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Breaking Spaces and Forms for the DPG Method and Applications Including Maxwell Equations

Carsten Carstensen
Humboldt-Universität zu Berlin

Leszek Demkowicz
University of Texas at Austin

Jay Gopalakrishnan
Portland State University, gjay@pdx.edu

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BREAKING SPACES AND FORMS FOR THE DPG METHOD AND APPLICATIONS INCLUDING MAXWELL EQUATIONS

C. CARSTENSEN, L. DEMKOWICZ, AND J. GOPALAKRISHNAN

Abstract. Discontinuous Petrov Galerkin (DPG) methods are made easily implementable using "broken" test spaces, i.e., spaces of functions with no continuity constraints across mesh element interfaces. Broken spaces derivable from a standard exact sequence of first order (unbroken) Sobolev spaces are of particular interest. A characterization of interface spaces that connect the broken spaces to their unbroken counterparts is provided. Stability of certain formulations using the broken spaces can be derived from the stability of analogues that use unbroken spaces. This technique is used to provide a complete error analysis of DPG methods for Maxwell equations with perfect electric boundary conditions. The technique also permits considerable simplifications of previous analyses of DPG methods for other equations. Reliability and efficiency estimates for an error indicator also follow. Finally, the equivalence of stability for various formulations of the same Maxwell problem is proved, including the strong form, the ultraweak form, and a spectrum of forms in between.

1. Introduction

When a domain Ω is partitioned into elements, a function in a Sobolev space like $H(\mathrm{curl}, \Omega)$ or $H(\mathrm{div}, \Omega)$ has continuity constraints across element interfaces, e.g, the former has tangential continuity, while the latter has continuity of its normal component. If these continuity constraints are removed from the space, then we obtain "broken" Sobolev spaces. Discontinuous Petrov Galerkin (DPG) methods introduced in [14, 16] used spaces of such discontinuous functions in broken Sobolev spaces to localize certain computations. The studies in this paper begin by clarifying this process of breaking Sobolev spaces. This process, sometimes called hybridization, has been well studied within a discrete setting. For instance, the hybridized Raviart-Thomas method [5, 11, 28] is obtained by discretizing a variational formulation and then removing the continuity constraints of the discrete space, i.e., by discretizing first and then hybridizing. In contrast, in this paper, we identify methods obtained by hybridizing first and then discretizing, a setting more natural for DPG methods. We then take this idea further by connecting the stability of formulations with broken spaces and unbroken spaces, leading to the first convergence proof of a DPG method for Maxwell equations.

The next section (Section 2) is devoted to a study of the interface spaces that arise when breaking Sobolev spaces. These infinite-dimensional interface spaces can be used to connect the broken and the unbroken spaces. The main result of Section 2, contained in Theorem 2.3, makes this connection precise and provides a surprisingly elementary characterization (by

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duality) of the natural norms on these interface spaces. This theorem can be viewed as a generalization of a similar result in [29].

Having discussed breaking spaces, we proceed to break variational formulations in Section 3. The motivation for the theory in that section is that some variational formulations set in broken spaces have another closely related variational formulation set in their unbroken counterpart. This is the case with all the formulations on which the DPG method is based. The main observation of Section 3 is a simple result (Theorem 3.1) which in its abstract form seems to be already known in other studies [22]. In the DPG context, it provides sufficient conditions under which stability of broken forms follow from stability of their unbroken relatives. As a consequence of this observation, we are able to drastically simplify many previous analyses of DPG methods. The content of Sections 2 and 3 can be understood without reference to the DPG method.

A quick introduction to the DPG method is given in Section 4, where known conditions needed for *a priori* and *a posteriori* error analysis are also presented. One of the conditions is the existence of a Fortin operator. Anticipating the needs of the Maxwell application, we then present, in Section 5, a sequence of Fortin operators for $H^1(K), H(\mathrm{curl}, K)$ and $H(\mathrm{div}, K)$, all on a single tetrahedral mesh element K. They are constructed to satisfy certain moment conditions required for analysis of DPG methods. They fit into a commuting diagram that help us prove the required norm estimates (see Theorem 5.1).

Time-harmonic Maxwell equations within a cavity are considered afterward in Section 6. Focusing first on a simple DPG method for Maxwell equation, called the primal DPG method, we provide a complete analysis using the tools developed in the previous section. To understand one of the novelties here, recall that the wellposedness of the Maxwell equations is guaranteed as soon as the excitation frequency of the harmonic wave is different from a cavity resonance. However, this wellposedness is not directly inherited by most standard discretizations, which are often known to be stable solely in an asymptotic regime [25]. The discrete spaces used must be sufficiently fine before one can even guarantee solvability of the discrete system, not to mention error guarantees. Furthermore, the analysis of the standard finite element method does not clarify how fine the mesh needs to be to ensure that the stable regime is reached. In contrast, the DPG schemes, having inherited their stability from the exact equations, are stable no matter how coarse the mesh is. This advantage is striking when attempting robust adaptive meshing strategies.

Mastery of the proliferation of formulations for the one Maxwell boundary value problem is another focus of Section 6. One may decide to treat individual equations of the Maxwell system differently, e.g., one equation may be imposed strongly, while another may be imposed weakly via integration by parts. Mixed methods make a particular choice, while primal methods make a different choice. We will show (see Theorem 6.3) that from the standpoint of wellposedness, one is no different from another, i.e., if any one of the six formulations considered in Section 6 is stable, then all the remaining are guaranteed to be stable. This result is an interesting application of the closed range theorem. However, when the DPG methodology is applied to discretize these formulations, the numerical results reported in Section 7, show that the various methods do exhibit differences. This is because the functional settings are different for different formulations, i.e., convergence to the solution occurs in different norms. Section 7 also provides results from numerical investigations on issues where the theory is currently silent.

2. Breaking Sobolev spaces

In this section, we discuss precisely what we mean by breaking Sobolev spaces using a mesh. We will define *broken spaces* and *interface spaces* and prove a duality result that clarifies the interplay between these spaces. We work with infinite-dimensional, but mesh-dependent spaces on an open bounded domain $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary. The mesh, denoted by Ω_h, is a disjoint partitioning of Ω into open elements K such that the union of their closures is the closure of Ω. The collection of element boundaries ∂K for all $K \in \Omega_h$, is denoted by $\partial\Omega_h$. We assume that each element boundary ∂K is Lipschitz. The shape of the elements is otherwise arbitrary for now.

We focus on the most commonly occurring first order Sobolev spaces of real or complex valed functions, namely $H^1(\Omega)$, $H(\mathrm{div},\Omega)$, and $H(\mathrm{curl},\Omega)$. Their *broken* versions are defined, respectively, by

$$
\begin{aligned}
H^1(\Omega_h) &= \{u \in L^2(\Omega) \,:\, u|_K \in H^1(K),\, K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H^1(K),\\
H(\mathrm{curl},\Omega_h) &= \{E \in (L^2(\Omega))^N \,:\, E|_K \in H(\mathrm{curl},K),\, K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H(\mathrm{curl},K),\\
H(\mathrm{div},\Omega_h) &= \{\sigma \in (L^2(\Omega))^N \,:\, \sigma|_K \in H(\mathrm{div},K),\, K \in \Omega_h\} &&= \prod_{K\in\Omega_h} H(\mathrm{div},K).
\end{aligned}
$$

As these broken spaces contain functions with no continuity requirements at element interfaces, their discretization is easier than globally conforming spaces.

To recover the original Sobolev spaces from these broken spaces, we need traces and interface variables. First, let us consider these traces on each element K in Ω_h.

$$
\begin{aligned}
\mathrm{tr}^K_{\mathrm{grad}}\, u &= u|_{\partial K} && u \in H^1(K),\\
\mathrm{tr}^K_{\mathrm{curl},\top}\, E &= (n_K \times E) \times n_K|_{\partial K} && E \in H(\mathrm{curl},K),\\
\mathrm{tr}^K_{\mathrm{curl},\dashv}\, E &= n_K \times E|_{\partial K} && E \in H(\mathrm{curl},K),\\
\mathrm{tr}^K_{\mathrm{div}}\, \sigma &= \sigma|_{\partial K} \cdot n_K && \sigma \in H(\mathrm{div},K).
\end{aligned}
$$

Here and throughout n_K denotes the unit outward normal on ∂K and is often simply written as n. Both n_K and these traces are well defined almost everywhere on ∂K, thanks to our assumption that ∂K is Lipschitz. The operators $\mathrm{tr}_{\mathrm{grad}}, \mathrm{tr}_{\mathrm{curl},\top}$, $\mathrm{tr}_{\mathrm{curl},\dashv}$, and $\mathrm{tr}_{\mathrm{div}}$ perform the above trace operation element by element on each of the broken spaces we defined previously, thus giving rise to linear maps

$$
\mathrm{tr}_{\mathrm{grad}} : H^1(\Omega_h) \to \prod_{K\in\Omega_h} H^{1/2}(\partial K), \qquad \mathrm{tr}_{\mathrm{curl},\top} : H(\mathrm{curl},\Omega_h) \to \prod_{K\in\Omega_h} H^{-1/2}(\mathrm{curl},\partial K),
$$
$$
\mathrm{tr}_{\mathrm{curl},\dashv} : H(\mathrm{curl},\Omega_h) \to \prod_{K\in\Omega_h} H^{-1/2}(\mathrm{div},\partial K), \qquad \mathrm{tr}_{\mathrm{div}} : H(\mathrm{div},\Omega_h) \to \prod_{K\in\Omega_h} H^{-1/2}(\partial K).
$$

It is well known that these maps are continuous and surjective. An element of $\prod_{K\in\Omega_h} H^{-1/2}(\partial K)$ is expressed using notations like $n \cdot \hat\sigma$ or $\hat\sigma_n$ (even when $\hat\sigma$ itself has not been assigned any separate meaning) that are evocative of their dependence on the interface normals. Similarly, the elements of the other trace map codomains are expressed using notations like

$$n \times \hat{E} \equiv \hat{E}_{\dashv} \in \prod_{K \in \Omega_h} H^{-1/2}(\mathrm{div}, \partial K), \qquad (n \times \hat{E}) \times n \equiv \hat{E}_{\top} \in \prod_{K \in \Omega_h} H^{-1/2}(\mathrm{curl}, \partial K).$$

Next, we need spaces of *interface* functions. We use the above trace operators to define them, after cautiously noting two issues that can arise on an interface piece $f = \partial K^+ \cap \partial K^-$ shared by two mesh elements $K^\pm$ in Ω_h. First, functions in the range of $\mathrm{tr}_{\mathrm{grad}}$ when restricted to F is generally multivalued, and we would like our interface functions to be single valued in some sense. Second, the range of the remaining trace operators consist of functionals whose restrictions to f are in general undefined. The following definitions circumvent these issues.

$$H^{1/2}(\partial\Omega_h) = \mathrm{tr}_{\mathrm{grad}}\, H^1(\Omega), \qquad H^{-1/2}(\mathrm{div}, \Omega_h) = \mathrm{tr}_{\mathrm{curl},\top}\, H(\mathrm{curl}, \Omega),$$
$$H^{-1/2}(\mathrm{curl}, \partial\Omega_h) = \mathrm{tr}_{\mathrm{curl},\dashv}\, H(\mathrm{curl}, \Omega), \qquad H^{-1/2}(\partial\Omega_h) = \mathrm{tr}_{\mathrm{div}}\, H(\mathrm{div}, \Omega).$$

Since $\mathrm{tr}_{\mathrm{grad}}\, H^1(\Omega) \subsetneq \mathrm{tr}_{\mathrm{grad}}\, H^1(\Omega_h)$, one could endow the subspace with the subspace topology (and similarly for other spaces). But we proceed differently and norm each of the above interface spaces by these quotient norms:

$$\|\hat{u}\|_{H^{1/2}(\partial\Omega_h)} = \inf_{u \in H^1(\Omega) \cap \mathrm{tr}_{\mathrm{grad}}^{-1}\{\hat{u}\}} \|u\|_{H^1(\Omega)}, \tag{2.1a}$$

$$\|\hat{E}_{\top}\|_{H^{-1/2}(\mathrm{div}, \partial\Omega_h)} = \inf_{E \in H(\mathrm{curl}, \Omega) \cap \mathrm{tr}_{\mathrm{curl},\dashv}^{-1}\{\hat{E}_{\dashv}\}} \|E\|_{H(\mathrm{curl}, \Omega)}, \tag{2.1b}$$

$$\|\hat{E}_{\dashv}\|_{H^{-1/2}(\mathrm{curl}, \partial\Omega_h)} = \inf_{E \in H(\mathrm{curl}, \Omega) \cap \mathrm{tr}_{\mathrm{curl},\top}^{-1}\{\hat{E}_{\top}\}} \|E\|_{H(\mathrm{curl}, \Omega)}, \tag{2.1c}$$

$$\|\hat{\sigma}_n\|_{H^{-1/2}(\partial\Omega_h)} = \inf_{\sigma \in H(\mathrm{div}, \Omega) \cap \mathrm{tr}_{\mathrm{div}}^{-1}\{\hat{\sigma}_n\}} \|\sigma\|_{H(\mathrm{div}, \Omega)}. \tag{2.1d}$$

These are indeed quotient norms because the infimums are over cosets generated by kernels of the trace maps. E.g., if w is any function in $H^1(\Omega)$ such that $\mathrm{tr}_{\mathrm{grad}}\, w = \hat{u}$, then the set where the minimization is carried out in (2.1a), namely $H^1(\Omega) \cap \mathrm{tr}_{\mathrm{grad}}^{-1}\{\hat{u}\}$, equals the coset $w + \prod_{K \in \Omega_h} \mathring{H}^1(K)$. Note that every element of this coset is an extension of $\hat{u}$. For this reason, such norms are also known as the "minimum energy extension" norms. For an alternate way to characterize the interface spaces, see [30].

Remark 2.1. The quotient norm in (2.1d) appeared in the literature as early as [29]. The word "hybrid" that appears in their title was used to refer to situations where, to quote [29], "the constraint of interelement continuity has been removed at the expense of introducing a Lagrange multiplier." The quote also summarizes the discussion of this section well. The above definitions of our four interface spaces are thus generalizations of a definition in [29] and each can be interpreted as an appropriate space of Lagrange multipliers.

We now show by elementary arguments that the quotient norms on the two pairs of trace spaces

$$\{H^{1/2}(\partial K), H^{-1/2}(\partial K)\} \quad \text{and} \quad \{H^{-1/2}(\mathrm{curl}, \partial K), H^{-1/2}(\mathrm{div}, \partial K)\},$$

are dual to each other. The duality pairing in any Hilbert space X, namely the action of a linear or conjugate linear (antilinear) functional $x' \in X'$ on $x \in X$ is denoted by $\langle x', x \rangle_{X' \times X}$ and we omit the subscript in this notation when no confusion can arise. We also adopt the convention that when taking supremum over vector spaces (such as in the next result) the zero element is omitted tacitly.

Lemma 2.2. *The following identities hold for any $n\cdot\hat\sigma$ in $H^{-1/2}(\partial K)$ and any $\hat u$ in $H^{1/2}(\partial K)$.*

$$\|\hat\sigma_n\|_{H^{-1/2}(\partial K)} = \sup_{u\in H^1(K)} \frac{|\langle\hat\sigma_n, u\rangle|}{\|u\|_{H^1(K)}} = \sup_{\hat u\in H^{1/2}(\partial K)} \frac{|\langle\hat\sigma_n,\hat u\rangle|}{\|\hat u\|_{H^{1/2}(\partial K)}}, \tag{2.2a}$$

$$\|\hat u\|_{H^{1/2}(\partial K)} = \sup_{\sigma\in H(\mathrm{div},K)} \frac{|\langle n\cdot\sigma,\hat u\rangle|}{\|\sigma\|_{H(\mathrm{div},K)}} = \sup_{\hat\sigma_n\in H^{-1/2}(\partial K)} \frac{|\langle\hat\sigma_n,\hat u\rangle|}{\|\hat\sigma_n\|_{H^{-1/2}(\partial K)}}. \tag{2.2b}$$

The next two identities hold for any $\hat E_\dashv$ in $H^{-1/2}(\mathrm{div},\partial K)$ and any $\hat F_\top$ in $H^{-1/2}(\mathrm{curl},\partial K)$.

$$\begin{aligned}\|\hat E_\dashv\|_{H^{-1/2}(\mathrm{div},\partial K)} &= \sup_{F\in H(\mathrm{curl},K)} \frac{|\langle\hat E_\dashv, F\rangle|}{\|F\|_{H(\mathrm{curl},K)}} \\ &= \sup_{\hat F_\top\in H^{-1/2}(\mathrm{curl},\partial K)} \frac{|\langle\hat E_\dashv,\hat F_\top\rangle|}{\|\hat F_\top\|_{H^{-1/2}(\mathrm{curl},\partial K)}},\end{aligned} \tag{2.2c}$$

$$\begin{aligned}\|\hat F_\top\|_{H^{-1/2}(\mathrm{curl},\partial K)} &= \sup_{E\in H(\mathrm{curl},K)} \frac{|\langle n\times E,\hat F_\top\rangle|}{\|E\|_{H(\mathrm{curl},K)}} \\ &= \sup_{\hat E_\dashv\in H^{-1/2}(\mathrm{div},\partial K)} \frac{|\langle\hat E_\dashv,\hat F_\top\rangle|}{\|\hat E_\dashv\|_{H^{-1/2}(\mathrm{div},\partial K)}}.\end{aligned} \tag{2.2d}$$

Proof. The first identity is proved using an equivalence between a Dirichlet and a Neumann problem. The Dirichlet problem is the problem of finding $\sigma\in H(\mathrm{div},K)$, given $\hat\sigma_n\in H^{-1/2}(\partial K)$, such that

$$\begin{cases} n\cdot\sigma = n\cdot\hat\sigma, & \text{on } \partial K,\\ -\,\mathrm{grad}(\mathrm{div}\;\sigma)+\sigma = 0, & \text{in } K.\end{cases} \tag{2.3}$$

The Neumann problem finds $w\in H^1(K)$ satisfying

$$\begin{cases} \dfrac{\partial w}{\partial n} = \hat\sigma_n, & \text{on } \partial K,\\ -\mathrm{div}(\mathrm{grad}\,w)+w = 0, & \text{in } K.\end{cases} \tag{2.4}$$

It is immediate that problems (2.4) and (2.3) are equivalent in the sense that w solves (2.4) if and only if $\sigma = \mathrm{grad}\,w$ solves (2.3) and moreover $\|w\|_{H^1(K)} = \|\sigma\|_{H(\mathrm{div},K)}$. It is also obvious from the calculus of variations that among all $H(\mathrm{div},K)$-extensions of $\hat\sigma_n$, the solution of (2.3) has the minimal $H(\mathrm{div},K)$ norm (i.e., σ is the "minimum energy extension" referred to earlier), so

$$\begin{aligned}\|\hat\sigma_n\|_{H^{-1/2}(\partial K)} = \|\sigma\|_{H(\mathrm{div},K)} = \|w\|_{H^1(K)} &= \sup_{v\in H^1(K)} \frac{|(\mathrm{grad}\,w,\mathrm{grad}\,v)_K + (w,v)_K|}{\|v\|_{H^1(K)}} \\ &= \sup_{v\in H^1(K)} \frac{|\langle\hat\sigma_n, v\rangle|}{\|v\|_{H^1(K)}},\end{aligned}$$

where we used the variational form of (2.4) in the last step. (Here and throughout, we use $(\cdot,\cdot)_K$ to denote the inner product in $L^2(K)$ or its Cartesian products.) This proves the first equality of (2.2a).

