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Stochastic Comparisons of Weighted Sums of Arrangement Increasing Random Variables

Xiaoqing Pan
University of Science and Technology of China

Min Yuan
University of Science and Technology of China

Subhash C. Kochar
Portland State University, kochar@pdx.edu

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Xiaoqing Pan
Department of Statistics and Finance, School of Management
University of Science and Technology of China
Hefei, Anhui 230026
People's Republic of China
panxq@ustc.edu.cn

Min Yuan
Department of Probability and Statistics, School of Mathematics
University of Science and Technology of China
Hefei, Anhui 230026
People's Republic of China
] myuan@ustc.edu.cn

Subhash C. Kochar
Fariborz Maseeh Department of Mathematics and Statistics
Portland State University, Portland, OR 97006, USA
kochar@pdx.edu

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Abstract

Assuming that the joint density of random variables $X_1, X_2, \ldots, X_n$ is arrangement increasing (AI), we obtain some stochastic comparison results on weighted sums of X_i's under some additional conditions. An application to optimal capital allocation is also given.

Mathematics Subject Classifications (2000): 60E15; 62N05; 62G30

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1 Introduction

During the past few decades, linear combinations of random variables have been extensively studied in statistics, operations research, reliability theory, actuarial science and other fields. Most of the related work restricts to some specific distributions such as Exponential, Weibull, Gamma and Uniform, among others. Karlin and Rinott (1983) and Yu (2011) studied the stochastic properties of linear combinations of independent and identically distributed (i.i.d) random variables without putting any distributional assumptions. Later on, Xu and Hu (2011, 2012), Pan et al. (2013) and Mao et al. (2013) weakened the i.i.d assumption to independent, yet possibly non-identically distributed (i.ni.d), random variables. It should be noted that most of the related work assumes that the random variables are mutually independent.

Recently, some work has appeared on stochastic comparisons of dependent random variables. Xu and Hu (2012) discussed stochastic comparisons of comonotonic random variables with applications to capital allocations. You and Li (2014) focused on linear combinations of random variables with Achimedean dependence structure. Cai and Wei (2014) proposed several new notions of dependence to measure dependence between risks. They proved that characterizations of these notions are related to properties of arrangement increasing (AI) functions (to be defined in Section 2). Motivated by the importance of AI functions, we study the problem of stochastic comparisons of weighted sums of AI random variables in this paper.

We say $X_1, \dots, X_n$ are AI random variables if their joint density $f(\boldsymbol{x})$ is an AI function. Ma (2000) proved the following result for AI random variables $X_1, \dots, X_n$:

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} a_{(i)} X_i \geq_{\mathrm{icx}} \sum_{i=1}^{n} b_{(i)} X_i, \quad \forall\, \mathbf{a}, \mathbf{b} \in \Re^n, \tag{1.1}$$

where $a_{(1)} \leq a_{(2)} \leq \cdots \leq a_{(n)}$ is the increasing arrangement of the components of the vector $\mathbf{a} = (a_1, a_2, \dots, a_n)$. The formal definitions of stochastic orders and majorization orders are given in Section 2.

Let $X_1, X_2, \dots, X_n$ be independent random variables satisfying

$$X_1 \geq_{\mathrm{hr}} X_2 \geq_{\mathrm{hr}} \cdots \geq_{\mathrm{hr}} X_n,$$

and let $\phi(x, a)$ be a convex function which is increasing in x for each a. Mao et al. (2013) proved that

(i) if ϕ is submodular, then

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_i, b_{(i)}\right); \tag{1.2}$$

(ii) if ϕ is supermodular, then

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(n-i+1)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_i, b_{(n-i+1)}\right). \tag{1.3}$$

The function ϕ in (1.2) and (1.3) could be interpreted as some appropriate distance measures in actuarial science. For more details, please refer to Xu and Hu (2012).

In this paper we further study the problem of stochastic comparisons of linear combinations of AI random variables not only for increasing convex ordering, but also for the usual stochastic ordering. The rest of this paper is organized as follows. Some preliminaries are given in Section 2. The main results are presented in Section 3. An application to optimal capital allocation is discussed in Section 4.

2 Preliminaries

In this section, we give definitions of some stochastic orders, majorization orders and supermodular [submodular] functions. Throughout the paper, the terms 'increasing' and 'decreasing' are used to mean 'non-decreasing' and 'non-increasing', respectively.

