\ell d\sigma_\theta d\sigma_\psi$$

on the space of functions obtained by restriction of functions from the space $H^1_2(M)$ to the subset $[0,\varepsilon] \times S^p_{r_p} \times S^q_{r_q} \subset M$ (with the conical standard metric) and the norm induced from $H^1_2(M)$.

Notice that each function from $H^1_2([0,\varepsilon] \times S^p_{r_p} \times S^q_{r_q})$ can be extended to the function from $H^1_2(M)$ (see [7]). Therefore, by restricting functions from M to the conical part we obtain the whole space $H^1_2([0,\varepsilon] \times S^p_{r_p} \times S^q_{r_q})$.

Now we decompose the function u into the Fourier series, using the coordinates (ℓ, θ, ψ)

$$u = \sum_{i,j} u_{ij}(\ell) \xi_i(\theta) \xi_j(\psi)$$

and use the notations:

$$u_{ij} := u_{ij}(\ell), \quad u_{ij,\ell} := \frac{\partial}{\partial \ell} u_{ij}(\ell), \quad u_{ij,\ell\ell} := \frac{\partial^2}{\partial \ell^2} u_{ij}(\ell).$$

We have

$$-u\Delta_{\psi,\theta} u$$

$$= -\left(\sum_{i,j} u_{ij} \xi_i(\theta) \xi_j(\psi) \right) \cdot \left(\sum_{i,j} \left[\frac{2u_{ij}}{r_p^2 \ell^2} \Delta_\theta \xi_i(\theta) \xi_j(\psi) + \frac{2u_{ij} \xi_i(\theta)}{r_q^2 \ell^2} \Delta_\psi \xi_j(\psi) \right] \right)$$

$$= \left(\sum_{i,j} u_{ij} \xi_i(\theta) \xi_j(\psi) \right) \cdot \left(\sum_{i,j} \left[-\frac{2u_{ij}}{r_p^2 \ell^2} \lambda_i^p \xi_i(\theta) \xi_j(\psi) - \frac{2u_{ij} \xi_i(\theta)}{r_q^2 \ell^2} \lambda_j^q \xi_j(\psi) \right] \right)$$

$$= \left(\sum_{i,j} u_{ij} \xi_i(\theta) \xi_j(\psi) \right) \cdot \left(\sum_{i,j} \left[\frac{2}{r_p^2 \ell^2} \lambda_i^p + \frac{2}{r_q^2 \ell^2} \lambda_j^q \right] u_{ij}(\ell) \xi_i(\theta) \xi_j(\psi) \right).$$

Now we add the term

$$|\nabla_r u|^2 + \frac{n-2}{4(n-1)} \frac{\Lambda}{\ell^2} u^2,$$

to the last expression, where u is decomposed into the same Fourier series by ψ, θ. In particular, we have

$$|\nabla_\ell u|^2 = |\sum_{i,j} [\nabla_\ell u_{ij}(\ell)] \xi_i(\theta) \xi_j(\psi)|^2.$$

Then we integrate the resulting expression over the product of spheres $S^p \times S^q$. We obtain

$$\langle \mathbb{L}_{\mathrm{cone}} u, u \rangle = \int_{[0,\varepsilon]} \ell^{n-1} d\ell \sum_{ij} \left[u_{ij,\ell}^2 + K_{ij} u_{ij}^2 \right] = \sum_{ij} \int_0^\varepsilon \ell^{n-1} d\ell [u_{ij,\ell}^2 + K_{ij} u^2],$$

where

$$K_{ij} = \frac{2\lambda_i^p}{r_p^2} + \frac{2\lambda_j^q}{r_q^2} + \frac{n-2}{4(n-1)} \Lambda.$$

The total quadratic form $\langle \mathbb{L} u, u \rangle$ is estimated from below by the integral over M_0 in $\langle \mathbb{L} u, u \rangle$ plus some positive constant times the form $\langle \mathbb{L}_{\mathrm{cone}} u, u \rangle$:

$$\langle \mathbb{L} u, u \rangle \geq c \langle \mathbb{L}_{\mathrm{cone}} u, u \rangle + \int_{M_0} [|\nabla u|^2 + R_g u^2] d\sigma_g.$$

Since $R_g > 0$ on M_0, second term is always positive. This is not true for the first term. Take for example $u = u_0 = const$. This function belongs to the space $H_2^1(M)$ and $\langle \mathbb{L}_{\mathrm{cone}} u_0, u_0 \rangle = K_{00} u_0^2 \frac{\varepsilon^n}{n}$, thus is negative for $\Lambda < 0$, since $\lambda_0^p = \lambda_0^q = 0$. This leads, therefore, to the following necessary condition for the form $\langle \mathbb{L} u, u \rangle$ to be positive (take $u = 1$):

$$\int_M R_g d\sigma_g > 0.$$

Now we find a lower bound for the quadratic form values of

$$\int_0^\varepsilon \left[\ell^{n-1} v_{,\ell}^2 + K \ell^{n-3} v^2 \right] d\ell$$

on the space $H_2^1(0, \varepsilon)$. Later we will specialize the results to the cases $v = u_{ij}, K = K_{ij}$.

Denote by $\bar{v}$ the function on $(0,1)$ obtained by the scaling $\ell = \varepsilon_3 t$: $v(\ell) = v(\varepsilon_3 t) = \bar{v}(t)$. We have

$$\int_0^\varepsilon [v_{,\ell}^2 + Kv^2\ell^{-2}]\ell^{n-1}d\ell = \varepsilon^{n-2}\int_0^1 [\bar{v}_{,t}^2 + K\bar{v}^2 t^{-2}]t^{n-1}dt.$$

Change of variables $t = s^a$; $dt = as^{a-1}ds$; $s = t^{\frac{1}{a}}$ gives

$$f_{,s} = f_{,t}t_{,s} = f_{,t}as^{a-1}; \quad f_{,t} = \frac{1}{a}s^{1-a}f_{,s}.$$

This leads to

$$\int_0^\varepsilon [v_{,\ell}^2 + Kv^2\ell^{-2}]\ell^{n-1}d\ell = \varepsilon^{n-2}\int s^{a(n-1)}as^{a-1}\left[\frac{1}{a^2}s^{2(1-a)}\bar{v}_{,s}^2 + Ks^{-2a}\bar{v}^2\right]ds$$

$$= \varepsilon^{n-2}a\int s^{a(n-1)+a-1+2-2a}\left[\frac{1}{a^2}\bar{v}_{,s}^2 + Ks^{-2}\bar{v}^2\right]ds$$

$$= -(n-2)\varepsilon^{n-2}\int_{+\infty}^1 \left[\bar{v}_{,s}^2 + a^2Ks^{-2}\bar{v}^2\right]ds$$

$$= (n-2)\varepsilon^{n-2}\int_1^{+\infty} \left[\bar{v}_{,s}^2 + \frac{K}{(n-2)^2}s^{-2}\bar{v}^2\right]ds.$$

Here we took $a = -\frac{1}{n-2}$. Thus it is enough to give a lower bound for

$$\int_1^{+\infty} \left[\bar{v}_{,s}^2 + K_1\bar{v}^2(s)\right]ds, \quad K_1 = \frac{K}{(n-2)^2},$$

on the space of $H_2^1(1,+\infty)$ (with the norm defined by the same quadratic expression with $K_1 = 1$).

Functions from Sobolev space $H_2^1(1,+\infty)$ are continuous at $s = 1$ and have the well-defined limit value $f(1) = \lim_{s\to 1} f(s)$. This value is the continuous linear functional on the space $H_2^1(1,+\infty)$. The kernel of this functional is the space $H_{2,0}^1(1,\infty)$ (which is the closure in $H_{2,0}^1(1,\infty)$ of the subspace $C_0^\infty(1,\infty)$ of smooth functions with compact support [7]). A function f from $H_{2,0}^1$ has a canonical extension $\hat{f}$ by zero to the function in $H_2^1(0,+\infty)$ with the norm defined by the same quadratic form

$$\|\hat{f}\|^2 = \int_0^{+\infty} \left[\hat{f}_{,s}^2 + \hat{f}^2(s)\right]ds.$$

Notice that the extension $\hat{f}$ has the same norm as the function f. For such a function, obtained by the extension to $(1, \infty)$ of a function from $H^1_{2,0}(1, \infty)$ and thus, being zero in a neighborhood of zero, we can use the simplest Hardy inequality (see [12]) to estimate

$$\int_0^{+\infty} \hat{f}^2(s) s^{-2} ds \leqq 4 \int_0^{+\infty} \hat{f}^2_{,s}(s) ds.$$

Because of the construction of the extension we get a similar inequality with f instead of $\hat{f}$ and the lower limit 1 replacing zero.

Using this estimate we get for $f \in H^1_{2,0}(1, \infty)$ and negative K_1:

$$\int_1^{+\infty} \left[f^2_{,s} + K_1 f^2(s) \right] ds \geq (1 + 4K_1) \int_1^{+\infty} f^2_{,s} ds.$$

Thus, if $1 + 4K_1 > 0$, the quadratic form

$$K(f, f) = \int_1^{+\infty} \left[f^2_{,s} + K_1 f^2(s) \right] ds$$

is positive definite on $H^1_{2,0}(1, \infty)$, where the norm induced from $H^1_2(1, \infty)$, is equivalent to the norm $\int_1^{+\infty} f^2_{,s} ds$ (this follows from the Hardy inequality). Applying this to the case where $K = K_{ij}$ we see that it is sufficient to check this condition for the case $i = j = 0$ (since $\lambda^p_i, \lambda^q_j > 0$). For $i = j = 0$,

$$K_1 = \frac{K}{(n-2)^2} = \frac{\Lambda}{4(n-1)(n-2)}.$$

Therefore

$$1 + 4K_1 = 1 + 4\frac{\Lambda}{4(n-1)(n-2)} = 1 + \frac{\Lambda}{(n-1)(n-2)} = \mu^2 > 0,$$

see Appendix. Therefore, on the subspace $H^1_{2,0}(1, \infty)$ the quadratic form $K(f, f)$ is positive definite.

To determine what happens at the complement to this subspace and to estimate our quadratic form from below we return to the quadratic form on the space $H^1_2(0, \varepsilon)$ with the (square) of the norm

$$\int_0^\varepsilon \ell^{n-1}(v^2_{,\ell} + \ell^{-2}v^2) d\ell = \|v_{,\ell}\|^2_{L_2(0,\varepsilon;\ell^{n-1}d\ell)} + \|\ell^{-1}v\|^2_{L_2(0,\varepsilon;\ell^{n-1}d\ell)}.$$

Let $K \leqq 0$. To find the lowest eignevalue of the quadratic form

$$\int_0^\varepsilon \ell^{n-1}(v_{,\ell}^2 + K\ell^{-2}v^2)d\ell$$

we calculate the minimum of the following fraction

$$\min_{u \neq 0} \frac{\|u_{,s}\|_{L_2} + K\|s^{-1}u\|_{L_2}}{\|u_{,s}\|_{L_2} + \|s^{-1}u\|_{L_2}}$$

over $u \in H_2^1(0, \varepsilon)$. We write this relation as

$$\frac{f(u) + K}{f(u) + 1}, \quad \text{with} \quad f(u) = \frac{\|v_{,l}\|^2_{L_2(0,\varepsilon;\ell^{n-1}d\ell}}{\|\ell^{-1}v\|^2_{L_2(0,\varepsilon;\ell^{n-1}d\ell}}.$$

Now we notice that the function

$$s \to \frac{s + K}{s + 1}$$

is increasing for $K \leqq 0$ and for $s \geqq 0$ takes its minimal value (equal to K) at $s = 0$. On the other hand $f(u)$ is well defined for all $u \in H_2^1(0, \varepsilon)$, $u \neq 0$, (since its denominator cannot be zero for $u \neq 0$). The function $f(u)$ is nonnegative and is equal to zero only if $u_{,\ell} = 0$, i.e. for constant functions $u = const$.

Therefore, the quadratic form

$$\int_0^\varepsilon \ell^{n-1}(v_{,\ell}^2 + K\ell^{-2}v^2)d\ell$$

has K as its minimal eigenvalue and the constant $u_0 = \sqrt{\frac{n-2}{\varepsilon^{n-2}}}$ as its eigen-vector of unit norm, corresponding to this eigenvalue.

As a result, the condition $\Lambda > 0$, or, what is the same, $\mu > 1$ is sufficient for the total quadratic form L, and the operator $\mathbb{L}_g$ to be positive definite. Recall that we have

$$\langle \mathbb{L}u, u \rangle \geq c \langle \mathbb{L}_{cone}u, u \rangle + \int_{M_0}[|\nabla u| + R_g u^2]d\sigma_g$$

$$= \sum_{ij} \int_0^\varepsilon \ell^{n-1}[u_{ij}^2 + K_{ij}u^2] + \int_{M_0}[|\nabla u| + R_g u^2]d\ell d\sigma_g.$$

The integral over M_0 is always nonnegative. In the sum, terms for which $K_{ij} \geq 0$ are also nonnegative. Since

$$K_{ij} = \frac{2\lambda_i^p}{r_p^2} + \frac{2\lambda_j^q}{r_q^2} + \frac{n-2}{4(n-1)}\Lambda = \frac{2\lambda_i^p}{r_p^2} + \frac{2\lambda_j^q}{r_q^2} + \frac{(n-2)^2(\mu^2-1)}{4},$$

for p, r_p, q, r_q fixed, all the terms in the sum over i, j are nonnegative except a finite number of them.

More than this, each term in the sum (quadratic form) for which $K_{ij} \lneqq 0$ has one and only one nonpositive eigenvalue with constant eigenfunction. For the form $\mathbb{L}_{\mathrm{cone}}$ on $H_2^1((0,\varepsilon) \times S^p \times S^q)$ this corresponds to the function(s) $u_{ij} = c_{ij}\xi_i^p(\theta)\xi_j^q(\psi)$ with some constants c_{ij}. The condition $K_{ij} \lneqq 0$ can be rewritten as follows

$$\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2} \leqq \frac{(n-2)}{(n-1)}|\Lambda|$$

and the number of negative modes of the form $\mathbb{L}_{\mathrm{cone}}$ can be found from here. Notice, in particular, that if $\Lambda < 0$, then for $i, j = 0$ that condition is always satisfied; thus, there is at least one negative mode for $\mathbb{L}_{\mathrm{cone}}$. That does not prevent, though, this negative input in the whole form $\langle \mathbb{L}u, u \rangle$ from being compensated by the input of the M_0-part.