Next, analogous to (2.3) and (2.4), we set up another pair of Dirichlet and Neumann problems. The first problem is to find u in $H^1(K)$, given any $\hat u \in H^{1/2}(\partial K)$, such that

$$\begin{cases} u = \hat u, & \text{on } \partial K, \\ -\operatorname{div}(\operatorname{grad} u) + u = 0, & \text{in } K. \end{cases} \tag{2.5}$$

The second is to find τ in $H(\operatorname{div}, K)$ such that

$$\begin{cases} \operatorname{div}\,\tau = \hat u, & \text{on } \partial K, \\ -\operatorname{grad}(\operatorname{div}\,\tau) + \tau = 0, & \text{in } K. \end{cases} \tag{2.6}$$

The solution u of (2.5) has the minimal $H^1(K)$ norm among all extensions of $\hat u$ into $H^1(K)$, i.e., $\|\hat u\|_{H^{1/2}(\partial K)} = \|u\|_{H^1(K)}$. Thus $\langle \hat\sigma_n, \hat u\rangle / \|\hat u\|_{H^{1/2}(\partial K)} = \langle \hat\sigma_n, u\rangle / \|u\|_{H^1(K)}$, so taking the supremum over all $\hat u$ in $H^{1/2}(\partial K)$, we obtain

$$\sup_{\hat u \in H^{1/2}(\partial K)} \frac{|\langle \hat\sigma_n, \hat u\rangle|}{\|\hat u\|_{H^{1/2}(\partial K)}} \leqslant \sup_{u \in H^1(K)} \frac{|\langle \hat\sigma_n, u\rangle|}{\|u\|_{H^1(K)}}.$$

Since the reverse inequality is obvious from the definition of the quotient norm in the denominator, we have established the second identity of (2.2a). To prove (2.2a), we begin, as above, by observing that τ is the solution to the Neumann problem (2.6) if and only if $u = \operatorname{div}\tau$ solves the Dirichlet problem (2.5). Moreover, $\|\tau\|_{H(\operatorname{div},K)} = \|u\|_{H^1(K)}$. Hence

$$\begin{aligned} \|\hat u\|_{H^{1/2}(\partial K)} = \|u\|_{H^1(K)} = \|\tau\|_{H(\operatorname{div},K)} &= \sup_{\rho \in H(\operatorname{div},K)} \frac{|(\operatorname{div}\tau, \operatorname{div}\rho)_K + (\tau,\rho)_K|}{\|\rho\|_{H(\operatorname{div},K)}} \\ &= \sup_{\rho \in H(\operatorname{div},K)} \frac{|\langle n\cdot\rho, \hat u\rangle|}{\|\rho\|_{H(\operatorname{div},K)}}, \end{aligned}$$

where we have used the variational form of (2.6) in the last step. The proof of (2.2b) can now be completed as before.

We follow exactly the same reasoning for the $H(\operatorname{curl})$ case, summarized as follows: On one hand, the norm of an interface function equals the norm of a minimum energy extension, while on the other hand, it equals the norm of the inverse of a Riesz map applied to a functional generated by the interface function. The minimum energy extension that yields the interface norm $\|\hat E_\dashv\|_{H^{-1/2}(\operatorname{div},\partial K)}$ is now the solution of the Dirichlet problem of finding $E \in H(\operatorname{curl}, K)$ satisfying

$$\begin{cases} n \times E = \hat E_\dashv, & \text{on } \partial K, \\ \operatorname{curl}(\operatorname{curl} E) + E = 0, & \text{in } K, \end{cases} \tag{2.7}$$

while the inverse of the Riesz map applied to the functional generated by $\hat E_\dashv$ is obtained by solving the Neumann problem

$$\begin{cases} n \times (\operatorname{curl} F) = \hat E_\dashv, & \text{on } \partial K \\ \operatorname{curl}(\operatorname{curl} F) + F = 0, & \text{in } K. \end{cases} \tag{2.8}$$

Again, the two problems are equivalent in the sense that F solves (2.8) if and only if $E = \operatorname{curl} F$ solves (2.7). Moreover, $\|E\|_{H(\operatorname{curl},K)} = \|F\|_{H(\operatorname{curl},K)}$. Hence

$$\begin{aligned}\|\hat{E}_{\dashv}\|_{H^{-1/2}(\operatorname{div},\partial K)} &= \|E\|_{H(\operatorname{curl},K)} = \|F\|_{H(\operatorname{curl},K)} \\ &= \sup_{G\in H(\operatorname{curl},K)} \frac{|(\operatorname{curl} F, \operatorname{curl} G)_K + (F,G)_K|}{\|G\|_{H(\operatorname{curl},K)}} \\ &= \sup_{G\in H(\operatorname{curl},K)} \frac{|\langle \hat{E}_{\dashv}, G\rangle|}{\|G\|_{H(\operatorname{curl},K)}}.\end{aligned}$$

The proof of (2.2c) follows from this. The proof of (2.2d) is similar and is left to the reader. □

Let us return to the product spaces like $H^1(\Omega_h), H(\operatorname{curl},\Omega_h)$, and $H(\operatorname{div},\Omega_h)$. Any Hilbert space V that is the Cartesian product of various Hilbert spaces $V(K)$ is normed in the standard fashion,

$$V = \prod_{K\in\Omega_h} V(K), \qquad \|v\|_V^2 = \sum_{K\in\Omega_h} \|v_K\|_{V(K)}^2,$$

where v_K denotes the K-component of any v in V. The dual space V' is the Cartesian product of component duals $V(K)'$. Writing an $\ell \in V'$ as $\ell(v) = \sum_{K\in\Omega_h} \ell_K(v_K)$, where $\ell_K \in V(K)'$, it is elementary to prove that $\|\ell\|_{V'}^2 = \sum_{K\in\Omega_h} \|\ell_K\|_{V(K)'}^2$, i.e.,

$$\left(\sup_{v\in V} \frac{|\ell(v)|}{\|v\|_V}\right)^2 = \sum_{K\in\Omega_h} \left(\sup_{v_K\in V(K)} \frac{|\ell_K(v_K)|}{\|v_K\|_{V(K)}}\right)^2. \tag{2.9}$$

Some of our interface spaces have such functionals, e.g., the function $\hat{\sigma}_n$ in $H^{-1/2}(\partial\Omega_h)$ gives rise to $\ell(v) = \langle \hat{\sigma}_n, v\rangle_h$ where

$$\langle \hat{\sigma}_n, v\rangle_h = \sum_{K\in\Omega_h} \langle \hat{\sigma}_n, v\rangle_{H^{-1/2}(\partial K)\times H^{1/2}(\partial K)},$$

is a functional acting on $v \in H^1(\Omega_h)$ which is the sum of component functionals $\ell_K(v) = \langle \hat{\sigma}_n, v\rangle_{H^{-1/2}(\partial K)\times H^{1/2}(\partial K)}$ acting on $v_K = v|_K$ over every $K \in \Omega_h$. Other functionals like $\langle \hat{E}_{\dashv}, F\rangle_h$ are defined similarly. We are now ready to state a few basic relationships between the interface and broken spaces.

Theorem 2.3. *The following identities hold for any interface space function $\hat{\sigma}_n$ in $H^{-1/2}(\partial\Omega_h)$, $\hat{u}$ in $H^{1/2}(\partial\Omega_h)$, $\hat{E}_{\dashv}$ in $H^{-1/2}(\operatorname{div},\partial K)$, and $\hat{F}_{\top}$ in $H^{-1/2}(\operatorname{curl},\partial K)$.*

$$\|\hat{\sigma}_n\|_{H^{-1/2}(\partial\Omega_h)} = \sup_{u\in H^1(\Omega_h)} \frac{|\langle \hat{\sigma}_n, u\rangle_h|}{\|u\|_{H^1(\Omega_h)}}, \tag{2.10a}$$

$$\|\hat{u}\|_{H^{1/2}(\partial\Omega_h)} = \sup_{\sigma\in H(\operatorname{div},\Omega_h)} \frac{|\langle n\cdot\sigma, \hat{u}\rangle_h|}{\|\sigma\|_{H(\operatorname{div},\Omega_h)}}, \tag{2.10b}$$

$$\|\hat{E}_{\dashv}\|_{H^{-1/2}(\operatorname{div},\partial\Omega_h)} = \sup_{F\in H(\operatorname{curl},\Omega_h)} \frac{|\langle \hat{E}_{\dashv}, F\rangle_h|}{\|F\|_{H(\operatorname{curl},\Omega_h)}}, \tag{2.10c}$$

$$\|\hat{F}_{\top}\|_{H^{-1/2}(\operatorname{curl},\partial\Omega_h)} = \sup_{E\in H(\operatorname{curl},\Omega_h)} \frac{|\langle n\times E, \hat{F}_{\top}\rangle_h|}{\|E\|_{H(\operatorname{curl},\Omega_h)}}. \tag{2.10d}$$

For any broken space function $v \in H^1(\Omega_h), \tau \in H(\mathrm{div}, \Omega_h)$, *and* $F \in H(\mathrm{curl}, \Omega_h)$,

$$v \in \mathring{H}^1(\Omega) \iff \langle \hat{\sigma}_n, v \rangle_h = 0 \qquad \forall\, \hat{\sigma}_n \in H^{-1/2}(\partial\Omega_h), \tag{2.11a}$$

$$\tau \in \mathring{H}(\mathrm{div}, \Omega) \iff \langle \tau \cdot n, \hat{u} \rangle_h = 0 \qquad \forall\, \hat{u} \in H^{1/2}(\partial\Omega_h), \tag{2.11b}$$

$$F \in \mathring{H}(\mathrm{curl}, \Omega) \iff \langle \hat{E}_\dashv, F \rangle_h = 0 \qquad \forall\, \hat{E}_\dashv \in H^{-1/2}(\mathrm{div}, \Omega_h). \tag{2.11c}$$

Proof. The identities immediately follow from Lemma 2.2 and (2.9). The proofs of the three equivalences in (2.11) are similar, so we will only detail the last one. If F is in $\mathring{H}(\mathrm{curl}, \Omega)$, then choosing any $E \in H(\mathrm{curl}, \Omega)$ such that $\mathrm{tr}_{\mathrm{curl},\top} E = \hat{E}_\dashv$ and integrating by parts over entire Ω,

$$(\mathrm{curl}\, E, F)_\Omega + (E, \mathrm{curl}\, F)_\Omega = 0,$$

because of the boundary conditions on F on $\partial\Omega$. Now, if the left hand side is integrated by parts again, this time element by element, then we find that $\langle \hat{E}_\dashv, F \rangle_h = 0$.

Conversely, given that $\langle \hat{E}_\dashv, F \rangle_h = 0$ for any F in $H(\mathrm{curl}, \Omega_h)$, consider $\mathrm{curl}\, F \in (\mathcal{D}(\Omega)^3)'$. As a distribution, $\mathrm{curl}\, F$ acts on $\phi \in \mathcal{D}(\Omega)^3$, and satisfies

$$(\mathrm{curl}\, F)(\phi) = (F, \mathrm{curl}\, \phi)_\Omega = (\mathrm{curl}\, F, \phi)_h - \langle n \times \phi, F \rangle_h = (\mathrm{curl}\, F, \phi)_h,$$

where we have integrated by parts element by element and denoted

$$(\cdot,\cdot)_h = \sum_{K \in \Omega_h} (\cdot,\cdot)_K.$$

This notation also serves to emphasize that the term $\mathrm{curl}\, F$ appearing on the right-hand side above is a derivative taken piecewise, element by element. Clearly $\mathrm{curl}\, F|_K$ is in $L^2(K)^3$ for all $K \in \Omega_h$ since $F \in H(\mathrm{curl}, \Omega_h)$, so the distribution $\mathrm{curl}\, F$ is in $L^2(\Omega)$. Having established that $F \in H(\mathrm{curl}, \Omega)$, we may now integrate by parts to get $\langle n \times E, F \rangle_{H^{-1/2}(\mathrm{div},\partial\Omega) \times H^{-1/2}(\mathrm{curl},\partial\Omega)} = (\mathrm{curl}\, E, F)_\Omega + (E, \mathrm{curl}\, F)_\Omega = \langle n \times E, F \rangle_h = 0$ for all $E \in H(\mathrm{curl}, \Omega)$. This shows that the trace $(n \times F) \times n|_{\partial\Omega} = 0$, i.e, $F \in \mathring{H}(\mathrm{curl}, \Omega)$. ☐

Remark 2.4. While $\prod_{K \in \Omega_h} H^{-1/2}(\partial K)$ and $\prod_{K \in \Omega_h} H^{1/2}(\partial K)$ are dual to each other, our interface spaces $H^{-1/2}(\partial\Omega_h)$ and $H^{1/2}(\partial\Omega_h)$ are *not* dual to each other in general.

Remark 2.5. Equivalences analogous to (2.11) hold with interface subspaces

$$\mathring{H}^{1/2}(\partial\Omega_h) = \mathrm{tr}_{\mathrm{grad}}\, \mathring{H}^1(\Omega), \qquad \mathring{H}^{-1/2}(\mathrm{div}, \Omega_h) = \mathrm{tr}_{\mathrm{curl},\dashv}\, \mathring{H}(\mathrm{curl}, \Omega),$$
$$\mathring{H}^{-1/2}(\mathrm{curl}, \partial\Omega_h) = \mathrm{tr}_{\mathrm{curl},\top}\, \mathring{H}(\mathrm{curl}, \Omega), \qquad \mathring{H}^{-1/2}(\partial\Omega_h) = \mathrm{tr}_{\mathrm{div}}\, \mathring{H}(\mathrm{div}, \Omega).$$

By a minor modification of the arguments in the proof in Theorem 2.3, we can prove that for any $v \in H^1(\Omega_h), \tau \in H(\mathrm{div}, \Omega_h)$, and $F \in H(\mathrm{curl}, \Omega_h)$,

$$v \in H^1(\Omega) \iff \langle \hat{\sigma}_n, v \rangle_h = 0 \qquad \forall\, \hat{\sigma}_n \in \mathring{H}^{-1/2}(\partial\Omega_h), \tag{2.12a}$$

$$\tau \in H(\mathrm{div}, \Omega) \iff \langle \tau \cdot n, \hat{u} \rangle_h = 0 \qquad \forall\, \hat{u} \in \mathring{H}^{1/2}(\partial\Omega_h), \tag{2.12b}$$

$$F \in H(\mathrm{curl}, \Omega) \iff \langle \hat{E}_\dashv, F \rangle_h = 0 \qquad \forall\, \hat{E}_\dashv \in \mathring{H}^{-1/2}(\mathrm{div}, \Omega_h). \tag{2.12c}$$

3. Breaking variational forms

The goal in this section is to investigate in what sense a variational formulation can be reformulated using broken spaces without losing stability. We will describe the main result in an abstract setting first and close the section with simple examples that use the results of the previous section.

Let X_0 and Y denote two Hilbert spaces and let Y_0 be a closed subspace of Y. For definiteness, we assume that all our spaces in this section are over $\mathbb{C}$ (but our results hold also for spaces over $\mathbb{R}$). In the examples we have in mind, Y will be a broken space, while Y_0 will be its unbroken analogue (but no such assumption is needed to understand the upcoming results abstractly). The abstract setting involves a continuous sesquilinear form $b_0 : X_0 \times Y \to \mathbb{C}$ satisfying the following assumption.

Assumption 1. There is a positive constant c_0 such that

$$c_0 \|x\|_{X_0} \leqslant \sup_{y \in Y_0} \frac{|b_0(x,y)|}{\|y\|_Y} \qquad \forall x \in X_0.$$

It is a well-known result of the Babuška and Nečas [1, 26] that Assumption 1 together with triviality of

$$Z_0 = \{y \in Y_0 : \; b_0(x,y) = 0 \text{ for all } x \in X_0\} \tag{3.1}$$

guarantees wellposedness of the following variational problem: Given $\ell \in Y_0'$ (the space of conjugate linear functionals on Y_0), find $x \in X_0$ satisfying

$$b_0(x,y) = \ell(y) \quad \forall y \in Y_0. \tag{3.2}$$

When Z_0 is non-trivial, we can still obtain existence of a solution x provided the load functional ℓ satisfies the compatibility condition $\ell(z) = 0$ for all $z \in Z_0$. In (3.2), the *trial space* X_0 need not be the same as the *test space* Y_0.

To describe a "broken" version of (3.2), we need another Hilbert space $\hat{X}$, together with a continuous sesquilinear form $\hat{b} : \hat{X} \times Y \to \mathbb{C}$. In applications Y and $\hat{X}$ will usually be set to a broken Sobolev space and an interface space, respectively. Define

$$b(\,(x,\hat{x}),\, y) = b_0(x,y) + \hat{b}(\hat{x},y).$$

Clearly $b : X \times Y \to \mathbb{C}$ is continuous, where

$$X = X_0 \times \hat{X} \tag{3.3}$$

is a Hilbert space under the Cartesian product norm. Now consider the following new broken variational formulation: Given $\ell \in Y'$, find $x \in X_0$ and $\hat{x} \in \hat{X}$ satisfying

$$b(\,(x,\hat{x}),\, y) = \ell(y) \quad \forall y \in Y. \tag{3.4}$$

The close relationship between problems (3.4) and (3.2) is readily revealed under the following assumption.

Assumption 2. The spaces $Y_0, Y,$ and $\hat{X}$ satisfy

$$Y_0 = \{y \in Y : \; \hat{b}(\hat{x},y) = 0 \text{ for all } \hat{x} \in \hat{X}\} \tag{3.5}$$

and there is a positive constant $\hat{c}$ such that

$$\hat{c}\,\|\hat{x}\|_{\hat{X}} \leqslant \sup_{y\in Y} \frac{|\hat{b}(\hat{x},y)|}{\|y\|_Y} \quad \forall \hat{x}\in\hat{X}. \tag{3.6}$$

Under this assumption, we present a simple result which shows that the broken form (3.4) inherits stability from the original unbroken form (3.2). A very similar such abstract result was formulated and proved in [22, Appendix A] and used for other applications. Our proof is simple, unsurprising, and uses the same type of arguments from the early days of mixed methods [5, p. 40]: stability of a larger system can be obtained in a triangular fashion by first restricting to a smaller subspace and obtaining stability there, followed by a backsubstitution-like step.

