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***Abstract*--The objective of this research is to establish consistency thresholds linked to alpha (α) levels for HDM's (Hierarchical Decision Model) judgment quantification method. Measuring consistency in order to control it is a crucial and inseparable part of any AHP/HDM experiment. The researchers on the subject recommend establishing thresholds that are statistically based on hypothesis testing, and are linked to the number of decision variables and α level. Such thresholds provide the means with which to evaluate the soundness and validity of an AHP/HDM decision. The linkage of thresholds to α levels allows the decision makers to set an appropriate inconsistency tolerance compatible with the situation at hand. The measurements of judgments are unreliable in the absence of an inconsistency measure that includes acceptable limits. All of this is essential to the credibility of the entire decision making process and hence is extremely useful for practitioners and researchers alike. This research includes distribution fitting for the inconsistencies. The superb fits obtained give confidence that all the statistical inferences based on the fitted distributions accurately reflect the HDM's inconsistency measure.**

I. BACKGROUND

Multi Criteria Decision Analysis (MCDA) is often used to analyze intricate and complex decision problems having multiple facets. It starts with identifying criteria and alternatives related to a decision objective. Numerical measures are then used to evaluate the relative importance of alternatives with regard to the criteria. Finally, the alternatives are prioritized and ranked [1]. By using such tools, users can analyze and evaluate complex problems having conflicting priorities and at the same time make sound decisions based on rational compromise.

The Hierarchical Decision Model (HDM) [2] which is a variant of Saaty's Analytic Hierarchy Process (AHP) [3] is a widely-accepted multi-criteria decision-making tool. The first step in the application of these methods involves structuring the decision problem into levels consisting of objectives and their associated criteria. The second step involves eliciting the preferences of the decision maker (DM) through pairwise comparisons. The third step is to process the DM's input and calculate the priorities of the objectives. The final step before analyzing the decision is to check the DM's consistency. This measure ensures that the pairwise comparisons are neither random nor illogical.

For the Hierarchical Decision Model (HDM), Cleland and Kocaoglu [4] use a variance-based approach to calculate the inconsistency, and recommend a 10% limit above which the reliability of the expert's judgment would be considered questionable. Similarly, for the Analytic Hierarchy Process (AHP), Saaty suggests using the consistency ratio (CR) and recommends an upper limit of 10% on CR [3].

Saaty's fixed 10% rule has been the subject of much criticism/dispute for being too restrictive, lacking statistical justification, and not being a function of the number of elements (decision variables) being compared.

II. RESEARCH OBJECTIVE

The objective of this research is to:

1. Establish the significance of analyzing inconsistency in decision making
2. Show the research gap for HDM with regard to establishing consistency thresholds that are:
   a. Linked to the number of variables.
   b. Based on statistical hypothesis testing.
   c. Linked to corresponding α levels.
3. Address the above-mentioned research gap.

The first 2 objectives are addressed by conducting a thorough literature review. The 3rd objective is covered by a new methodology.

The methodology used in this research is based on testing the null hypothesis that the judgmental responses obtained from a respondent are random. Rejecting this null hypothesis will mean that the inconsistency of the respondent is significantly lower than what would be expected from random judgement responses.

III. DEFINITION OF INCONSISTENCY MEASURE IN HDM

In HDM, the relative value of each of the *n* variables is calculated *n* factorial times based on the ratios among them. The arithmetic mean of the values is the weight of the variable. The normalized weights of the variables make up the weight vector. The variance of the mean among the *n*! values of a single variable is calculated and the sum of variances for *n* values is computed. The inconsistency measure for HDM proposed in this research is the square Root of the Sum of Variances (RSV) defined as follows for *n* decision elements:

$$RSV = \sqrt{\sum_{i=1}^{n} \sigma_i^2} \quad (1)$$

where σ_i^2 is the variance of the mean of the i^{th} decision element, and *n* is the number of decison elements:

$$\sigma_i^2 = \frac{1}{n!} \sum_{j=1}^{n!} \left(x_{ij} - \bar{x}_{ij}\right)^2 \ \forall\ i = 1, \cdots, n \quad (2)$$

where x_{ij} is the normalized relative value of the variable *i* for the j^{th} orientation in *n* factorial orientations, and $\bar{x}_{ij}$ is the mean of the normalized relative value of the variable *i* for the j^{th} orientation:

$$\bar{x}_{ij} = \frac{1}{n!} \sum_{j=1}^{n!} x_{ij} \tag{3}$$

IV. WIDESPREAD USE OF HDM AND AHP

There is widespread evidence that AHP and its variants such as HDM are important research areas in the field of decision making. Merely a decade after its proposal, even an AHP critic admitted that AHP has established itself as a "major tool in multi-criteria decision analysis" [5]. The widespread acceptance of AHP in the US and worldwide is often attributed to the power and simplicity of AHP [6, 7]. The applicability and flexibility of AHP has also contributed to its great popularity and has helped make it one of the most widely-used decision-making tools [7-10]. AHP and its variants have been applied in a multitude of fields across all sectors where decision-making is needed [7, 10-13]. All of this has given AHP "an impressive record of success" [14]. It is of great importance and relevance to point out that the popularity and success of AHP has also made it a heavily researched area in decision making [7, 8, 10, 15]. The sheer volume of research articles on AHP and its variants [10] and numerous literature reviews on the same subject [10, 15-17] clearly establish this as one of the most important areas of research in decision making science.