Definition 2.1 (Stochastic orders)
Let X and Y be two random variables with probability (mass) density functions f and g; and survival functions $\overline{F}$ and $\overline{G}$ respectively. We say that X is smaller than Y

(1) *in the usual stochastic order, denoted by $X \leq_{\mathrm{st}} Y$, if $\overline{F}(t) \leq \overline{G}(t)$ for all t or, equivalently, if $\mathsf{E}[h(X)] \leq \mathsf{E}[h(Y)]$ for all increasing functions h;*

(2) *in the hazard rate order, denoted by $X \leq_{\mathrm{hr}} Y$, if $\overline{G}(t)/\overline{F}(t)$ is increasing in t for which the ratio is well defined;*

(3) *in the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if $g(t)/f(t)$ is increasing in t for which the ratio is well defined;*

(4) *in the increasing convex order, denoted by $X \leq_{\mathrm{icx}} Y$, if $\mathsf{E}[h(X)] \leq \mathsf{E}[h(Y)]$ for all increasing convex functions h for which the expectations exist.*

The relationships among these orders are shown in the following diagram (see Shaked and Shanthikumar, 2007; Müller and Stoyan, 2002):

$$X \leq_{\mathrm{lr}} Y \Longrightarrow X \leq_{\mathrm{hr}} Y \Longrightarrow X \leq_{\mathrm{st}} Y \Longrightarrow X \leq_{\mathrm{icx}} Y.$$

Shanthikumar and Yao (1991) considered the problem of extending the above concepts to compare the components of dependent random variables. In this paper we will focus only on extension of likelihood ratio ordering to the case of dependent random variables. Let (X, Y) be a continuous bivariate random vector on $[0, \infty]^2$ with joint density (or mass) function $f(x, y)$.

Definition 2.2 *For a bivariate random variable (X, Y), X is said to be smaller than Y according to joint likelihood ordering, denoted by $X \le_{\ell r:j} Y$, if and only if*

$$E[\Psi(X, Y)] \ge E[\Psi(Y, X)], \quad \Psi \in \mathcal{G}_{\ell r},$$

where

$$\mathcal{G}_{\ell r} : \{\Psi : \Psi(x, y) \ge \Psi(y, x), \quad x \le y\}.$$

It can be seen that

$$X \le_{\ell r:j} Y \quad \Leftrightarrow \quad f \in \mathcal{G}_{\ell r},$$

where $f(\cdot, \cdot)$ denotes the joint density of (X, Y).

As pointed out by Shanthikumar and Yao (1991), joint likelihood ratio ordering between the components of a bivariate random vector may not imply likelihood ratio ordering between their marginal distributions unless the random variables are independent, but it does imply stochastic ordering between them, that is,

$$X \le_{\ell r:j} Y \Rightarrow X \le_{\text{st}} Y.$$

A bivariate function $\Psi \in \mathcal{G}_{\ell r}$ is called *arrangement increasing* (AI). Hollander et al. (1977) have studied many interesting properties of such functions, though, apparently, they did not relate it to the notion of likelihood ratio ordering. We can extend this concept to compare more than two random variables in the following way.

Let $\pi = (\pi(1), \ldots, \pi(n))$ be any permutation of $\{1, \ldots, n\}$ and let $\pi(\mathbf{x}) = (x_{\pi(1)}, \ldots, x_{\pi(n)})$. For any $1 \le i \ne j \le n$, we denote $\pi_{ij} = (\pi_{ij}(1), \ldots, \pi_{ij}(n))$ with $\pi_{ij}(i) = j$, $\pi_{ij}(j) = i$ and $\pi_{ij}(k) = k$ for $k \ne i, j$.

Definition 2.3 (AI function)
A real-valued function $g(\mathbf{x})$ defined on $\Re^n$ is said to be an arrangement increasing (AI) function if

$$(x_i - x_j)\,[g(\mathbf{x}) - g(\boldsymbol{\pi}_{ij}(\mathbf{x}))] \le 0,$$

for any pair (i, j) such that $1 \le i < j \le n$.

We say $X_1, \ldots, X_n$ are AI random variables if their joint density $f(\boldsymbol{x})$ is an AI function.

Definition 2.4 *We say that a function $h(x, y)$ is Totally Positive of order 2 (TP_2) if $h(x, y) \ge 0$ and*

$$\begin{vmatrix} h(x_1, y_1) & h(x_1, y_2) \\ h(x_2, y_1) & h(x_2, y_2) \end{vmatrix} \ge 0, \tag{2.1}$$

whenever $x_1 < x_2$, $y_1 < y_2$.

Hollander et al. (1977) and Marshall et al. (2011) gave many examples of AI random variables. The following vectors of random variables $\mathbf{X} = (X_1, ...X_n)$ are arrangement increasing.

(1) $X_1, \ldots, X_n$ are identically independent distributed random variables.

(2) $X_1, \ldots, X_n$ are exchangeable random variables.

(3) Suppose $X_1, \ldots, X_n$ are independent random variables with density functions $h(\lambda_i, x_i)$, $i = 1, \ldots, n$, respectively. Then $f(\boldsymbol{x}) = \prod h(\lambda_i, x_i)$ is arrangement increasing if and only if $h(\lambda, x)$ is TP_2 in λ and x.

