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Set-Theoretic Reconstructability of Elementary Cellular Automata

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Set-theoretic reconstructability analysis is used to characterize the structures of the mappings of elementary cellular automata. The minimum complexity structure for each ECA mapping, indexed by parameter σ, is more effective than the λ parameter of Langton as a predictor of chaotic dynamics.

1. Introduction

This paper reports a study of elementary cellular automata (ECA), using the set-theoretic reconstructability analysis (SRA) methods of Klir (1985), Conant (1981), Broekstra (1979) , and others. Cellular automata (CA) are discrete dynamic systems, defined in terms of mappings of qualitative variables, which exhibit the same dynamic behaviors – including chaos – seen in systems of continuous differential equations. The object of the present study is to ascertain the degree to which the attractors of such systems can be deduced strictly from the mappings which govern them, i.e., (in the absence of closed form solutions) without actually "running" the dynamics.

Though ECAs are too simple to represent fully the general behavior of cellular automata, the small number of ECAs offer the possibility and the challenge of a "complete," i.e., non-statistical, understanding of their behavior. The property of interest here is the attractor governing the dynamics, most simply whether the attractor is chaotic or not. Langton (1992) has shown that a λ parameter defined on a CA rule space allows one to partially predict whether the dynamic system will show fixed point or limit cycle behavior or show chaotic behavior. Two questions naturally arise: what is the limit of this predictability and how does this limit illuminate the relationship between the "complexity" of a dynamic law and the "complexity" of the temporal behaviors which it produces.

Because of space limitations, only a partial SRA is presented here. In later reports, the full SRA, a related information-theoretic reconstructability analysis, and other CA parameterizations, e.g., the Z parameter of Wuensche (1992) will be discussed.

2. Elementary Cellular Automata

An ECA consists of an array of cells in one dimension, where each cell can take on one of 2 states (0,1) and where the binary string representing the array changes at discrete

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Table 1. An example of an ECA rule (Rule #150)

t: i-1	i	i+1	t+1: i
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

time intervals. The dynamics of such systems can be plotted on a plane: the cell array is commonly represented by a horizontal sequence of 0's and 1's or of light and dark cells with time (successive rows) as the vertical axis. The next state of any cell depends only upon its present "neighborhood," which includes the state of the cell itself and those of its immediate neighbors to the left and right. That is, if $s_t(i)$ is the state of cell i at time t, the dynamic law governing the ECA is described by the boolean function, $s_{t+1}(i) = f(s_t(i-1), s_t(i), s_t(i+1))$. Since there are 2^3 possible neighborhoods and since each neighborhood can map into either of the two states of $s_{t+1}(i)$, there are $2^{2^3} = 256$ mappings (ECA rules). An example is given in Table 1.

The 256 possible mappings are indexed by the binary number defined by $s_{t+1}(i)$ for the set of all neighborhoods, where the lowest order bit of this index is f(0,0,0) and the highest order bit is f(1,1,1). Thus, for example, the mapping of Table 1 is indexed by the number 10010110 and is Rule #150. The 256 mappings divide up into 88 equivalence classes Wolfram (1986), given that one considers mappings to be equivalent if they are related (1) by reflection, i.e., by left-right inversion of their arguments (which, if the dynamics were shown on a transparency, would merely involve turning the transparency over), (2) by complementing, i.e., negating the arguments and the function (which merely produces a photographic negative reversal), or (3) by both reflection and complementing. In general, an equivalence class will have 4 members, but f may generate itself under reflection and/or complementing, so that an equivalence class may have 1, 2, or 4 members. A representative rule (chosen consistently) is used to label the classes.

The dynamics of these 88 classes are governed by different attractors. Wolfram (1986) identifies four attractor types: I. homogeneous (where the dynamics settle down to a fixed point which is uniform, i.e., which consists of all 0's or 1's), II. fixed point (but not uniform) or periodic, III. chaotic, or IV. "complex." Strictly speaking, a finite state machine cannot be chaotic, but is necessarily periodic, but if transient lengths increase with the number of states of the system, one can consider the dynamics to be chaotic. Li and Packard (1990) have used an alternative classification to that of Wolfram: A. null, B. fixed point, C. periodic, D. locally chaotic (chaotic in some parts of the cell array but regular in other parts), and E. chaotic.

In this paper, we have used a reduced form of Li and Packard's classification which categorizes each rule as either *nonchaotic* (A or B or C) or *chaotic* (D or E); the two categories are given the labels $\mathcal{N}$ and $\mathcal{C}$. Also, we have adopted the assignments by these authors of the 88 classes to these categories. We are interested in ascertaining the degree to which the attractor type ($\mathcal{N}$ or $\mathcal{C}$) can be predicted given only the ECA mapping,

Table 2. Possible structures for ECA mappings

σ	structures			
6	ABCD			mapping
5	ABD:ACD:BCD			3 relations → mapping
4	ABD:ACD	ABD:BCD	ACD:BCD	2 relations → mapping
3	ABD	ACD	BCD	mapping
2	AD	BD	CD	mapping
1	D			constant

i.e., without generating the actual dynamics for any particular initial state of the cellular array. For the purposes of predicting the attractor, we use two parameterizations of the ECA rules: a standard measure and a new measure based on SRA.