V. THE TOPIC OF CONSISTENCY IN HDM/AHP

Before the synthesis of single-level priorities or aggregate priorities across multiple levels, HDM/AHP models require assignment of preference to the various elements being compared. This is done by eliciting the input of a decision maker or an expert in pairwise comparison of the elements. Inconsistency in the choices made by a decision maker is the contradiction in terms of order of preference (ordinal inconsistency) or in terms of relative degree of preference (cardinal inconsistency). Compliance with cardinal consistency leads necessarily to compliance with ordinal consistency but not vice versa. Since inconsistency in pairwise comparisons can directly affect the quality and integrity of the order and degree of preference in the final result, there is consensus among decision scientists that inconsistency should be measured and controlled within an upper limit.

The importance of consistency in AHP is well stated by AHP's original author, Thomas Saaty, "how to measure inconsistency and improve the judgments to obtain better consistency is a concern of the AHP" [18]. Because the soundness of the result of an AHP model, or any pairwise comparison for that matter, is directly related to consistency, the analysis of this parameter is a critical step [19, 20], and an important consideration in AHP [21, 22]. In AHP, improving consistency improves the validity of judgments [1]. All of these considerations have made the topic of consistency one of the most researched topics in AHP [23-27].

VI. EXISTING CONSISTENCY MEASURES

To establish the significance of the research topic and to identify the research gaps, several books and more than 180 journal articles were reviewed. A large number of researchers have focused on the topic and developed consistency measures. Most of the studies are for improvement of CI (Consistency Index) measure of Saaty's REV (Right Eigenvector method) or for suggesting alternatives to it. Some are applicable to other judgment quantification methods, but none is addressed to the inconsistency in HDM's measurements.

The literature on consistency measures is summarized below.

Barzilai and Golany [28] advocate the geometric mean method and claim it is the only acceptable method for multiplicative Pairwise Comparison Matrices (PCMs) to simultaneously satisfy immunity to rank reversals, independence of problem description, independence of scale inversion, left-right eigenvector asymmetry, uniqueness, independence of order of operations and inter-level consistency. For additive PCMs, the authors suggest using the arithmetic mean method.

Golden and Wang [29] advocate using the Row Geometric Mean Method. They seek to develop a measure of consistency that is easy to use, is a function of matrix size, and has an intuitively appealing probability distribution.

Ra [30] recommends the Sum of Inverse Column Sums (SICS) to measure consistency. The procedure is very simple, the columns of the PCM are summed, and then their reciprocals are added. SICS ranges in value from 0 to 1 with 1 indicating perfect consistency. Based on a simulation study using 1,000 matrices, Ra provides thresholds for SISC for $n = 3 - 13$. The "Standard" limits ensure very good consistency compliance, while the "Average" limits ensure ordinal consistency with minimal cardinal consistency violation. In a later study by Kretchik and Ra [31], SICS is presented as a consistency measure that is easy to use, is independent of the prioritization procedure, and is well bounded. Beta distribution is shown to be a good fit for SICS probability distribution.

Jensen and Hicks [32] indicate that when considering tradeoffs regarding validity, reliability, consistency, and solution determinacy, being "finely cardinal" is not necessarily better or worse than being "coarsely ordinal". Strictly for ordinal consistency, they propose to use Kendall's Coefficient of Consistency, and provide computation formulas for the cases of with or without preference equivalence.

Koczkodaj [33] suggests a consistency measure that allows selection of thresholds, and can link inconsistency to a particular element rather than an abstract value such as λ_{max}.

His consistency measure is computed among each triplet of the PCM elements.

Bozoki and Rapcsak [34] extend Koczkodaj's [33] definition of consistency measure to the entire PCM based on triplets.

Takeda [35] proposes a Measure of Consistency (MC) for the Row Geometric Mean Method.

Wedley [36] suggests that for the sake of efficiency, instead of having to fill n(n-1)/2 paired comparisons, a DM needs only to do n-1 comparisons. The rest of the comparisons are redundant and can be filled by a computer algorithm. Such a procedure, in addition to achieving efficiency, also builds good ordinal consistency. Wedley defines Average Absolute Deviation in Consistency Indexes as the consistency measure for filling incomplete PCMs.