Theorem 3.1. *Assumptions 1 and 2 imply*

$$c_1\|(x,\hat{x})\|_X \leqslant \sup_{y\in Y} \frac{|b(\,(x,\hat{x}),\,y)|}{\|y\|_Y},$$

where c_1 is defined by

$$\frac{1}{c_1^2} = \frac{1}{c_0^2} + \frac{1}{\hat{c}^2}\left(\frac{\|b_0\|}{c_0}+1\right)^2.$$

Moreover, if $Z=\{y\in Y:\ b(\,(x,\hat{x}),y)=0$ for all $x\in X_0$ and $\hat{x}$ in $\hat{X}\}$, then

$$Z = Z_0.$$

Consequently, if $Z_0=\{0\}$, then (3.4) is uniquely solvable and moreover the solution component x from (3.4) coincides with the solution of (3.2).

Proof. We need to bound $\|x\|_{X_0}$ and $\|\hat{x}\|_{\hat{X}}$. First,

$$\begin{aligned}
c_0\|x\|_{X_0} &\leqslant \sup_{y_0\in Y_0} \frac{|b_0(x,y)|}{\|y\|_Y} && \text{by Assumption 1,}\\
&\leqslant \sup_{y_0\in Y_0} \frac{|b_0(x,y)+\hat{b}(\hat{x},y)|}{\|y\|_Y} && \text{by Assumption 2, (3.5)}\\
&\leqslant \sup_{y\in Y} \frac{|b(\,(x,\hat{x}),\,y)|}{\|y\|_Y} && \text{as } Y_0\subseteq Y.
\end{aligned}$$

Next, to bound $\|\hat{x}\|_{\hat{X}}$, using (3.6) of Assumption 2,

$$\begin{aligned}
\hat{c}\,\|\hat{x}\|_{\hat{X}} &\leqslant \sup_{y\in Y}\frac{|\hat{b}(\hat{x},y)|}{\|y\|_Y} = \sup_{y\in Y}\frac{|b(\,(x,\hat{x}),\,y)-b_0(x,y)|}{\|y\|_Y}\\
&\leqslant \|b_0\|\,\|x\|_{X_0} + \sup_{y\in Y}\frac{|b(\,(x,\hat{x}),\,y)|}{\|y\|_Y},
\end{aligned}$$

where $\|b_0\|$ is the smallest number C for which the inequality $|b_0(x,y)|\leqslant C\|x\|_{X_0}\|y\|_Y$ holds for all $x\in X_0$ and all $y\in Y$. Using the already proved bound for $\|x\|_{X_0}$ in the last inequality and combining,

$$\begin{aligned}\|(x,\hat{x})\|_X^2 &= \|x\|_{X_0}^2 + \|\hat{x}\|_{\hat{X}}^2 \\ &\leqslant \left(\frac{1}{c_0} \sup_{y\in Y} \frac{|b(\,(x,\hat{x}),\, y)|}{\|y\|_Y} \right)^2 + \left(\frac{1}{\hat{c}} \left(\frac{\|b_0\|}{c_0} + 1 \right) \sup_{y\in Y} \frac{|b(\,(x,\hat{x}),\, y)|}{\|y\|_Y} \right)^2\end{aligned}$$

from which the inequality of the theorem follows.

Finally, to prove that $Z = Z_0$, using (3.5),

$$\begin{aligned} y \in Z &\iff b_0(x,y) = 0 \text{ for all } x \in X_0 \text{ and } \hat{b}(\hat{x},y) = 0 \text{ for all } \hat{x} \in \hat{X} \\ &\iff b_0(x,y) = 0 \text{ for all } x \in X_0 \text{ and } y \in Y_0, \end{aligned}$$

which holds if and only if $y \in Z_0$. □

Remark 3.2. Note that in the proof of the inf-sup condition, we did not fully use (3.5). We only needed $Y_0 \subseteq \{y \in Y : \hat{b}(\hat{x},y) = 0 \text{ for all } \hat{x} \in \hat{X}\}$. The reverse inclusion was needed to conclude that $Z = Z_0$.

Remark 3.3. It is natural to ask, in the same spirit as Theorem 3.1, if the numerical solutions of DPG methods using discretizations of the broken formulations coincide with those of discretizations of the original unbroken formulation. A result addressing this question is given in [3, Theorem 2.6].

In the remainder of this section, we illustrate how to apply this theorem on some examples.

Example 3.4 (Primal DPG formulation). Suppose $f \in L^2(\Omega)$ and u satisfies

$$-\Delta u = f \qquad \text{in } \Omega, \tag{3.7a}$$

$$u = 0 \qquad \text{on } \partial\Omega. \tag{3.7b}$$

The standard variational formulation for this problem, finds u in $\mathring{H}^1(\Omega)$ such that

$$(\operatorname{grad} u, \operatorname{grad} v)_\Omega = (f, v)_\Omega \qquad \forall v \in \mathring{H}^1(\Omega). \tag{3.8}$$

This form is obtained by multiplying (3.7a) by $v \in \mathring{H}^1(\Omega)$ and integrating by parts over the entire domain Ω. If on the other hand, we multiply (3.7a) by a $v \in H^1(\Omega_h)$ and integrate by parts element by element, then we obtain another variational formulation proposed in [17]: Solve for u in $\mathring{H}^1(\Omega)$ as well as a separate unknown $\hat{\sigma}_n \in H^{-1/2}(\partial\Omega_h)$ (representing the fluxes $n \cdot \operatorname{grad} u$ along mesh interfaces) satisfying

$$(\operatorname{grad} u, \operatorname{grad} y)_h + \langle \hat{\sigma}_n, y \rangle_h = (f, y)_\Omega \qquad \forall y \in H^1(\Omega_h). \tag{3.9}$$

We can view this as the broken version of (3.8) by setting

$$\begin{aligned} X_0 &= \mathring{H}^1(\Omega), & Y_0 &= \mathring{H}^1(\Omega), \\ \hat{X} &= H^{-1/2}(\partial\Omega_h), & Y &= H^1(\Omega_h), \\ b_0(u,y) &= (\operatorname{grad} u, \operatorname{grad} y)_h, & \hat{b}(\hat{\sigma}_n, y) &= \langle \hat{\sigma}_n, y \rangle_h. \end{aligned}$$

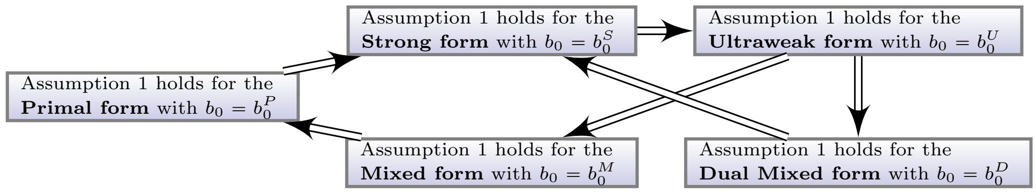


FIGURE 1. Chains of implications of inf-sup conditions

For these settings, the conditions required to apply Theorem 3.1 are verified as follows.

$$\begin{aligned} \text{Coercivity of } b_0(\cdot,\cdot) \text{ on } Y_0 &\implies \text{Assumption 1 holds.} \\ \text{Theorem 2.3, (2.10a)} &\implies \text{(3.6) of Assumption 2 holds with } \hat{c} = 1. \\ \text{Theorem 2.3, (2.11a)} &\implies \text{(3.5) of Assumption 2 holds.} \end{aligned}$$

Noting that $Z_0 = \{0\}$, an application of Theorem 3.1 implies that problem (3.9) is wellposed. This wellposedness result also shows that (3.9) is uniquely solvable with a more general right-hand side f in $H^1(\Omega_h)'$.

An alternate (and longer) proof of this wellposedness result can be found in [17]. The classical work of [29] also uses the spaces $H^1(\Omega_h)$ and $H^{-1/2}(\partial\Omega_h)$, but proceeds to develop a Bubnov-Galerkin hybrid formulation different from the Petrov-Galerkin formulation (3.9). ///

Example 3.5 (Many formulations of an elliptic problem). Considering a model problem involving diffusion, convection, and reaction terms, we now show how to analyze, all at once, its various variational formulations. The diffusion coefficient $a = \alpha^{-1} : \Omega \to \mathbb{R}^{3\times 3}$ is a symmetric matrix function which is uniformly bounded and positive definite on Ω, the convection coefficient is $\beta \in L^\infty(\Omega)^3$ which satisfies $\operatorname{div}(a\beta) = 0$, and reaction is incorporated through a non-negative $\gamma \in L^\infty(\Omega)$. The classical form of the equations on Ω are $\sigma = a \operatorname{grad} u + a\beta u + f_1$ and $-\operatorname{div}\sigma + \gamma u = f_2$ (for some given $f_1 \in L^2(\Omega)^3$ and $f_2 \in L^2(\Omega)$) together with the boundary condition $u|_{\partial\Omega} = 0$. This can be written in operator form using

$$A\begin{bmatrix}\sigma \\ u\end{bmatrix} = \begin{bmatrix}\alpha\sigma - \operatorname{grad} u - \beta u \\ \operatorname{div}\sigma - \gamma u\end{bmatrix}, \qquad A^*\begin{bmatrix}\sigma \\ u\end{bmatrix} = \begin{bmatrix}\alpha\sigma - \operatorname{grad} u \\ \operatorname{div}\sigma - \beta\cdot\sigma - \gamma u\end{bmatrix}. \tag{3.10}$$

We begin with the formulation closest to the classical form.

Strong form: Let $x = (\sigma, u)$ be a group variable. Set spaces by

$$X_0 = H(\operatorname{div},\Omega) \times \mathring{H}^1(\Omega), \quad Y = Y_0 = L^2(\Omega)^3 \times L^2(\Omega),$$

and consider the problem of finding $x \in X_0$, given $f \in Y$, satisfying $Ax = f$. We can trivially fit this into our variational framework (3.2) by setting b_0 to

$$b_0^S(x,y) = (Ax, y)_\Omega.$$

Unlike the remaining formulations below, there is no need to discuss a broken version of the above strong form as the test space already admits discontinuous functions. The next formulation is often derived directly from a second order equation obtained by eliminating σ from the strong form.

Primal form: First, set spaces by

$$X_0 = \mathring{H}^1(\Omega), \qquad Y_0 = \mathring{H}^1(\Omega),$$
$$\hat{X} = H^{-1/2}(\partial\Omega_h), \quad Y = H^1(\Omega_h).$$

Then, with $\hat{b}$ set to $\hat{b}^P(\hat{\sigma}_n, v) = \langle n \cdot \hat{\sigma}, y \rangle_h$ and b_0 set to

$$b_0^P(u, v) = (a \operatorname{grad} u, \operatorname{grad} v)_h - (a\beta u, \operatorname{grad} v)_h + (\gamma u, v)_\Omega,$$

the standard primal formulation is (3.2) and its broken version is (3.4).

Next, consider the formulation derived by multiplying each equation in the strong form by a test function and integrating *both* equations by parts, i.e., both equations are imposed weakly. It was previously studied in [9, 15], but we can now simplify its analysis considerably using Theorem 3.1.

Ultraweak form: Set group variables $x = (\sigma, u)$, $y = (\tau, v)$, $\hat{x} = (\hat{\sigma}_n, \hat{u})$, and

$$X_0 = L^2(\Omega)^3 \times L^2(\Omega), \qquad Y_0 = H(\operatorname{div}, \Omega) \times \mathring{H}^1(\Omega),$$
$$\hat{X} = \mathring{H}^{1/2}(\partial\Omega_h) \times H^{-1/2}(\partial\Omega_h), \quad Y = H(\operatorname{div}, \Omega_h) \times H^1(\Omega_h),$$
$$b_0^U(x, y) = (x, A^* y)_{\Omega_h}, \qquad \hat{b}^U(\hat{x}, y) = \langle \hat{\sigma}_n, v \rangle_h + \langle \hat{u}, n \cdot \tau \rangle_h.$$

Formulations (3.2) and (3.4) with $b_0 = b_0^U$ and $\hat{b} = \hat{b}^U$ are of the ultraweak type.

The fourth formulation, well-known as the mixed form [5], is derived by weakly imposing (via integration by parts) the first equation of the strong form, but strongly imposing the second equation.

Dual Mixed form: Set the spaces by

$$X_0 = H(\operatorname{div}, \Omega) \times L^2(\Omega), \qquad Y_0 = H(\operatorname{div}, \Omega) \times L^2(\Omega),$$
$$\hat{X} = \mathring{H}^{1/2}(\partial\Omega_h), \qquad Y = H(\operatorname{div}, \Omega_h) \times L^2(\Omega).$$

The well-known mixed formulation is then (3.2) with $b_0 = b_0^D$,

$$\begin{aligned} b_0^D(\,(\sigma, u),\, (\tau, v)\,) = {} & (\alpha\sigma, \tau)_h + (u, \operatorname{div} \tau)_h - (\beta u, \tau)_h \\ & + (\operatorname{div} \sigma, v)_h - (\gamma u, v)_h. \end{aligned}$$

Its broken version is (3.4) with $\hat{b}$ set to $\hat{b}^D(\hat{u}, \tau) = \langle \hat{u}, n \cdot \tau \rangle_h$.

Note that the well-known discrete hybrid mixed method [5, 11] is also derived from b_0^D. That method however works with a Bubnov-Galerkin formulation obtained by breaking both the trial and the test $H(\operatorname{div}, \Omega)$ components, while above we have broken only the test space. The last formulation in this example reverses the roles by weakly imposing the second equation of the strong form and strongly imposing the first equation:

Mixed form: Set

$$X_0 = L^2(\Omega)^3 \times \mathring{H}^1(\Omega), \qquad Y_0 = L^2(\Omega)^3 \times \mathring{H}^1(\Omega),$$
$$\hat{X} = H^{-1/2}(\partial\Omega_h), \qquad Y = L^2(\Omega)^3 \times H^1(\Omega_h).$$

The dual mixed formulation is (3.2) with $b_0 = b_0^M$,

$$b_0^M(\,(\sigma, u),\,(\tau, v)\,) = (\alpha\sigma, \tau)_\Omega - (\operatorname{grad} u, \tau)_\Omega - (\beta u, \tau)_\Omega - (\sigma, \operatorname{grad} v)_\Omega - (\gamma u, v)_\Omega.$$

and its broken version is (3.4) with $\hat{b}$ set to $\hat{b}^M(\hat{\sigma}_n, v) = \langle \hat{\sigma}_n, v\rangle_h$.

The variational problem (3.2) with b_0^M is sometimes called [4] the *primal mixed* form to differentiate it with the *dual mixed* form given by b_0^D. The broken formulation (3.4) with b_0^M and $\hat{b}^M$ was called the *mild weak DPG formulation* in [6]. Their analysis can also be simplified now using Theorem 3.1.

In order to apply Theorem 3.1 to all these formulations, we need to verify Assumption 1. This can be done for all the formulations at once, because the six implications displayed in Figure 1 are proved in [12] for the model problem of this example. We will not detail this proof here because we provide full proofs of similar implications for Maxwell equations in Section 6 (and this example is simpler than the Maxwell case). To apply these implications for the current example, we pick a formulation for which Assumption 1 is easy to prove: That the primal form is coercive $b_0^P(u, u) \geqslant C\|u\|^2_{H^1(\Omega)}$ follows immediately by integration by parts and the Poincaré inequality (under the simplifying assumptions we placed on the coefficients). This verifies Assumption 1 for the primal form, which in turn verifies it for all the formulations by the above chain of equivalences. Assumption 2 can be immediately verified for all the formulations using either (2.11) or (2.12). Together with the easily verified triviality of Z_0 in each case, we have proven the wellposedness of all the formulations above, including the broken ones. ///

4. The DPG method

In this section, we quickly introduce the DPG method, indicate why the broken spaces are needed for practical reasons within the DPG method, and recall known abstract conditions under which an error analysis can be conducted.

Let X and Y be Hilbert spaces and let $b : X \times Y \to \mathbb{C}$ be a continuous sesquilinear form. In the applications we have in mind, X will always be of the form (3.3) (but we need not assume it for the theory in this section). The variational problem is to find x in X, given $\ell \in Y'$, satisfying

$$b(x, y) = \ell(y) \qquad \forall y \in Y. \tag{4.1}$$

The DPG method uses finite-dimensional subspaces $X_h \subset X$ and $Y_h \subset Y$. The test space used in the method is a subspace $Y_h^{\text{opt}} \subseteq Y_h$ of *approximately optimal test functions* computed for any arbitrarily given trial space X_h. It is defined by $Y_h^{\text{opt}} = T_h(X_h)$ where $T_h : X_h \to Y_h$ is given by

$$(T_h z, y)_Y = b(z, y) \qquad \forall y \in Y_h. \tag{4.2}$$

Here $(\cdot,\cdot)_Y$ is the inner product in Y, hence by Riesz representation theorem on Y_h, the operator T_h is well defined. The discrete problem posed by the DPG discretization is to find $x_h \in X_h$ satisfying

$$b(x_h, y) = \ell(y) \qquad \forall y \in Y_h^{\text{opt}}. \tag{4.3}$$

For practical implementation purposes, it is important to note that T_h can be easily and inexpensively computed via (4.2) provided the space Y_h is a subspace of a broken space. Then (4.2) becomes a series of small decoupled problems on each element.

For *a posteriori* error estimation, we use an estimator $\tilde{\eta}$ that actually works for any $\tilde{x}_h$ in X_h, computed as follows. (Note that $\tilde{x}_h$ need not equal the solution x_h of (4.3).) First we solve for $\tilde{\varepsilon}_h$ in Y_h by

$$(\tilde{\varepsilon}_h, y)_Y = \ell(y) - b(\tilde{x}_h, y), \qquad \forall y \in Y_h. \tag{4.4}$$

Again, this amounts to a local computation if Y_h is a subspace of a broken space. Then, set

$$\tilde{\eta} = \|\tilde{\varepsilon}_h\|_Y.$$

When Y_h is a broken space, the element-wise norms of $\tilde{\varepsilon}_h$ serve as good error estimators [18]. The notations η and ε_h (without tilde) refer to similarly computed quantities with x_h in place of $\tilde{x}_h$. An analysis of errors and error estimators of the DPG method can be conducted using the following assumption introduced in [23]. In accordance with the traditions in the theory mixed methods [5], we will call the operator Π in the assumption a *Fortin operator*.