Hollander et al. (1977) have shown that the following multivariate distributions are AI: Multinomial, Negative multinomial, Multivariate hypergeometric, Dirichlet, Inverted Dirichlet, Negative multivariate hypergeometric, Dirichlet compound negative multinomal, Multivariate logarithmic series distribution, Multivariate F distribution, Multivariate Pareto distribution, Multivariate normal distribution with common variance and common covariance.

Majorization defines a partial ordering of the diversity of the components of vectors. For extensive and comprehensive details on the theory of majorization order and their applications, please refer to Marshall et al. (2011).

Definition 2.5 (Majorization, Schur-concave [Schur-convex] and log-concave)
For vectors $\mathbf{x}, \mathbf{y} \in \Re^n$, $\mathbf{x}$ is said to be majorized by $\mathbf{y}$, denoted by $\mathbf{x} \preceq_{\mathrm{m}} \mathbf{y}$, if $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$ and

$$\sum_{i=1}^{j} x_{(i)} \geq \sum_{i=1}^{j} y_{(i)} \text{ for } j = 1, \ldots, n-1.$$

A real-valued function ϕ defined on a set $A \subseteq \Re^n$ is said to be Schur-concave [Schur-convex] on A if, for any $\mathbf{x}, \mathbf{y} \in A$,

$$\mathbf{x} \succeq_{\mathrm{m}} \mathbf{y} \Longrightarrow \phi(\mathbf{x}) \leq [\geq] \, \phi(\mathbf{y});$$

and ϕ is said to be log-concave on $A \subset \Re^n$ if A is a convex set and, for any $\mathbf{x}, \mathbf{y} \in A$ and $\alpha \in [0, 1]$,

$$\phi(\alpha \mathbf{x} + (1-\alpha)\mathbf{y}) \geq [\phi(\mathbf{x})]^{\alpha} [\phi(\mathbf{y})]^{1-\alpha}.$$

Definition 2.6 (Supermodular [Submodular] function)
A real-valued function $\varphi : \Re^n \to \Re$ is said to be supermodular [submodular] if

$$\varphi(\mathbf{x} \vee \mathbf{y}) + \varphi(\mathbf{x} \wedge \mathbf{y}) \geq [\leq] \; \varphi(\mathbf{x}) + \varphi(\mathbf{y}) \quad \text{for all } \mathbf{x}, \mathbf{y} \in \Re^n.$$

Here, $\vee$ and $\wedge$ denote the componentwise maximum and the componentwise minimum, respectively. If $\varphi : \Re^n \to \Re$ has second partial derivatives, then it is supermodular [submodular] if and *only if*

$$\frac{\partial^2}{\partial x_i \partial x_j}\varphi(\mathbf{x}) \geq [\leq]\ 0 \text{ for all } i \neq j \text{ and } \mathbf{x} \in \Re^n.$$

Marshall et al. (2011) gave several examples of supermodelar [submodular] functions. Below we give some examples of linear functions which are supermodular [submodular] .

(1) $\varphi(\mathbf{a}, \mathbf{x}) = \sum_{i=1}^n \phi_1(a_i)\phi_2(x_i)$: if ϕ_1 and ϕ_2 are increasing, then $\varphi(\mathbf{a}, \mathbf{x})$ is submodular; and if if ϕ_1 is decreasing and ϕ_2 is increasing, then $\varphi(\mathbf{a}, \mathbf{x})$ is supermodular.

(2) $\varphi(\mathbf{a}, \mathbf{x}) = \sum_{i=1}^n \phi(x_i - a_i)$: if ϕ is convex, then $\varphi(\mathbf{a}, \mathbf{x})$ is submodular; and if ϕ is concave, then $\varphi(\mathbf{a}, \mathbf{x})$ is supermodular.

(3) $\varphi(\mathbf{a}, \mathbf{x}) = \sum_{i=1}^n \max\{a_i, x_i\}$ is submodular.

(4) $\varphi(\mathbf{a}, \mathbf{x}) = \sum_{i=1}^n \sup\left\{c_1 a_i + c_2 x_i : (c_1, c_2) \in C\right\}$ is submodular, where $C \subseteq \Re^2$.

3 Main results

In this section, we study stochastic comparisons of weighted sums of the form $\sum_{i=1}^n \phi(X_i, a_i)$ where $X_1, \ldots, X_n$ are random variables with joint density function $f(\mathbf{x})$. In what follows, we make the following assumptions:

(A1) $f(\mathbf{x})$ is log-concave,

(A2) $f(\mathbf{x})$ is arrangement increasing,

(A3) $\phi : \Re^2 \to \Re$ is a convex function.

We consider both usual stochastic order as well as increasing convex order for comparison purposes.

3.1 Usual stochastic ordering

Before we give the main result, we list several lemmas, which will be used in the sequel.