Takeda and Yu [37] propose a consistency measure based on the geometric mean of the pairwise comparisons.

Monsuur [38] has a consistency measure, k, which is scale independent, and can be adjusted to the decision situation. He recommends an upper limit for the consistency measure of $k \geq 0.9$. For $n = 3 - 9$ these thresholds closely match λ_{max} upper limits for consistent PCMs calculated by Vargas [39].

Salo and Hämäläinen [40] use a scale-invariant Consistency Measure (CM).

Barzilai [41] formulates a consistency measure RC (Relative Consistency) for the multiplicative case based on his earlier measure for the additive case [28]. First, the multiplicative PCM (M) is transformed to its "equivalent" additive matrix (A). Next the consistent components of A are computed in determining RC.

Shiraishi, *et al.* [42] define the characteristic polynomial of a PCM in developing a consistency measure *"c_3"* as the coefficient of the characteristic polynomial.

Crawford and Williams [43] use a consistency measure for the Row Geometric Mean Method (RGMM). Aguaron and Moreno-Jimenez [44] formalize the measure, call it the Geometric Consistency Index (GCI), and provided the thresholds associated with it.

Peláez and Lamata [45] have a Consistency Index (CI*) that is a function of matrix size, and is applicable to reciprocal matrices.

Gass and Rapcsák [46] use the Singular Value Decomposition (SVD) method as a prioritization procedure. They suggest the Frobenius norm of the difference between the original PCM and one formed by the SVD as an Inconsistency Measure (IM). The authors note that linking this measure to practical application and the DM's confidence still needs to be developed.

Alonso and Lamata [47] have a statistical consistency acceptance criterion that is linked to matrix size and has thresholds based on α levels.

Fedrizzi and Giove [48] have a method for calculating the missing elements of an additive incomplete PCM.

Stein and Mizzi [49] suggest using the Harmonic Consistency Index. The harmonic sum $HM(s)$ is simply the sum of the inverse sum of the columns which is identical to what Ra [30], and Kretchik and Ra [31] have proposed years earlier.

Fedrizzi and Brunelli [50] show that the further the pairwise judgments are from the neutral "indifference" position, the harder it is for the DM to achieve consistency, and vice versa. This phenomenon which the authors call "strength of preference effect" results in the DM with strong preference choices being penalized. The authors state that almost all consistency measures suffer from this shortcoming. To remedy this situation, they offer an approach for assessing consistency which they call "consistency equivalence classes".

Čaklović [51] utilizes the Potential Method (PM) to develop a consistency measure. In PM, inconsistency is defined as the angle between the original preference flow and its consistent approximation.

Matteo, *et al.* [52] compared 2 pairs of the above mentioned consistency indices: CI^* from Peláez and Lamata [45] to c_3 from Shiraishi, *et al.* [42], as well as GCI from Aguaron and Moreno-Jimenez [44] to ρ from Fedrizzi and Giove [48]. The authors show proportionality between the first pair as well as the second, and suggest that their results should be used by researchers before embarking on developing new consistency measures in order to avoid duplication of effort.

Siraj, *et al.* [53] have a prioritization procedure based on all possible element combinations from tree spanning of pairwise comparisons. In the new method, Enumerating All Spanning Trees (EAST), the weight vector is composed of the average of individual weights computed for each tree. The consistency measure is the variance among the weight vector.

There is consensus in the literature on the importance of defining, understanding, controlling, and improving consistency in AHP in order to build reliability, confidence and meaningfulness in the entire process of AHP decision making [3, 29, 33, 38, 40, 41, 45, 49, 54]. Considerable research effort on achieving these goals for AHP consistency spans the 3 decades since the introduction of this decision-making tool. This proves the significance of this topic for both researchers and practitioners.

VII. RESEARCH GAPS

The table below summarizes the research gaps identified.