Assumption 3. There is a continuous linear operator $\Pi : Y \to Y_h$ such that for all $w_h \in X_h$ and all $v \in Y$,

$$b(w_h, v - \Pi v) = 0.$$

Theorem 4.1. *Suppose Assumption 3 holds. Assume also that there is a positive constant c_1 such that*

$$c_1 \|x\|_X \leqslant \sup_{y \in Y} \frac{|b(x, y)|}{\|y\|_Y} \qquad \forall x \in X, \tag{4.5}$$

and the set $Z = \{y \in Y : \ b(x, y) = 0$ for all $x \in X\}$ equals $\{0\}$. Then the DPG method (4.3) *is uniquely solvable for x_h and the a priori error estimate*

$$\|x - x_h\|_X \leqslant \frac{\|b\| \|\Pi\|}{c_1} \inf_{z_h \in X_h} \|x - z_h\|_X \qquad \textit{(quasi-optimality)} \tag{4.6}$$

holds, where x is the unique exact solution of (4.1)*. Moreover, we have the following inequalities for any $\tilde{x}_h$ in X_h and its corresponding error estimator $\tilde{\eta}$, with the data-approximation error* $\operatorname{osc}(\ell) = \|\ell \circ (1 - \Pi)\|_{Y'}$.

$$c_1 \|x - \tilde{x}_h\|_X \leqslant \|\Pi\| \, \tilde{\eta} + \operatorname{osc}(\ell), \qquad \textit{(reliability)} \tag{4.7a}$$

$$\tilde{\eta} \leqslant \|b\| \, \|x - \tilde{x}_h\|_X, \qquad \textit{(efficiency)} \tag{4.7b}$$

$$\operatorname{osc}(\ell) \leqslant \|b\| \, \|1 - \Pi\| \min_{z_h \in X_h} \|x - z_h\|_X. \tag{4.7c}$$

Here $\|\Pi\|$ and $\|b\|$ are any constants that satisfy $\|\Pi y\|_Y \leqslant \|\Pi\| \|y\|_Y$ and $|b(w, y)| \leqslant \|b\| \|w\|_X \|y\|_Y$, respectively, for all $w \in X$ and $y \in Y$. To apply the theorem to specific examples of DPG methods, we must verify (4.5). This will usually be done by appealing to Theorem 3.1 and verifying Assumptions 1 and 2. The previous sections provided tools for verifying Assumptions 1 and 2. In the next section, we will provide some tools to verify the remaining major condition in the theorem, namely Assumption 3.

Remark 4.2. A proof of Theorem 4.1 is available in existing literature. The *a priori* error bound (4.6) was proved in [23]. The inequalities of (4.7), useful for *a posteriori* error estimation, were proved in [7]. In particular, a reliability estimate slightly different from (4.7a) (with worse constants) was proved in [7], but the same ideas yield (4.7a) easily (for example, cf. [8, proof of Lemma 3.6]).

Remark 4.3. The operator T_h is an approximation to an idealized trial-to-test operator $T : X \to Y$ given by

$$(Tx, y)_Y = b(x, y) \qquad \forall y \in Y.$$

If $B : X \to Y'$ is the operator defined by the form satisfying $(Bx)(y) = b(x, y)$ for all $x \in X$ and $y \in Y$, then clearly $T = R_Y^{-1} B$, where $R_Y : Y \to Y'$ is the Riesz map defined by $(R_Y y)(v) = (y, v)_Y$. In some examples [15], it is possible to analytically compute T and then one may substitute $Y_h^{\rm opt}$ with the *exactly optimal test space* $Y^{\rm opt} = T(X_h)$.

Remark 4.4. The above-mentioned trial-to-test operator $T = R_Y^{-1} B$ should not be confused with another trial-to-test operator $S = (B')^{-1} R_X$ of [2] (also cf. [20]):

$$\begin{array}{ccc} X & \xrightarrow{\;B\;} & Y' \\ {\scriptstyle R_X}\big\downarrow & & \big\uparrow{\scriptstyle R_Y} \\ X' & \xleftarrow{\;B'\;} & Y \end{array}$$

Application of S requires the inversion of the dual operator B'.

5. Fortin operators

The Fortin operator Π appearing in Assumption 3 is problem specific since it depends on the form b and the spaces. However, there are a few Fortin operators that have proved widely useful for analyzing DPG methods, including one for $Y = H^1(\Omega_h)$ and another for $Y = H(\mathrm{div}, \Omega_h)$, both given in [23]. In this section, we complete this collection by adding another operator for $Y = H(\mathrm{curl}, \Omega_h)$ intimately connected to the other two operators. Its utility will be clear in a subsequent section.

Since the Fortin operators for DPG methods are to be defined on broken Sobolev spaces, their construction can be done focusing solely on one element. We will now assume that the mesh Ω_h is a geometrically conforming finite element mesh of tetrahedral elements. Let $P_p(D)$ denote the set of polynomials of degree at most p on a domain D and let $N_p(D) = P_{p-1}(D)^3 + x \times P_{p-1}(D)^3$ denote the Nédélec [27] space. For domains $D \subset \mathbb{R}^n$, $n = 2, 3$, let $R_p(D) = P_{p-1}(D)^n + x P_{p-1}(D)$ denote the Raviart-Thomas [28] space. We use Π_p to denote the L^2 orthogonal projection onto $P_p(K)$. From now on, let us use C to denote a generic constant independent of $h_K = \operatorname{diam} K$. Its value at different occurrences may differ and may possibly depend on the shape regularity of K and the polynomial degree p.

Theorem 5.1. *On any tetrahedron K, there are operators*

$$\begin{aligned} \Pi_{p+3}^{\rm grad} &: H^1(K) \to P_{p+3}(K), \\ \Pi_{p+3}^{\rm curl} &: H(\mathrm{curl}, K) \to N_{p+3}(K), \\ \Pi_{p+3}^{\rm div} &: H(\mathrm{div}, K) \to R_{p+3}(K), \end{aligned}$$

such that the norm estimates

$$\| \Pi_{p+3}^{\mathrm{grad}} v\|_{H^1(K)} \leqslant C\|v\|_{H^1(K)}, \tag{5.1a}$$

$$\| \Pi_{p+3}^{\mathrm{curl}} F\|_{H(\mathrm{curl},K)} \leqslant C\|F\|_{H(\mathrm{curl},K)}, \tag{5.1b}$$

$$\| \Pi_{p+3}^{\mathrm{div}} q\|_{H(\mathrm{div},K)} \leqslant C\|q\|_{H(\mathrm{div},K)} \tag{5.1c}$$

hold, the diagram

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & H^1(K)/\mathbb{R} & \xrightarrow{\mathrm{grad}} & H(\mathrm{curl},K) & \xrightarrow{\mathrm{curl}} & H(\mathrm{div},K) & \xrightarrow{\mathrm{div}} & L^2(K) & \longrightarrow & 0 \\
 & & \Big\downarrow \Pi_{p+3}^{\mathrm{grad}} & & \Big\downarrow \Pi_{p+3}^{\mathrm{curl}} & & \Big\downarrow \Pi_{p+3}^{\mathrm{div}} & & \Big\downarrow \Pi_{p+2} & & \\
0 & \longrightarrow & P_{p+3}(K)/\mathbb{R} & \xrightarrow{\mathrm{grad}} & N_{p+3}(K) & \xrightarrow{\mathrm{curl}} & R_{p+3}(K) & \xrightarrow{\mathrm{div}} & P_{p+2}(K) & \longrightarrow & 0
\end{array}$$

commutes, and these identities hold for any $v \in H^1(K)$, $E \in H(\mathrm{curl},K)$, and $\tau \in H(\mathrm{div},K)$:

$$(q, \Pi_{p+3}^{\mathrm{grad}} v - v)_K = 0 \qquad \forall\, q \in P_{p-1}(K), \tag{5.2a}$$

$$\langle n\cdot\sigma, \Pi_{p+3}^{\mathrm{grad}} v - v\rangle = 0 \qquad \forall\, n\cdot\sigma \in \mathrm{tr}_{\mathrm{div}}^K R_{p+1}(K), \tag{5.2b}$$

$$(q, \Pi_{p+3}^{\mathrm{curl}} E - E)_K = 0 \qquad \forall\, q \in P_p(K)^3, \tag{5.2c}$$

$$\langle (n\times F)\times n, n\times(\Pi_{p+3}^{\mathrm{curl}} E - E)\rangle = 0 \qquad \forall\, (n\times F)\times n \in \mathrm{tr}_{\mathrm{curl},\top}^K P_{p+1}(K)^3, \tag{5.2d}$$

$$(q, \Pi_{p+3}^{\mathrm{div}} \tau - \tau)_K = 0 \qquad \forall\, q \in P_{p+1}(K)^3, \tag{5.2e}$$

$$\langle n\cdot(\Pi_{p+3}^{\mathrm{div}} \tau - \tau), \mu\rangle = 0 \qquad \forall\, \mu \in \mathrm{tr}_{\mathrm{grad}}^K P_{p+2}(K). \tag{5.2f}$$

To provide a constructive proof of Theorem 5.1, we will exhibit Fortin operators. We will use the exact sequence properties of the finite element spaces appearing as codomains of the operators in the theorem. We cannot use the canonical interpolation operators in these finite element spaces because they do not satisfy (5.1). Hence we will restrict the codomains of our operators to the following subspaces whose construction is motivated by zeroing out the unbounded degrees of freedom.

$$B_{p+3}^{\mathrm{grad}}(K) = \{v \in P_{p+3}(K) :\ v \text{ vanishes on all edges and vertices of } K\},$$

$$\begin{aligned}
B_{p+3}^{\mathrm{curl}}(K) = \Big\{ E \in N_{p+3}(K) :\ & t\cdot E = 0 \text{ on all edges of } K, \\
& \int_{\partial K} \phi\, n\cdot\mathrm{curl}\, E = 0 \quad \forall \phi \in P_{p+2}^{0,\perp}(\partial K) \text{ and} \\
& \int_{\partial K} ((n\times E)\times n)\cdot r = 0 \quad \forall r \in R_1(\partial K) \Big\},
\end{aligned}$$

$$B_{p+3}^{\mathrm{div}}(K) = \Big\{ \tau \in R_{p+3}(K) :\ \int_{\partial K} \phi\, n\cdot\tau = 0 \quad \forall \phi \in P_{p+2}^{\perp}(\partial K) \Big\}.$$

Here t denotes a tangent vector along the underlying edge, $R_1(\partial K) = \{r : r|_f \in R_1(f)$ for all faces f of $\partial K\}$, and $P_{p+2}^{0,\perp}(\partial K)$ and $P_{p+2}^{\perp}(\partial K)$ are defined as follows. To simplify notation, let $P_p(\partial K) = \mathrm{tr}_{\mathrm{div}}^K R_{p+1}(K)$ (the space of functions on ∂K that are polynomials of degree at most p on each face of ∂K) and let $P_p^{\mathrm{c}}(\partial K) = \mathrm{tr}_{\mathrm{grad}}^K P_p(K)$. Let $P_p^{0,\perp}(\partial K)$ denote the $L^2(\partial K)$-orthogonal complement of $P_p^{\mathrm{c}}(\partial K) + P_0(\partial K)$ in $P_p(\partial K)$ and let $P_p^{\perp}(\partial K)$ denote the

$L^2(\partial K)$-orthogonal complement of $P_p^{\rm c}(\partial K)$ in $P_p(\partial K)$. The following result is proved in [23, Lemma 3.2].

Lemma 5.2. *For any $v \in H^1(K)$, there is a unique $\Pi_0^{\rm g} v$ in $B_{p+3}^{\rm grad}(K)$ satisfying*

$$(q, \Pi_0^{\rm g} v - v)_K = 0 \quad \text{for all } q \in P_{p-1}(K), \tag{5.3a}$$

$$\langle n\cdot\sigma, \Pi_0^{\rm g} v - v\rangle = 0 \quad \text{for all } n\cdot\sigma \in \operatorname{tr}_{\rm div}^K R_{p+1}(K), \text{ and} \tag{5.3b}$$

$$\| \Pi_0^{\rm g} v\|_{L^2(K)} + h_K \|\operatorname{grad} \Pi_0^{\rm g} v\|_{L^2(K)} \leqslant C\left(\|v\|_{L^2(K)} + h_K \|\operatorname{grad} v\|_{L^2(K)}\right). \tag{5.3c}$$

We define $\Pi_{p+3}^{\rm grad}$ as a minor modification of the analogous operator in [23]. Given any $v \in H^1(K)$, first compute its mean value on the boundary

$$m_{\partial K}(v) = \frac{1}{|\partial K|}\int_{\partial K} v,$$

then split $v = v_0 + m_{\partial K}(v)$ where $v_0 = v - m_{\partial K}(v)$ has zero mean trace, and finally define

$$\Pi_{p+3}^{\rm grad} v = \Pi_0^{\rm g} v_0 + m_{\partial K}(v). \tag{5.4}$$

Lemma 5.3. *$\Pi_{p+3}^{\rm grad} v$ satisfies* (5.2a)–(5.2b) *and* (5.1a) *for any $v \in H^1(K)$*

Proof. Since $\Pi_{p+3}^{\rm grad} v - v = \Pi_0^{\rm g} v_0 - v_0$, equations (5.3a) and (5.3b) immediately yield (5.2a) and (5.2b). To prove the norm estimate (5.1a), note that standard scaling arguments imply

$$\|m_{\partial K}(v)\|^2_{L^2(K)} \leqslant \|v\|^2_{L^2(\partial K)} \frac{|K|}{|\partial K|} \leqslant C(\|v\|^2_{L^2(K)} + h_K^2 \|\operatorname{grad} v\|^2_{L^2(K)}), \tag{5.5}$$

$$\|v - m_{\partial K}(v)\|_{L^2(K)} \leqslant C h_K \|\operatorname{grad} v\|_{L^2(K)}. \tag{5.6}$$

for all v in $H^1(K)$. Combining (5.5) and (5.3c), we get

$$\| \Pi_{p+3}^{\rm grad} v\|_{L^2(K)} \leqslant C\left(\|v\|_{L^2(K)} + h_K \|\operatorname{grad} v\|_{L^2(K)}\right),$$

while combining (5.6) and (5.3c),

$$\begin{aligned} h_K \|\operatorname{grad}(\Pi_{p+3}^{\rm grad} v)\|_{L^2(K)} &= h_K \|\operatorname{grad} \Pi_0^{\rm g}(v - m_{\partial K}(v))\|_{L^2(K)}, && \text{by (5.4)}\\ &\leqslant C\left(\|v - m_{\partial K}(v)\|_{L^2(K)} + h_K \|\operatorname{grad}(v - m_{\partial K}(v))\|_{L^2(K)}\right) && \text{by (5.3c)}\\ &\leqslant C h_K \|\operatorname{grad} v\|_{L^2(K)} && \text{by (5.6).} \end{aligned}$$

These estimates together prove (5.1a). □

The next lemma is proved in [23, Lemma 3.3]. It defines $\Pi_{p+3}^{\rm div}$ exactly as in [23].

Lemma 5.4. *Any $\sigma \in B_{p+3}^{\rm div}(K)$ satisfying*

$$(q, \sigma)_K = 0 \qquad \forall\, q \in P_{p+1}(K)^3, \tag{5.7a}$$

$$\langle n\cdot\sigma, \mu\rangle = 0 \qquad \forall\, \mu \in \operatorname{tr}_{\rm grad}^K P_{p+2}(K), \tag{5.7b}$$

vanishes. Moreover, for any $\tau \in H(\operatorname{div}, K)$, there is a unique function $\Pi_{p+3}^{\rm div}\tau$ in $B_{p+3}^{\rm div}(K)$ satisfying (5.2e)–(5.2f)*. It also satisfies* (5.1c).

The remaining operator $\Pi_{p+3}^{\rm curl}$ will be defined after the next result. It is modeled after the previous two lemmas, but requires considerably more work.

Lemma 5.5. *Any $E \in B^{\mathrm{curl}}_{p+3}(K)$ satisfying*

$$(\phi, E)_K = 0 \qquad \forall \phi \in P_p(K)^3 \tag{5.8a}$$

$$\langle \mu, n \times E \rangle = 0 \qquad \forall \mu \in \mathrm{tr}^K_{\mathrm{curl},\top} P_{p+1}(K)^3, \tag{5.8b}$$

vanishes.

Proof. Integrating by parts twice and using (5.8b), we have

$$\int_{\partial K} \psi\, n \cdot \mathrm{curl}\, E = (\mathrm{curl}\, E, \mathrm{grad}\, \psi)_K = \langle n \times E, \mathrm{grad}\, \psi \rangle = 0 \qquad \forall \psi \in P_{p+2}(K). \tag{5.9}$$

In addition, by Stokes theorem applied to one face f of K, we have

$$\int_f \kappa\, n \cdot \mathrm{curl}\, E = \int_{\partial f} \kappa\, E \cdot t = 0 \qquad \forall \kappa \in P_0(\partial K), \tag{5.10}$$

since $E \cdot t = 0$ on all edges by the definition of $B^{\mathrm{curl}}_{p+3}(K)$. The definition of $B^{\mathrm{curl}}_{p+3}(K)$ also gives

$$\int_{\partial K} \phi\, n \cdot \mathrm{curl}\, E = 0 \qquad \forall \phi \in P^{0,\perp}_{p+2}(\partial K). \tag{5.11}$$

Since $P_{p+2}(\partial K) = P^{0,\perp}_{p+2}(\partial K) + P^c_{p+2}(\partial K) + P_0(\partial K)$, equations (5.9),(5.10) and (5.11) together imply

$$\int_{\partial K} \psi\, n \cdot \mathrm{curl}\, E = 0 \qquad \forall \psi \in P_{p+2}(\partial K). \tag{5.12}$$

Since $n \cdot \mathrm{curl}\, E \in P_{p+2}(\partial K)$, we thus find that $n \cdot \mathrm{curl}\, E = 0$ on ∂K. This implies that the tangential component of E on ∂K, namely $E_\top = (n \times E) \times n$, has vanishing surface curl, so it must equal a surface gradient, i.e., $E_\top = \mathrm{grad}_\top v$ for some $v \in P^{\mathrm{c}}_{p+3}(\partial K)$. Moreover, since $E_\top$ vanishes on all edges, v may be chosen to be of the form $v = b_f v_p$ for some $v_p \in P_p(\partial K)$, where b_f is the product of all barycentric coordinates of K that do not vanish a.e. on f.