Lemma 3.1 (Prékopa, 1973; Eaton, 1982)
Suppose that $h : \Re^m \times \Re^k \to \Re_+$ is a log-concave function and that

$$g(\mathbf{x}) = \int_{\Re^k} h(\mathbf{x}, \mathbf{z})d\mathbf{z}$$

is finite for each $\mathbf{x} \in \Re^m$. Then g is log-concave on $\Re^m$.

Lemma 3.2 (Pan et al., 2013)
If $g : \Re^2 \to \Re_+$ is log-concave and $-g$ is AI, i.e.

$$g(x_{(2)}, x_{(1)}) \geq g(x_{(1)}, x_{(2)}) \quad for\ all\ (x_1, x_2) \in \Re^2, \tag{3.1}$$

then

$$(x_1, x_2) \preceq_{\rm m} (y_1, y_2) \Longrightarrow g(x_{(1)}, x_{(2)}) \geq g(y_{(1)}, y_{(2)}).$$

We list Theorem 23 in Karlin and Rinott (1983) as a lemma, and we give a new proof as follows.

Lemma 3.3 (Karlin and Rinott, 1983)
Let $\mathbf{X} \in \Re^n$ have a log-concave density and let $\phi(\mathbf{x}, \mathbf{a})$ be convex in $(\boldsymbol{x}, \boldsymbol{a}) \in \Re^{n+m}$. Then $g(\mathbf{a}) = P(\phi(\mathbf{X}, \mathbf{a}) \leq t)$ is log-concave on $\Re^m$.

Proof. If we denote

$$A = \left\{(\mathbf{x}, \mathbf{a}) \in \Re^{n+m} : \phi(\mathbf{x}, \mathbf{a}) \leq t\right\},$$

then A is a convex set due to the convexity of ϕ. Thus, I_A is log-concave. If $f(\mathbf{x})$ denotes the joint density function of $\mathbf{X}$, then we have

$$g(\mathbf{a}) = \int_{\Re^n} f(\mathbf{x})\ \mathrm{I}_A\ d\mathbf{x}.$$

Since $f(\mathbf{x})$ is log-concave, $f(\mathbf{x})\mathrm{I}_{\mathbf{A}}$ is log-concave. By Lemma 3.1, $g(\mathbf{a})$ is log-concave. ∎

From Lemma 3.3, we have

Lemma 3.4 *Let $\mathbf{X} \in \Re^n$ have a log-concave density and let $\phi(\mathbf{x}, \mathbf{a})$ be concave in $(\boldsymbol{x}, \boldsymbol{a}) \in \Re^{n+m}$. Then $g(\mathbf{a}) = P(\phi(\mathbf{X}, \mathbf{a}) \geq t)$ is log-concave on $\Re^m$.*

Theorem 3.5 *Under the assumptions (A1), (A2) and (A3),*
(i) *if ϕ is supermodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\rm m} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\rm st} \sum_{i=1}^{n} \phi\left(X_i, b_{(i)}\right).$$

(ii) *if ϕ is submodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\rm m} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_{n-i+1}, a_{(i)}\right) \geq_{\rm st} \sum_{i=1}^{n} \phi\left(X_{n-i+1}, b_{(i)}\right).$$

Proof. Since the proof of part (ii) is quite similar to part (i), we only prove part (i). By the nature of majorization, we only need to prove that

$$\phi(X_1, a_{(1)}) + \phi(X_2, a_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\rm st} \phi(X_1, b_{(1)}) + \phi(X_2, b_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i)$$

holds, for all

$$(a_{(1)}, a_{(2)}, c_3, \dots, c_n) \succeq_{\mathrm{m}} (b_{(1)}, b_{(2)}, c_3, \dots, c_n),$$

with $(a_{(1)}, a_{(2)}) \succeq_{\mathrm{m}} (b_{(1)}, b_{(2)})$. For any fixed t, we denote

$$g(a_1, a_2) = P\left(\varphi(\mathbf{X}, \mathbf{a}) \le t\right)$$

with

$$\varphi(\mathbf{x}, \mathbf{a}) = \phi(x_1, a_1) + \phi(x_2, a_2) + \sum_{i=3}^{n} \phi(x_i, c_i)$$

Since ϕ is convex, $\varphi(\mathbf{x}, \mathbf{a})$ is convex on $\Re^{2n}$. By Lemma 3.3, $g(a_1, a_2)$ is log-concave. From Lemma 3.2, it is sufficient to prove (3.1). Hence, we only need to prove

$$\varphi\left(\mathbf{X}, \mathbf{a}_{21}\right) \le_{\mathrm{st}} \varphi\left(\mathbf{X}, \mathbf{a}_{12}\right) \tag{3.2}$$

where $\mathbf{a}_{12} = (a_{(1)}, a_{(2)})$ and $\mathbf{a}_{21} = (a_{(2)}, a_{(1)})$. It is sufficient to prove that, for all increasing function h,

$$\mathrm{E}\left[h\left(\varphi(\mathbf{X}, \mathbf{a}_{12})\right)\right] \ge \mathrm{E}\left[h\left(\varphi(\mathbf{X}, \mathbf{a}_{21})\right)\right]$$