TABLE 1: SUMMARY OF RESEARCH GAPS

No	Article	Critical Issue Discussed	Research Gaps
1	Barzilai and Golany [28]	Provide consistency measure for additive Pairwise Comparison Matrices (PCMs)	The multiplicative case is not addressed, no thresholds for the measure are provided, and the measure is not linked to number of elements or α levels. The results are not in ratio scale.
2	Golden and Wang [29]	Provide consistency measure for Row Geometric Mean Method (RGMM)	The measure is applicable only to RGMM. Although the measure is linked to number of elements, thresholds and α levels would be hard to establish due to the rough fit to normal distribution.
3	Ra [30], Kretchik and Ra [31]	Provide consistency measure for HDM's Constant Sum (CS)	The fit of the measure to a beta distribution is quite rough. This made linking it to number of elements weak and consequently multiple thresholds and α levels were not established. No theoretical justification for the measure is given, and therefore its interpretation is unknown. Finally, the upper bound for the measure is unproven particularly for higher order matrices.
4	Jensen and Hicks [32]	Provide ordinal consistency measure for pairwise comparisons	Measures only ordinal consistency. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
5	Koczkodaj [33]	Provide consistency measure for pairwise comparison matrices (PCMs)	Good measure with the added benefit of locating inconsistency within a triplet. Few recommended thresholds have been established for low order matrices. Extending these to higher orders is yet to be done. Thresholds are not linked to α levels.
6	Takeda [35]	Provide consistency measure for RGMM	The measure is applicable only to RGMM. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
7	Wedley [36]	Provide a per step consistency check for use while filling incomplete matrices	The measure is specifically for filling incomplete matrices. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
8	Takeda and Yu [37]	Provide a consistency measure for a subset of a Pairwise Comparison Matrix (PCM)	The measure is specifically for a subset of a PCM. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
9	Monsuur [38]	Provide an intrinsic consistency measure that is scale independent	The measure is linked to the abstract quantity of maximum eigenvalue. No statistically based thresholds or corresponding α levels are provided.
10	Salo and Hämäläinen [40]	Provide consistency measure that is scale invariant	The measure is more suitable for distance-minimizing methods. The measure is not linked to the matrix order and no statistically based thresholds or corresponding α levels are provided.
11	Barzilai [41]	Provide consistency measure that ensures immunity to rank reversal, independence of problem description, independence of scale inversion, left-right eigenvector asymmetry, uniqueness, independence of order of operations and inter-level consistency	The measure has many advantages. However, it is cumbersome to calculate (involves transforming multiplicative PCMs to their additive equivalents), is unbounded, and lacks statistically based thresholds and their corresponding α levels.
12	Shiraishi*, et al.* [42]	Provide consistency measure for positive reciprocal matrices	The measure is not linked to the matrix order and no statistically based thresholds or corresponding α levels are provided.
13	Crawford and Williams [43], Aguaron and Moreno-Jimenez [44]	Provide consistency measure for RGMM and provide thresholds for the measure	The measure is applicable only to RGMM. Because of GCI's independence of order, the thresholds were approximated by establishing a relationship to CR. The thresholds are provided for n = 3, 4, and all matrices > 4. This makes the measure's link to the number of elements quite weak. Only 4 α levels are given.
14	Peláez and Lamata [45]	Provide a consistency index that is easy to use, is a function of matrix size, and is applicable to other types of reciprocal matrices	The measure is a function of matrix size, has statistically based thresholds, and corresponding α levels. However, the thresholds and the α levels are for Saaty's scale only, the measure is mathematical and its applicability to judgment quantification in HDM is questionable. Is this measure proportional to HDM's inconsistency measure? Will it work in concert with HDM's statistical prioritization procedure?
15	Gass and Rapcsák [46]	Provide consistency measure for Singular Value Decomposition (SVD) method	The measure is applicable only to SVD. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
16	Alonso and Lamata [47]	Provide consistency measure for REV method	The measure is a function of matrix size, has statistically based thresholds, and corresponding α levels. However, it is applicable only to REV.
17	Fedrizzi and Giove [48]	Provide consistency measure for additive PCMs	The measure is applicable only to additive PCMs. No thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
18	Stein and Mizzi [49]	Provide consistency measure for PCMs	The measure is theoretically applicable to all PCMs. However, no thresholds for the measure are provided, and the measure is not linked to number of elements or α levels.
19	Fedrizzi and Brunelli [50]	Provide a consistency approach that takes into account "strength of preference effect"	A consistency approach rather than a measure.
20	Čaklović [51]	Provide consistency measure for the Potential Method	The measure is a function of matrix size, has statistically based thresholds, and corresponding α levels. However, it is only applicable to the Potential Method.
21	Siraj*, et al.* [53]	Provide consistency measure for the method of Enumerating All Spanning Trees (EAST)	The method is similar to HDM's: The weight vector is the average of variable weights which are computed for many "orientations". However, the consistency measure is applicable only to "EAST". It is not linked to the matrix order and no statistically based thresholds or corresponding α levels are provided.

In summary, many of the prioritization procedures lack an inconsistency measure, and many of the ones that do provide inconsistency measures have global limits defined without considering the number of elements involved or the α levels required [44, 47].

As a variant of AHP, HDM's procedure for judgment quantification, which was developed by D.F. Kocaoglu, defines inconsistency as follows:

$$Inconsistency = \frac{1}{n}\sum_{i=1}^{n}\sigma_i \quad (4)$$

where σ_i is the standard deviation of the mean of n factorial normalized relative values for the i^{th} decision element, and n is the number of decisison elements.