We also notice that as it follows from the proof, in the case where $\Lambda \leqq 0$, there is the lower bound for the form $\langle \mathbb{L}u, u \rangle$, i. e.

$$\langle \mathbb{L}u, u \rangle \geqq D\|u\|_{H_2^1(M)}.$$

Here D is a finite constant, and

$$|D| \leq \frac{4(n-2)}{n-2}|\Lambda|c_4,$$

where c_4 is the constant above. This proves Theorem 4.2. $\qquad\square$

4.4. Necessary condition for positivity of $\langle \mathbb{L}u, u \rangle$. We examine the case when $\Lambda \leq 0$ but $K_{ij} > 0$ for all $i > 0$ and $j > 0$. Since

$$\Lambda = -(n-1)(n-2) + \frac{2p(p-1)}{r_p^2} + \frac{2q(q-1)}{r_q^2},$$

then the condition $\Lambda \leq 0$ is equivalent to

$$\frac{(n-1)(n-2)}{2} \geq \frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2}. \tag{16}$$

On the other hand the conditions $K_{10} > 0$ and $K_{01} > 0$ imply that $K_{ij} > 0$ for all $i > 0$ and $j > 0$. We recall that zero has multiplicity 1 for Δ_θ and Δ_ψ, and the next eigenvalue is p for Δ_θ and q for Δ_ψ respectively. This gives that $K_{10} > 0$ is equivalent to

$$\frac{2p}{r_p^2} + \frac{(n-2)^2\Lambda}{4(n-1)(n-2)} > 0 \ \text{ or } \ \Lambda > -\frac{8(n-1)}{n-2}\cdot\frac{p}{r_p^2}.$$

We use the above formula for Λ to get that

$$K_{10} > 0 \iff \frac{4(n-1)}{n-2}\cdot\frac{p}{r_p^2} + \frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} > \frac{(n-1)(n-2)}{2} \quad (17)$$

Similarly,

$$K_{01} > 0 \iff \frac{4(n-1)}{n-2}\cdot\frac{q}{r_q^2} + \frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} > \frac{(n-1)(n-2)}{2} \quad (18)$$

We have proved the following.

Proposition 4.7 *Assume $\Lambda \le 0$ (which is equivalent to (16)) and the conditions (17), (18) are satisfied. Then the form $\langle \mathbb{L}u, u \rangle$ is positive if and only if*

$$\int_M R_g d\sigma_g > 0.$$

5   Weak Yamabe Theorem

5.1. Yamabe functional. Now we define the Yamabe functional on M and study its properties. Let $\alpha \in [1, \alpha^*]$, where $\alpha^* = \frac{n+2}{n-2}$. For each α we consider the functional

$$I_\alpha(\varphi) = \frac{E(\varphi)}{\left(\int_M |\varphi|^{\alpha+1} dV_g\right)^{\frac{2}{\alpha+1}}}, \quad \varphi \in H^1(M, dV_g), \ \ \varphi \ne 0.$$

$$E(\varphi) = \int \left(|\nabla\varphi|^2 + \frac{n-2}{4(n-1)} R_g \varphi^2 \right) dV_g.$$

This is the Yamabe functional if $\alpha = \alpha^*$. One can prove the following fact by using Theorem 3.3.

Proposition 5.1 *The functional $I_\alpha : H_2^1(M, g) \longrightarrow \mathbf{R}$ is defined and continuous on the space $H_2^1(M, g)$ for all $\alpha \in [1, \alpha^*)$.*

We denote by C_μ the *norm of the embedding* $H_2^1(M) \subset L_\mu(M)$, i.e.

$$C_\mu = \inf_{\substack{\varphi \neq 0 \\ \varphi \in H_2^1(M)}} \frac{\|\varphi\|_{L_\mu}}{\|\varphi\|_{H_2^1}},$$

where it is assumed that $\mu \leq 1 + \alpha^*$.

Proposition 5.2 *Let $\Lambda > 0$. Then the functional I_α is bounded from below, i.e. for all $\varphi \in H_2^1(M)$, $\varphi \neq 0$ we have*

$$I_\alpha(\varphi) \geq \frac{C_1}{C_{\alpha+1}^2}$$

with the constant $C_1 > 0$ given in Theorem 4.2, and $C_{\alpha+1}$ as above.

Proof. Theorem 4.2 gives that $\langle \mathbb{L}_g u, u \rangle \geq C_1 \|u\|_{H_2^1}^2$. Thus

$$I_\alpha(\varphi) = \frac{\langle \mathbb{L}_g \varphi, \varphi \rangle}{\|\varphi\|_{L_{\alpha+1}}^2} \geq \frac{C_1 \|\varphi\|_{H_2^1}^2}{\|\varphi\|_{L_{\alpha+1}}^2}$$

for any $\varphi \in C_*^\infty(M)$. The embedding Theorem 3.3 gives the continuous embedding $H_2^1(M) \subset L_{\alpha+1}(M)$ with $\alpha \leq \alpha^*$, and $\|\varphi\|_{L_{\alpha+1}} \leq C_{\alpha+1} \|\varphi\|_{H_2^1}$. Thus

$$\frac{\|\varphi\|_{H_2^1}^2}{\|\varphi\|_{L_{\alpha+1}}^2} \geq \frac{1}{C_{\alpha+1}} \quad \text{giving} \quad I_\alpha(\varphi) \geq \frac{C_1}{C_{\alpha+1}^2}.$$

Since the space $C_*^\infty(M)$ is dense in $H_2^1(M)$, it gives the result.   □

Here is the "easy" version of the Yamabe theorem. To prove it we follow the course of the corresponding result for closed manifolds, see [4, Theorem 5.5].

Theorem 5.3 *Let M be a compact closed manifold with a metric g and a conical singularity as above. Let $\Lambda > 0$. For any $\alpha \in [1, \alpha^*)$ there exists a function $u_\alpha \geq 0$ minimizing the functional I_α, so that*

$$\int_M u_\alpha^{\alpha+1} dV_g = 1$$

giving $I_\alpha(u_\alpha) = \min \left\{ I_\alpha(\varphi) \mid \varphi \in H_2^1(M), \ \varphi \neq 0 \right\}.$

Denote this value $Q_\alpha = I_\alpha(M, [g]) = I_\alpha(u_\alpha)$. The function u_α is a weak (in $H_2^1(M)$) solution of the equation:

$$-\Delta u_\alpha + \frac{n-2}{4(n-1)}R_g u_\alpha = Q_\alpha u_\alpha^\alpha.$$

5.1. Proof of Theorem 5.3. (a) First we prove that the functional I_α is bounded, and thus Q_α is finite. Let $q = 1 + \alpha$, $2 \le q < 1 + \alpha^*$. The conformal Laplacian $\mathbb{L}_g$ is positive by Theorem 4.2 (Recall that $\Lambda > 0$). Thus $I_\alpha(\varphi) \ge 0$ for any $\varphi \in H_2^1(M, g)$, therefore $Q_\alpha \ge 0$ (moreover, $Q_\alpha > \frac{C_1}{C_{\alpha+1}^2}$, see Proposition 5.2). On the other hand,

$$Q_\alpha \le I_\alpha(1) \quad = \quad \frac{n-2}{4(n-1)}\int_M R_g d\sigma_g$$

$$= \quad \frac{n-2}{4(n-1)}\int_{M\backslash K} R_g d\sigma_g + \frac{n-2}{4(n-1)}\int_K R_g d\sigma_g.$$

The first integral on the right is bounded since R_g is continuous in $M\backslash\{x^*\}$. The second integral is bounded since $n > 2$ and

$$R_r \sim \frac{C}{\ell^2} \quad \text{as} \ \ \ell \to 0, \quad \text{and} \quad \int_0^{\varepsilon_0} \ell^{n-3} d\ell < \infty.$$

(a$'$) Now we have that $\mathrm{Vol}_g(M) < \infty$, and $1 \in L_s(M)$ for any $s \ge 2$. Thus we use the inequality

$$\int_M fg d\sigma_g \le \left(\int_M f^s d\sigma_g\right)^{1/s}\left(\int_M g^{s'} d\sigma_g\right)^{1/s'}$$

(which holds for positive functions $f \in L_s(M)$, $g \in L'_s(M)$, with $\frac{1}{s} + \frac{1}{s'} = 1$, $s, s' > 0$). We apply this for $f = \varphi^q$, $g = 1$, $s = \frac{q}{2}$, $s' = \frac{q}{q-2} > 0$. Thus we get

$$\int_M \varphi^2 \cdot 1 d\sigma_g \le \left(\int_M (\varphi^2)^{\frac{q}{2}} d\sigma_g\right)^{\frac{2}{q}} \cdot \left(\int_M 1^{\frac{q}{q-2}} d\sigma_g\right)^{\frac{q-2}{q}}, \quad \text{or}$$

$$\|\varphi\|_{L_2(M)}^2 \le \|\varphi\|_{L_q(M)}^2 \cdot (\mathrm{Vol}_g(M))^{\frac{q-2}{q}}, \quad \text{so}$$

$$\|\varphi\|_{L_2(M)} \le \|\varphi\|_{L_q(M)} \cdot (\mathrm{Vol}_g(M))^{\frac{q-2}{2q}}.$$

(b) Now let $\{\varphi_i\}$ be a minimizing sequence such that

$$\int_M \varphi_i^q d\sigma_g = 1, \quad \varphi_i \in H_2^1(M, g), \quad \text{and} \quad \lim_{i \to \infty} I_\alpha(\varphi_i) = Q_\alpha.$$

First we prove that the set $\{\varphi_i\}$ is bounded in $H_2^1(M, g)$. We have

$$
\begin{aligned}
\|\varphi_i\|_{H_2^1(M)} &= \|\nabla\varphi_i\|_{L_2(M)}^2 + \|\chi\varphi_i\|_{L_2(M)}^2 \\
&= I_\alpha(\varphi_i) - \frac{n-2}{4(n-1)}\int_M R_g \varphi_i^2 d\sigma_g + \int_M \chi^2 \varphi_i^2 d\sigma_g.
\end{aligned}
\tag{19}
$$

Since $\{\varphi_i\}$ is a minimizing sequence, we can assume that $I_\alpha(\varphi_i) \leq Q_\alpha + 1$.

Now we consider the case when $R_g > 0$ everywhere (i.e. $\Lambda > 0$). Then we have that $R_g = \frac{\Lambda}{\ell^2}$ on the cone K, thus $\chi^2(x) < CR_g(x)$ for some positive constant C and any $x \in M$. The sum of the first two terms in (19) coincides with $\|\nabla\varphi_i\|_{L_2(M)}^2$, so it is positive. Therefore

$$\|\varphi_i\|_{H_2^1(M)} \leq A \cdot I_\alpha(\varphi_i) - A \cdot \frac{n-2}{4(n-1)}\int_M R_g \varphi_i^2 d\sigma_g + C\int_M R_g \varphi_i^2 d\sigma_g$$

for any $A \geq 1$. We choose A large enough, so that

$$C - A\frac{n-2}{4(n-1)} < 0,$$

to get the estimate

$$\|\varphi_i\|_{H_2^1(M)} \leq A \cdot I_\alpha(\varphi_i) \leq A \cdot (Q_\alpha + 1).$$

Notice that on $M \setminus K$ both integrals are estimated by the norm $\|\varphi\|_{L_2(M)}$.

(c) We follow the proof of **(c)** in [4, Theorem 5.5] to find a subsequence $\{\varphi_j\}$ of $\{\varphi_i\}$ and a nonnegative function $u_\alpha \in H_2^1(M)$ such that

(α) $\varphi_j \to u_\alpha$ in $L_{\alpha+1}(M)$;

(β) $\varphi_j \to u_\alpha$ weakly in $H_2^1(M)$;

(γ) $\varphi_j \to u_\alpha$ almost everywhere.

One may satisfy (γ) since $L_{\alpha+1}(M) \subset L_2(M)$ continuously and any sequence converging in $L_2(M)$ has a subsequence that converges almost everywhere. To satisfy (β) one uses that the embeddings $H_1^2(M) \subset L_{\alpha+1}(M) \subset L_2(M) \subset H_2^1(M)$ are continuous. Then $u_\alpha \in H_1^1(M)$ because of weak compactness in reflexible Banach spaces, see [32, Chapter V, Section 2]. Finally [32, Chapter V, Section 1] gives

$$\|u_\alpha\|_{H_1^1} \leq \lim_{j\to\infty} = \inf \|\varphi_j\|_{H_1^1}.$$

This proves **(c)**.

(d) Here we prove that u_α is a weak solution of the Yamabe equation. It means that for all $\varphi \in H_2^1(M)$

$$\int_M (\nabla u_\alpha) \cdot (\nabla \varphi) d\sigma_g + \frac{n-2}{4(n-1)} \int_M R_g u_\alpha \varphi d\sigma_g = Q_\alpha \int_M u_\alpha^\alpha \varphi d\sigma_g.$$

The proof is literally the same as in [4, Theorem 5.5]. The only difference is the use of the space $C_*^\infty(M)$ instead of $C_0^\infty(M)$. This ends the proof of theorem 5.3 for $\lambda > 0$. $\qquad\square$

Corollary 5.4 *The solution $\varphi_\alpha \neq 0$ on M_*.*

Remark. We will see later, in Secion 7, that there are cases where the minimizer (Yamabe solution) belongs to the space $H_2^1(M)$ but not to $H_2^2(M)$. This is very different from the case of compact manifolds (cf. [3]).

Remark. The case $\Lambda \leq 0$ is also very interesting. One can prove the result similar to Theorem 5.3 under the same restrictions as in Proposition 4.7.