To use the remaining (as yet unused) condition in the definition of $B^{\mathrm{curl}}_{p+3}(K)$, note that the tangential component of the coordinate vector x, namely $x_\top$ is in $R_1(\partial K)$. Combining this with (5.8b), we find that for all $\mu \in \mathrm{tr}^K_{\mathrm{curl},\top} P_{p+1}(K)^3$ and any $\kappa \in \mathbb{R}$,

$$\begin{aligned} 0 = \langle n \times E, \mu \rangle + \kappa \int_{\partial K} E_\top \cdot x &= \int_{\partial K} E_\top \cdot (\mu \times n + \kappa x_\top) \\ &= \sum_f \int_f \mathrm{grad}_\top (b_f v_p) \cdot (\mu \times n + \kappa x_\top) \\ &= \sum_f \int_f b_f v_p\, \mathrm{div}_\top (\mu \times n + \kappa x_\top), \end{aligned} \tag{5.13}$$

where the sums run over all faces f of ∂K. For any $\mu \in \mathrm{tr}^K_{\mathrm{curl},\top} P_{p+1}(K)^3$, the function $\mu \times n$ is in the Raviart-Thomas space on the closed manifold ∂K denoted by $R^c_{p+1}(\partial K)$. (Note that unlike $R_1(\partial K)$, this space consist of functions with the appropriate compatibility conditions across edges of ∂K.) The surface divergence map

$$\mathrm{div}_\top : R^c_{p+1}(\partial K) \to \left\{ w \in P_p(\partial K) : \int_{\partial K} w = 0 \right\}$$

is surjective. Hence the term $\operatorname{div}_\top(\mu \times n + \kappa x_\top)$ appearing in (5.13) spans all of $P_p(\partial K)$ as μ and κ are varied. Choosing μ and κ so that $\operatorname{div}_\top(\mu \times n + \kappa x_\top) = v_p$, we conclude that v_p vanishes and hence $E_\top = \operatorname{grad}_\top(b_f v_p) = 0$, i.e.,

$$n \times E = 0 \qquad \text{on } \partial K. \tag{5.14}$$

Next, setting $\phi = \operatorname{curl} r$ in (5.8a) and integrating by parts, we obtain

$$\int_K r \cdot \operatorname{curl} E = 0 \qquad \forall r \in P_{p+1}(K)^3. \tag{5.15}$$

From (5.12) and (5.15), it follows that $\tau = \operatorname{curl} E$ is in $B^{\text{div}}_{p+3}(K)$, and furthermore, τ satisfies (5.7a) and (5.7b). Hence, by Lemma 5.4, τ vanishes. Thus $\operatorname{curl} E = 0$ and consequently $E = \operatorname{grad} v$ for some $v \in P_{p+3}(K)$. Furthermore, by (5.14), we may choose $v = b_K v_{p-1}$ for some $v_{p-1} \in P_{p-1}(K)$, where b_K is the product of all barycentric coordinates of K. Then (5.8a) implies

$$\int_K \operatorname{grad}(b_K v_{p-1}) \cdot \phi = \int_K b_K v_{p-1} \cdot \operatorname{div} \phi = 0 \qquad \forall \phi \in P_p(K)^3.$$

It now follows from the surjectivity of $\operatorname{div} : P_p(K)^3 \to P_{p-1}(K)$ that v_{p-1}, and in turn $E = \operatorname{grad}(b_K v_{p-1})$, vanishes on K. $\square$

The next lemma defines the operator Π^{curl}_{p+3}. It will be useful to observe now that for any $E \in B^{\text{curl}}_{p+3}(K)$,

$$\int_{\partial K} ((n \times E) \times n) \cdot r = 0 \quad \forall r \in R_1(\partial K) \iff \int_{\partial K} ((n \times E) \times n) \cdot x_\top = 0. \tag{5.16}$$

Indeed, while the forward implication is obvious, the converse follows from (5.10). This shows that the condition that appears both in the definition of $B^{\text{curl}}_{p+3}(K)$ and in (5.16) above, actually amounts to just one constraint.

Lemma 5.6. *Given any $E \in H(\operatorname{curl}, K)$, there is a unique $\Pi^{\text{curl}}_{p+3} E$ in $B^{\text{curl}}_{p+3}(K)$ satisfying* (5.2c)–(5.2d).

Proof. We need to estimate $n = \dim B^{\text{curl}}_{p+3}(K)$. First, note that since $P_0(\partial K) \cap P^{\text{c}}_{p+2}(\partial K)$ is a one-dimensional space of constant functions on ∂K,

$$\begin{aligned}
\dim P^{0,\perp}_{p+2}(\partial K) &= \dim P_{p+2}(\partial K) - \dim(P^c_{p+2}(\partial K) + P_0(\partial K)) \\
&= \dim P_{p+2}(\partial K) - \dim P^c_{p+2}(\partial K) - \dim P_0(\partial K) + 1 \\
&= 6p + 11.
\end{aligned}$$

The tangential component $E \cdot t$ of any $E \in N_{p+3}(K)$ is a polynomial of degree at most $p+2$ on each edge, so $E \cdot t$ represents $p+3$ constraints per edge. Hence, counting the number of constraints in the definition of $B^{\text{curl}}_{p+3}(K)$,

$$\begin{aligned}
n = \dim B^{\text{curl}}_{p+3}(K) &\geqslant \dim N_{p+3}(K) - 6(p+3) - \dim(P^{0,\perp}_{p+2}(\partial K)) - 1 \\
&= \dim N_{p+3}(K) - 6(p+3) - (6p+11) - 1
\end{aligned}$$

where we have used (5.16). Thus,

$$n \geqslant \dim N_{p+3}(K) - 12p - 30. \tag{5.17}$$

Next, we count the number of equations in (5.2c)–(5.2d), namely

$$\begin{aligned} m &= \dim(\operatorname{tr}^K_{\mathrm{curl},\top} P_{p+1}(K)^3) + \dim P_p(K)^3 \\ &= 2\dim P_{p+1}(\partial K) - 6(p+2) + \dim P_p(K)^3 \\ &= \dim N_{p+3}(K) - 6(p+2) - 6(p+3). \end{aligned}$$

This together with (5.17) implies that $m = \dim N_{p+3}(K) - 12p - 30 \leqslant n$ Thus, the system (5.2c)–(5.2d), after using a basis, is an $m \times n$ matrix system of the form $Ax = d$, where $x \in \mathbb{R}^n$ is the vector of coefficients in a basis expansion of $\mathit{\Pi}^{\mathrm{curl}}_{p+3} E$ and d is the right-hand side vector made using the given E. By Lemma 5.5, $\mathrm{nul}(A) = 0$. Hence $m \leqslant n = \mathrm{rank}(A) + \mathrm{nul}(A) = \mathrm{rank}(A) \leqslant \min(m,n)$ shows that $m = n$. The system determining $\mathit{\Pi}^{\mathrm{curl}}_{p+3} E$ is therefore a square invertible system. ☐

Lemma 5.7. *For all $v \in H^1(K)/\mathbb{R}$ and $E \in H(\mathrm{curl}, K)$,*

$$\mathrm{grad}(\mathit{\Pi}^{\mathrm{grad}}_{p+3}\, v) \in B^{\mathrm{curl}}_{p+3}(K), \qquad \mathrm{curl}(\mathit{\Pi}^{\mathrm{curl}}_{p+3}\, E) \in B^{\mathrm{div}}_{p+3}(K).$$

Proof. Let $e = \mathrm{grad}(\mathit{\Pi}^{\mathrm{grad}}_{p+3}\, v) = \mathrm{grad}(\mathit{\Pi}^{\mathrm{g}}_0\, v_0)$ where $\mathit{\Pi}^{\mathrm{g}}_0\, v_0 \in B^{\mathrm{grad}}_{p+3}(K)$ is as in (5.4). Since $\mathit{\Pi}^{\mathrm{g}}_0\, v_0$ is constant along edges of K, e must satisfy $t \cdot e = 0$ along the edges. Moreover,

$$m_{\partial K}(\mathit{\Pi}^{\mathrm{g}}_0\, v_0) = 0 \tag{5.18}$$

due to (5.3b). Hence, integrating by parts on on any face f of ∂K,

$$\int_f e_\top \cdot x_\top = \int_f x_\top \cdot \mathrm{grad}_\top(\mathit{\Pi}^{\mathrm{g}}_0\, v_0) = -\int_f \mathit{\Pi}^{\mathrm{g}}_0\, v_0 \,\mathrm{div}_\top(x_\top) = -2\int_f \mathit{\Pi}^{\mathrm{g}}_0\, v_0.$$

Summing over all faces f of ∂K and using (5.18), we conclude that

$$\int_{\partial K} e_\top \cdot x = 0.$$

Therefore, to finish proving that $e \in B^{\mathrm{curl}}_{p+3}(K)$, it only remains to show that $\int_{\partial K} \phi\, n \cdot \mathrm{curl}\, e = 0$ for all $\phi \in P^{0,\perp}_{p+2}(\partial K)$. But this is obvious from the fact that e is a gradient.

Next, we need to show that $\sigma = \mathrm{curl}(\mathit{\Pi}^{\mathrm{curl}}_{p+3}\, E)$ is in $B^{\mathrm{div}}_{p+3}(K)$. Since it is obvious that $\sigma \in R_{p+3}(K)$, it suffices to prove that

$$\int_{\partial K} \phi\, n \cdot \mathrm{curl}(\mathit{\Pi}^{\mathrm{curl}}_{p+3}\, E) = 0 \tag{5.19}$$

for all ϕ in $P^{\perp}_{p+2}(\partial K)$. Note that $P^{\perp}_{p+2}(\partial K)$ can be orthogonally decomposed into its subspace $P_0(\partial K) \cap P^{\perp}_{p+2}(\partial K)$ and its $L^2(\partial K)$-orthogonal complement. The latter is a subspace of $P^{0,\perp}_{p+2}(\partial K)$ where (5.19) holds (since $\mathit{\Pi}^{\mathrm{curl}}_{p+3}\, E \in B^{\mathrm{curl}}_{p+3}(K)$). Hence it only remains to prove that (5.19) holds for ϕ in $P_0(\partial K) \cap P^{\perp}_{p+2}(\partial K)$. But Stokes theorem shows that (5.19) actually holds for all $\phi \in P_0(\partial K)$ – cf. (5.10). ☐

Lemma 5.8. *For all $v \in H^1(K), E \in H(\mathrm{curl}, K)$, and $\sigma \in H(\mathrm{div}, K)$,*

$$\mathrm{grad}\, \mathit{\Pi}^{\mathrm{grad}}_{p+3}\, v = \mathit{\Pi}^{\mathrm{curl}}_{p+3}\, \mathrm{grad}\, v, \tag{5.20a}$$

$$\mathrm{curl}\, \mathit{\Pi}^{\mathrm{curl}}_{p+3}\, E = \mathit{\Pi}^{\mathrm{div}}_{p+3}\, \mathrm{curl}\, E, \tag{5.20b}$$

$$\mathrm{div}\, \mathit{\Pi}^{\mathrm{div}}_{p+3}\, \sigma = \mathit{\Pi}_{p+2}\, \mathrm{div}\, \sigma. \tag{5.20c}$$

Proof. By Lemma 5.7, $\delta_1 = \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v) - \varPi^{\mathrm{curl}}_{p+3} \operatorname{grad} v$ is in $B^{\mathrm{curl}}_{p+3}(K)$. We will now show that δ_1 satisfies (5.8). Let $\phi \in P_p(K)^3$ and consider

$$(\phi, \delta_1)_K = (\phi, \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v - v))_K - (\phi, \varPi^{\mathrm{curl}}_{p+3}(\operatorname{grad} v) - \operatorname{grad} v)_K.$$

By Lemma 5.6, $\varPi^{\mathrm{curl}}_{p+3} \operatorname{grad} v$ satisfies (5.2c)–(5.2d), so the last term above vanishes. Integrating the remaining term on the right-hand side by parts, and using (5.2a)–(5.2b), we find that

$$(\phi, \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v - v))_K = 0 \qquad \forall \phi \in P_p(K)^3. \tag{5.21}$$

This proves that $(\phi, \delta_1)_K = 0$, i.e., (5.8a) holds. Next, for any $\mu = \operatorname{tr}^K_{\mathrm{curl},\top}(F)$, $F \in P_{p+1}(K)^3$, we have

$$\langle \mu, n \times \delta_1 \rangle = \langle F \times n, \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v - v) \rangle - \langle \mu, n \times (\varPi^{\mathrm{curl}}_{p+3}(\operatorname{grad} v) - \operatorname{grad} v) \rangle.$$

The last term vanishes due to (5.2d). Moreover,

$$\langle F \times n, \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v - v) \rangle = -(\operatorname{curl} F, \operatorname{grad}(\varPi^{\mathrm{grad}}_{p+3} v - v))_K = 0$$

due to (5.21), so we have proven that (5.8b) holds as well. Hence by Lemma 5.5, $\delta_1 = 0$. This proves (5.20a).

To prove (5.20b), we proceed similarly and show that $\delta_2 = \operatorname{curl} \varPi^{\mathrm{curl}}_{p+3} E - \varPi^{\mathrm{div}}_{p+3} \operatorname{curl} E$ is zero. By Lemma 5.7, we know that $\delta_2 \in B^{\mathrm{div}}_{p+3}(K)$, so if we prove that

$$(\delta_2, \phi)_K = 0 \qquad \forall \phi \in P_{p+1}(K)^3, \tag{5.22a}$$

$$\langle n \cdot \delta_2, \mu \rangle_{\partial K} = 0 \qquad \forall \mu \in \operatorname{tr}^K_{\mathrm{grad}} P_{p+2}(K). \tag{5.22b}$$

then Lemma 5.4 would yield $\delta_2 = 0$. To prove (5.22b),

$$\begin{aligned}
\langle n \cdot \delta_2, \mu \rangle_{\partial K} &= \langle n \cdot (\operatorname{curl}(\varPi^{\mathrm{curl}}_{p+3} E - E) + (I - \varPi^{\mathrm{div}}_{p+3}) \operatorname{curl} E), \mu \rangle_{\partial K} \\
&= \langle n \cdot \operatorname{curl}(\varPi^{\mathrm{curl}}_{p+3} E - E), \mu \rangle_{\partial K} && \text{by (5.7b)} \\
&= \langle (\varPi^{\mathrm{curl}}_{p+3} E - E) \times n, \operatorname{grad}_\top \mu \rangle_{\partial K} = 0 && \text{by (5.2d)}.
\end{aligned}$$

To prove (5.22a),

$$\begin{aligned}
(\delta_2, \phi)_K &= (\operatorname{curl}(\varPi^{\mathrm{curl}}_{p+3} E - E) + (I - \varPi^{\mathrm{div}}_{p+3}) \operatorname{curl} E, \phi)_K \\
&= (\operatorname{curl}(\varPi^{\mathrm{curl}}_{p+3} E - E), \phi)_K && \text{by (5.2e)} \\
&= (\varPi^{\mathrm{curl}}_{p+3} E - E, \operatorname{curl} \phi)_K + \langle (\varPi^{\mathrm{curl}}_{p+3} E - E) \times n, \phi \rangle = 0 && \text{by (5.2c)–(5.2d)}.
\end{aligned}$$

This finishes the proof of (5.22) and hence (5.20b) follows.

Finally, to prove (5.20c), let $\delta_3 = \operatorname{div} \varPi^{\mathrm{div}}_{p+3} \sigma - \varPi_{p+2} \operatorname{div} \sigma$ in $P_{p+2}(K)$. For any $w \in P_{p+2}(K)$, integrating by parts,

$$\begin{aligned}
(\delta_3, w)_K &= (\operatorname{div}(\varPi^{\mathrm{div}}_{p+3} \sigma - \sigma), w)_K \\
&= -(\varPi^{\mathrm{div}}_{p+3} \sigma - \sigma, \operatorname{grad} w)_K + \langle n \cdot (\varPi^{\mathrm{div}}_{p+3} \sigma - \sigma), w \rangle,
\end{aligned}$$

which vanishes by (5.2e)–(5.2f). Hence $\delta_3 = 0$ and (5.20c) is proved. $\square$

Proof of Theorem 5.1. The lemmas of this section prove all statements of Theorem 5.1 except (5.1b). To prove (5.1b), we use a scaling argument and the commutativity properties of Lemma 5.8. Let $\hat{K}$ denote the unit tetrahedron and let the $H(\text{curl})$-Fortin operator on $\hat{K}$, defined as above, be denoted by $\hat{\Pi}^{\text{curl}}_{p+3}$. By the unisolvency result of Lemma 5.6 and finite dimensionality, there is a $C_0 > 0$ such that

$$\| \hat{\Pi}^{\text{curl}}_{p+3} \hat{E} \|^2_{H(\text{curl},\hat{K})} \leqslant C_0 \|\hat{E}\|^2_{H(\text{curl},\hat{K})}$$

for all $\hat{E} \in H(\text{curl}, \hat{K})$. Let $S_K : \hat{K} \to K$ be the one-to-one affine map that maps $\hat{K}$ onto a general tetrahedron K. For $E : K \to \mathbb{R}^3$, define $\Phi(E) = (S_K')^t (E \circ S_K)$, and $\|E\|^2_{K,\text{curl}} = h_K^{-2} \|E\|^2_{L^2(K)} + \|\operatorname{curl} E\|^2_{L^2(K)}$. The elementary proofs of the following assertions (i)–(iii) are left to the reader.

(i) $E \in B^{\text{curl}}_{p+3}(K)$ if and only if $\Phi(E) \in B^{\text{curl}}_{p+3}(\hat{K})$.
(ii) There are constants C_1, C_2 depending only on the shape regularity of K (but not on h_K) such that $C_1 \|E\|^2_{K,\text{curl}} \leqslant h_K \|\Phi(E)\|^2_{\hat{K},\text{curl}} \leqslant C_2 \|E\|^2_{K,\text{curl}}$.
(iii) $\Phi(\Pi^{\text{curl}}_{p+3} E) = \hat{\Pi}^{\text{curl}}_{p+3} \Phi(E)$.

These three statements imply that

$$C_1 \|\Pi^{\text{curl}}_{p+3} E\|^2_{K,\text{curl}} \leqslant h_K \|\hat{\Pi}^{\text{curl}}_{p+3} \Phi(E)\|^2_{\hat{K},\text{curl}} \leqslant h_K C_0 \|\Phi(E)\|^2_{\hat{K},\text{curl}} \leqslant C_0 C_2 \|E\|^2_{K,\text{curl}}.$$

While this immediately gives the needed estimate for the L^2-part, namely

$$\| \Pi^{\text{curl}}_{p+3} E\|_{L^2(K)} \leqslant C \|E\|_{H(\text{curl},K)}, \tag{5.23}$$

we need to improve the estimate on the curl to finish the proof: For this, we use the commutativity property

$$\begin{aligned} \| \operatorname{curl} \Pi^{\text{curl}}_{p+3} E\|_{L^2(K)} &= \| \Pi^{\text{div}}_{p+3} \operatorname{curl} E\|_{L^2(K)} && \text{by Lemma 5.8,} \\ &\leqslant C \| \operatorname{curl} E\|_{H(\text{div},K)} && \text{by Lemma 5.4.} \end{aligned} \tag{5.24}$$

The required estimate (5.1b) follows from (5.23) and (5.24). □

Before concluding this section, let us illustrate how to use Theorem 5.1 for error analysis of DPG methods by an example.