The acceptable limit is 0.1. It does not vary with number of elements, and is not linked to a α levels.

Clearly, this research gap presents an opportunity to complete the development of this important metric.

Wide-spread research [29, 31, 44, 47, 51, 55] indicates that a statistical approach built on the estimated distribution of the inconsistency parameter is the way to achieve the desirable inconsistency properties of

a) Being a function of the number of elements
b) Having limits linked to α levels

The research presented in this paper does this for the Hierarchical Decision Model (HDM) procedure.

VIII. RESEARCH APPROACH

A method has been developed, in this study, for analyzing decision inconsistencies using the HDM's judgment quantification method in response to the key gap that has been identified in the literature. The research question is: How can HDM's consistency thresholds be defined to comply with the requirements of:

1. Being a function of the size of the decision problem.
2. Being subjected to hypothesis testing.
3. Being defined as a distribution.
4. Being linked to α levels.

The literature review shows that the method of choice among researchers for defining consistency thresholds with the above desired properties is through computer simulation of randomly generated inputs into the judgment quantification methods.

HDM inconsistency is defined in this research as the square root of the sum of variances (RSV) of the means of n variables calculated in n factorial orientations:

$$RSV = \sqrt{\sum_{i=1}^{n}\sigma_i^2} \quad (5)$$

It is a modified version of the current inconsistency measure used in HDM, which is

$$Inconsistency = \frac{1}{n}\sum_{i=1}^{n}\sigma_i \quad (6)$$

This modification was necessary because the numerical values for the current measure were very small and the precision was being lost when large numbers of randomly generated input matrices were analyzed.

Below is the simulation procedure used for defining the consistency thresholds for HDM's judgment quantification method:

1. Setup input data structure: This involves building Matrix "A" of the Constant Sum method, which is an $n \times n$ matrix.
2. Fill in the data structure: This will be done by populating either side of the left diagonal of Matrix "A" with randomly generated numbers in the range of 1 – 99. The other half of the matrix will be filled with the 100-compliment of the mirror positions on the other diagonal side.
3. Perform necessary calculations: This will involve building matrices "B" and "C" of the Constant Sum Method, defining the n factorial orientations for all the elements, computing elements' values for all orientations using the direct and indirect ratios derived from Matrix "C", calculating the mean and variance of each element from all orientations, and finally computing the RSV (Square Root of the Sum of Variances) for all elements as the measure of inconsistency.
4. Store results: Save the results from each run.
5. Repeat the above steps: The above represents the computations for one set of simulated judgment inputs. The process is repeated for 100,000 sets of input data. Initial testing shows that stability is reached well before that level. Also, literature shows that there is no statistically significant difference in repeating the simulation beyond 100,000 cycles under any condition.
6. Analyze the results: This involves plotting the sample's histogram, determining the sample's statistical parameters such as the minimum, maximum, mean, percentiles, cumulative distribution function (CDF), and Quantile function.
7. Perform curve fitting and test goodness-of-fit (GOF) using the Kolmogorov–Smirnov test (K–S test).

IX. SUMMARY OF RESULTS

Following is a summary of the highlights of the research results.

- The inconsistency thresholds were defined for $n = 3 - 12$ and corresponding fitted distributions were obtained.
- For each of the fitted distributions, the equations for the cumulative distribution and the quantile functions along with their specific set of parameters were identified.
- For $n = 3$, the fitted distribution is 3-parameter generalized gamma.
- For $n = 4 - 12$, the fitted distribution is Johnson SB.

- The GOF results are very good:
  - For $n = 3$, the GOF is "Do Not Reject" at all significance (α) levels (0.01, 0.02, .05, 0.1, and 0.2) for the K-S GOF test.
  - For $n = 4 - 12$, the GOF is "Do Not Reject" at all significance (α) levels (0.01, 0.02, .05, 0.1, and 0.2) for all GOF tests (K-S, Anderson-Darling, and Chi-Squared)
- All data verifications were performed with satisfactory outcome:
  - No significant difference was found between the 10k and 100k simulation data.
  - No significant difference was found between the 100k and 500k simulation data.
  - No significant difference was found between the results from the fit equations and 500k simulation data.

RSV is based on the sum of the variances of the decision variables calculated for each variable in $n!$ orientations of the variables . The variance in the relative value of a variable decreases as the number of variables increases. Consequently, when the number of variables reaches 13, the required growth in the sum of variances is no longer sufficient to provide the necessary increase for a new set of RSV values suitable for the new level (13). Therefore, the RSV measure cannot be used for calculations involving variables higher than 12.

Figure 1 shows the inconsistency limits for $n = 3 - 12$ variables at α from 0.01 to 0.50. The inconsistency value shown on the y-axis is the threshold limit, defining the maximum inconsistency that would be observed with the probability α shown on the x-axis if the judgmental values for the decision element were obtained randomly. In other words, the probability of randomness in the judgmental values obtained for decision elements is less than or equal to α for the number of elements (n) being compared pairwise.