6   Asymptotic of solutions: the linear case

In this section we study the asymptotic behavior of solutions of the linear equation $\mathbb{L}_g u = Q_1 u$ near the point x_*. We use the polar coordinates (ℓ, θ, φ) on the conical part K. Then the equation $\mathbb{L}_g u = Q_1 u$ has the form

$$\left(\frac{\partial}{\partial \ell^2} + \frac{n-1}{\ell} \frac{\partial}{\partial \ell} + \frac{2}{r_p^2 \ell^2} \Delta_\theta + \frac{2}{r_q^2 \ell^2} \Delta_\psi \right) u + \left[-\frac{n-2}{4(n-1)} \frac{\Lambda}{\ell^2} + Q_1 \right] u = 0. \quad (20)$$

First, we recall some basic information on the Laplacian operator on spheres ([11]). Let $\left\{ \lambda_j^p, \chi_j^p \right\}$ be the spectrum of the Laplacian $\Delta_{S^p} = -\Delta_\theta$, and,

respectively, $\{\lambda_i^q, \chi_i^q\}$ of $\Delta_{S^q} = -\Delta_\psi$. It is well-known ([11]) that

$$\lambda_0^p = 0, \quad \lambda_1^p = \cdots \lambda_{p+1}^p = p, \quad \lambda_{p+2}^p = 2(p+1), \dots,$$

$$\lambda_0^q = 0, \quad \lambda_1^q = \cdots \lambda_{q+1}^q = q, \quad \lambda_{q+2}^q = 2(q+1), \dots.$$

Any L^2-function on S^p (correspondingly on S^q) decomposes into Fourier series with respect to the orthonormal basis (χ_i^p) (correspondingly (χ_j^q)). On the cone K we have

$$u(\ell, \theta, \psi) = \sum_{ij} u_{ij}(\ell) \chi_i^p(\theta) \chi_j^q(\psi). \tag{21}$$

We decompose u as in (21) to obtain the following system of equations for the coefficient functions $u_{ij}(\ell)$ on the half-line $\ell \geq 0$:

$$\frac{\partial^2 u_{ij}}{\partial \ell^2} + \frac{n-1}{\ell}\frac{\partial u_{ij}}{\partial \ell} + \left[Q_1 - \left(\frac{n-2}{4(n-1)}\Lambda + \frac{2}{r_p^2}\lambda_i^p + \frac{2}{r_q^2}\lambda_j^q \right)\frac{1}{\ell^2} \right] u_{ij} = 0, \quad \text{or}$$

$$\frac{\partial^2 u_{ij}}{\partial \ell^2} + \frac{n-1}{\ell}\frac{\partial u_{ij}}{\partial \ell} + \left[Q_1 - \frac{K_{ij}}{\ell^2} \right] u_{ij} = 0. \tag{22}$$

$$\text{with} \quad K_{ij} = \frac{n-2}{4(n-1)}\Lambda + \frac{2}{r_p^2}\lambda_i^p + \frac{2}{r_q^2}\lambda_j^q.$$

The equations (22) are known as *degenerate hypergeometric* or *Whitteker equations* (see [8, Vol. 1, Chapter 6]). Such an equation can be reduced, via an appropriate substitution, to the Bessel equations with the pure imaginary parameter ν. Their solutions can be explicitly written in terms of the corresponding Bessel functions.

Here we are interested in asymptotic behavior of solutions as $\ell \to 0$. Thus, we are looking for solutions in the form of power series

$$u_{ij} = \ell^{\nu_{ij}} \sum_{k=0}^{\infty} a_k \ell^k.$$

We have the first and the second derivatives:

$$u'_{ij} = \nu_{ij}\ell^{\nu_{ij}-1}\sum_{k=0}^{\infty}a_k\ell^k + \ell^{\nu_{ij}}\sum_{k=0}^{\infty}(k+1)a_{k+1}\ell^k,$$

$$u''_{ij} = \nu_{ij}(\nu_{ij}-1)\ell^{\nu_{ij}-2}\sum_{k=0}^{\infty}a_k\ell^k + 2\nu_{ij}\ell^{\nu_{ij}-1}\sum_{k=0}^{\infty}(k+1)a_{k+1}$$

$$= \ell^k + \ell^{\nu_{ij}}\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}\ell^k.$$

We collect the coefficients for the different powers of ℓ in the equation (22):

$$\ell^{\nu_{ij}-2} \quad : \quad \nu_{ij}(\nu_{ij}-1)a_0 + (n-1)\nu_{ij}a_0 - K_{ij}a_0 = 0,$$

$$\ell^{\nu_{ij}-1} \quad : \quad \nu_{ij}(\nu_{ij}-1)a_1 + 2\nu_{ij}a_1 + (n-1)(\nu_{ij}+1)a_1 - K_{ij}a_1 = 0.$$

For the (general) coefficient of $\ell^{\nu_{ij}+m}$ we obtain the following equation:

$$\ell^{\nu_{ij}+m} \quad : \quad \nu_{ij}(\nu_{ij}-1)a_{m+2} + 2\nu_{ij}(m+2)a_{m+2} + (m+2)(m+1)a_{m+2}$$

$$+(n-1)(\nu_{ij}+m+2)a_{m+2} + Q_1a_m - K_{ij}a_{m+2} = 0.$$

Thus we get the recursive equation for the coefficients a_m which we denote by (Y_{m+2}):

$$[(\nu_{ij}+m+2)^2 + (n-2)(\nu_{ij}+m+2) - K_{ij}]a_{m+2} = -Q_1a_m \quad (Y_{m+2})$$

Denote by $K_{\nu_{ij}+m}$ the left side of the previous equation. We consider the equation (Y_0):

$$(\nu_{ij}^2 + (n-2)\nu_{ij} - K_{ij})a_0 = 0. \qquad (Y_0)$$

Here we have either $a_0 = 0$ or

$$\nu_{ij}^{(e,\pm)} = -\frac{n-2}{2} \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + K_{ij}}.$$

The second equation (Y_1) is as follows:

$$(\nu_{ij}^2 + n\nu_{ij} + (n-1-K_{ij}))a_1 = 0. \qquad (Y_1)$$

Here we have either $a_1 = 0$ or

$$\nu_{ij}^{(o,\pm)} = -\frac{n}{2} \pm \sqrt{\left(\frac{n}{2}\right)^2 + K_{ij} - (n-1)}.$$

Notice now that $\nu_{ij}^{e\pm} = \nu_{ij}^{o\pm} + 1$ and, more then this, $K_{\nu_{ij}^{e\pm}+m+1} = K_{\nu_{ij}^{e\pm}+m}$. Comparing the power series solution $u_{ij}^{e\pm}(\ell)$ with even indices m and $u_{ij}^{o\pm}(\ell)$ with odd indices m we see that $u_{ij}^{e\pm}(\ell) = u_{ij}^{o\pm}(\ell)$. Thus it is enough to study the solution $u_{ij}^{e\pm}(\ell)$ only. We have for these solutions:

$$
\begin{aligned}
\frac{n}{2} + \nu_{ij}^{e\pm} &= 1 \pm \sqrt{\left(\frac{n-2}{2}\right)^2 + K_{ij}} \\[2mm]
&= 1 \pm \frac{n-2}{2} \sqrt{1 + \frac{\Lambda}{(n-1)(n-2)} + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right)} \\[2mm]
&= 1 \pm \frac{n-2}{2} \sqrt{\mu^2 + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right)}.
\end{aligned}
$$

Here

$$\mu^2 = 1 + \frac{\Lambda}{(n-1)(n-2)} = \frac{2}{(n-1)(n-2)}\left[\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2}\right],$$

see Appendix.

From this we conclude that the solution u_{ij}^{e-} with the leading term $\ell^{\nu_{ij}^{e-}}$ never belongs to $H_2^1(K)$ but belongs to $L_2(K)$ if (and only if)

$$\frac{n-2}{2}\sqrt{\mu^2 + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right)} < 1,$$

which is equivalent to

$$\mu^2 + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right) < \frac{4}{(n-2)^2}.$$

For $i = j = 0$ (radial solution) this condition, after substitution of value for μ^2, takes the form

$$\left[\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2}\right] < \frac{2(n-1)}{n-2}.$$

These conditions on n, p, r_p, r_q are met in the nonlinear case as well; we call this situation the *"minus-case"*. Thus, in the minus-case, the solution u_{00}^{e-} belongs to $L_2(K)$. On the other hand, the solution u_{ij}^{e+} always belongs to $H_2^1(K)$ (see Proposition 3.7 and Appendix). It belongs to $H_2^2(K)$ (classical solution) if and only if

$$\frac{n-2}{2}\sqrt{\mu^2 + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right)} > 1, \quad \text{that is if}$$

$$\mu^2 + \frac{8}{(n-2)^2}\left(\frac{\lambda_i^p}{r_p^2} + \frac{\lambda_j^q}{r_q^2}\right) > \frac{4}{(n-2)^2}.$$

For $i = j = 0$ (the radial solution) this condition, after substitution of μ^2, takes the form

$$\left[\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2}\right] > \frac{2(n-1)}{n-2}.$$

We will work with the same condition on n, p, r_p, r_q in the nonlinear case; we call this situation the *"plus-case"*. Thus, in the plus-case, the solution u_{00}^{e+} belongs to $H_2^2(K)$.

Now we return to the recursive equation (Y_{m+2}) for the coefficients a_m in the case ν_{ij}^{e+}. We write it in the form:

$$K_{\nu_{ij}^{e+}+m}a_{m+2} = -Q_1 a_m, \quad \text{which gives} \quad a_{m+2} = \frac{-Q_1 a_m}{K_{\nu_{ij}^{e+}+m}}$$

provided the denominator is not zero.

Notice that, provided equation (Y_0) is satisfied, the expression given above for $K_{\nu_{ij}^{e+}+m}$ can be rewritten as

$$K_{\nu_{ij}^{e+}+m} = (m+2)(2\nu_{ij} + m + n).$$

It follows from this formula that these coefficients are always nonzero. We use the previous formula for a_{m+2} recursively to obtain

$$a_{2m} = \frac{(-Q_1)^m}{\prod_{t=1}^m K_{\nu_{ij}^{e+}+2t}}a_0 = \frac{(-Q_1)^m}{\prod_{t=1}^m (2t+2)(2\nu_{ij} + 2t + n)}a_0,$$

and for the solution u_{ij}^{e+}, which we redenote to be u_{ij}, we have

$$u_{ij} = a_{ij} \ell^{\nu_{ij}} \sum_{m=0}^{\infty} \frac{(-Q_1)^m}{\prod_{t=1}^{m}(2t+2)(2\nu_{ij}+2t+n)} \ell^{2m}$$

with arbitrary constants $a_{ij} \in R$.

We combine all calculations in the following theorem.

Theorem 6.1 *Let M be a manifold with tame conical singularity as above, with $\dim M \geq 5$, and $Q_1 > 0$.*

(1) *There exists a solution u_{ij} of the equation (22), restricted on the cone K, is given by*

$$u_{ij} = \ell^{\nu_{ij}} \cdot \left(\sum_{m=0}^{\infty} \frac{(-1)^m Q_1^m \ell^{2m}}{\prod_{t=1}^{m}(2t+2)(2\nu_{ij}+2t+n)} \right) \cdot a_0, \quad \text{with}$$

$$\nu_{ij} = \frac{n-2}{2} \left[\sqrt{ \mu^2 + \frac{4}{(n-2)^2} \left(\frac{2\lambda_i^p}{r_p^2 + \frac{2\lambda_j^q}{r_q^2}} \right) } - 1 \right], \quad \text{where} \quad (23)$$

$$\mu^2 = \frac{2}{(n-1)(n-2)} \left[\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} \right].$$

This solution belongs to the Sobolev space $H_2^1(M)$ and also to the space $H_2^2(M)$ in the plus-case (see (30)).

(2) *The second linearly independent solution (denoted above as $u_{ij}^{e;-}$) of the equation (22) only belongs to $L_2(K)$ in the minus-case (see (31)).*

(3) *The general solution $u(\ell, \theta, \psi)$ of (22) in $H_2^1(K)$ has the form*

$$u(\ell, \theta, \psi) = \sum_{ij} a_{ij} \ell^{\nu_{ij}^{e+}} f_{ij}(\ell) \kappa_i^p(\theta) \kappa_j^q(\psi).$$

Here the functions $f_{ij}(x)$ are defined by

$$f_{ij}(\sqrt{Q_1}\ell) = \sum_{m=0}^{\infty} \frac{(-Q_1)^m}{\prod_{t=1}^{m}(2t+2)(2\nu_{ij}+2t+n)} \ell^{2m},$$

so that $u_{ij}^{e+} = a_{ij} \ell^{\nu_{ij}^{e+}} f_{ij}(\ell)$ and the coefficients a_{ij} ensure convergence of this series with respect to $H_2^1(K)$-norm.

We notice that the exponent ν_{ij} of the solution u_{ij} does not depend on the eigenvalue Q_1.

Remark. We notice that the radial solution $u_{00}^{e+}(\ell)$ of the equation (20) has the following asymptotic

$$u_{00}(\ell) \sim \ell^{\nu_{00}}, \quad \nu_{00} = \frac{n-2}{2}\left(\sqrt{1+\frac{\Lambda}{(n-1)(n-2)}}-1\right) = \frac{n-2}{2}(\mu-1),$$

and that for any solution $u(\ell, \theta, \psi)$ of (20), there exists the radial solution $u_0(\ell)$ of (20) (radial part of u) such that as $\ell \to 0$,

$$|u(\ell, \theta, \psi) - u_0(\ell)| \leqq C\ell^{\sigma}, \quad \text{for some } \sigma > 0.$$

7 Asymptotic of solutions: the nonlinear case

In this section we study the nonlinear Yamabe equation near the singular point:

$$\frac{\partial^2 u}{\partial \ell^2} + \frac{n-1}{\ell}\frac{\partial u}{\partial \ell} + \frac{2}{r_p^2 \ell^2}\Delta_\theta u + \frac{2}{r_q^2 \ell^2}\Delta_\psi u - \frac{n-2}{4(n-1)}\frac{\Lambda}{\ell^2}u + Q_\alpha u^\alpha = 0 \quad (24)$$

defined on the open set $(0, \varepsilon) \times S^p \times S^q$. Here $3 \leq p \leq n-3$, $\Lambda = \Lambda(p, q, r_p, r_q)$ is the curvature factor, see Appendix for the details.

We restrict our attention to the *radial solutions*, and return to the general case at the end of this section. Thus, we study *positive* solutions $u = u(\ell)$ of the equation (24).

