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# On a Conjecture of Furstenberg

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## 1 Estimation of the Hausdorff Dimension

### 1.1 Statement of the problem

Consider an iterated function system  $\Psi_t$  given by three generators:

$$\begin{aligned}\psi_0(x) &= \frac{x}{3}, \\ \psi_1(x) &= \frac{x+1}{3},\end{aligned}$$

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$$\psi_t(x) = \frac{x+t}{3}$$

where  $t \in \mathbf{R}$  is a fixed parameter.

By [1], for every  $t$  there is a unique compact set  $Z_t$  which is invariant under  $\Psi_t$  and such that the orbit of any compact set under  $\Psi_t$  converges to  $Z_t$  in the Hausdorff metric. An elementary interpretation of  $Z_t$  is as the set of number which can be represented by generally infinite expressions in base 3 which use digits  $0, 1, t$ .

In this paper we are proving the following:

**Theorem 1** *For every  $t$  irrational,  $HD(Z_t) \geq 1 - \frac{\log(5/3)}{2\log 3} > 0.767$ .*

Since the Hausdorff dimension is an affine invariant, from now we will assume without loss of generality that  $|t| \leq 1$ . Theorem 1 will be derived from a technical Theorem 2 which is stated later.

**Conjectures of Furstenberg.** Let's quote three related conjectures of Furstenberg.

**Conjecture 1** *For every  $t$  irrational,  $HD(Z_t) = 1$ .*

Let  $W$  be the limit set of the iterated function systems in  $\mathbf{R}^2$  which is generated by  $x \rightarrow x/3$ ,  $x \rightarrow [x + (1, 0)]/3$  and  $x \rightarrow [x + (0, 1)]/3$ .

**Conjecture 2** *For every  $t$  irrational almost every  $\beta \in \mathbf{R}$  the line  $v = tu + \beta$  intersects  $W$  along a set with Hausdorff dimension 0.*

Let  $T$  denote the operator

$$Tf(x) = \frac{1}{3} [f(x) + f(x-1) + f(x-t)]$$

acting on the space of continuous functions with compact support.

**Conjecture 3** *For every  $t$  irrational the spectral radius of the adjoint  $T^*$  is equal to 1.*

**Historical remarks.** Theorem 1 is a step towards proving Conjecture 1. Conjecture 1 was the subject of work by several authors. One should mention [3] where it was established that for almost every  $t$ , both in the topological and category sense,  $HD(Z_t) = 1$ , and that  $|Z_t| = 0$  (Lebesgue measure) for every  $t$  irrational, see also [4]. In [2] a study of the continuity properties of the function  $t \rightarrow HD(Z_t)$  was undertaken, while [6] contains numerical data mostly in support of Conjecture 1.

## 1.2 Energy estimate

Given a positive probabilistic measure  $\mu$  on  $\mathbf{R}$  and  $\alpha \geq 0$ , we define its *energy integral*

$$I_\alpha(\mu) := \int \frac{d\mu(x) d\mu(y)}{|x - y|^\alpha}.$$

For a Borel set  $Z \subset \mathbf{R}$  consider the set  $A(Z)$  which consists of those  $\alpha \geq 0$  for which there exists a Radon measure  $\mu_\alpha$  supported on  $Z$  and  $I_\alpha(\mu_\alpha) < \infty$ . It is known that  $HD(Z) = \sup A(Z)$ , see [5]. Hence, each time we get a measure  $\mu$  supported on  $Z$  and  $I_\alpha(\mu) < \infty$ , we have bounded the Hausdorff dimension of  $Z$  by  $\alpha$  from below.

**Natural measures.** We will work with a concrete measure  $\mu^t$  supported on  $Z_t$ . Consider a sequence of measures

$$\mu_0^t = \delta_0, \mu_1^t = \frac{1}{3}(\delta_0 + \delta_1 + \delta_t)$$

and

$$\mu_n^t = \mu_1^t * (\psi_{0*} \mu_1^t) * \cdots * (\psi_{0*}^{n-1} \mu_1^t)$$

for  $n > 0$ . The choice of equal weighting of the measures transferred by all generators was of course arbitrary. One easily verifies that

$$\mu_n^t = \mu_k^t * \phi_{0*}^k \mu_{n-k}^t$$

for  $0 \leq k \leq n$ . Hence, measures  $\mu_n^t$  converge weakly to  $\mu^t$  which is supported on the interval  $[0, 1/2]$ .