Example 5.9 (Primal DPG method for the Dirichlet problem). Consider the broken variational problem of Example 3.4: Find $(u, \hat{\sigma}_n) \in X = \mathring{H}^1(\Omega) \times H^{-1/2}(\partial \Omega_h)$ such that (4.1) holds with

$$b(\,(u, \hat{\sigma}_n), y\,) = (\operatorname{grad} u, \operatorname{grad} y)_h + \langle \hat{\sigma}_n, y \rangle_h, \qquad Y = H^1(\Omega_h).$$

We want to analyze the DPG method given by (4.3) with

$$\begin{aligned} X_h &= \{(w_h, n \cdot \hat{\tau}_h) \in X : \; w_h|_K \in P_{p+1}(K) \text{ and } n \cdot \hat{\tau}_h|_{\partial K} \in \operatorname{tr}^K_{\text{div}} R_{p+1}(K) \\ &\qquad\qquad \text{for all (tetrahedral mesh elements) } K \in \Omega_h\}, \\ Y_h &= \{y_h \in H^1(\Omega_h) : y_h|_K \in P_{p+3}(K) \text{ for all } K \in \Omega_h\}. \end{aligned}$$

We have already shown in Example 3.4 that the inf-sup condition required for application of Theorem 4.1 holds (and $Z = \{0\}$). Hence to obtain optimal error estimates from Theorem 4.1, it suffices to verify Assumption 3. We claim that Assumption 3 holds with $\Pi = \Pi^{\mathrm{grad}}_{p+3}$. Indeed,

$$\begin{aligned} b(\,(w_h, n\cdot\hat\tau_h), y - \Pi^{\mathrm{grad}}_{p+3}\, y) &= (\operatorname{grad} w_h, \operatorname{grad}(y - \Pi^{\mathrm{grad}}_{p+3}\, y)\,)_h + \langle n\cdot\hat\tau_h, y - \Pi^{\mathrm{grad}}_{p+3}\, y\rangle_h \\ &= -(\Delta w_h, y - \Pi^{\mathrm{grad}}_{p+3}\, y)_h + \langle n\cdot\hat\tau_h - \frac{\partial w_h}{\partial n}, y - \Pi^{\mathrm{grad}}_{p+3}\, y\rangle_h = 0 \end{aligned}$$

by applying (5.2a)–(5.2b) element by element. Note that here we have used the fact that the discrete spaces have been set so that $-\Delta w_h|_K \in P_{p-1}(K)$ and $n\cdot\hat\tau_h - n\cdot\operatorname{grad} w_h$ is a polynomial of degree at most p on each face of ∂K (i.e., it is in $\mathrm{tr}^K_{\mathrm{div}}\, R_{p+1}(K)$), allowing us to apply (5.2a)–(5.2b). Applying Theorem 4.1, we recover the error estimates for this method, originally proved in [17]. ///

6. Maxwell equations

In this section, we combine the various tools developed in the previous sections to analyze the DPG method for a model problem in time-harmonic electromagnetic wave propagation.

6.1. **The cavity problem.** Consider a cavity Ω, an open bounded connected and contractible domain in $\mathbb{R}^3$, shielded from its complement by a perfect electric conductor throughout its boundary $\partial\Omega$. If all time variations are harmonic of frequency $\omega > 0$, then Maxwell equations within the cavity reduce to these:

(6.1a) $$-\imath\omega\mu H + \operatorname{curl} E = 0 \qquad \text{in } \Omega,$$

(6.1b) $$-\imath\omega\epsilon E - \operatorname{curl} H = -J \qquad \text{in } \Omega,$$

(6.1c) $$n \times E = 0 \qquad \text{on } \partial\Omega.$$

The functions $E, H, J : \Omega \to \mathbb{C}^3$ represent electric field, magnetic field, and imposed current, respectively, and $\imath$ denotes the imaginary unit. For simplicity we assume that the electromagnetic properties ϵ and μ are positive and constant on each element of the tetrahedral mesh Ω_h. The number $\omega > 0$ denotes a fixed wavenumber. In this section we develop and analyze a DPG method for (6.1).

Eliminating H from (6.1a) and (6.1b), we obtain the following second order (non-elliptic) equation

(6.2) $$\operatorname{curl} \mu^{-1} \operatorname{curl} E - \omega^2 \epsilon E = f,$$

where $f = \imath\omega J$. The standard variational formulation for this problem is obtained by multiplying (6.2) by a test function $F \in \mathring{H}(\mathrm{curl}, \Omega)$, integrating by parts and using the boundary condition (6.1c): Find $E \in \mathring{H}(\mathrm{curl}, \Omega)$ satisfying

(6.3) $$(\mu^{-1} \operatorname{curl} E, \operatorname{curl} F)_\Omega - \omega^2 (\epsilon E, F)_\Omega = \langle f, F\rangle$$

for any given $f \in \mathring{H}(\mathrm{curl}, \Omega)'$. It is well-known [25] that (6.3) has a unique solution for every $f \in \mathring{H}(\mathrm{curl}, \Omega)'$ whenever ω is not in the countably infinite set Σ of resonances of the cavity Ω. Throughout this section, we assume $\omega \notin \Sigma$. This wellposedness result provides an accompanying stability estimate, namely there is a constant $C_\omega > 0$ such that

(6.4) $$\|E\|_{H(\mathrm{curl},\Omega)} \leqslant C_\omega \|f\|_{\mathring{H}(\mathrm{curl},\Omega)'}$$

for any $f \in \mathring{H}(\operatorname{curl}, \Omega)'$ and $E \in \mathring{H}(\operatorname{curl}, \Omega)$ satisfying (6.3). Note that the stability constant C_ω may blow up as ω approaches a resonance. We continue to use C to denote a generic mesh-independent constant, which in this section may depend on ω, μ, and ϵ as well.

6.2. **Primal DPG method for the cavity problem.** The primal DPG method for the cavity problem is obtained by breaking (6.3). Multiply (6.2) by a (broken) test function $F \in H(\operatorname{curl}, \Omega_h)$ and integrate by parts, element by element, to get

$$(\mu^{-1} \operatorname{curl} E, \operatorname{curl} F)_h - \langle n \times \mu^{-1} \operatorname{curl} E, F \rangle_h - \omega^2 (\varepsilon E, F)_h = (f, F)_h.$$

Now set $n \times \hat{H} = (\imath\omega)^{-1} n \times \mu^{-1} \operatorname{curl} E$ to be an independent interface unknown which is to be found in $H^{-1/2}(\operatorname{div}, \Omega_h)$. This leads to the variational problem (3.4) with the following spaces and forms:

$$X_0 = \mathring{H}(\operatorname{curl}, \Omega), \qquad Y = H(\operatorname{curl}, \Omega_h), \tag{6.5a}$$

$$\hat{X} = H^{-1/2}(\operatorname{div}, \partial\Omega_h), \qquad Y_0 = \mathring{H}(\operatorname{curl}, \Omega), \tag{6.5b}$$

$$b_0(E, F) = (\mu^{-1} \operatorname{curl} E, \operatorname{curl} F)_h - \omega^2 (\varepsilon E, F)_h, \tag{6.5c}$$

$$\hat{b}(\hat{H}_\dashv, F) = -\imath\omega \langle \hat{H}_\dashv, F \rangle_h. \tag{6.5d}$$

This is the primal DPG formulation for the Maxwell cavity problem.

The numerical method discretizes the above variational problem using subspaces $X_h \subset X = X_0 \times \hat{X}$ and $Y_h \subset Y$ defined by

$$X_h = \{(E_h, n \times \hat{H}_h) \in \mathring{H}(\operatorname{curl}, \Omega) \times H^{-1/2}(\operatorname{div}, \partial\Omega_h) : n \times \hat{H}_h|_{\partial K} \in \operatorname{tr}^K_{\operatorname{curl},\dashv} P_{p+1}(K)^3, \text{ and } E_h|_K \in P_p(K)^3 \text{ for all } K \in \Omega_h\}, \tag{6.6a}$$

$$Y_h = \{F_h \in H(\operatorname{curl}, \Omega_h) : F_h|_K \in N_{p+3}(K) \text{ for all } K \in \Omega_h\}. \tag{6.6b}$$

We have the following error bound for the numerical solution in terms of the mesh size $h = \max_{K \in \Omega_h} h_K$ and polynomial degree $p \geqslant 1$.

Corollary 6.1. *Suppose $(E_h, n \times \hat{H}_h) \in X_h$ is the DPG solution given by* (4.3) *with forms and spaces set by* (6.5) *and let $(E, n \times \hat{H}) \in X$ be the exact solution of* (4.1)*. Then, there exists a C depending only on ω, p, and the shape regularity of the mesh such that*

$$\begin{aligned}\|E - E_h\|_{H(\operatorname{curl},\Omega)} &+ \|n \times (\hat{H} - \hat{H}_h)\|_{H^{-1/2}(\operatorname{div},\partial\Omega_h)} \\ &\leqslant Ch^p \left(|E|_{H^{p+1}(\Omega)} + |\operatorname{curl} E|_{H^{p+1}(\Omega)} + |\operatorname{curl} H|_{H^{p+1}(\Omega)} \right).\end{aligned}$$

Proof. To apply Theorem 4.1, we must verify the inf-sup condition (4.5) for the broken form. As in the previous examples, as a first step, we verify the inf-sup condition for the unbroken form stated in Assumption 1. Given any $E \in \mathring{H}(\operatorname{curl}, \Omega)$, let $f_E \in \mathring{H}(\operatorname{curl}, \Omega)'$ be defined by $\langle f_E, F \rangle = (\mu^{-1} \operatorname{curl} E, \operatorname{curl} F)_\Omega - \omega^2 (\epsilon E, F)_\Omega$ for all $F \in \mathring{H}(\operatorname{curl}, \Omega)$. Then, (6.2) and (6.4) implies

$$\|E\|_{H(\operatorname{curl},\Omega)} \leqslant C_\omega \|f_E\|_{\mathring{H}(\operatorname{curl},\Omega)'} = C_\omega \sup_{F \in H(\operatorname{curl},\Omega)} \frac{|b_0(E, F)|}{\|F\|_{H(\operatorname{curl},\Omega)}},$$

i.e., Assumption 1 holds with $c_0 = C_\omega^{-1}$. Assumption 2, with $\hat{c} = \omega^{-1}$ is immediately verified by (2.11c) and (2.10c) of Theorem 2.3. Hence Theorem 3.1 verifies (4.5) and also shows that $Z = \{0\}$. The only remaining condition to verify before applying Theorem 4.1 is Assumption 3, which immediately follows by the choice of spaces and Theorem 5.1.

Applying Theorem 4.1, we find that

$$\|E - E_h\|^2_{H(\mathrm{curl},\Omega)} + \|n \times (\hat{H} - \hat{H}_h)\|^2_{H^{-1/2}(\mathrm{div},\partial\Omega_h)}$$
$$\leqslant C \inf_{(G_h, n\times\hat{R}_h)\in X_h} \left[\|E - G_h\|^2_{H(\mathrm{curl},\Omega)} + \|n \times \hat{H} - n \times \hat{R}_h\|^2_{H^{-1/2}(\mathrm{div},\partial\Omega_h)} \right].$$

Now, $H = (i\omega\mu)^{-1} \operatorname{curl} E$ is an extension to Ω of the exact interface solution $n \times \hat{H}$. Moreover, the interface function $n \times \hat{R}_h$ appearing above can be extended into $X^p_{0,h} = \{r \in \mathring{H}(\mathrm{curl}, \Omega) : r_h|_K \in P_{p+1}(K)^3\}$. Since the interface norm is the minimum over all extensions, by standard approximation estimates (see e.g., [19, Theorem 8.1]),

$$\|E - E_h\|^2_{H(\mathrm{curl},\Omega)} + \|n \times (\hat{H} - \hat{H}_h)\|^2_{H^{-1/2}(\mathrm{div},\partial\Omega_h)}$$
$$\leqslant C \left[\inf_{G_h \in X^p_{0,h}} \|E - G_h\|^2_{H(\mathrm{curl},\Omega)} + \inf_{R_h \in X^{p+1}_{0,h}} \|H - R_h\|^2_{H(\mathrm{curl},\Omega)} \right]$$
$$\leqslant C \sum_{K\in\Omega_h} \Big[h_K^{2(s_1+1)} |E|^2_{H^{s_1+1}(K)} + h_K^{2s_1} |\operatorname{curl} E|^2_{H^{s_1+1}(K)} +$$
$$h_K^{2(s_2+1)} |H|^2_{H^{s_2+1}(K)} + h_K^{2s_2} |\operatorname{curl} H|^2_{H^{s_2+1}(K)} \Big],$$

where $1/2 < s_1 \leqslant p$ and $1/2 < s_2 \leqslant p + 1$. Hence the corollary follows. □

Remark 6.2. Unlike the standard finite element method, for the DPG method, there is no need for h to be "sufficiently small" to assert a convergence estimate as in Corollary 6.1.

6.3. **Alternative formulations of the same problem.** In Example 3.5, we saw that a single diffusion-convection-reaction equation admits various different formulations. The situation is similar with Maxwell equations. First, let us write (6.1) in operator form using an operator A (analogous to the one in (3.10), but now) defined by

$$A \begin{bmatrix} H \\ E \end{bmatrix} = \begin{bmatrix} \imath\omega\mu & -\operatorname{curl} \\ \operatorname{curl} & \imath\omega\epsilon \end{bmatrix} \begin{bmatrix} H \\ E \end{bmatrix} = \begin{bmatrix} \imath\omega\mu H - \operatorname{curl} E \\ \imath\omega\epsilon E + \operatorname{curl} H \end{bmatrix}$$

as $A(H, E) = (0, J)$ for some given J in $L^2(\Omega)^3$. However, we will not restrict to right-hand sides of this form as we will need to allow the most general data possible in the ensuing wellposedness studies.

We view A as an unbounded closed operator on $L^2(\Omega)^6$ whose domain is

$$\operatorname{dom}(A) = \{(H, E) \in H(\mathrm{curl}, \Omega)^2 : n \times E = 0 \text{ on } \partial\Omega\}.$$

It is easy to show that its adjoint (in the sense of closed operators) is the closed operator A^* given by

$$A^* \begin{bmatrix} H \\ E \end{bmatrix} = \begin{bmatrix} -\imath\omega\mu & \operatorname{curl} \\ -\operatorname{curl} & -\imath\omega\epsilon \end{bmatrix} \begin{bmatrix} H \\ E \end{bmatrix} = \begin{bmatrix} -\imath\omega\mu H + \operatorname{curl} E \\ -\imath\omega\epsilon E - \operatorname{curl} H \end{bmatrix},$$

whose domain is the following subspace of $L^2(\Omega)^6$:

$$\operatorname{dom}(A^*) = \{(H, E) \in H(\mathrm{curl}, \Omega)^2 : E_\top = 0 \text{ on } \partial\Omega\}.$$

Classical arguments show that both A and A^* are injective. To facilitate comparison, we list all our formulations at once, including the already studied primal form.

Strong form: Let $x = (H, E)$ be a group variable. Set

$$X_0 = H(\mathrm{curl}, \Omega) \times \mathring{H}(\mathrm{curl}, \Omega), \quad Y = Y_0 = L^2(\Omega)^6.$$

Note that $X_0 = \{\mathrm{dom}(A), \|\cdot\|_{H(\mathrm{curl},\Omega)}\}$, i.e., $\mathrm{dom}(A)$ considered as a subspace of $H(\mathrm{curl}, \Omega)^2$ (rather than as a subspace of $L^2(\Omega)^6$). The Maxwell problem is to find $x \in X_0$, given $f \in Y_0$, such that $Ax = f$. This fits into our variational framework (3.2) by setting b_0 to

$$b_0^S(x, y) = (Ax, y)_\Omega.$$

Primal form for E: This is the same as in (6.5), i.e., with the spaces as set there, with $\hat{b}$ set to $\hat{b}^E(\hat{H}_\dashv, F) = -\imath\omega\langle \hat{H}_\dashv, F\rangle_h$ and b_0 set to

$$b_0^E(E, F) = (\mu^{-1} \mathrm{curl}\, E, \mathrm{curl}\, F)_\Omega - \omega^2(\epsilon E, F)_\Omega,$$

the electric primal formulation is (3.2) and its broken version is (3.4).

Primal form for H: Eliminating E from (6.1), we obtain $\mathrm{curl}\, \epsilon^{-1} \mathrm{curl}\, H - \omega^2 \mu H = \mathrm{curl}\, \epsilon^{-1} J$ and a (possibly nonhomogeneous) boundary condition on $n \times \epsilon^{-1} \mathrm{curl}\, H$. With this in place of (6.3) as the starting point and repeating the derivation that led to (6.5), we obtain the following magnetic primal form. Set

$$\begin{aligned} &X_0 = H(\mathrm{curl}, \Omega), && Y_0 = X_0,\\ &\hat{X} = \mathring{H}^{-1/2}(\mathrm{div}, \Omega_h), && Y = H(\mathrm{curl}, \Omega_h),\\ &b_0^H(H, F) = (\epsilon^{-1} \mathrm{curl}\, H, \mathrm{curl}\, F)_h - \omega^2(\mu H, F)_h, && \hat{b}^H(\hat{E}_\dashv, F) = \imath\omega\langle \hat{E}_\dashv, F\rangle_h. \end{aligned}$$

With $\hat{b}$ set to $\hat{b}^H$ and b_0 set to b_0^H, the magnetic primal formulation is (3.2) and its broken version is (3.4).

Ultraweak form: This form is obtained by integrating by parts all equations of the strong form. Using group variables $x = (H, E)$, $y = (R, S)$ and $\hat{x} = (\hat{H}_\top, \hat{E}_\top)$, set

$$X_0 = L^2(\Omega)^6, \qquad Y_0 = H(\mathrm{curl}, \Omega) \times \mathring{H}(\mathrm{curl}, \Omega), \tag{6.7a}$$

$$Y = H(\mathrm{curl}, \Omega_h)^2, \qquad \hat{X} = H^{-1/2}(\mathrm{curl}, \partial\Omega_h) \times \mathring{H}^{-1/2}(\mathrm{curl}, \partial\Omega_h), \tag{6.7b}$$

$$b_0^U(x, y) = (x, A^* y)_h, \quad \hat{b}^U(\hat{x}, y) = \langle \hat{H}_\top, n \times S\rangle_h - \langle \hat{E}_\top, n \times R\rangle_h \tag{6.7c}$$

and consider formulations (3.2) and (3.4) with $b_0 = b_0^U$ and $\hat{b} = \hat{b}^U$. Note that in the definition of b_0^U, the operator A^* is applied element by element, per our tacit conventions when using the $(\cdot,\cdot)_h$-notation.

Dual Mixed form: Among the two equations in the strong form, if one weakly imposes (by integrating by parts) the first equation and strongly imposes the second, then we get the following dual mixed form. Set

$$\begin{aligned} &X_0 = H(\mathrm{curl}, \Omega) \times L^2(\Omega)^3, && Y_0 = X_0,\\ &\hat{X} = \mathring{H}^{-1/2}(\mathrm{curl}, \partial\Omega_h), && Y = H(\mathrm{curl}, \Omega_h) \times L^2(\Omega)^3 \end{aligned}$$

and consider (3.2) with $b_0 = b_0^D$,

$$\begin{aligned} b_0^D(\,(H, E), (R, S)\,) = {} & (\imath\omega\mu H, R)_\Omega - (E, \mathrm{curl}\, R)_h \\ & + (\imath\omega\epsilon E + \mathrm{curl}\, H, S)_h. \end{aligned}$$

Its broken version is (3.4) with $\hat{b}$ set to $\hat{b}^D(\hat{E}_\top, (R, s)) = \langle \hat{E}_\top, n \times R \rangle_h$.