7.1. Reduction to a dynamical system. We use the cylindrical coordinates $t = -\ln \ell$, or $\ell = e^{-t}$, so that $t \longrightarrow \infty$ as $\ell \longrightarrow 0$. Then the equation (24) becomes

$$u_{tt} - (n-2)u_t - \frac{n-2}{4(n-1)}\Lambda u + Q_\alpha e^{-2t}u^\alpha = 0. \quad (25)$$

We are looking for solutions of (25) defined on the set

$$(-\ln(\varepsilon), +\infty) \times S^p \times S^q.$$

We use the substitution $u(t) = e^{\lambda t}w(t)$ in (25), where $\lambda = \frac{2}{\alpha-1}$. Then the time-dependence of the coefficients in (25) disappears, and one obtains the following equation:

$$w'' + \left(\frac{4}{\alpha-1} - (n-2)\right)w' + \left(\frac{4}{(\alpha-1)^2} - \frac{2(n-2)}{\alpha-1} - \frac{n-2}{4(n-1)}\Lambda\right)w + Q_\alpha w^\alpha = 0. \quad (26)$$

We exclude the case $\alpha = 1$. Notice that the functions $u(\ell)$ and $w(t)$ are related as follows: $u(\ell) = \ell^{-\frac{2}{\alpha-1}} w(-\ln\ell)$. We denote

$$\bar{b} = -\frac{4}{\alpha-1} + (n-2), \quad \bar{a} = -\frac{4}{(\alpha-1)^2} + \frac{2(n-2)}{\alpha-1} + \frac{n-2}{4(n-1)}\Lambda. \quad (27)$$

Let $x = w$, $y = w'$. Then (26) is equivalent to the dynamical system

$$\begin{cases} x' &= y \\ y' &= \bar{a}x + \bar{b}y - Q_\alpha x^\alpha. \end{cases} \quad (28)$$

7.2. The equilibrium points. We find the equilibrium points of (28) by solving the system:

$$\begin{cases} y = 0, \\ \bar{a}x - Q_\alpha x^\alpha = x(\bar{a} - Q_\alpha x^{\alpha-1}) = 0. \end{cases}$$

Since $y = 0$, the second equation has the solution $x_1 = 0$ for all values of the parameters and, in addition, the solution

$$x_2 = \left(\frac{\bar{a}}{Q_\alpha}\right)^{\frac{1}{\alpha-1}} > 0 \quad (29)$$

provided $\frac{\bar{a}}{Q_\alpha} > 0$. Thus the system (28) has one equilibrium point $w_1 = (0,0)$ if $\frac{\bar{a}}{Q_\alpha} \le 0$, and an additional one, $w_2 = (x_2, 0)$ with x_2 given by (29) if $\frac{\bar{a}}{Q_\alpha} > 0$.

Remark. Notice that we consider only positive values for the root x_2 since we are looking for the positive solutions of (26). In terms of the dynamical system (28) this means that a solution has to stay in the right half-plane for $t > T$ for some T.

7.3. The equilibrium point w_1. To study a behavior of (28) near $(0,0)$, we analyze its linear approximation:

$$A_{(0,0)} = \begin{pmatrix} 0 & 1 \\ \bar{a} & \bar{b} \end{pmatrix}.$$

The parameters $\bar{b}$ and $\bar{a}$ are defined in (27). Thus, we have the characteristic equation

$$\lambda^2 - \bar{b}\lambda - \bar{a} = 0, \quad \text{with the roots} \quad \lambda_\pm = \frac{\bar{b}}{2} \pm \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}.$$

The corresponding eigenvectors $\vec{v}_\pm$ are given by

$$\vec{v}_\pm = \begin{pmatrix} 1 \\ \lambda_\pm \end{pmatrix}, \quad \text{so that} \quad A_{(0,0)}\vec{v}_\pm = \lambda_\pm \vec{v}_\pm.$$

We show (see Appendix, Claim 8.1) that $\bar{b} < 0$ for all $1 < \alpha \le \alpha^*$, and that $\frac{\bar{b}^2}{4} + \bar{a} > 0$ if $n \ge 3$. Therefore, the point $(0,0)$ is a *saddle point* if $\bar{a} > 0$, and a *stable focus* if $\bar{a} < 0$.

7.4. The equilibrium point w_2. Now we study the second equilibrium point w_2:

$$w_2 = (x_2, 0), \quad \text{with} \quad x_2 = \left(\frac{\bar{a}}{Q_\alpha} \right)^{\frac{1}{\alpha-1}},$$

which exists provided $\bar{a}Q_\alpha > 0$. Here we have the following linear approximation of (28) near the point w_2:

$$A_{w_2} = \begin{pmatrix} 0 & 1 \\ \bar{a} - \alpha Q_\alpha \frac{\bar{a}}{Q_\alpha} & \bar{b} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \bar{a}(1-\alpha) & \bar{b} \end{pmatrix}$$

with the characteristic equation $r^2 - \bar{b}r + \bar{a}(\alpha-1) = 0$. We have the following eigenvalues $r_\pm$ and the eigenvectors $\vec{z}_\pm$:

$$r_\pm = \frac{\bar{b}}{2} \pm \sqrt{\frac{\bar{b}^2}{4} - \bar{a}(\alpha-1)}, \quad \vec{z}_\pm = \begin{pmatrix} 1 \\ r_\pm \end{pmatrix}, \quad \text{so that} \quad A_{(x_2,0)}\vec{z}_\pm = r_\pm \vec{z}_\pm.$$

Thus, we have the following alternatives for the equilibrium point w_2.

1. If $\bar{a} < 0$, then w_2 is a saddle.

2. If $\bar{a} > 0$, but $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) < 0$, then the w_2 is a stable focus.

3. If $\bar{a} > 0$, and $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) > 0$, then w_2 is a stable node.

It is shown in Appendix that all three cases are realized for different values of α.

7.5. The phase pictures. Now we study the critical points of the system (28) for different values of parameters α, Q_α, and others. We determine asymptotic behavior of the solutions of system (28) by comparing them with the corresponding solutions of the linearized system at the points w_1, w_2. For all $\alpha < \alpha^*$ the points w_1, w_2 are hyperbolic. Thus, locally

(near critical points) linear and nonlinear phase pictures are trajectory-equivalent. Moreover, those solutions $w(t)$ of the nonlinear system which go to w_2 as $t \to +\infty$ have the asymptotic behavior $w(t) \simeq w_2$, and this determines to which Sobolev spaces they belong. For the unbounded solutions $(w(t) \to \infty)$, one has $x(t) \to \infty$. Thus, the corresponding asymptotic behavior of $u(\ell)$ is worse than that of an α-basic function. This allows us to decide, in most cases, to which Sobolev space $H_2^k(M)$ those solutions belong.

Finally, there are solutions $w(t)$ that tend to $w_1 = (0,0)$ as $t \to +\infty$. Locally (near w_1), the nonlinear system may be thought of as a perturbation of the linear one. We use results by Lettenmeyer, Hartman and Wintner (see [10, Ch. 4, Theorems 5,9], and [13, Ch. X, Theorem 13.1, Corollary 16.3]). We check below that in our case the conditions of those theorems are met. These results guarantee that the principal term of asymptotic behavior is the same for a solution tending to the origin for the system (28) and its linearization.

The results presented below depend on the inequalities below. We will also describe these cases as the *"plus-case"* and the *"minus-case"* respectively:

The plus-case: $$\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} > \frac{2(n-1)}{n-2}. \tag{30}$$

The minus-case: $$\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} < \frac{2(n-1)}{n-2}. \tag{31}$$

We have the following alternative cases.

Case 1: $\bar{a} < 0$, $Q_\alpha < 0$. In this case the phase picture is given in Fig. 7.1. Here we have two families and three separate solutions:

(1) The family C_∞ consists of the solutions $w(t)$ going to ∞ asymptotically as $t \to +\infty$ nestling at the unstable separatrix trajectory of the point w_2. We notice that for given α the corresponding solution $u(\ell)$ approaches $+\infty$ faster, compared to an α-basic function $\ell^{-\frac{2}{\alpha-1}}$ as $\ell \to 0$.

As it is proved in the Appendix, in the plus-case (30), $\alpha_0 < \frac{n+4}{n}$ and, since $\bar{a} < 0$, $\alpha < \alpha_0$. Thus, an α-basic function does not belong to $L_2(K)$. Therefore, the solutions from the family C_∞ do not belong to $L_2(K)$.

In the minus case (31) if $\alpha < \frac{n+4}{n}$, the same argument leads to the same conclusion, i.e. that the solutions from the family C_∞ do not belong

B. Botvinnik & S. Preston, Conformal Laplacian and Conical Singularities   40

to $L_2(K)$. But if $\frac{n+4}{n} < \alpha < \alpha_0$, then an α-basic function belongs to $L_2(K)$ and we do not know whether the solutions from the family C_∞ belong to $L_2(K)$.
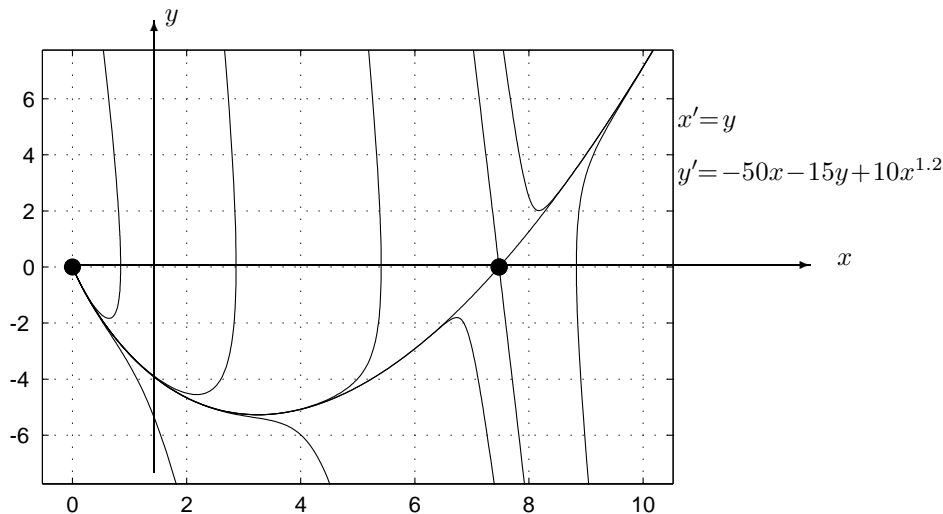

Fig. 7.1. The phase picture for the case $\bar{a} < 0$, $Q_\alpha < 0$, here $s = 1/4$.

(2) The family C_0. Each solution of the family $w(t) \in C_0$ goes to $w_1 = (0,0)$ as $t \to +\infty$, *nestling at the direction of the eigenvector* v_+. Notice that all but one (denoted by w_s, see below) of the admissible solutions tending to w_1 belong to the family C_0. The solutions $w(t)$ and the corresponding solutions $u(\ell)$ have the following asymptotic behavior:

$$w(t) \sim e^{\lambda_+ t} = e^{\left(\frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)t}, \ t \to \infty \ \ u(\ell) \sim \ell^{-\frac{2}{\alpha-1} - \left(\frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)}, \ \ell \to 0.$$

To analyze the behavior of solutions from the family C_0, we use Lemma 3.6. We see that $u(\ell) \sim \ell^q$ with $q = -\frac{2}{\alpha-1} - (\frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}})$.

Recall that $u(\ell) \in H_2^k(M)$ if and only if $k < \frac{n}{2} + q$. Since $\bar{b} = (n-2) - \frac{4}{\alpha-1}$, we have $\frac{2}{\alpha-1} + \frac{\bar{b}}{2} = \frac{n-2}{2}$; therefore, in this case

$$\tfrac{n}{2} + q = \tfrac{n}{2} - \tfrac{n-2}{2} - \sqrt{\tfrac{\bar{b}^2}{4} + \bar{a}} = 1 - \sqrt{\tfrac{\bar{b}^2}{4} + \bar{a}} = 1 - \tfrac{n-2}{2}\mu, \ \ \text{where}$$

$$\mu = \sqrt{1 + \tfrac{\Lambda}{(n-1)(n-2)}}.$$

As it is proved in the Appendix, such a solution never belongs to $H_2^1(M)$. It belongs to $L_2(M)$ only in the minus case (31).

(3) Two incoming separatrix trajectories of the saddle point w_2. For both these trajectories, $w_{sep}(t) \to w_2$. Thus, the corresponding solution $u_{sep}(\ell)$ is exactly an α-basic function near x^*. If in this case $\alpha < \alpha_0 < \frac{n+4}{n}$ (in the plus-case (30), see the Appendix), the solutions $u_{sep}(\ell)$ do not belong to $L_2(M)$ (recall that here $\alpha < \alpha_0$ since $\bar{a} < 0$).

(4) The solution $w_s(t) \to w_1 = (0,0)$, corresponding to the eigenvector v_-. This solution, obtained by a C^1-diffeomorphic twist of the corresponding solution of the linearization of (29) at w_1, has the asymptotic behavior $w_s(t) \sim e^{\lambda_- t}$. The corresponding solution $u_s(\ell) \sim l^q$, $q = -\frac{2}{\alpha-1} - \lambda_-$, where

$$q = -\frac{2}{\alpha-1} - \frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} = -\frac{n-2}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}},$$

see above. Therefore,

$$\frac{n}{2} + q = 1 + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} = 1 + \frac{n-2}{2}\mu.$$

As it is shown in the Appendix, the solution $u_s(t)$ always belongs to $H_2^1(M)$. Furthermore, $u_s(\ell) \in H_2^2(M)$ if and only if the plus case (30) condition is met.

Case 2: $\bar{a} < 0$, $Q_\alpha > 0$. In this case the phase picture is given in Fig. 7.2. Here we have one family C_0 of admissible solutions and a special solution $u_s(\ell)$. Solutions of the family C_0 tend to $w_1 = (0,0)$ as $t \to +\infty$. Such solutions $w(t)$ asymptotically nestle in the direction of the eigenvector v_+.