Tables 2 and 3 list the numerical values of the inconsistency threshold limits for n (number of decision variables) varying between 3 and 12 with α from 0.01 to 0.25 and 0.26 to 0.50 at 0.01 intervals, respectively.

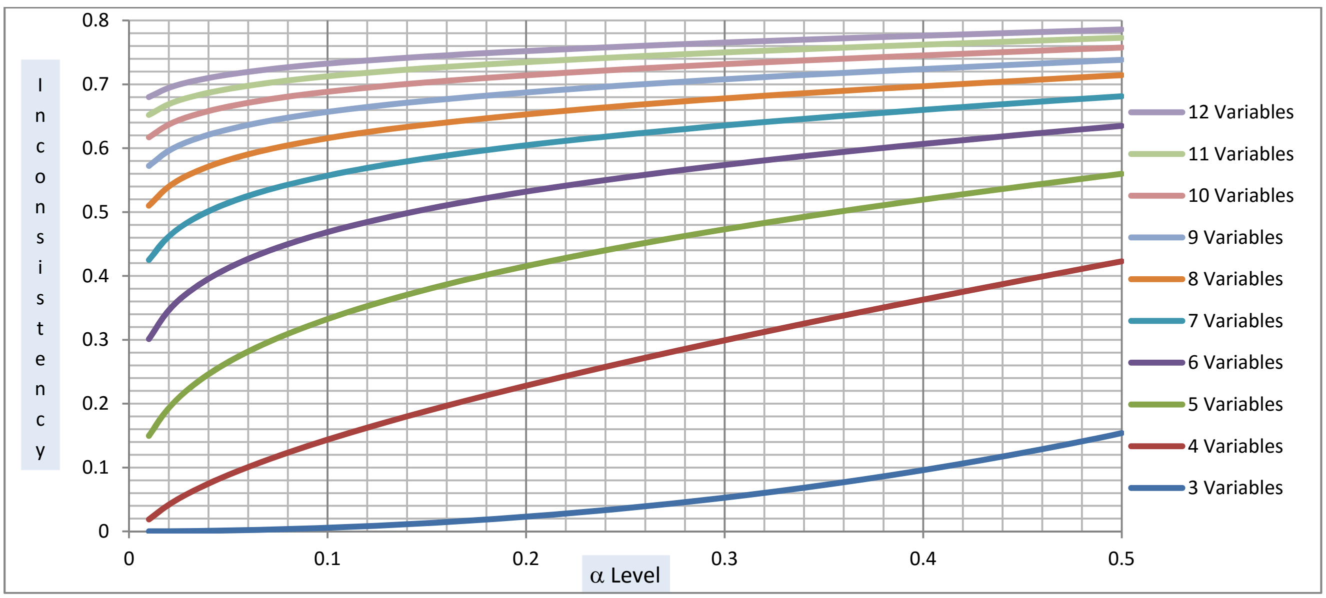

Figure 1: Inconsistency Threshold Limits for 3 – 12 Decision Variables

TABLE 2: INCONSISTENCY THRESHOLD LIMITS FOR 3 – 12 DECISION VARIABLES AT α = 0.01 TO α = 0.25

	Number of variables									
Percentile (α)	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0.01	0.0001	0.0188	0.1495	0.3012	0.4249	0.5100	0.5723	0.6170	0.6521	0.6800
0.02	0.0002	0.0418	0.1934	0.3462	0.4620	0.5400	0.5961	0.6371	0.6690	0.6945
0.03	0.0005	0.0596	0.2230	0.3745	0.4847	0.5581	0.6105	0.6493	0.6793	0.7034
0.04	0.0009	0.0748	0.2460	0.3955	0.5012	0.5713	0.6211	0.6582	0.6868	0.7100
0.05	0.0014	0.0884	0.2651	0.4124	0.5143	0.5818	0.6295	0.6653	0.6928	0.7152
0.06	0.0021	0.1008	0.2816	0.4266	0.5253	0.5904	0.6365	0.6712	0.6978	0.7196
0.07	0.0028	0.1124	0.2963	0.4390	0.5347	0.5979	0.6425	0.6763	0.7022	0.7234
0.08	0.0037	0.1233	0.3095	0.4499	0.5430	0.6045	0.6478	0.6807	0.7060	0.7267
0.09	0.0046	0.1337	0.3215	0.4597	0.5505	0.6104	0.6526	0.6848	0.7095	0.7298
0.10	0.0057	0.1437	0.3327	0.4686	0.5572	0.6157	0.6569	0.6884	0.7126	0.7325
0.11	0.0069	0.1532	0.3430	0.4769	0.5634	0.6206	0.6609	0.6918	0.7155	0.7350
0.12	0.0083	0.1625	0.3528	0.4845	0.5691	0.6252	0.6646	0.6949	0.7182	0.7374
0.13	0.0097	0.1714	0.3620	0.4916	0.5745	0.6294	0.6681	0.6978	0.7207	0.7396
0.14	0.0113	0.1801	0.3706	0.4983	0.5795	0.6334	0.6713	0.7005	0.7231	0.7417
0.15	0.0129	0.1886	0.3789	0.5047	0.5842	0.6371	0.6743	0.7031	0.7253	0.7436
0.16	0.0147	0.1969	0.3868	0.5106	0.5887	0.6406	0.6772	0.7056	0.7274	0.7455
0.17	0.0166	0.2050	0.3944	0.5163	0.5929	0.6440	0.6800	0.7079	0.7294	0.7473
0.18	0.0187	0.2129	0.4016	0.5218	0.5970	0.6472	0.6826	0.7101	0.7313	0.7490
0.19	0.0207	0.2207	0.4086	0.5270	0.6009	0.6502	0.6851	0.7122	0.7332	0.7506
0.20	0.0230	0.2283	0.4154	0.5320	0.6046	0.6532	0.6875	0.7143	0.7350	0.7521
0.21	0.0254	0.2359	0.4219	0.5368	0.6081	0.6560	0.6899	0.7163	0.7367	0.7537
0.22	0.0281	0.2433	0.4282	0.5414	0.6116	0.6587	0.6921	0.7182	0.7383	0.7551
0.23	0.0306	0.2506	0.4343	0.5459	0.6149	0.6614	0.6943	0.7200	0.7399	0.7565