Now,

$$\begin{aligned}
&\mathrm{E}\left[h\left(\varphi(\mathbf{X}, \mathbf{a}_{12})\right)\right] - \mathrm{E}\left[h\left(\varphi(\mathbf{X}, \mathbf{a}_{21})\right)\right] \\
&\quad = \left(\int \cdots \int_{x_1 \le x_2} + \int \cdots \int_{x_1 \ge x_2}\right) h\left(\varphi(\mathbf{x}, \mathbf{a}_{12})\right) f(\mathbf{x}) dx_1 \dots dx_n \\
&\qquad - \left(\int \cdots \int_{x_1 \le x_2} + \int \cdots \int_{x_1 \ge x_2}\right) h\left(\varphi(\mathbf{x}, \mathbf{a}_{21})\right) f(\mathbf{x}) dx_1 \dots dx_n \\
&\quad = \int \cdots \int_{x_1 \le x_2} \left[h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{12}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{21}\right)\right)\right] \left[f(\mathbf{x}) - f(\boldsymbol{\pi}_{12}(\mathbf{x}))\right] dx_1 \dots dx_n.
\end{aligned}$$

Since $f(\mathbf{x})$ is AI, it follows that

$$f(\mathbf{x}) - f(\boldsymbol{\pi}_{12}(\mathbf{x})) \ge 0, \quad \forall\ x_1 \le x_2.$$

Meanwhile, since ϕ is supermodular, for all $x_1 \le x_2$ and $a_{(1)} \le a_{(2)}$, we have

$$\phi(x_1, a_{(1)}) + \phi(x_2, a_{(2)}) \ge \phi(x_1, a_{(2)}) + \phi(x_2, a_{(1)}).$$

Thus,

$$\phi(x_1, a_{(1)}) + \phi(x_2, a_{(2)}) + \sum_{i=3}^{n} \phi(x_i, c_i) \ge \phi(x_1, a_{(2)}) + \phi(x_2, a_{(1)}) + \sum_{i=3}^{n} \phi(x_i, c_i).$$

Since h is increasing, we have

$$h(\varphi(\mathbf{x}, \mathbf{a}_{12})) - h(\varphi(\mathbf{x}, \mathbf{a}_{21})) \ge 0.$$

Therefore, (3.2) holds and the desired result follows. ∎

Since exchangeable random variables are arrangement increasing, the following results follows immediately from this theorem.

Corollary 3.6 *Let $X_1, \ldots, X_n$ be exchangeable random variables satisfying assumption (A1). If ϕ is a convex function on $\Re$, then, for any $\mathbf{a}$, $\mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\rm m} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi(X_i, a_i) \geq_{\rm st} \sum_{i=1}^{n} \phi(X_i, b_i).$$

Proposition 3.7 *Under the assumptions (A1), (A2) and (A3),*
(i) *if ϕ is supermodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\rm m} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\rm st} \sum_{i=1}^{n} \phi\left(X_i, b_i\right).$$

(ii) *if ϕ is submodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\rm m} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_{n-i+1}, a_{(i)}\right) \geq_{\rm st} \sum_{i=1}^{n} \phi\left(X_{n-i+1}, b_i\right).$$

Proof. The proof of (ii) is quite similar to that of (i), so we only prove (i). By the nature of majorization, we only need to prove that for all

$$(a_{(1)}, a_{(2)}, c_3, \ldots, c_n) \succeq_{\rm m} (b_{(1)}, b_{(2)}, c_3, \ldots, c_n),$$

with $(a_{(1)}, a_{(2)}) \succeq_{\rm m} (b_{(1)}, b_{(2)})$, we have

$$\phi(X_1, a_{(1)}) + \phi(X_2, a_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\rm st} \phi(X_1, b_1) + \phi(X_2, b_2) + \sum_{i=3}^{n} \phi(X_i, c_i).$$

From (3.2), we have

$$\phi(X_1, b_{(1)}) + \phi(X_2, b_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\rm st} \phi(X_1, b_{(2)}) + \phi(X_2, b_{(1)}) + \sum_{i=3}^{n} \phi(X_i, c_i). \tag{3.3}$$

Meanwhile, from Theorem 3.5, we have

$$\phi(X_1, a_{(1)}) + \phi(X_2, a_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\rm st} \phi(X_1, b_{(1)}) + \phi(X_2, b_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i). \tag{3.4}$$

Combing (3.3) and (3.4), we have

$$\phi(X_1, a_{(1)}) + \phi(X_2, a_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\rm st} \phi(X_1, b_{(2)}) + \phi(X_2, b_{(1)}) + \sum_{i=3}^{n} \phi(X_i, c_i). \tag{3.5}$$