The asymptotic behavior of $w(t)$ and of the corresponding solutions $u(\ell)$ is the following:

$$w(t) \sim e^{\lambda_+ t} = e^{\left(\frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4}+\bar{a}}\right)t}, \ t \to \infty, \quad u(\ell) \sim \ell^{-\frac{2}{\alpha-1}-\left(\frac{\bar{b}}{2}+\sqrt{\frac{\bar{b}^2}{4}+\bar{a}}\right)}, \ \ell \to 0.$$

The same argument as in Case 1 proves that solutions from the family C_0 belong to $L_2(M)$ only in the minus-case (31) and that none of these solutions belong to $H_2^1(M)$. Similar to Case 1, the solution $u_s(\ell)$ always belongs to $H_2^1(M)$ and belongs to $H_2^2(M)$ if and only if the plus-case condition (30)
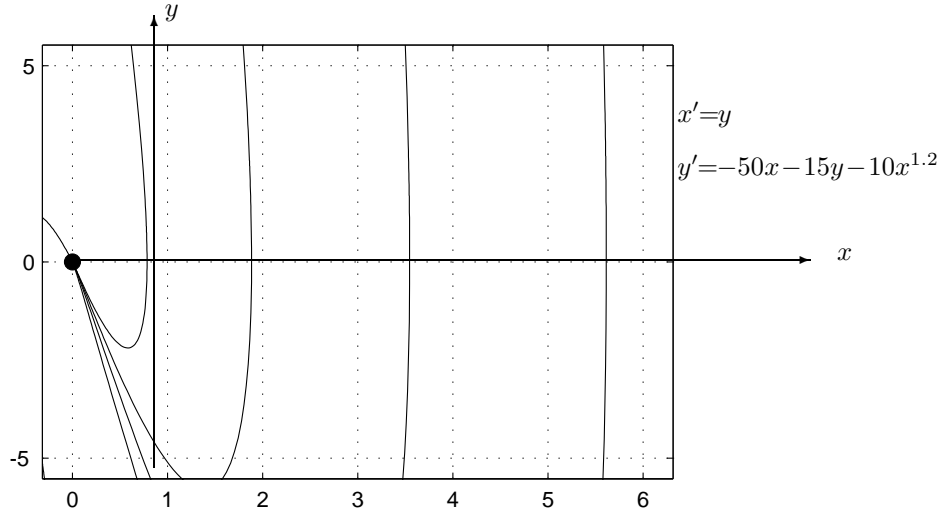
is met.

Fig. 7.2. The phase picture for the case $\bar{a} < 0$, $Q_\alpha > 0$, here $s = 1/4$.

Case 3: $\bar{a} > 0$, $Q_\alpha > 0$, $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) < 0$. In this case the phase picture is given in Fig. 7.3.
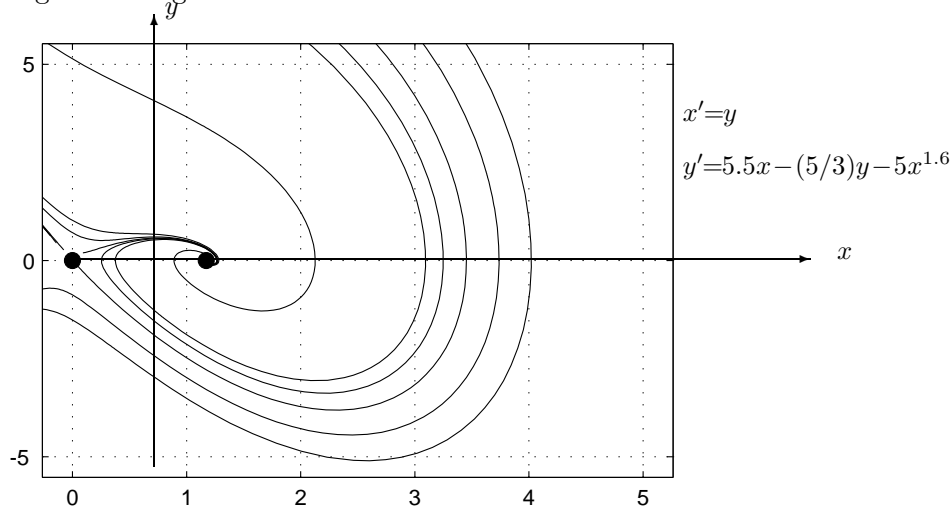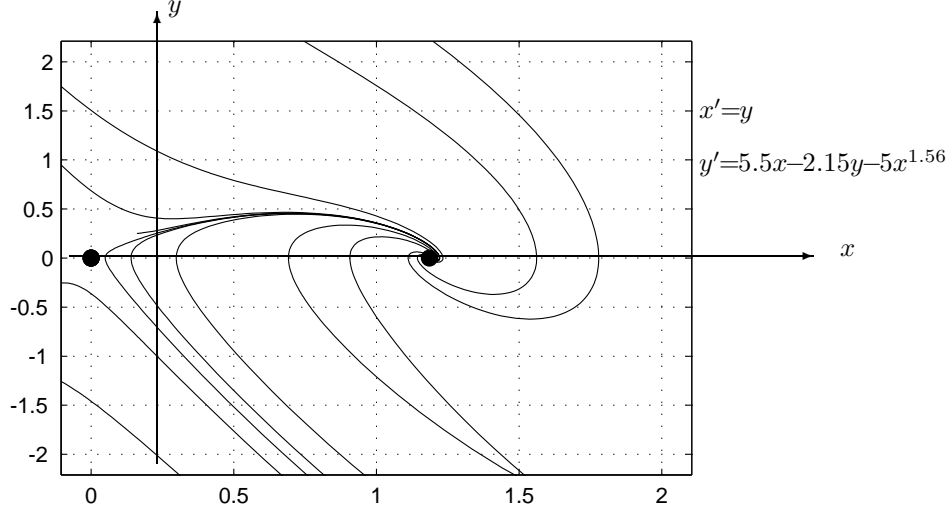

Fig. 7.3. The phase picture for the case $\bar{a} > 0$, $Q_\alpha > 0$,
$\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) < 0$, here $s = 1/4$.

Case 3': $\bar{a} > 0$, $Q_\alpha > 0$, $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) > 0$. We have the following phase

given at Fig. 7.3′.

Fig. 7.3′. The phase picture for the case $\bar{a} > 0$, $Q_\alpha > 0$, $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) > 0$.

We analyze the cases 3, 3′ together since they have very similar classes of admissible solutions.

Here we have the class C_F of Fowler solutions (see [8]): Here $w(t) \to w_2$ as $t \to +\infty$, and

$$u(\ell) \sim \ell^{-\frac{2}{\alpha-1}}, \quad \ell \to 0.$$

Also we have a separatrix solution $w_s(t)$ that approaches $w_1 = (0,0)$ as $t \to +\infty$. This solution asymptotically nestles at the eigendirection v_-. It has the following asymptotic:

$$w_s(t) \sim e^{\lambda_- t} = e^{\left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)t}, \ t \to +\infty, \quad u_s(\ell) \sim \ell^{-\frac{2}{\alpha-1} - \left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)}, \ \ell \to 0.$$

Remark. In fact, the solution $u_s(\ell)$ gives a minimum for the corresponding Yamabe functional. This asymptotic behavior refines the general result on the Yamabe minimizer for cylindrical manifolds for the particular "slice" $S^p \times S^q$, see [1].

The Fowler solutions in both cases 3 and 3′ do not belong to $H_2^1(M)$, by Proposition 3.7. Moreover, they even do not belong to $L_2(M)$ for $\alpha_0 < \alpha \leq \frac{n+4}{n}$ (the minus-case (31)); however, they belong to $L_2(M)$ for $\frac{n+4}{n} < \alpha$.

The separatrix solution $u_s(\ell)$ has the asymptotical behavior ℓ^q as $\ell \to$

$+\infty$, with $q = -\frac{2}{\alpha-1} - \left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)$. Using $\frac{2}{\alpha-1} + \frac{\bar{b}}{2} = \frac{n-2}{2}$, we obtain

$$q = -\frac{n-2}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} = -\frac{n-2}{2} + \frac{n-2}{2}\mu.$$

Thus, $\frac{n}{2} + q = 1 + \frac{n-2}{2}\mu$. Similarly to the Case 1, solution $u_s(\ell)$ always belongs to $H_2^1(M)$ but belongs to $H_2^2(M)$ if and only if the plus-case condition (30) is met.
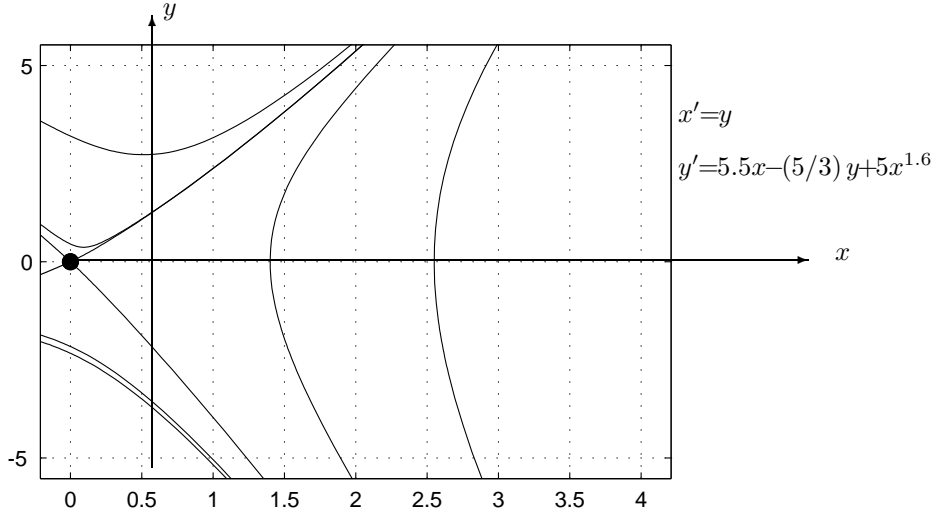

Fig. 7.4. The phase picture for the case $\bar{a} > 0$, $Q_\alpha < 0$, here $s = 3/4$.

Case 4: $\bar{a} > 0$, $Q_\alpha < 0$. Here the phase picture is given in Fig. 7.4. In this case we have the family C_∞ of admissible solutions $w(t)$, such that $w(t) \to \infty$ as $t \to \infty$. The corresponding solution $u(\ell)$ goes to ∞ faster then the α-*basic function* $\ell^{-\frac{2}{\alpha-1}}$ for a given α. Thus, these solutions do not belong to $H_2^1(M)$.

Here, $\alpha > \alpha_0$, and in the plus case (30), or in the minus case (31) when $\alpha > \frac{n+4}{n}$, an α-basic function belongs to $L_2(K)$. But, we do not know if solutions from C_∞ belong to this space. On the other hand, in the minus-case (31), if $\alpha < \frac{n+4}{n}$, such solutions do not belong to $L_2(K)$.

In addition, we have the separatrix solution $w_s(t)$ that tends to $w_1 = (0,0)$ as $t \to +\infty$. This solution asymptotically nestles in the direction of the eigenvector v_-. The asymptotic of $w_s(t)$ and $u_s(\ell)$ is the following:

$$w_s(t) \sim e^{\lambda_- t} = e^{\left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)t}, \ t \to \infty, \quad u_s(\ell) \sim \ell^{-\frac{2}{\alpha-1} - \left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)}, \ \ell \to 0.$$

Thus, the separatrix solution $u_s(\ell)$ has asymptotic ℓ^q with

$$q = -\frac{2}{\alpha - 1} - \left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right) = -\frac{n-2}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}},$$

and $\frac{n}{2} + q = 1 + \frac{n-2}{2}\mu$. Similarly to the Case 1, solution $u_s(\ell)$ always belongs to $H_2^1(M)$ but belongs to $H_2^2(M)$ if and only if the plus-case condition (30) is met.

Now we consider the special cases that were previously left outside of the scope.

Case 5. $Q_\alpha = 0$. In this case, we have a linear system with the matrix

$$A = \begin{pmatrix} 0 & 1 \\ \bar{a} & \bar{b} \end{pmatrix}.$$

Thus, the solutions can be written out explicitly. Depending on the sign of $\bar{a}$, we have either a stable node (if $\bar{a} < 0$, Case 5−, Fig. 7.5−) or a saddle point (if $\bar{a} > 0$, Case 5+, Fig. 7.5+). Correspondingly, we have linear versions of cases 2 and 4. Conclusions that were made in these cases regarding asymptotics of admissible solutions are true in this case ($Q_\alpha = 0$) as well.

In the Case 5+, where $\bar{a} > 0$, the point w_1 is a saddle. A generic admissible solution from the family C_∞ is growing faster than an α-basic function. More specifically, we have $w(t) \sim e^{\lambda_+ t}$ as $t \to +\infty$. Thus, the corresponding solution $u(\ell)$ has the asymptotic:

$$u(\ell) \sim \ell^{-\frac{2}{\alpha-1} - \lambda_+}, \quad \ell \to 0.$$

We use Lemma 3.6: we have $q = -\frac{2}{\alpha-1} - \left(\frac{\bar{b}}{2} + \sqrt{\frac{\bar{b}^2}{4} + \bar{a}}\right)$. The same argument as before gives that $\frac{n}{2} + q = 1 - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} = 1 - \frac{2}{n-2}\mu$. Thus, this solution never belongs to $H_2^1(K)$, and it does belong to $L_2(K)$ if and only if the
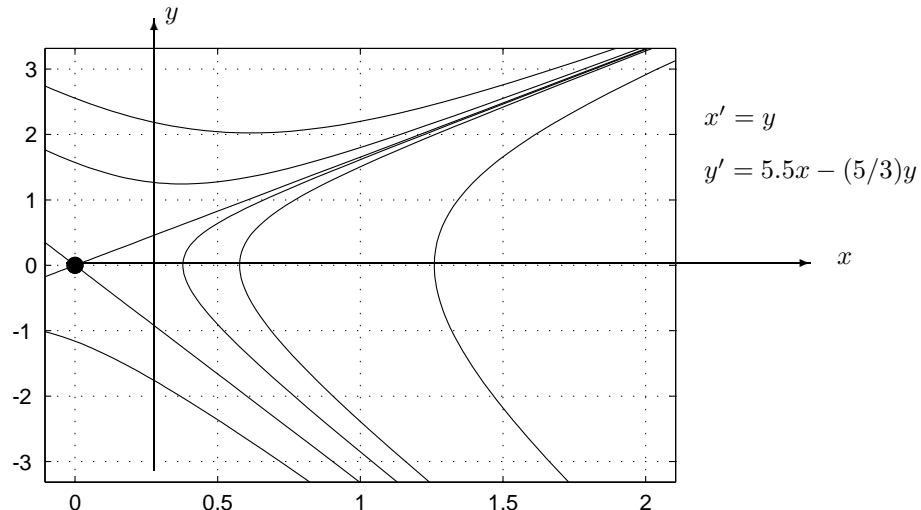
minus-case condition (31) is met.

$$x' = y$$

$$y' = 5.5x - (5/3)y$$

Fig. 7.5+. The phase picture for the case $\bar{a} > 0$, $Q_\alpha = 0$, here $s = 3/4$.
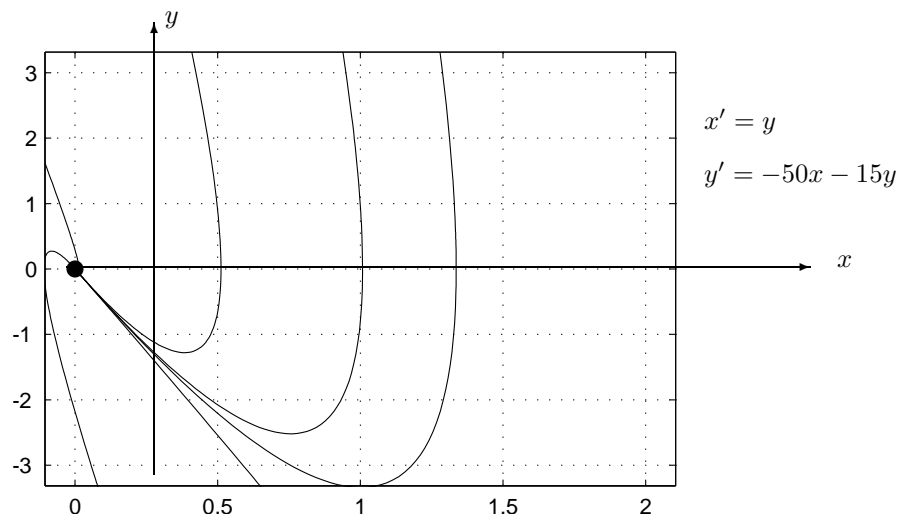

$$x' = y$$

$$y' = -50x - 15y$$

Fig. 7.5−. The phase picture for the case $\bar{a} < 0$, $Q_\alpha = 0$, here $s = 1/4$.