0.24	0.0334	0.2578	0.4403	0.5502	0.6181	0.6639	0.6964	0.7218	0.7415	0.7579
0.25	0.0363	0.2648	0.4461	0.5544	0.6212	0.6664	0.6984	0.7235	0.7430	0.7592

TABLE 3: INCONSISTENCY THRESHOLD LIMITS FOR 3 – 12 DECISION VARIABLES AT α = 0.26 TO α = 0.50

	Number of variables									
Percentile (α)	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
0.26	0.0393	0.2719	0.4517	0.5585	0.6243	0.6688	0.7004	0.7252	0.7445	0.7605
0.27	0.0425	0.2788	0.4572	0.5625	0.6272	0.6711	0.7023	0.7268	0.7459	0.7618
0.28	0.0457	0.2856	0.4626	0.5664	0.6301	0.6734	0.7042	0.7284	0.7473	0.7630
0.29	0.0491	0.2924	0.4679	0.5701	0.6329	0.6756	0.7061	0.7300	0.7487	0.7643
0.30	0.0526	0.2991	0.4730	0.5738	0.6356	0.6778	0.7079	0.7315	0.7500	0.7654
0.31	0.0564	0.3057	0.4780	0.5774	0.6383	0.6799	0.7096	0.7330	0.7513	0.7666
0.32	0.0602	0.3123	0.4830	0.5809	0.6409	0.6820	0.7113	0.7345	0.7526	0.7677
0.33	0.0641	0.3188	0.4878	0.5843	0.6434	0.6840	0.7130	0.7359	0.7538	0.7689
0.34	0.0682	0.3253	0.4926	0.5877	0.6460	0.6860	0.7147	0.7374	0.7551	0.7700
0.35	0.0725	0.3317	0.4972	0.5910	0.6484	0.6880	0.7163	0.7388	0.7563	0.7710
0.36	0.0769	0.3380	0.5018	0.5942	0.6508	0.6899	0.7179	0.7401	0.7575	0.7721
0.37	0.0815	0.3443	0.5063	0.5974	0.6532	0.6918	0.7195	0.7415	0.7587	0.7732
0.38	0.0862	0.3506	0.5108	0.6006	0.6555	0.6937	0.7211	0.7428	0.7599	0.7742
0.39	0.0911	0.3568	0.5152	0.6037	0.6579	0.6955	0.7226	0.7441	0.7610	0.7752
0.40	0.0960	0.3630	0.5195	0.6067	0.6601	0.6973	0.7241	0.7454	0.7621	0.7762
0.41	0.1012	0.3691	0.5237	0.6097	0.6624	0.6991	0.7256	0.7467	0.7633	0.7772
0.42	0.1065	0.3752	0.5279	0.6126	0.6646	0.7009	0.7271	0.7480	0.7644	0.7782
0.43	0.1119	0.3813	0.5321	0.6155	0.6668	0.7026	0.7286	0.7492	0.7655	0.7792
0.44	0.1174	0.3873	0.5362	0.6184	0.6689	0.7044	0.7300	0.7505	0.7666	0.7802
0.45	0.1233	0.3933	0.5402	0.6213	0.6710	0.7061	0.7314	0.7517	0.7677	0.7812
0.46	0.1292	0.3993	0.5443	0.6241	0.6732	0.7078	0.7329	0.7529	0.7687	0.7821
0.47	0.1351	0.4052	0.5482	0.6268	0.6752	0.7094	0.7343	0.7541	0.7698	0.7831
0.48	0.1411	0.4112	0.5521	0.6296	0.6773	0.7111	0.7357	0.7553	0.7709	0.7840
0.49	0.1473	0.4170	0.5560	0.6323	0.6794	0.7128	0.7371	0.7565	0.7719	0.7849
0.50	0.1539	0.4229	0.5599	0.6350	0.6814	0.7144	0.7384	0.7577	0.7729	0.7859