From (3.4) and (3.5), we get the desired result. ∎

Corollary 3.8 *Under the assumptions (A1), (A2) and (A3), for any* $\mathbf{a}$, $\mathbf{b} \in \Re^n$,

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi(X_{n-i+1} - a_{(i)}) \geq_{\mathrm{st}} \sum_{i=1}^{n} \phi(X_{n-i+1} - b_i).$$

Proof. It is easy to prove that $\phi(x-a)$ is convex and submodular on $\Re^2$. By Proposition 3.7, the result follows. ∎

If the function ϕ is concave, we get a similar result in the same way..

Theorem 3.9 *Let* ϕ *be a concave function on* $\Re$. *Then under the assumptions (A1) and (A2),*
(i) *if* ϕ *is supermodular, then, for any* $\mathbf{a}, \mathbf{b} \in \Re^n$,

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_{n-i+1}, a_{(i)}\right) \geq_{\mathrm{st}} \sum_{i=1}^{n} \phi\left(X_{n-i+1}, b_i\right).$$

(ii) *if* ϕ *is submodular, then, for any* $\mathbf{a}, \mathbf{b} \in \Re^n$,

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\mathrm{st}} \sum_{i=1}^{n} \phi\left(X_i, b_i\right).$$

Corollary 3.10 *Let* ϕ *be a concave function on* $\Re$. *Under the assumptions (A1) and (A2), for any* $\mathbf{a}$, $\mathbf{b} \in \Re^n$,

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi(X_i - a_{(i)}) \geq_{\mathrm{st}} \sum_{i=1}^{n} \phi(X_i - b_i).$$

Finally, we give an example where the assumptions (A1) and (A2) are satisfied.

Example 3.11 *Let* $\boldsymbol{Y} = (Y_1, \ldots, Y_n)$ *be an* n*-dimensional normal variable with mean vector* $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_n)$ *satisfying* $\mu_1 \leq \mu_2 \leq \ldots \leq \mu_n$, *and*

$$\mathrm{Cov}(Y_i, Y_j) = [\rho + (1-\rho)\delta_{ij}]\, \sigma^2, \quad \forall i, j,$$

where $\rho > -\frac{1}{n-1}$ *and* δ_{ij} *is Kronecker delta, that is* $\delta_{ii} = 1$ *and* $\delta_{ij} = 0$, $\forall\, i \neq j$. *Hollander et al. (1977) proved that the joint density function of* $\boldsymbol{Y}$ *is AI. It is easy to see that the joint density function of* $\boldsymbol{Y}$ *is log-concave. Thus, the assumptions (A1) and (A2) hold.*

3.2 Increasing convex ordering

So far, we have obtained all the results under the assumption (A1) that the joint density function is log-concave. However, there exist cases where the joint density functions are not log-concave, even if the marginal density functions are log-concave (cf. You and Li, 2014). An (1998) remarked that if X has a log-concave density, then its density has at most an exponential tail, i.e.,

$$f(x) = O\left(\exp(-\lambda x)\right), \quad \lambda > 0, \quad x \to \infty.$$

Thus, all the power moments $\mathsf{E}|X|^\gamma$, $\gamma > 0$, of the random variable X exist. In this section, we prove the following theorem without the assumption (A1).

Theorem 3.12 *Under the assumptions (A2) and (A3),*
(i) *if ϕ is supermodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_i, a_{(i)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_i, b_{(i)}\right).$$

(ii) *if ϕ is submodular, then, for any $\mathbf{a}, \mathbf{b} \in \Re^n$,*

$$\mathbf{a} \succeq_{\mathrm{m}} \mathbf{b} \Longrightarrow \sum_{i=1}^{n} \phi\left(X_{n-i+1}, a_{(i)}\right) \geq_{\mathrm{icx}} \sum_{i=1}^{n} \phi\left(X_{n-i+1}, b_{(i)}\right).$$

Proof. We only prove part (i) as the proof of part (ii) follows on the same lines. By the nature of majorization, we only need to prove that for all

$$(a_{(1)}, a_{(2)}, c_3, \ldots, c_n) \succeq_{\mathrm{m}} (b_{(1)}, b_{(2)}, c_3, \ldots, c_n),$$

with $(a_{(1)}, a_{(2)}) \succeq_{\mathrm{m}} (b_{(1)}, b_{(2)})$, we have

$$\phi(X_1, a_{(1)}) + \phi(X_2, a_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i) \geq_{\mathrm{icx}} \phi(X_1, b_{(1)}) + \phi(X_2, b_{(2)}) + \sum_{i=3}^{n} \phi(X_i, c_i).$$