We also have the stable separatrix solution $u_s(\ell)$, which has the asymptotic ℓ^q with

$$q = -\frac{2}{\alpha - 1} - \left(\frac{\bar{b}}{2} - \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} \right).$$

Similarly to the above cases, we conclude that $u_s(\ell) \in H^1_2(M)$, and it also

belongs to $H_2^2(M)$ if and only if the plus-case condition (30) is met.

In the Case 5− (where $\bar{a} < 0$) we have the linear version of the Case 2. Here, the generic (slow) admissible solutions from the family C_0, which are nestling in the eigendirection v_+, never belong to $H_2^2(M)$ and belong to $L_2(M)$ if and only if the minus-case condition (31) is met.

The fast-decreasing solution $u_s(t) = \ell^{-\frac{2}{\alpha-1}-\lambda_-}$, corresponding to the smaller eigenvalue, has the asymptotic $\sim \ell^{1+\sqrt{\frac{\bar{b}^2}{4}+\bar{a}}} = \ell^{1+\frac{2}{n-2}\mu}$. Thus, according to the discussion in the Appendix, it always belongs to $H_2^1(M)$ and belongs to $H_2^2(M)$ if and only if the plus-case condition (30) is met.

Case 6: $\bar{a} = 0$. In this case, (when $\alpha = \alpha_0$, see Appendix) w_1 is the only singular point of the system (28). This point is degenerate: $\lambda_- = \bar{b}_0 = -(n-2)\mu$, $\lambda_+ = 0$. Depending on the sign of Q_α we have the cases 6+ or 6−.

Case 6+: $(Q_\alpha > 0$, a weak stable node). In this case the phase picture is the following:
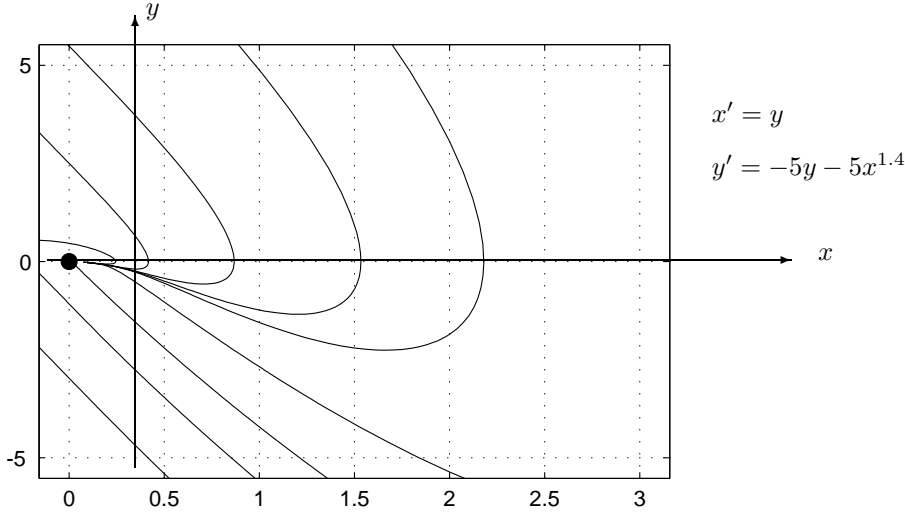

$$x' = y$$
$$y' = -5y - 5x^{1.4}$$

Fig. 7.6+. The phase picture for $\bar{a} = 0$, $Q_\alpha > 0$, here $s = 1/2$.

This case is similar to Case 2. The difference is in the fact that $\lambda_+ = 0$. As a result, the solutions of the family C_0 tend to $w_1 = (0,0)$ slowly and, similar to the Case 2, do not belong to $L_2(M)$.

The fast solution $w_s(t)$ corresponds to the second eigenvalue $\lambda_- =$

$\bar{b}$, where the corresponding eigenfunction $u_s(\ell)$ behaves asymptotically as $u_s(\ell) \sim l^q$, with $q = -\frac{2}{\alpha_0-1} - \bar{b}$. Thus, we have: $\frac{n}{2} + q = \frac{n}{2} - \frac{2}{\alpha_0-1} - \bar{b} = 1 - \frac{\bar{b}}{2} = 2 - \frac{n}{2} + \frac{2}{\alpha_0-1} = 1 + \frac{n-2}{2}\mu$. Therefore, $u_s(\ell) \in H_2^1(M)$ and, in addition, it belongs to $H_2^2(M)$ if the plus-case condition (30) is met.

Case 6-: ($Q_\alpha < 0$, weak saddle). In this case the phase picture is given at Fig. 7.6-. The Case 6− is similar to the Case 4. We have the family of solutions C_∞ and a separatrix solution $w_s(t)$.

The family C_∞ consists of solutions $u(\ell)$ which grow faster than the basic solutions $\ell^{-\frac{2}{\alpha_0-1}}$. Thus, these solutions do not belong to $L_2(M)$ if the value α_0 (for which $\bar{a} = 0$) is smaller then the critical value $\frac{n+4}{n}$ from Proposition 3.7, that is, in the minus-case (31). In the opposite case, we do not know if solutions from the family C_∞ belong to $L_2(K)$ or not.

On the other hand, separatrix solution $u_s(\ell)$ has the asymptotic ℓ^q with $q = -\frac{2}{\alpha-1} - \bar{b}$. Similarly to the above consideration we conclude that separatrix solution $u_s(\ell) \in H_2^1(M)$ and it belongs to $H_2^2(K)$ if the plus-case condition (30) is met.
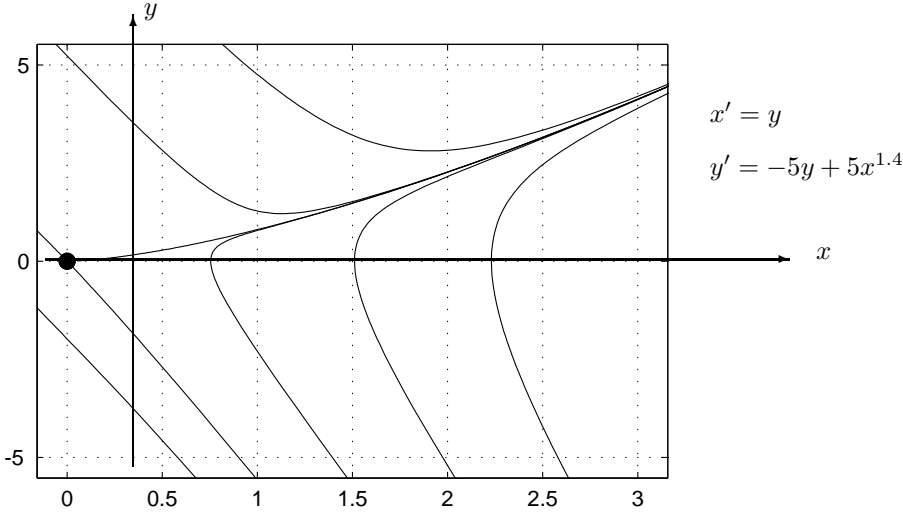

$$x' = y$$
$$y' = -5y + 5x^{1.4}$$

Fig. 7.6-. The phase picture for $\bar{a} = 0$, $Q_\alpha < 0$, here $s = 1/2$.

Case 7. The critical: $\alpha = \alpha^* = \frac{n+2}{n-2}$. In this case $\alpha - 1 = \frac{4}{n-2}$, $s = 1$. We have here $\bar{b} = 0$, and $\bar{a}(1) = \bar{a}^* = \frac{(n-2)^2}{4}(1 + \frac{\Lambda}{(n-2)(n-1)}) = \frac{(n-2)^2}{4}\mu^2$.

Therefore (see Appendix) $\bar{a}^* > 0$. Our system takes the form

$$\begin{cases} x' &= y \\ y' &= \bar{a}^* x - Q_{\alpha^*} x^{\alpha^*}. \end{cases} \tag{32}$$

We have the corresponding equation of the second order

$$w'' - \bar{a}^* w + Q_{\alpha^*} w^{\alpha^*} = 0.$$

This equation has a first integral (a Hamiltonian function of system (32)):

$$I(x,y) = \frac{y^2}{2} - \frac{\bar{a}^* x^2}{2} + \frac{Q_{\alpha^*}}{\alpha^* + 1} x^{\alpha^* + 1}$$

For the singular point $w_1 = (0,0)$ we have $\lambda_{\pm} = \pm\sqrt{\bar{a}^*}$, thus w_1 is saddle point.

Case 7−: $Q_{\alpha^*} < 0$. In this case w_1 is the only singular point. The phase picture is given at Fig. 7.7-.
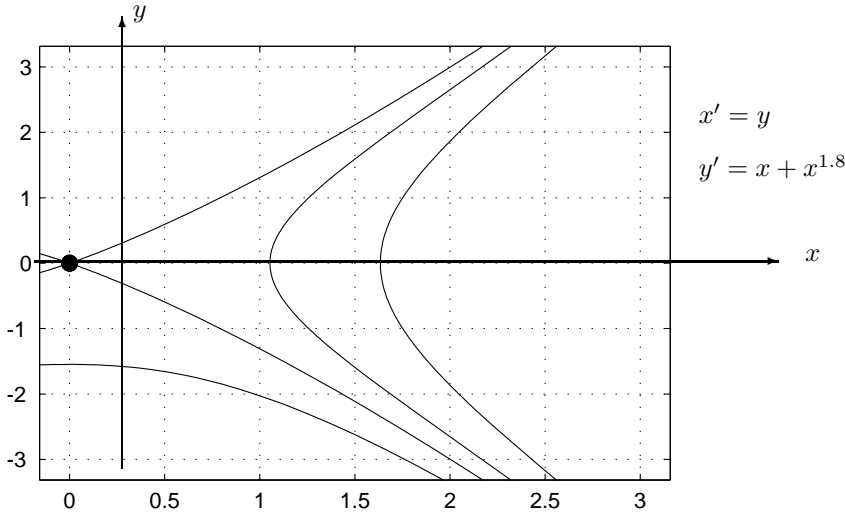

$x' = y$

$y' = x + x^{1.8}$

Fig. 7.7-. The phase picture for the case $\alpha = \alpha^*$, $Q_{\alpha^*} < 0$.

Here we have the family C_∞ of admissible solutions and the incoming separatrix $w_s(t)$. All solutions of the family C_∞ go to infinity as $t \to +\infty$. For such solutions $w(t) \to +\infty$ as $t \to +\infty$, and $u(\ell)$ goes to $+\infty$ faster than the basic function $\ell^{-\frac{2}{\alpha^* - 1}}$ as $\ell \to 0$. Thus, none of the solutions $u(\ell) \in C_\infty$ belong to $H_2^1(M)$. Furthermore, since $\alpha^* > \frac{n+4}{4}$, an α^*-basic function does

belong to $L_2(K)$, and in order to determine whether $u(\ell)$ belongs to $L_2(M)$, we have to study its asymptotic behavior in more detail.

The separatrix solution $w_s(t)$ that tends to $w_1 = (0,0)$ as $t \to +\infty$. This solution asymptotically nestles in the direction of the eigenvector v_-. We have the asymptotic:

$$w_s(t) \sim e^{\lambda_- t} = e^{-\sqrt{\bar{a}^*} t}, \quad t \to \infty, \quad u_s(\ell) \sim \ell^{-\frac{n-2}{2} + \sqrt{\bar{a}^*}} \quad \ell \to 0.$$

For this solution $\frac{n}{2} + q = 1 + \frac{n-2}{2}\mu$. Thus, this solution always belongs to $H_2^1(M)$ and does belong to $H_2^2(M)$ if the plus-case condition (30) is fulfilled.