The standard parameterization of CA rules is λ, proposed originally under the name "internal homogeneity" by Walker and Ashby (1966), and given its current name and extensively studied by Langton (1992). λ in the present context is defined as follows. Let r_0 be the number of 0's in the binary representation of the rule and r_1 be the number of 1's; then, $\lambda = \min(r_0, r_1)$. Langton has shown that CAs with small λ tend to have homogeneous, fixed point, or limit cycle behavior, while those with large λ tend to have chaotic dynamics. (Wolfram's Class IV automata occur at intermediate λ values, but for ECAs there are only two Class IV rules.)

The new measure based on SRA is described in Section 3; it indexes each rule with a structural parameter σ. In Section 4, σ is compared to λ for the prediction of ECA dynamics.

3. Set-Theoretic Structure Analysis

For convenience, let $A = s_t(i-1); B = s_t(i); C = s_t(i+1); D = s_{t+1}(i)$. An ECA rule is a mapping of A $\otimes$ B $\otimes$ C onto D (where $\otimes$ = cartesian product). The set of 12 structures possible for such mappings is given in Table 2. (A structure is defined as a set of relations, relations being defined as subsets of cartesian products.) These structures are arrayed on 6 levels indexed by parameter σ. The structures actually constitute a lattice, but the parent-child relationships of descent are not explicitly shown. Three levels (σ = 2, 3, and 6) specify simple (deterministic) mappings of either one, two, or three variables onto D. On one level ($\sigma = 1$), the mapping is trivial and independent of A,B,C; i.e., the rule is either D=0 or D=1. The remaining two levels (σ = 4, 5) specify structures containing either two or three overlapping (stochastic) relations, which taken jointly yield a mapping.

Each rule will be assigned a σ value according to the lowest level structure which can satisfactorily represent it. σ is a measure of the complexity of the rule in the sense of its non-decomposability. Higher level structures are required for mappings which cannot be decomposed to lower level structures. For example, ABCD can represent any mapping at all, while D represents only two mappings (Rules #0 and #255).

Note that classification by level, as opposed to specific structure, treats A, B, and C, equivalently: no discrimination is made between the *adjacent* cells, A and C, and the *center* cell, B, whose future state, D, is generated by the rule. If one preserves this discrimination, though, there are really 9 (not 12) qualitatively distinct specific

Table 3. Structural levels of the 88 ECA equivalence classes; $\mathcal{C}$ rules are dotted

σ		rules									
6	(47)	1	2	4	6	8	9	·18	·22	24	25
		·26	28	·30	32	33	36	37	38	40	41
		44	·45	·54	56	57	72	·73	74	104	105
		·106	108	128	·129	130	131	132	133	134	·137
		·146	150	152	·154	156	·161	164			
5	(20)	7	11	13	14	19	23	35	42	43	50
		76	77	138	140	142	162	168	178	200	232
4	(7)	27	29	46	58	78	172	184			
3	(9)	3	5	10	12	34	·60	·90	136	160	
2	(4)	15	51	170	204						
1	(1)	0									

structures. For example, for $\sigma = 4$, structure ABD:BCD differs from ABD:ACD and ACD:BCD, but the latter two are identical under reflection. The σ classification does not preserve this discrimination. However, as shall be shown later, doing so would probably not appreciably improve the predictions of this structural analysis.

The structure of an ECA rule is determined as follows. Any given relation, R–in our case, an ECA mapping–defines the top level of the structural lattice, i.e., ABCD. We assess whether a simpler structure can model R without error. Given structure, $S = P_1 : P_2 : \ldots : P_n$, where P_i is a projection (embedded relation) of R and M_i is the cartesian product of variables absent in P_i, then the (maximum likelihood) reconstructed relation is (Conant 1981, Klir 1985, Broekstra 1979)

$$R' = (P_1 \otimes M_1) \cap (P_2 \otimes M_2) \cdots \cap (P_n \otimes M_n).$$

For example, for $R =$ ABCD and $S =$ ABD:ACD: $M_1 =$ C, $M_2 =$ B, and

$$R' = (ABD \otimes C) \cap (ACD \otimes B).$$

(ABD $\otimes$ C) combines the ABD tuples with both values of C; (ACD $\otimes$ B) combines the ACD tuples with both values of B; the intersection selects tuples allowed by both expanded relations and thus yields the maximum uncertainty solution consistent with ABD and ACD. S fits the relation if and only if $R' = R$.

Reconstructability analysis, as just defined, finds the lowest level structure, S, for which $R' = R$, but a fuller analysis (to be reported elsewhere) would index each rule by the vector of errors, $|R'(S_j)\text{-}R|$, for all S_j. The results of the minimal analysis are given in Table 3, which lists the σ values for the 88 equivalence classes. In parenthesis and in smaller type are the number of classes at each level; chaotic rules are also indicated. As might be expected most rules are completely non-decomposable.