We denote

$$\varphi(\mathbf{x}, \mathbf{a}) = \phi(x_1, a_1) + \phi(x_2, a_2) + \sum_{i=3}^{n} \phi(x_i, c_i),$$

$\mathbf{a}_{12} = (a_{(1)}, a_{(2)})$ and $\mathbf{a}_{21} = (a_{(2)}, a_{(1)})$. It is sufficient to prove that for all increasing convex function h,

$$\mathrm{E}[h(\varphi(\mathbf{X}, \mathbf{a}_{12}))] \geq \mathrm{E}[h(\varphi(\mathbf{X}, \mathbf{b}_{12}))] \text{ holds.}$$

Now,

$$\begin{aligned}
&\mathrm{E}[h(\varphi(\mathbf{X}, \mathbf{a}_{12}))] - \mathrm{E}[h(\varphi(\mathbf{X}, \mathbf{b}_{12}))] \\
&= \left(\int \cdots \int_{x_1 \leq x_2} + \int \cdots \int_{x_1 \geq x_2}\right) h\left(\varphi(\mathbf{x}, \mathbf{a}_{12})\right) f(\mathbf{x}) dx_1 \ldots dx_n \\
&\quad - \left(\int \cdots \int_{x_1 \leq x_2} + \int \cdots \int_{x_1 \geq x_2}\right) h\left(\varphi(\mathbf{x}, \mathbf{b}_{12})\right) f(\mathbf{x}) dx_1 \ldots dx_n \\
&= \int \cdots \int_{x_1 \leq x_2} \left[h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{12}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{12}\right)\right)\right] f(\mathbf{x}) dx_1 \ldots dx_n \\
&\quad + \int \cdots \int_{x_1 \leq x_2} \left[h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{21}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{21}\right)\right)\right] f(\boldsymbol{\pi}_{12}(\mathbf{x})) dx_1 \ldots dx_n \\
&\geq \int \cdots \int_{x_1 \leq x_2} \left[h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{12}\right)\right) + h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{21}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{12}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{21}\right)\right)\right] f(\boldsymbol{\pi}_{12}(\mathbf{x})) dx_1 \ldots dx_n.
\end{aligned}$$

The last inequality follows from the fact that $f(\mathbf{x})$ is AI, i.e.,

$$f(\mathbf{x}) - f(\boldsymbol{\pi}_{12}(\mathbf{x})) \geq 0, \quad \forall\, x_1 \leq x_2.$$

Thus, it is sufficient to prove

$$h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{12}\right)\right) + h\left(\varphi\left(\mathbf{x}, \mathbf{a}_{21}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{12}\right)\right) - h\left(\varphi\left(\mathbf{x}, \mathbf{b}_{21}\right)\right) \geq 0. \tag{3.6}$$

Since ϕ is convex and supermodular, for $a_{(1)} \leq b_{(1)} \leq b_{(2)} \leq a_{(2)}$ and $x_1 \leq x_2$, we have

$$\begin{aligned}
&\varphi(\mathbf{x}, \mathbf{a}_{12}) - \varphi(\mathbf{x}, \mathbf{b}_{12}) = \phi(x_1, a_{(1)}) + \phi(x_2, a_{(2)}) - \phi(x_1, b_{(1)}) - \phi(x_2, b_{(2)}) \\
&\quad = \left[\phi(x_1, a_{(1)}) + \phi(x_1, a_{(2)}) - \phi(x_1, b_{(1)}) - \phi(x_1, b_{(2)})\right] \\
&\qquad + \left[\phi(x_2, a_{(2)}) - \phi(x_1, a_{(2)}) + \phi(x_1, b_{(2)}) - \phi(x_2, b_{(2)})\right] \geq 0,
\end{aligned}$$

$$\begin{aligned}
&\varphi(\mathbf{x}, \mathbf{a}_{12}) - \varphi(\mathbf{x}, \mathbf{b}_{21}) = \phi(x_1, a_{(1)}) + \phi(x_2, a_{(2)}) - \phi(x_2, b_{(1)}) - \phi(x_1, b_{(2)}) \\
&\quad = \left[\phi(x_1, a_{(1)}) + \phi(x_1, a_{(2)}) - \phi(x_1, b_{(1)}) - \phi(x_1, b_{(2)})\right] \\
&\qquad + \left[\phi(x_2, a_{(2)}) - \phi(x_1, a_{(2)}) + \phi(x_1, b_{(1)}) - \phi(x_2, b_{(1)})\right] \geq 0
\end{aligned}$$

and

$$\begin{aligned}
&\varphi(\mathbf{x}, \mathbf{a}_{12}) + \varphi(\mathbf{x}, \mathbf{a}_{21}) - \varphi(\mathbf{x}, \mathbf{b}_{12}) - \varphi(\mathbf{x}, \mathbf{b}_{21}) \\
&\quad = \phi(x_1, a_{(1)}) + \phi(x_1, a_{(2)}) - \phi(x_1, b_{(1)}) - \phi(x_1, b_{(2)}) \\
&\qquad + \phi(x_2, a_{(2)}) + \phi(x_2, a_{(1)}) - \phi(x_2, b_{(2)}) - \phi(x_2, b_{(1)}) \geq 0.
\end{aligned}$$