Case 7+: $Q_{\alpha^*} > 0$. If $Q_{\alpha^*} > 0$, we have the second singular point $w_2 = x_2 = ((\frac{\bar{a}^*}{Q_{\alpha^*}})^{\frac{n-2}{4}}, 0)$. The eigenvalues of the linearization at this point are

$$r_\pm = \pm\sqrt{-\frac{4\bar{a}^*}{n-2}}.$$

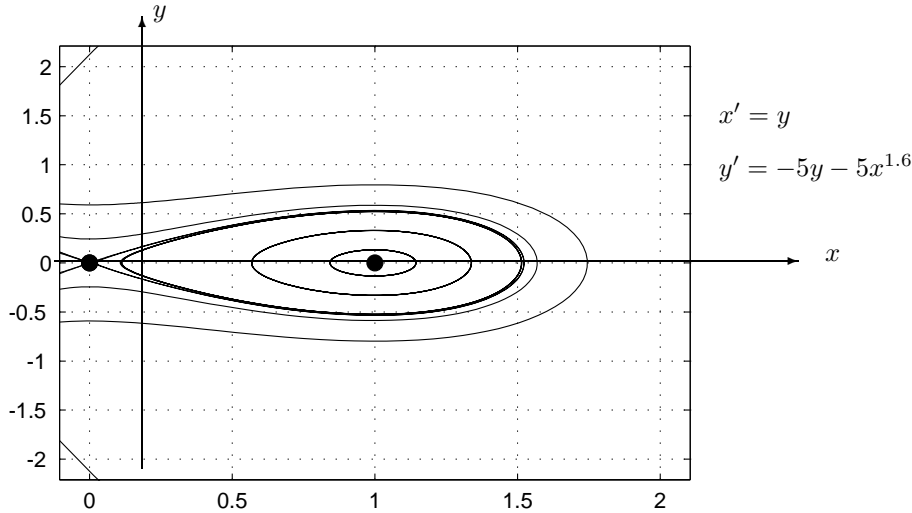Thus, w_2 is a center, see Fig. 7.7+ for the phase picture.

Fig. 7.7+. The phase picture for the case $\alpha = \alpha^*$, $Q_{\alpha^*} > 0$.

Thus, inside the homoclinic loop we have a family of periodic solutions. These solutions are known as Fowler or Delaunay solutions (see [15]). Notice that the value of the first integral I on the separatrix loop is zero, and, at w_2,

$$I(w_2) = -\frac{Q_{\alpha^*}}{n}\left(\frac{\bar{a}^*}{Q_{\alpha^*}}\right)^{\frac{n}{2}} < 0.$$

Therefore, a Fowler solution is determined by the value of the integral I in the interval $(I(w_2), 0)$, or by the minimal value $w_{\min}$ of $w(t) = (x(t), y(t))$ along its trajectory. The integral $I(x, y)$ takes values in $[I(w_2), 0]$. Such a solution is also determined by the minimum value $x_{\min}$ of $x(t)$, $x_{\min} \in (0, (\frac{\bar{a}^*}{Q_{\alpha^*}})^{\frac{n-2}{4}})$. For such a trajectory $x_{\min} \le x(t) \le x_{\max}$ for all t. As a result, for the corresponding function $u(\ell)$ we have

$$x_{\min}\ell^{-\frac{n-2}{2}} \le u(\ell) \le x_{\max}\ell^{-\frac{n-2}{2}}.$$

Therefore, the Fowler solutions have the asymptotic $\ell^{-\frac{n-2}{2}}$ as $\ell \to 0$ and (see Proposition 3.7) do belong to $L_2(M)$, but not to $H_2^1(M)$.

In addition, we have the incoming separatrix solution $w_s(t)$ that tends to $w_1 = (0, 0)$ as $t \to +\infty$. This solution asymptotically nestles in the direction of the eigenvector v_-. We have the asymptotic

$$w_s(t) \sim e^{\lambda_- t} = e^{-\sqrt{\bar{a}^*}t}, \quad t \to \infty \quad u_s(\ell) \sim \ell^{-\frac{n-2}{2}+\sqrt{\bar{a}^*}}, \quad \ell \to 0.$$

The separatrix solution $u_s(\ell) \sim \ell^q$ with $q = -\frac{n-2}{2} + \sqrt{\bar{a}^*} = \frac{n-2}{2}(\mu - 1)$. Thus, $\frac{n}{2} + q = 1 + \frac{n-2}{2}\mu$, and this solution always belongs to $H_2^1(M)$ and belongs to $H_2^2(M)$ if and only if the plus-case condition (30) is fulfilled.

We denote

$$\sigma = \frac{n-2}{2}(\mu - 1), \quad \mu^2 = \frac{2}{(n-1)(n-2)}\left[\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2}\right]. \quad (33)$$

7.6. Perturbation near the point w_1**.** Now we consider the system (28) as a perturbation of its linearization near w_1. We will apply [10, Ch. 4, Theorems 5 and 9] or [13, Ch. X, Corollary 16.3] to those solutions, which tend to zero as $t \to \infty$. We claim that two following conditions for the nonlinear term $Q_\alpha x^\alpha$ (with $\alpha > 1$) are satisfied.

(1) Indeed, the perturbation $f(t, x)$ must satisfy

$$|f(t, x) \le L|x|^{1+\rho}$$

for $t \ge t_0, |x| \le \delta$, where L, δ, ρ are positive constants. In our case, it is enough to take $\rho = \alpha - 1$.

(2) The Lipschitz condition for $f(t, x)$ is satisfied since x^α is differentiable for $x \ge 0$ and its derivative $\alpha x^{\alpha-1}$ can be made arbitrarily small for $|x| < \delta$ with small enough δ.

Thus [10, Ch. 4, Theorems 5 and 9] imply that there is one-to-one correspondence between those solutions of the system (28) that tend to zero as $t \to +\infty$ and of the corresponding solution of the linearization of (28). This shows that the norms $|w(t)|$ of corresponding solutions have the same asymptotical behavior as $t \to +\infty$. Using more refined results of (Hartman, Theorem 13.1 and Corollary 16.2) we may conclude the same about the asymptotic behavior of solutions $w(t)$ themselves, rather then their norms).

We have proved the following results.

Theorem 7.1 *Let M be a manifold with tame conical singularities as above, $n \geq 5$, $K \subset M$ be its conical part, and $1 \leq \alpha \leq \alpha^*$.*

(1) *Then the equation (25) has unique radial solution $u_s(\ell)$ which belongs to the space $H_2^1(K)$. Moreover, $u_s(\ell) \sim \ell^{-\sigma}$, where $\sigma = \frac{n-2}{2}(\mu - 1)$ (see (33)), and this solution is classical, i.e. belongs to the Sobolev space $H_2^2(M)$ if and only if the condition*

$$\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} > \frac{2(n-1)}{n-2}$$

is fulfilled.

(2) *If $\alpha \leq \frac{n+4}{4} < \alpha^*$, the equation (25) does not have other radial solutions in $L_2(K)$.*

(3) *For $Q_\alpha > 0$, $\alpha > \frac{n+4}{4}$, there exists a family of solutions C_F (Fowler solutions) which belong to the space $L_2(M)$, but not to $H_2^1(M)$.*

We denote, as above, $Q_\alpha = \inf\limits_{\varphi \in H_2^1(M),\ \varphi \neq 0} I_\alpha(\varphi)$.

Theorem 7.2 *Let M be a manifold with tame conical singularity as above.*

(1) *If $1 < \alpha < \alpha^*$, then a minimizing function $u_\alpha(\ell)$ exists, belongs to the space $H_2^1(M)$ and to the space $H_2^2(M)$ if the plus-case condition (30) is met. Asymptotically, $u_\alpha(\ell) \sim \ell^{-\sigma}$ near x_*, where $\sigma = \frac{n-2}{2}(\mu - 1)$.*

(2) *If $\alpha = \alpha^*$, then a minimizing function $u_\alpha(\ell)$ (existing in $H_2^1(M)$) belongs to $H_2^1(M)$, and, if the plus-case condition (30) is met, to the space $H_2^2(M)$ and $u_\alpha(\ell) \sim \ell^{-\sigma}$ near x_*.*

Remark. Notice that the asymptotic behavior of the solution $u_s(\ell) \in H_2^1(M)$ **is the same for all** α, $1 \leqq \alpha \leqq \alpha^*$, including the linear case $\alpha = 1$, compare Section 6.

7.7. Nonradial solutions. Theorem 7.2, describes the asymptotic behavior of those radial solutions of (24) which belong to the Sobolev space $H_2^2(M)$. It is important to determine whether general (non-radial) solutions have better or worse asymptotics then that of $u_s(\ell)$.

Similar questions have been studied extensively for the solutions of the Yamabe equation on S^n with a finite number of singularities (in our terms, when $p = n$, $\alpha = \alpha^*$, see [15] for the most recent results and references to earlier works). It is shown in [15] that any nonradial solution asymptotically behaves as a shift (by t) of uniquely defined radial solutions of the same equation. There is clear evidence that the same holds in our case.

However, for our purposes we do not need such a result in full strength. We restrict our attention to the case when a solution tends to zero as $t \to +\infty$. Then a modification of the proof of [10, Ch. 4,Theorem 5] may be done, so that it will work in our case. That gives the following result:

Theorem 7.3 *Let* $u(\ell, \theta, \psi)$ *be a solution of* $(51)_\alpha$ *such that*

$$\|u\|_{L_2(S^p \times S^q)}(\ell) \to 0$$

as $\ell \to 0$. *Then there exists a radial solution* $u_0(\ell)$ *of (24) with the same property such that*

$$\|u(\ell, \theta, \psi) - u_0(\ell)\| = o(\|u_0(\ell)\|) \qquad as \ \ell \to 0.$$

As a result, the nonradial solutions of (24) that tend to zero, approaching the singular point x_* have, for all $\alpha > 1$, the same asymptotic behavior as corresponding radial solutions (which also approach zero).

Recall that the original metric g on the cone is given by $g_K = d\ell^2 + \frac{r_p^2 \ell^2}{2} d^2\theta + \frac{r_q^2 \ell^2}{2} d^2\psi$. According to [1], there exists a Yamabe metric $\check{g}$ on M.

Corollary 7.4 *Let* (M, g) *be a manifold with tame conical singularity as above, and let* $\check{g} \in [g]$ *be a Yamabe metric on* M. *Then*

$$\check{g} \sim \ell^{-\sigma} \left(d\ell^2 + \frac{r_p^2 \ell^2}{2} d^2\theta + \frac{r_q^2 \ell^2}{2} d^2 \right)$$

near the singular point x_*.

8   Appendix

Here we collect some necessary computations, most of which are very simple.

8.1. Function $\Lambda = \Lambda(p)$. Recall that in the appropriate coordinates the scalar curvature on the cone $C(S^p \times S^q)$ is given as

$$R_{g_K}(\theta, \psi, \ell) = \tfrac{\Lambda}{\ell^2}, \quad \text{with} \quad \Lambda = p(p-1)\tfrac{2-r_p^2}{r_p^2} + q(q-1)\tfrac{2-r_q^2}{r_q^2} - 2pq.$$

We let n be fixed and study Λ as function of p only, $\Lambda = \Lambda(p)$.

Substituting $q = n - p - 1$, we transform Λ to the expression

$$
\begin{aligned}
\Lambda(p) \ &= \ -(n-1)(n-2) + 2\left[\tfrac{p(p-1)}{r_p^2} + \tfrac{q(q-1)}{r_q^2}\right] \\[2mm]
&= \ p(p-1)\tfrac{2-r_p^2}{r_p^2} + q(q-1)\tfrac{2-r_q^2}{r_q^2} - 2pq \\[2mm]
&= \ 2p^2\left(r_p^{-2} + r_q^{-2}\right) - 2p\left[\left(r_p^{-2} + r_q^{-2}\right) + (n-2)\tfrac{2}{r_q^2}\right] \\[2mm]
&\qquad\qquad\qquad + (n-1)(n-2)\left(\tfrac{2}{r_q^2} - 1\right).
\end{aligned}
$$

To get the first equality we split $\tfrac{2-r_p^2}{r_p^2} = \tfrac{2}{r_p^2} - 1$ and use the fact that $p(p-1) + q(q-1) - 2pq = (n-1)(n-2)$.

8.2. Minimal value of $\Lambda(p)$. To find the minimum of function $\Lambda(p)$ for $1 < p < n - 1$ we calculate

$$\frac{d\Lambda}{dp} = 4p\left(r_p^{-2} + r_q^{-2}\right) - 2\left[\left(r_p^{-2} + r_q^{-2}\right) + (n-2)\tfrac{2}{r_q^2}\right],$$

and find

$$p_{\min} = \frac{\left(r_p^{-2} + r_q^{-2}\right) + (n-2)\tfrac{2}{r+q^2}}{2\left(r_p^{-2} + r_q^{-2}\right)} = \frac{1}{2} + \frac{n-2}{1 + \tfrac{r_q^2}{r_p^2}}.$$

Correspondingly, q takes the minimal value

$$q_{\min} = \frac{1}{2} + \frac{n-2}{1 + \tfrac{r_p^2}{r_q^2}}.$$

Notice that for all possible values of $r_p > 0$, $r_q > 0$, we have

$$\frac{1}{2} < p_{\min} < n - \frac{3}{2}.$$

For the value of $\Lambda(p_{\min})$ we have

$$
\begin{aligned}
\Lambda(p_{\min}) &= -(n-1)(n-2) + \frac{2}{r_p^2}\left(\frac{1}{2} + \frac{n-2}{1+\frac{r_q^2}{r_p^2}}\right)\left(\frac{-1}{2} + \frac{n-2}{1+\frac{r_q^2}{r_p^2}}\right) \\[2mm]
&\quad + \frac{2}{r_q^2}\left(\frac{1}{2} + \frac{n-2}{1+\frac{r_p^2}{r_q^2}}\right)\left(\frac{-1}{2} + \frac{n-2}{1+\frac{r_p^2}{r_q^2}}\right) \\[2mm]
&= -(n-1)(n-2) - \frac{1}{2}(r_p^{-2} + r_q^{-2}) + \frac{2(n-2)^2}{r_p^2+r_q^2}.
\end{aligned}
$$

In Section 7.5 we use the parameter μ^2. We have:

$$
\begin{aligned}
\mu^2 &= 1 + \frac{\Lambda}{(n-1)(n-2)} = \\[2mm]
&= 1 + \frac{p(p-1)\frac{2-r_p^2}{r_p^2} + q(q-1)\frac{2-r_q^2}{r_q^2} - 2pq}{(n-1)(n-2)} = \\[2mm]
&= \frac{p(p-1)\frac{2}{r_p^2} + q(q-1)\frac{2}{r_q^2}}{(n-1)(n-2)} > 0
\end{aligned}
$$

if $p, q > 0$. Here we have used $p(p-1) + q(q-1) + 2pq = (n-1)(n-2)$.