XI. APPLICATION EXAMPLE

As an illustrative example, two experts (expert I and expert II) are asked to rank 6 criteria (A, B, ⋯, F) in a pairwise comparison fashion. The inconsistency tolerance is set at a maximum of 0.05 by the decision maker conducting the experiment. Below are the results.

Expert I

Inputs:

A : B	A : C	A : D	A : E	A : F	B : C	B : D	B : E
80 : 20	62 : 38	38 : 62	64 : 36	80 : 20	88 : 12	68 : 32	17 : 83

B : F	C : D	C : E	C : F	D : E	D : F	E : F
50 : 50	28 : 72	17 : 83	14 : 86	59 : 41	25 : 75	83 : 17

Weights:

A	B	C	D	E	F
0.24	0.15	0.06	0.15	0.25	0.15

Inconsistency:

$$RSV = \sqrt{0.037947 + 0.018851 + 0.003697 + 0.021697 + 0.04008 + 0.020164} = 0.3774$$

Referring to table 2 under values for 6 decision variables, 0.37741 corresponds to α between 0.03 and 0.04. This means the inconsistency of expert I is within the acceptable set limit, and the choices should be considered valid.

Expert II

Inputs:

A : B	A : C	A : D	A : E	A : F	B : C	B : D	B : E
85 : 15	71 : 29	30 : 70	64 : 36	62 : 38	88 : 12	68 : 32	9 : 91

B : F	C : D	C : E	C : F	D : E	D : F	E : F
67 : 33	28 : 72	17 : 83	14 : 86	59 : 41	25 : 75	83 : 17

Weights:

A	B	C	D	E	F
0.22	0.14	0.05	0.16	0.27	0.15

Inconsistency:

$$RSV = \sqrt{0.04554 + 0.02301 + 0.004083 + 0.03140 + 0.05684 + 0.02589} = 0.4322$$

Referring to table 2 under values for 6 decision variables, 0.4322 corresponds to α between 0.06 and 0.07. This means the inconsistency of expert II is higher than the acceptable set limit, and expert II should be asked to revise the choices and make the pairwise assignments more consistent.

XII. CONCLUSION

The importance of measuring and controlling consistency in any AHP/HDM application cannot be overemphasized. Nonetheless, any consistency measure without meaningful thresholds remains mainly abstract and offers limited practical benefit. HDM's proposed inconsistency measure, RSV, along with the thresholds established as a result of this research fulfill all the requirements previously established for a robust, useful, and practical consistency measure. RSV and its thresholds are:

- A function of the number of decision variables
- Derived using statistical hypothesis testing
- Linked to any desired α levels

The thresholds allow decision makers who provide data through pairwise comparisons as well as decision makers who use or apply the decisions based on those pairwise choices to assess the soundness and validity of their decisions. Moreover, the thresholds allow the various decision makers to select a particular level of α which is appropriate to the specific circumstances of the decision problem.

XIII. CONTRIBUTION

The contribution of this research is to fill an important research gap identified through the literature review by defining the acceptable limits of inconsistency for any number of decision elements from 3 to 12 at any given α level in HDM calculations.

A byproduct of this research includes two fundamental improvements to HDM's judgement quantification method to enhance its speed and efficiency while maintaining a high degree of accuracy. This is done by the development of new computation algorithms that drastically reduce computational burden thereby greatly increasing the method's speed and consequently making it truly practical.

XIV. FUTURE WORK

The development of RSV as a measure of inconsistency in HDM, with thresholds at desired α levels, and defined as a function of the number of decision variables has met all the goals set out in the proposal of this paper. It has also identified the opportunity for future work to extend the inconsistency measure to one which:

- Can be used for calculations involving more than twelve decision variables
- Is independent of the judgment quantification method, thereby eliminating the need for the calculation of the variance of the means of *n* variables in *n* factorial orientations.
- Is universally applicable to any pairwise-comparison based method.

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