Thus, for any increasing convex function h, if $\varphi(\mathbf{x}, \mathbf{a}_{21}) \geq \varphi(\mathbf{x}, \mathbf{b}_{21})$, then $h(\varphi(\mathbf{x}, \mathbf{a}_{21})) \geq h(\varphi(\mathbf{x}, \mathbf{b}_{21}))$ and $h(\varphi(\mathbf{x}, \mathbf{a}_{12})) \geq h(\varphi(\mathbf{x}, \mathbf{b}_{12}))$, which implies (3.6). Otherwise, if $\varphi(\mathbf{x}, \mathbf{a}_{21}) \leq \varphi(\mathbf{x}, \mathbf{b}_{21})$, we have

$$(\varphi(\mathbf{x}, \mathbf{a}_{21}), \varphi(\mathbf{x}, \mathbf{a}_{12})) \succeq_{\mathrm{m}} (\varphi(\mathbf{x}, \mathbf{b}_{21}), \varphi(\mathbf{x}, \mathbf{a}_{21}) - (\varphi(\mathbf{x}, \mathbf{b}_{21}) - \varphi(\mathbf{x}, \mathbf{a}_{12}))).$$

Therefore,

$$\begin{aligned}
&h(\varphi(\mathbf{x}, \mathbf{a}_{12})) + h(\varphi(\mathbf{x}, \mathbf{a}_{21})) \\
&\quad \geq h(\varphi(\mathbf{x}, \mathbf{b}_{21})) + h(\varphi(\mathbf{x}, \mathbf{a}_{21}) - (\varphi(\mathbf{x}, \mathbf{b}_{21}) - \varphi(\mathbf{x}, \mathbf{a}_{12}))) \\
&\quad \geq h(\varphi(\mathbf{x}, \mathbf{b}_{21})) + h(\varphi(\mathbf{x}, \mathbf{b}_{12})),
\end{aligned}$$

where the first inequality is due to the convexity of h. Therefore, (3.6) holds and the desired result follows. ∎

4 An application to optimal capital allocation

In this section, we outline an application of our main results. Let $X_1, \ldots, X_n$ be n risks in a portfolio. Assume that a company wishes to allocate the total capital $p = p_1 + \ldots + p_n$ to the corresponding risks. As defined in Xu and Hu (2012), the loss function

$$L(\boldsymbol{p}) = \sum_{i=1}^{n} \phi(X_i - p_i), \quad \boldsymbol{p} \in A = \{\boldsymbol{p} \in \Re_+^n : p_1 + \ldots + p_n = p\}$$

is a reasonable criterion to set the capital amount p_i to X_i, where ϕ is convex. A good capital allocation strategy is to make the loss function $L(\boldsymbol{p})$ as small as possible in some sense. Besides, the different capital allocation strategies affect the general loss function via stochastic comparisons. Therefore, it is meaningful for us to find the best capital allocation strategy if it exists via the methods in Section 3.

Theorem 4.1 *If the joint density function of $X_1, \ldots, X_n$ satisfies assumptions (A1) and (A2) of Section 3, and if $\boldsymbol{p}^* = (p_1^*, \ldots, p_n^*)$ is the solution to the best capital allocation strategy, then, we have $p_1^* \leq p_2^* \leq \ldots \leq p_n^*$.*

Proof. Let $\boldsymbol{p} = (p_1, p_2, p_3, \ldots, p_n)$ be any admissible allocation, and let $\hat{\boldsymbol{p}} = (p_2, p_1, p_3, \ldots, p_n)$. Without loss of generality, we assume $p_1 \leq p_2$. By the nature of majorization, we only need to prove that

$$P\left(L(\hat{\boldsymbol{p}}) \geq t\right) \geq P\left(L(\boldsymbol{p}) \geq t\right), \quad \forall\ t.$$

That means

$$\phi(X_1 - p_2) + \phi(X_2 - p_1) + \sum_{i=3}^{n} \phi(X_i - p_i) \geq_{\rm st} \phi(X_1 - p_1) + \phi(X_2 - p_2) + \sum_{i=3}^{n} \phi(X_i - p_i).$$

Since the usual stochastic order is closed under convolution, we only need to prove

$$\phi(X_1 - p_2) + \phi(X_2 - p_1) \geq_{\rm st} \phi(X_1 - p_1) + \phi(X_2 - p_2). \tag{4.1}$$

Under assumptions (A1) and (A2), (4.1) holds due to Corollary 3.8. Therefore, the desired conclusion follows. ∎

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