The maximal value is achieved at the end point of the interval $[3, n-3]$ of the admissible values of p. We have for $p = 1$,

$$\Lambda(1) = -(n-1)(n-2) + (n-2)(n-3)\frac{2}{r_q^2},$$

$$\Lambda(n-2) = -(n-1)(n-2) + (n-2)(n-3)\frac{2}{r_p^2}.$$

Correspondingly, maximal value of $\Lambda(p)$ is achieved at one of these ends.

8.2. Parameters $\bar{a}$, $\bar{b}$ in (28). In order to analyze the "phase portraits" of the system (28) for different values of Q_α and Λ, we need particular information on the parameters $\bar{a}$, $\bar{b}$. Introduce the parameter $s = \frac{(\alpha-1)(n-2)}{4}$. Then it is easy to see that $0 < s \leq 1$ since $1 < \alpha \leq \alpha^* = \frac{n+2}{n-2}$.

Claim 8.1 *For $n \geq 3$, $\bar{b} < 0$.*

Indeed, for $n \geq 3$, we have

$$\bar{b} = (n-2) - \frac{4}{\alpha - 1} = (n-2)\left(1 - \frac{4}{(\alpha - 1)(n - 2)}\right) = -\frac{(n-2)(s-1)}{s} < 0.$$

Now we study dependence of $\bar{a}$ on s and Λ. We have

$$
\begin{aligned}
\bar{a} &= \frac{(n-2)}{4(n-1)}\Lambda + \frac{2(n-2)}{\alpha - 1} - \frac{4}{(\alpha - 1)^2} \\[2mm]
&= \frac{(n-2)}{4}\cdot\frac{\Lambda}{n-1} + \frac{4(n-2)^2}{2(\alpha-1)(n-2)} - \frac{4\cdot 4(n-2)^2}{4(\alpha-1)^2(n-2)^2} \\[2mm]
&= \frac{(n-2)}{4}\cdot\frac{\Lambda}{n-1} + \frac{(n-2)^2}{2s} - \frac{(n-2)^2}{4s^2} \\[2mm]
&= \frac{(n-2)^2}{4}\left[\frac{\Lambda}{(n-1)(n-2)} + \frac{2}{s} - \frac{1}{s^2}\right].
\end{aligned}
$$

To analyze the equilibrium point $(0,0)$, we need to know the sign of expression $\frac{\bar{b}^2}{4} + \bar{a}$. We have (using expressions for $\bar{b}$ an $\bar{a}$ obtained above):

$$
\begin{aligned}
\frac{\bar{b}^2}{4} + \bar{a} &= \frac{(n-2)^2(1-s)^2}{4s^2} + \frac{(n-2)^2}{4}\left[\frac{\Lambda}{(n-1)(n-2)} + \frac{2}{s} - \frac{1}{s^2}\right] \\[2mm]
&= \frac{(n-2)^2}{4}\left[1 + \frac{\Lambda}{(n-1)(n-2)}\right] = \frac{(n-2)^2}{4}\mu^2 \geq 0.
\end{aligned}
$$

Thus, we have that $\bar{a}(s) \to -\infty$ as $s \to 0$, and

$$\bar{a}(1) = \frac{(n-2)^2}{4}\left[\frac{\Lambda}{(n-1)(n-2)} + 1\right] = \frac{(n-2)^2}{4}\mu^2 > 0$$

for all Λ. Also, we have

$$\frac{d\bar{a}}{ds} = \frac{(n-2)^2}{4}\left[-\frac{2}{s^2} + \frac{2}{s^3}\right] = \frac{(n-2)^2}{2s^3}[1 - s]$$
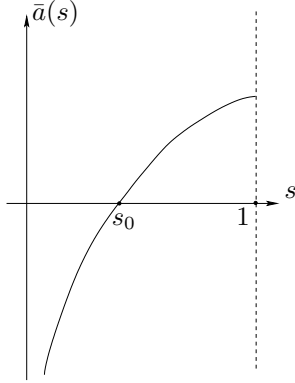
which is positive for all $s < 1$, and zero for $s = 1$.

Thus a graph of the function $\bar{a}(s)$ has a form given at Fig. 8.1. The point s_0 (when $\bar{a}(s_0) = 0$ is a root of the equation

$$\frac{\Lambda}{(n-1)(n-2)}s^2 + 2s - 1 = 0.$$

One has the following positive root:

$$s_0 = \frac{1}{1 + \sqrt{1 + \frac{\Lambda}{(n-1)(n-2)}}} = \frac{1}{1 + \mu}$$

Fig. 8.1.

for all Λ.

In terms of the parameter α, $\bar{a}$ changes sign from negative to positive if $\alpha = \alpha_0$, with

$$\alpha_0 = 1 + \frac{4}{(n-2)\left(1 + \sqrt{1 + \frac{\Lambda}{(n-1)(n-2)}}\right)} = 1 + \frac{4}{(n-2)(1+\mu)}. \qquad (34)$$

Claim 8.2 *The parameter $\bar{a} < 0$ if $\alpha \in (1, \alpha_0)$, $\bar{a} > 0$ if $\alpha \in (\alpha_0, \alpha^*)$, and $\bar{a} = 0$ if $\alpha = \alpha_0$, where α_0 is given by (34).*

For the expression $\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1)$ we get

$$\frac{\bar{b}^2}{4} - \bar{a}(\alpha - 1) = \frac{(n-2)^2(s-1)^2}{4s^2} - \frac{4s}{(n-2)}\frac{(n-2)^2}{4}\left[\frac{\Lambda}{(n-1)(n-2)} + \frac{2}{s} - \frac{1}{s^2}\right]$$

$$= \left[\frac{(n-2)^2}{4} - 2(n-2)\right] + \frac{(n-2)^2}{4}\frac{1}{s^2} + \left[(n-2) - \frac{(n-2)^2}{2}\right]\frac{1}{s} - (n-2)(\mu^2 - 1)s.$$

As $s \to 0$, the leading term is the first one, and the value of the function goes to $+\infty$. On the other hand, at $s = 1$, the value of this function is equal to $-(n-2)\mu^2 < 0$. Thus, this expression takes both negative and positive values.

It was shown, in Lemma 3.6, that a function with the α-basic asymptotic behavior at x_* belongs to $L_2(M)$ iff $\frac{n+4}{n} < \alpha$. It is instructive to compare this condition with the condition on the $sign(\bar{a})$.

The condition $\frac{n+4}{n} > \alpha_0$ is satisfied if and only if $\frac{n-2}{n} > \frac{1}{1+\mu}$, i. e. iff

$$\mu > \frac{2}{n-2}.$$

Taking the square of this inequality and using the expression for μ we see that this condition is equivalent to

$$\Lambda > (n-1)(n-2)\left[\frac{4}{(n-2)^2 - 1}\right] = -\frac{n(n-1)(n-4)}{n-2}.$$

Now we analyze the necessary and sufficient condition from Proposition 3.7 when a solution u belongs to the Sobolev space H_2^k. The condition is

$$k < 1 \pm \sqrt{\frac{\bar{b}^2}{4} + \bar{a}} = 1 \pm \frac{n-2}{2}\mu.$$

Here we reformulate this condition in terms of p, P, q, Q. Consider first the minus-solutions, for which the condition takes the form

$$k < 1 - \frac{n-2}{2}\mu.$$

Since $\mu > 0$, such solution can not belong to H_2^1. On the other hand, it belongs to L_2 if and only if $\frac{n-2}{2}\mu < 1$, or when $\mu < \frac{2}{n-2}$. Substituting this expression for μ, we get the result

Claim 8.3 *The solution u belongs to L_2 iff*

$$\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_q^2} < \frac{2(n-1)}{n-2}.$$

Example: For $r_p = r_q = 1$ this condition takes the form $(n-1-p)^2 < 2(n-1)$, or $p < n-1 - \sqrt{2(n-1)}$.

Consider now plus-solutions where the condition is $k < 1 + \frac{n-2}{2}\mu$. It is clear ($\mu > 0$) that the solution u always belongs to H_2^1. This solution belongs to H_2^2 (i.e. is the classical solution) iff $\mu < \frac{2}{n-2}$, or, substituting the expression for μ, when

$$\frac{p(p-1)}{r_p^2} + \frac{q(q-1)}{r_p^2} > \frac{2(n-1)}{n-2}.$$

Lemma 8.4 *If $\bar{a} < 0$, a function with the α-basic asymptotic behavior at x_* does not belong to $L_2(M)$.*

References

[1] K. Akutagawa, B. Botvinnik, Yamabe metrics on cylindrical manifolds, to appear in GAFA, a preliminary version is availble at math.DG/0107164.

[2] T. Aubin, Équations différentielles non linéaires et problème de Yamabe concernant la courbure scalaire, J. Math. Pures Appl. (9) 55 (1976), 269–296.

[3] T. Aubin, Some nonlinear problems in Riemannian geometry. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 1998. xviii+395 pp.

[4] T. Aubin, Le problème de Yamabe concernant la courbure scalaire. (French) C. R. Acad. Sci. Paris Sér. A-B 280 (1975), Aii, A721–A724.

[5] A. Besse, Einstein manifolds, Springer, Berlin, 1987.

[6] T. Branson, B. Örsted, Conformal geometry and global invariants. Differential Geom. Appl. 1 (1991), no. 3, 279–308.

[7] Y. Egorov, V. Kondratiev, On spectral theory of elliptic operators. (English. English summary) Operator Theory: Advances and Applications, 89. Birkhäuser Verlag, Basel, 1996. x+328 pp.

[8] A. Erdélyi, W.M. Wilhelm; F. Oberhettinger, F. Tricomi, Higher transcendental functions. Vols. I, II. Based, in part, on notes left by Harry Bateman. McGraw-Hill Book Company, Inc.,

[9] B. Botvinnik, J. Rosenberg, The Yamabe invariant for non-simply connected manifolds, math.DG/0104186.

[10] W. A. Coppel, Stability and asymptotic behavior of differential equations. D. C. Heath and Co., Boston, Mass. 1965.

[11] P. Gilkey, Invariance theory, the heat equation, and the Atiyah-Singer index theorem. Second edition. Studies in Advanced Mathematics. CRC Press, Boca Raton, FL, 1995. x+516 pp.

[12] G. Hardy, J. Littlewood, Pólya, G. Inequalities. 2d ed. Cambridge University Press, 1952. xii+324 pp.

[13] P. Hartman, Ordinary differential equations. Reprint of the second edition. Birkhuser, Boston, Mass., 1982.

[14] O. Kobayashi, Scalar curvature of a metric with unit volume. Math. Ann. 279 (1987), no. 2, 253–265.

[15] N. Korevaar, R. Mazzeo, F. Pacard, R. Schoen, Refined asymptotics for constant scalar curvature metrics with isolated singularities, Invent. Math. 135 (1999) 233-272.

[16] V. Kozlov, V. Maz'ya, J. Rossmann, Elliptic boundary value problems in domains with point singularities. Mathematical Surveys and Monographs, 52. American Mathematical Society, Providence, RI, 1997. x+414 pp.

[17] M. Lesch, Operators of Fuchs type, conical singularities, and asymptotic methods. Teubner-Texte zur Mathematik [Teubner Texts in Mathematics], 136. B. G. Teubner Verlagsgesellschaft mbH, Stuttgart, 1997. 190 pp.

[18] V. Maz'ya, S. Nasarow, B. Plamenewski, Asymptotische Theorie elliptischer Randwertaufgaben in singulär gestörten Gebieten. I. (German) [Asymptotic theory of elliptic boundary value problems in singularly perturbed domains. I.] Störungen isolierter Randsingularitäten. [Perturbations of isolated boundary singularities] Mathematische Lehrbücher und Monographien, II. Abteilung: Mathematische Monographien [Mathematical Textbooks and Monographs, Part II: Mathematical Monographs], 82. Akademie-Verlag, Berlin, 1991. 432 pp.

[19] R. Mazzeo, F. Pacard, Constant scalar curvature metrics with isolated singularities, Duke Math. J. 99 (1999), 353-418.

[20] R. Mazzeo, D. Pollack, K. Uhlenbeck, Connected sum construction for constant scalar curvature metrics, Topol. Methods Nonlinear Anal. 6 (1995), 207-233.

[21] S. Mizohata, The theory of partial differential equations. Translated from the Japanese by Katsumi Miyahara. Cambridge University Press, New York, 1973. xii+490 pp.

[22] T. Parker, S. Rosenberg, Invariants of conformal Laplacians. J. Differential Geom. 25 (1987), no. 2, 199–222.

[23] J. Petean, G. Yun, Surgery and the Yamabe invariant. Geom. Funct. Anal. 9 (1999), no. 6, 1189–1199.

[24] J. Petean, Computations of the Yamabe invariant. Math. Res. Lett. 5 (1998), no. 6, 703–709.

[25] P. Petersen, Riemannian geometry. Graduate Texts in Mathematics, 171. Springer-Verlag, New York, 1998. xvi+432 pp.

[26] R. Schoen, Conformal deformation of a Riemannian metric to constant scalar curvature. J. Differential Geom. 20 (1984), no. 2, 479–495.

[27] R. Schoen, Variational theory for the total scalar curvature functional for Riemannian metrics and related topics. Topics in calculus of variations (Montecatini Terme, 1987), 120–154, Lecture Notes in Math., 1365, Springer, Berlin-New York, 1989.

[28] R. Schoen, S. T. Yau, On the structure of manifolds with positive scalar curvature. Manuscripta Math. 28 (1979), no. 1-3, 159–183.

[29] H. Triebel, Interpolation theory, function spaces, differential operators. North-Holland Mathematical Library, 18. North-Holland Publishing Co., Amsterdam-New York, 1978. 528 pp.

[30] N. Trudinger, Remarks concerning the conformal deformation of Riemannian structures on compact manifolds, Ann. Scuola Norm. Sup. Pisa (3) 22 (1968), 265–274.

[31] H. Yamabe, On a deformation of Riemannian structures on compact manifolds, Osaka Math. J. 12 (1960), 21–37.

[32] K. Yosida, Functional analysis. Sixth edition. Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], 123. Springer-Verlag, Berlin-New York, 1980.

Boris Botvinnik, University of Oregon, Eugene
e-mail: botvinn@math.uoregon.edu

Serge Preston, Portland State University,
serge@mth.pdx.edu