Mixed form: Reversing the roles above and weakly imposing the second equation while strongly imposing the first, we get another mixed formulation. Set

$$\begin{aligned} X_0 &= L^2(\Omega)^3 \times \mathring{H}(\mathrm{curl}, \Omega), & Y_0 &= X_0, \\ \hat{X} &= H^{-1/2}(\mathrm{curl}, \partial\Omega_h), & Y &= L^2(\Omega)^3 \times H(\mathrm{curl}, \Omega_h) \end{aligned}$$

and consider (3.2) with $b_0 = b_0^M$,

$$\begin{aligned} b_0^M(\,(H, E), (R, S)\,) = {} & (\imath\omega\mu H - \mathrm{curl}\, E, R)_h \\ & + (\imath\omega\epsilon E, S)_\Omega + (H, \mathrm{curl}\, S)_h. \end{aligned}$$

Its broken version is (3.4) with $\hat{b}$ set to $\hat{b}^M(\hat{H}_\top, (R, S)) = \langle \hat{H}_\top, n \times S \rangle_h$.

These form a total of six unbroken and five broken formulations, counting the already discussed broken and unbroken electric primal formulation. To analyze the remaining formulations, let us begin by verifying Assumption 1 for all the unbroken formulations. To this end, label the statement of Assumption 1 with b_0 set to the above-defined b_0^I as "(I)" for all $I \in \{E, H, S, U, D, M\}$. Then (analogous to the equivalences in Figure 1 for the elliptic example) we now have equivalence of statements $(D), (E), \ldots$ as proved next.

Theorem 6.3. *The following implications hold:*

$$\begin{tikzcd} & (H) \arrow[dl, Rightarrow] & (D) \arrow[l, Rightarrow] & \\ (S) \arrow[rrr, Rightarrow] & & & (U) \arrow[ul, Rightarrow] \arrow[dl, Rightarrow] \\ & (E) \arrow[ul, Rightarrow] & (M) \arrow[l, Rightarrow] & \end{tikzcd}$$

Proof. We begin with the most substantial of all the implications, which allows us to go from the strongest to the weakest formulation. When there can be no confusion, let us abbreviate Cartesian products of $L^2(\Omega)$ as simply L and write W for $H(\mathrm{curl}, \Omega) \times \mathring{H}(\mathrm{curl}, \Omega)$. Clearly, W is complete in the $H(\mathrm{curl}, \Omega)^2$-norm. It is easy to see that the graph norms $(\|x\|_L^2 + \|Ax\|_L^2)^{1/2}$ and $(\|x\|_L^2 + \|A^*x\|_L^2)^{1/2}$ are both equivalent to the $H(\mathrm{curl}, \Omega)^2$-norm, so W is a Hilbert space in any of these norms. These norm equivalences show that the inf-sup condition (S) holds if and only if

$$C\|x\|_L \leqslant \|Ax\|_L \qquad \forall x \in \mathrm{dom}(A). \tag{6.8}$$

$\underline{(S) \implies (U)}$: The bound (6.8) implied by (S) shows that the range of A is closed. By the closed range theorem for closed operators, range of A^* is closed. Since A^* is also injective, it follows that

$$C\|y\|_L \leqslant \|A^*y\|_L \qquad \forall y \in \mathrm{dom}(A^*) \tag{6.9}$$

holds with the same constant as in (6.8). This in turn implies that the following inf-sup condition holds:

$$C\|y\|_W \leqslant \sup_{x \in L} \frac{|(x, A^*y)_\Omega|}{\|x\|_L} \qquad \forall y \in W.$$

Thus, to complete the proof of (U), it suffices to show that

$$\inf_{x\in L}\sup_{y\in W}\frac{|(x,A^*y)_\Omega|}{\|x\|_L\|y\|_W}=\inf_{y\in W}\sup_{x\in L}\frac{|(x,A^*y)_\Omega|}{\|x\|_L\|y\|_W}. \tag{6.10}$$

For completeness, we now describe the standard argument that shows that one may reverse the order of inf and sup to prove (6.10). Viewing $A^*: W\to L$ as a bounded linear operator, we know that it is a bijection because of (6.8) and (6.9). Hence $(A^*)^{-1}: L\to W$ is bounded. The right-hand side of (6.10) equals its operator norm $\|(A^*)^{-1}\|$. The left hand side of (6.10) equals the operator norm of the dual of $(A^*)^{-1}$ (considered as the dual operator of a continuous linear operator with L as the pivot space identified to be the same as its dual space). The norms of a continuous linear operator and its dual are equal, so (6.10) follows.

$\underline{(U)\implies(D)}$: Let $x=(H,E)$ and $y=(R,S)$ be in $W=H(\operatorname{curl},\Omega)\times\mathring{H}(\operatorname{curl},\Omega)$. Clearly, W is contained in $Y_0^D=H(\operatorname{curl},\Omega)\times L^2(\Omega)^3$. Because of the extra regularity of S, we may integrate by parts the last term in the definition of $b_0^D((H,E),(R,S))$ to get that $b_0^D(x,y)=b_0^U(x,y)$. Hence using (U),

$$\sup_{y\in Y_0^D}\frac{|b_0^D(x,y)|}{\|y\|_{Y_0^D}}\geqslant\sup_{y\in W}\frac{|b_0^D(x,y)|}{\|y\|_W}=\sup_{y\in W}\frac{|b_0^U(x,y)|}{\|y\|_W}\geqslant C\|x\|_L. \tag{6.11}$$

Thus, to finish the proof of (D), we only need to control $\operatorname{curl} H$ using the last term of b_0^D.

$$\begin{aligned}\|\operatorname{curl} H\|_L&=\sup_{S\in L}\frac{|(\operatorname{curl} H,S)_\Omega|}{\|S\|_L}=\sup_{S\in L}\frac{|b_0^D((H,E),(0,S))-(\imath\omega\varepsilon E,S)|}{\|S\|_L}\\&\leqslant\sup_{y\in Y_0^D}\frac{|b_0^D(x,y)|}{\|y\|_{Y_0^D}}+\|\imath\omega\varepsilon E\|_L.\end{aligned}$$

Using (6.11) to bound the last term, the proof of (D) is finished.

$\underline{(D)\implies(H)}$: For any $H\in H(\operatorname{curl},\Omega)$, set $\ell_H(R)=(\epsilon^{-1}\operatorname{curl} H,\operatorname{curl} R)_\Omega-\omega^2(\mu H,R)_\Omega$. We need to prove (H), which is equivalent to

$$\|H\|_{H(\operatorname{curl},\Omega)}\leqslant C\|\ell_H\|_{H(\operatorname{curl},\Omega)'}. \tag{6.12}$$

Introducing a new variable $E=-(\imath\omega\epsilon)^{-1}\operatorname{curl} H$, we find that $(\imath\omega)^{-1}\ell_H(R)=-(E,\operatorname{curl} R)_\Omega+(\imath\omega\mu H,R)_\Omega$. Hence

$$b_0^D((H,E),(R,S))=(\imath\omega)^{-1}\ell_H(R)\qquad\forall(R,S)\in H(\operatorname{curl},\Omega)\times L^2(\Omega)^3.$$

Hence (6.12) immediately follows from (D).

$\underline{(H)\implies(S)}$: To prove the inf-sup condition (S), it is enough to prove (6.8) for all $x=(H,E)\in W$. Given any $F,G\in L^2(\Omega)^3$, the equation $Ax=(F,G)$ is the same as the system

$$\imath\omega\mu H-\operatorname{curl} E=F, \tag{6.13a}$$

$$\imath\omega\epsilon E+\operatorname{curl} H=G. \tag{6.13b}$$

We multiply (6.13b) by the conjugate of $\epsilon^{-1}\operatorname{curl} R$, for some $R\in H(\operatorname{curl},\Omega)$ and integrate by parts, while we multiply (6.13a) by $-\imath\omega R$ and solely integrate. The result is

$$\begin{aligned}(\epsilon^{-1}\operatorname{curl} H,\operatorname{curl} R)_\Omega+(\imath\omega\operatorname{curl} E,R)_\Omega&=(G,\varepsilon^{-1}\operatorname{curl} R)_\Omega,\\-(\omega^2\mu H,R)_\Omega-(\imath\omega\operatorname{curl} E,R)_\Omega&=-(F,\imath\omega R)_\Omega.\end{aligned}$$

Adding the above two equations together, we get the primal form $b_0^H(H,R) = \ell(R)$ where $\ell(R) = (G, \varepsilon^{-1} \operatorname{curl} R)_\Omega - (F, \imath\omega R)_\Omega$. Hence the given inf-sup condition (H) implies

$$C\|H\|_{H(\operatorname{curl},\Omega)} \leqslant \sup_{R\in H(\operatorname{curl},\Omega)} \frac{|b_0^H(H,R)|}{\|R\|_{H(\operatorname{curl},\Omega)}} = \sup_{R\in H(\operatorname{curl},\Omega)} \frac{|\ell(R)|}{\|R\|_{H(\operatorname{curl},\Omega)}}.$$

Since $|\ell(R)| \leqslant C(\|F\|_L + \|G\|_L)\|R\|_{H(\operatorname{curl},\Omega)}$, this provides the required bound for $\|H\|_L$. Since $\operatorname{curl} H$ is also bounded, equation (6.13b) yields a bound for $\|E\|_L$. Combining these bounds, (6.8) follows.

To conclude the proof of the theorem, we note that the proofs of the implications $(U) \implies (M)$, $(M) \implies (S)$, $(E) \implies (S)$ are similar to the proofs of $(U) \implies (D)$, $(D) \implies (H)$, and $(H) \implies (S)$, respectively. □

Theorem 6.3 verifies Assumption 1 for all the formulations because we know from (6.4) that (E) holds. Assumption 2 can be easily verified for all the broken formulations using Theorem 2.3. Assumption 3 can be verified using Theorem 5.1. Hence convergence rate estimates like in Corollary 6.1 can be derived for each of the broken formulations. We omit the repetitive details.

7. Numerical studies

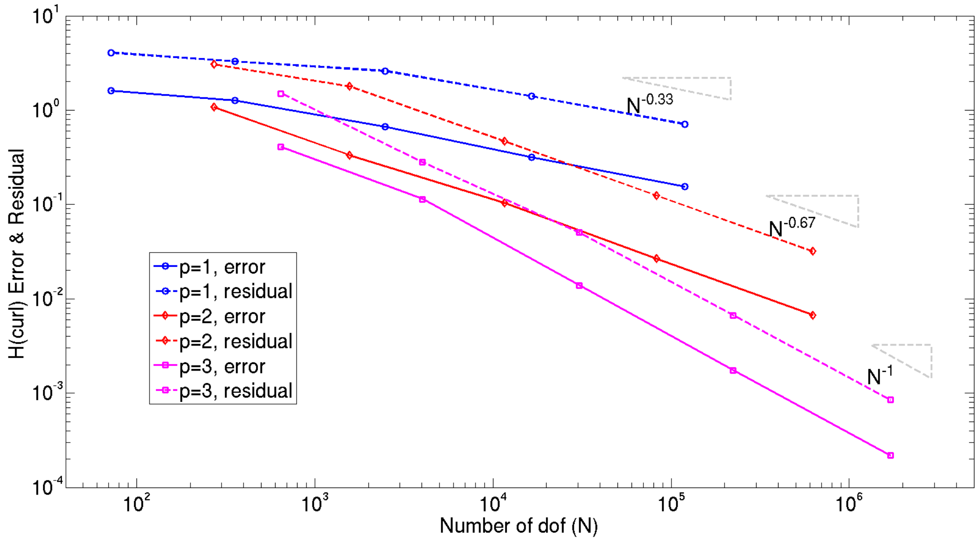
In this section, we present some numerical studies focusing on the Maxwell example. Numerical results for other examples, including the diffusion-convection-reaction example, can be found elsewhere [10, 13]. The numerical studies are not aimed at verifying the already proved convergence results, but rather at investigations of the performance of the DPG method beyond the limited range of applicability permitted by the theorems. All numerical examples presented in this section have been obtained with *hp3d*, a 3D finite element code supporting anisotropic h and p refinements and solution of multi-physics problems involving variables discretized compatibly with the $H^1(\Omega)$-$H(\operatorname{curl},\Omega)$-$H(\operatorname{div},\Omega)$ exact sequence of spaces. The code has recently been equipped with a complete family of orientation embedded shape functions for elements of many shapes [21]. The remainder of this section is divided into results from two numerical examples.

Example 7.1 (Smooth solution). We numerically solve the time-harmonic Maxwell equations setting material data to

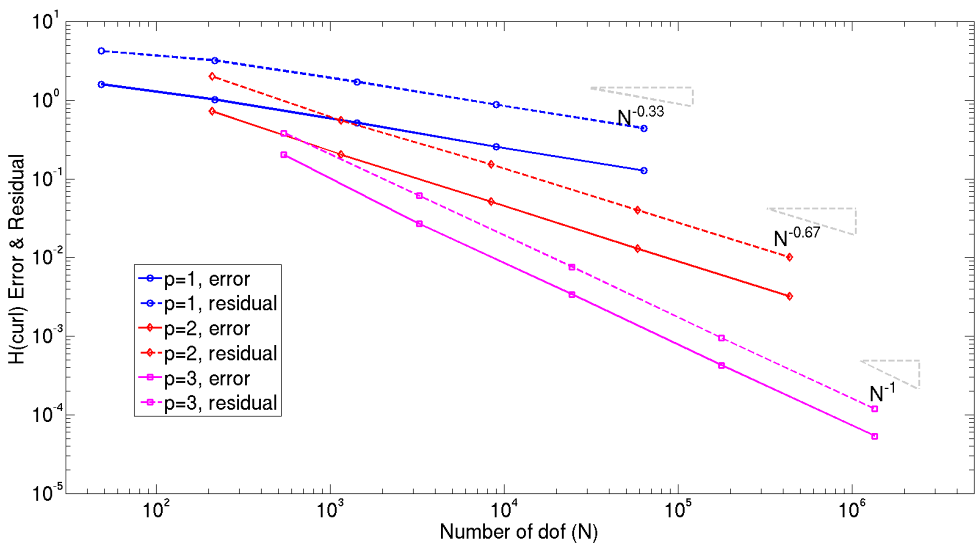
$$\epsilon = \mu = 1, \quad \omega = 1,$$

and Ω to the unit cube. To obtain Ω_h, the unit cube was partitioned first into five tetrahedra: four similar ones adjacent to the faces of the cube, and a fifth inside of the cube. We have used the refinement strategy of [24] to generate a sequence of successive uniform refinements. On these meshes, consider the primal DPG method for E, described by (6.5), with data set so that the exact solution is the following smooth function.

$$E_1 = \sin\pi x_1 \, \sin\pi x_2 \, \sin\pi x_3, \quad E_2 = E_3 = 0\,.$$



(A) Tetrahedral meshes.



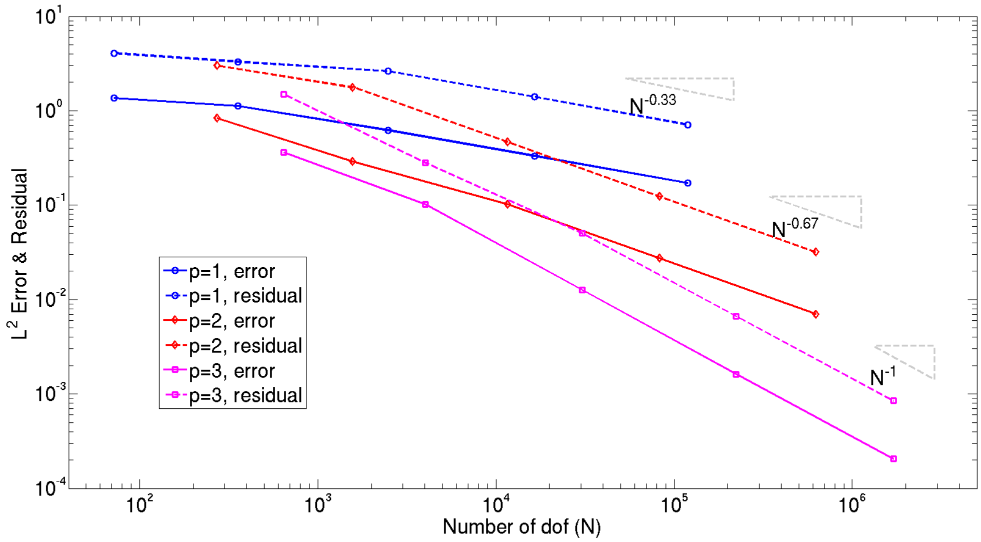
(B) Hexahedral meshes.

FIGURE 2. Rates for the case of smooth solution and primal formulation.

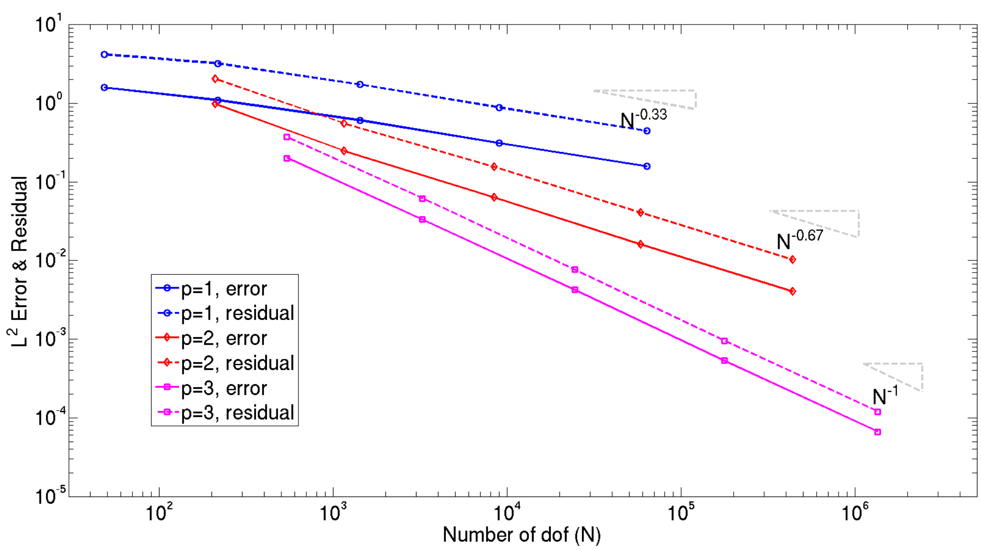
Instead of the pair of discrete spaces (6.6) that we know is guaranteed to work by our theoretical results, we experiment with these discrete spaces:

$$X_h = \{(E_h, n \times \hat{H}_h) \in \mathring{H}(\mathrm{curl}, \Omega) \times H^{-1/2}(\mathrm{div}, \partial\Omega_h) :\; n \times \hat{H}_h|_{\partial K} \in \mathrm{tr}^K_{\mathrm{curl},\dashv} N_p(K), \quad \text{and } E_h|_K \in N_p(K) \text{ for all } K \in \Omega_h\}, \tag{7.1a}$$

$$Y_h = \{F_h \in H(\mathrm{curl}, \Omega_h) : F_h|_K \in N_{p+2}(K) \text{ for all } K \in \Omega_h\}. \tag{7.1b}$$



(A) Tetrahedral meshes.