4. Predicting Dynamics

We are interested in the degree to which knowing λ or σ gives information about the attractor governing the dynamics. Table 4 gives the contingency tables relating the dependent variable, a, and the independent variables, λ and σ. (The tables are computed for the 256 rules, rather than for the 88 equivalence classes, to weight class multiplicity properly.)

Chaotic attractors are found only for $\lambda \geq 2$, but for these λ values, both chaotic and

Table 4. Contingency tables: λ or σ vs. attractor, a

λ	a: $\mathcal{N}$	a: $\mathcal{C}$		σ	a: $\mathcal{N}$	a: $\mathcal{C}$	
0	2		(2)	1	2		(2)
1	16		(16)	2	6		(6)
2	52	4	(56)	3	24	6	(30)
3	96	16	(112)	4	24		(24)
4	44	26	(70)	5	56		(56)
	(210)	(46)	(256)	6	98	40	(138)
					(210)	(46)	(256)

non-chaotic attractors are observed, so λ cannot be a perfect predictor of the attractor. Chaotic attractors are found only for σ=3 and σ=6, though mostly for the latter value, for which both chaotic and non-chaotic attractors are found. What is intriguing is that even though σ=4 and σ=5 structures are "more complex" than σ=3 structures, chaotic dynamics do not occur. σ=4 and σ=5 structures are distinctive in being intersections of two or three relations which yield mappings but whether this accounts for the fact that chaos occurs for σ=3 and 6, but not 4 or 5, is unclear.

A parameterization based on 9 structure types rather than 6 levels is unlikely to give better results. σ=3 includes 6 chaotic rules which distribute into 2 equivalence classes (60, 90) which have the distinct structures ABD and ACD. The remaining 24 non-chaotic rules distribute into 7 classes (3, 5, 10, 12, 34, 136, 160) which include 2 occurrences each of ABD or BCD (the same under reflection) and 3 occurrences of ACD. σ=6 has only one structure, so the issue doesn't arise.

To obtain a quantitative assessment, we calculate from the above contingency tables the uncertainty (Shannon entropy) of the attractor type and the reduction of this uncertainty knowing the rule property, λ or σ. This is a "conservative" approach to assessing association. One might consider a, λ, and σ to be ordinal variables and use some ordinal measure of association. However, the use of the nominal measure, uncertainty, allows us not to insist upon a specific ordering of rule types or upon a monotonic association between independent and dependent variables. The results vindicate this choice, as chaoticity does not vary monotonically with σ.

Uncertainties and uncertainty reductions are evaluated from the frequencies given in Table 4 as follows.

$$H(a) = -\sum p(a_i)\, log_2\, p(a_i)$$

$$H(\lambda) = -\sum p(\lambda_j)\, log_2\, p(\lambda_j)$$

$$H(a,\lambda) = -\sum\sum p(a_i,\lambda_j)\, log_2\, p(a_i,\lambda_j)$$

$$H(a|\lambda) = H(a,\lambda) - H(\lambda)$$

Similar equations give H(σ), H(a,σ), and H(a|σ). We compare the uncertainty reductions achieved by λ and σ. Obviously a parameter which carries more information can more readily reduce H(a), so we normalize the uncertainty reduction by the information used for this reduction, e.g., $(H(a) - H(a|\lambda))\ /\ H(\lambda)$. These "efficiency" calculations play a role similar to tests of statistical significance.

Table 5. Reduction of attractor uncertainty

		%△H
H(a)	0.679	
H(a|λ)	0.600	-11.6%
H(a|σ)	0.553	-18.6%
H(a|λ,σ)	0.350	-48.5%
(H(a)-H(a|λ)) /H(λ)	0.044	
(H(a)-H(a|σ)) /H(σ)	0.069	
(H(a)-H(a|λ,σ)/H(λ,σ)	0.109	
H(λ)	1.818	-35.3%
H(λ|σ)	1.200	
H(σ)	1.822	-35.1%
H(σ|λ)	1.196	

Table 5 compares the two rule parameters. The table shows that σ is a better predictor of the attractor than λ, both absolutely (reducing H by 18.6% as compared to 11.6%) and when normalized by input information (achieving an efficiency of reduction of .069 as compared to .044). It is also clear that λ and σ do not reflect the same properties of the rule, since together they predict much better (48.5% reduction and an efficiency of .109) than either measure does individually, and since each measure only reduces about 35% of the uncertainty of the other. Also, the .109 efficiency with both measures, which is much larger than either individual efficiency, shows a synergistic effect between the two parameters.

5. Conclusions

The attractor type is partially predictable from attributes defined directly from the rule mappings. This suggests that, while simple laws generate "complex" (here meaning chaotic, not Wolfram Class IV) behavior, more complex (high λ or σ) rules are more likely to be associated with complex (chaotic) behavior. Nonetheless, even though predictability can be improved with σ over what is obtainable from λ, predictability is still quite limited. The questions of the limit of attractor predictability and the underlying basis for this limit remain unanswered.

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