(B) Hexahedral meshes.

FIGURE 3. Rates for the case of smooth solution and ultraweak formulation.

The observed rates of convergence of the error $\|E - E_h\|_{H(\text{curl},\Omega_h)}$ and the residual norm η are shown in Figure 3a. The rates are optimal. This suggests that the results of Corollary 6.1 may hold with other choices of spaces. The problem of finding the minimal DPG test space for which optimal convergence rates can be obtained for the Maxwell problem remains unsolved.

We also present similar results obtained using cubic meshes using $H(\text{curl}, \Omega)$-conforming Nédélec hexahedron of the first type. Namely, X_h and Y_h are set by (7.1) after revising $N_p(K)$ to $Q_{p-1,p,p}(K) \times Q_{p,p-1,p}(K) \times Q_{p,p,p-1}(K)$ where $Q_{l,m,n}(K)$ denotes the set of polynomials

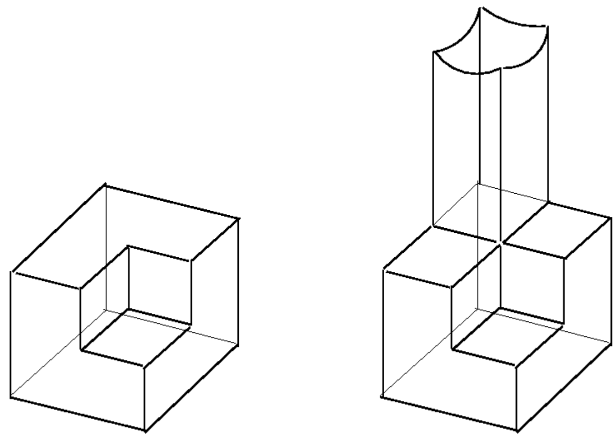


FIGURE 4. Construction of Fichera oven.

of degree at most l, m, and n in the x_1, x_2 and x_3 directions, respectively. The convergence rates reported in Figure 3b are again optimal.

Before concluding this example, we also report convergence rates obtained from the ultraweak formulation of (6.7). The discrete spaces are now set by

$$\begin{aligned} \text{(7.2a)}\quad X_h = \{&(E, H, \hat{E}_\top, \hat{H}_\top) \in L^2(\Omega)^3 \times L^2(\Omega)^3 \times H^{-1/2}(\mathrm{curl}, \partial\Omega_h) \times H^{-1/2}(\mathrm{curl}, \partial\Omega_h) : \\ &E|_K, H|_K \in P_{p-1}(K)^3,\ \hat{E}_\top|_{\partial K}, \hat{H}_\top|_{\partial K} \in \mathrm{tr}^K_{\mathrm{curl},\top} N_p(K) \text{ for all } K \in \Omega_h\}, \\ \text{(7.2b)}\quad Y_h = \{&(F, G) \in H(\mathrm{curl}, \Omega_h) \times H(\mathrm{curl}, \Omega_h) :\ F|_K, G|_K \in N_{p+2}(K) \text{ for all } K \in \Omega_h\}. \end{aligned}$$

Recall that the DPG computations require a specification of the Y-norm. Using the observation (made in the proof of Theorem 6.3) that the adjoint graph norm is equivalent to the natural norm in $H(\mathrm{curl}, \Omega)^2$, we set

$$\|(E, H)\|_Y^2 = \sum_{K \in \Omega_h} \left(\|E\|^2_{L^2(\Omega)} + \|H\|^2_{L^2(\Omega)} + \|\imath\omega\mu H - \mathrm{curl}\, E\|^2_{L^2(\Omega)} + \|\imath\omega\epsilon E + \mathrm{curl}\, H\|^2_{L^2(\Omega)} \right)$$

in all computations involving the ultraweak formulation. The results reported in Figure 3 again show optimal convergence rates. Note that only the errors in the interior variables E and H (in $L^2(\Omega)$-norm) are reported in the figure. To compute errors in the interface variables, we must compute approximations to fractional norms carefully (see [7] for such computations in two dimensions). Since the code does not yet have this capability in three dimensions, we have not reported the errors in interface variables. ///

Example 7.2 (Singular solution). To illustrate adaptive possibilities of DPG method and the difference between different variational formulations, we now present results from a "Fichera oven" problem. We start with with the standard domain with a Fichera corner obtained by refining a cube $(0, 2)^3$ into eight congruent cubes and removing one of them. We then attach an infinite waveguide to the top of the oven and truncate it at a unit distance from the Fichera corner, as shown in Figure 4. Setting

$$\epsilon = \mu = 1, \quad \omega = 5,$$

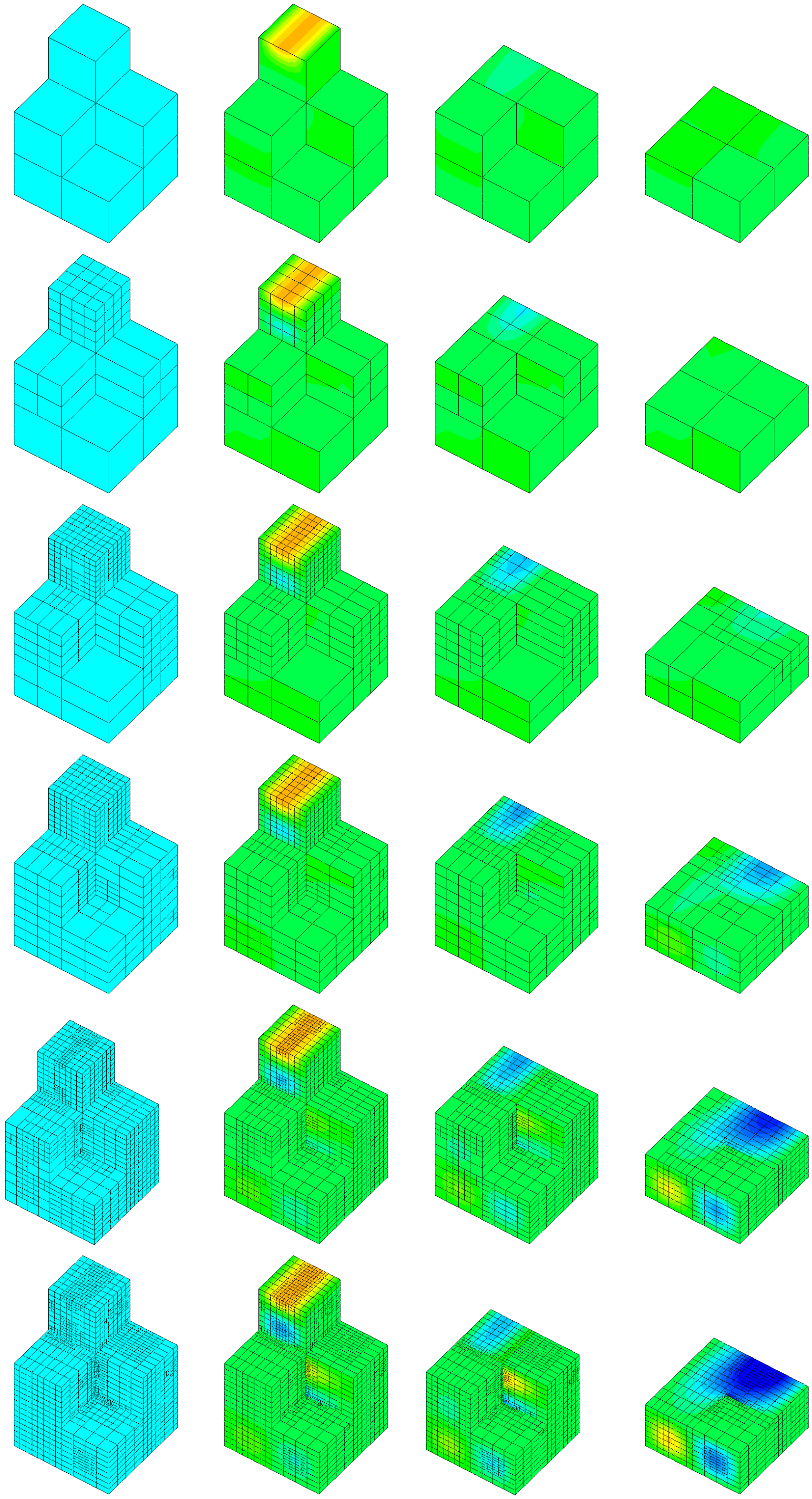
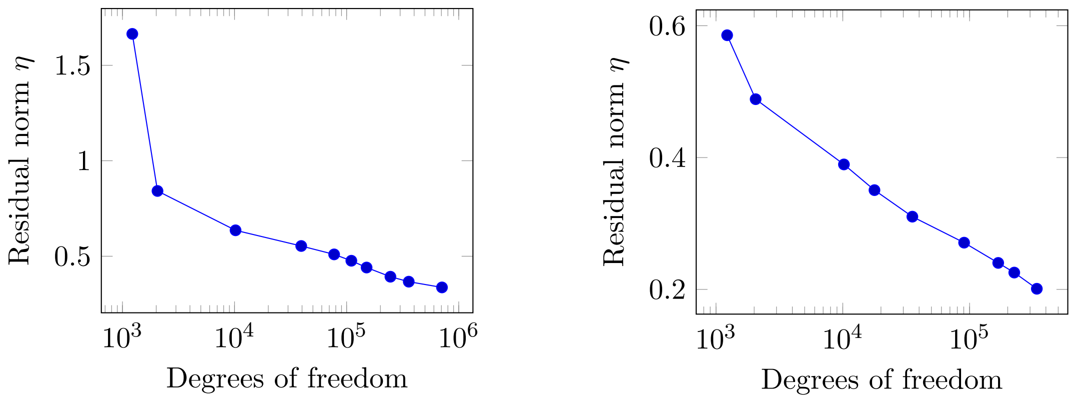


FIGURE 5. Every other iterate (iterates 1,3,5,7,9 and 11) from adaptive algorithm applied to solve the Fichera oven problem with the primal formulation. Meshes (left) and the corresponding real part of E_1 are shown.



(A) Adaptivity for primal formulation. (B) Adaptivity for ultraweak formulation.

FIGURE 6. Convergence of residual during adaptive iterations for the Fichera oven.

we drive the problem with the first propagating waveguide mode,

$$E_1^D = \sin \pi x_2, \quad E_2^D = E_3^D = 0$$

which is used for non-homogenous electric boundary condition $n \times E = n \times E^D$ across the waveguide section. Analogous to a microwave oven model, we set the homogeneous perfect electric boundary condition $n \times E = 0$ everywhere else on the boundary. The above material data correspond to about 0.8 wavelengths per unit domain. In all the reported computations, we start with a uniform mesh of eight quadratic elements that clearly does not even meet the Nyquist criterion. We expect the solution to develop strong singularities at the reentrant corner and edges, but we do not know the exact solution.

First, we report the results from the electric primal formulation, choosing spaces again as in (7.1). Figure 5 presents the evolution of the mesh along with the corresponding real part of the first component of electric field E_1. Since we do not have the exact solution for this problem, we display convergence history using a plot of the evolution of the computed residual η in Figure 6a. (Recall that theoretical guidance on the similarity of behaviors of error estimator η and the error is provided by Theorem 4.1.) Clearly, the figure shows the residual is being driven to zero during the adaptive iteration.

Next, we solve the same problem using the ultraweak formulation with the spaces set as in (7.2) and the Y-norm set to the adjoint graph norm as in the previous example. The convergence history of the residual norm η is displayed in Figure 6b. The evolution of the mesh along with the real part of E_1 is illustrated in Figure 7.

It is illustrative to visualize the difference between the two different DPG formulations and the accompanying convergence in different norms. Figure 8 presents a side-by-side comparison of the real part of the electric field component E_1 obtained using the primal (left) and ultraweak (right) formulations. The same color scale (min $= -1$, max $= 1$) is applied to both solutions in this figure (whereas the scales of Figures 5 and 7 are not identical). Obviously, the meshes are different, but they are of comparable size, so we believe the comparison is fair. The primal method, which delivers solution converging in the stronger $H(\text{curl}, \Omega)$-norm, "grows" the unknown solution slower, whereas the ultraweak formulation converging in the weaker

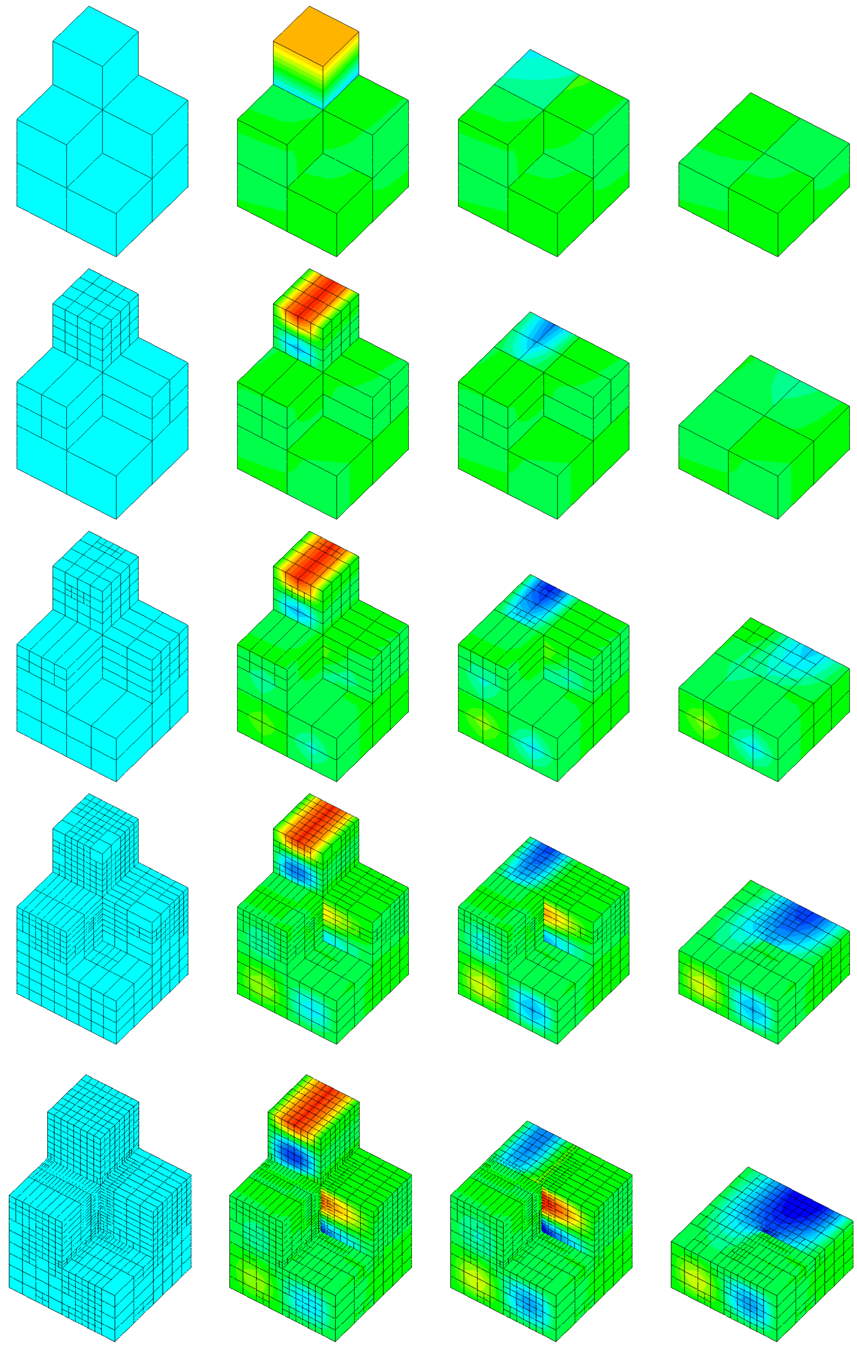
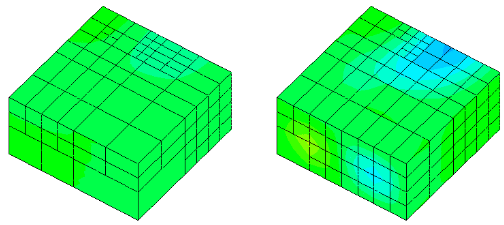
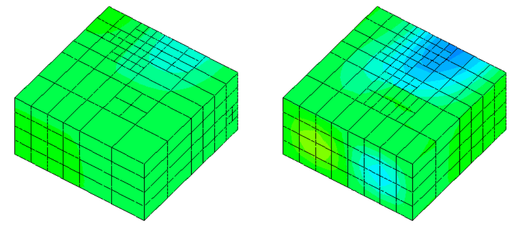


FIGURE 7. Iterates 1,3,5,7, and 9 from adaptive algorithm for Fichera oven problem with the ultraweak formulation. Meshes (left) and the corresponding real part of E_1 are shown.

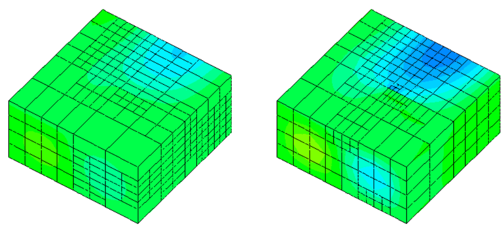
$L^2(\Omega)$-norm seems to capture the same solution features faster. Both methods ultimately approximate the same solution but at different speeds and the ultraweak formulation seems to be a winner. Recall that the number of interface unknowns for both formulations is identical,



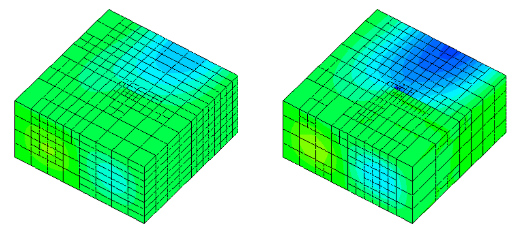
(A) Primal (left) and ultraweak (right) iterate 6.



(B) Primal (left) and ultraweak (right) iterate 7.



(C) Primal (left) and ultraweak (right) iterate 8.



(D) Primal (left) and ultraweak (right) iterate 9.

FIGURE 8. Same scale comparison of real part of E_1 computed by the primal and the ultraweak formulations. A middle slice (through singular vertex) of adaptive iterates 6,7,8 and 9 are shown.

but the total number of unknowns for the ultraweak formulation is higher, i.e., the ultraweak formulation requires a larger number of local (element-by-element) computations. ///

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Department of Mathematics, Humboldt-Universität zu Berlin, Unter den Linden 6, 10099 Berlin, Germany
E-mail address: cc@math.hu-berlin.de

Institute for Computational Engineering and Sciences, The University of Texas at Austin, Austin, TX 78712, USA
E-mail address: leszek@ices.utexas.edu

PO Box 751, Portland State University, Portland, OR 97207-0751, USA
E-mail address: gjay@pdx.edu