**Estimates.** Let us begin to estimate the energy integral. Let  $0 < \alpha < 1$ . For  $n \geq 0$  denote  $L_n := \{(x, y) : 2 \cdot 3^{-n} < |x - y| \leq 2 \cdot 3^{1-n}\}$ .

$$I_\alpha(\mu^t) = \int \frac{d\mu^t(x) d\mu^t(y)}{|x - y|^\alpha} = \sum_{n=1}^{\infty} \int_{L_n} \frac{d\mu^t(x) d\mu^t(y)}{|x - y|^\alpha}.$$

Since  $\mu^t = \mu_n^t * \psi_{0*}^n \mu^t$  and the support of  $\psi_{0*}^n \mu^t$  is contained in  $[-3^{-n}/2, 3^{-n}/2]$ , we can write

$$\int_{L_n} \frac{d\mu^t(x) d\mu^t(y)}{|x - y|^\alpha} = \int_{L_n} |x - y|^{-\alpha} * h(x) * h(y) d\mu_n^t(x) d\mu_n^t(y)$$

where  $h$  is a non-negative function with total mass 1 and support contained in  $[-3^{-2}/2, 3^{-n}/2]$ . Because of that, for  $(x, y) \in L_n$  we get

$$|x - y|^{-\alpha} * h(x) * h(y) \leq 3^{n\alpha}$$

and

$$I_\alpha(\mu^t) \leq \sum_{n=1}^{\infty} 3^{n\alpha} \int_{L_n} d\mu_n^t(x) d\mu_n^t(y). \quad (1)$$

Denote

$$s(n, \beta) = 3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(x - y - \beta) d\mu_n^t(x) d\mu_n^t(y) = \quad (2)$$

$$3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(z - a) d(\mu_n^t * (\mu_n^t)')(z)$$

where the apostrophe means the measure transported by the map  $x \rightarrow -x$ . Then

$$\frac{3^{n\alpha} \int_{L_n} d\mu_n^t(x) d\mu_n^t(y) \leq s(n, \frac{5}{2}3^{-n}) + \dots + s(n, \frac{11}{2}3^{-n}) + s(n, -\frac{5}{2}3^{-n}) + \dots + s(n, -\frac{11}{2}3^{-n})}{3^{n(1-\alpha)}}.$$

If we write  $S_n = \sup_{\beta \in \mathbf{R}} s(n, \beta)$ , then we get from estimate (1) that

$$I_\alpha(\mu^t) \leq 8 \sum_{n=1}^{\infty} 3^{n(\alpha-1)} S_n. \quad (3)$$

So the task is reduced to estimating the exponential rate of increase for  $S_n$ .

### 1.3 Projection measure

Consider a measure  $\nu_1$  is  $\mathbf{R}^2$  defined by

$$\nu_1 = \frac{1}{3}(\delta_{(0,0)} + \delta_{(0,1/3)} + \delta_{(1/3,0)}) .$$

If  $\psi$  denotes the homothety with scale  $1/3$ , then we define

$$\nu_n = \nu_1 * (\psi_* \nu_1) \cdots * (\psi_*^{n-1} \nu_1) .$$

If  $\pi_t : \mathbf{R}^2 \rightarrow \mathbf{R}$  denotes the linear projection given by  $\pi_t(u, v) = tu + v$ , then we have  $\mu_n^t = \pi_{t*} \nu_n$ . Hence

$$\mu_n^t * (\mu_n^t)' = \pi_{t*} \left[ \nu_1 * \nu_1' * \psi_*(\nu_1 * \nu_1') * \cdots * \psi_*^{n-1}(\nu_1 * \nu_1') \right] .$$

Measure  $\nu_1 * (\nu_1)'$  is obtained explicitly and equals

$$\frac{1}{9} \sum_{k, \ell = -1, 0, 1} b(k, \ell) \delta_{(k/3, \ell/3)} \quad (4)$$

where  $b(k, \ell) = 1$  if  $k \neq \ell$ ,  $3$  if  $k = \ell = 0$  and  $0$  otherwise. Function  $b$  extends to  $\mathbf{Z} \times \mathbf{Z}$  by  $b(k_1, \ell_1) = b(k, \ell)$  where  $k, \ell = -1, 0, 1$  and  $k - k_1, \ell - \ell_1 \in 3\mathbf{Z}$ .

From the defining formula (2),

$$\begin{aligned} s(n, \beta) &= 3^n \int \chi_{(-3^{-n}/2, 3^{-n}/2]}(z - \beta) d(\mu_n^t * (\mu_n^t)')(z) = \\ &= 3^n \sum_{k, \ell \in \mathbf{Z}} (\nu_n * \nu_n')(k3^{-n}, \ell3^{-n}) \chi_{(-3^{-n}/2, 3^{-n}/2]}(tk3^{-n} + \ell3^{-n} - \beta) . \end{aligned}$$

For every  $k \in \mathbf{Z}$  and  $\beta$ , a non-zero contribution is obtained only when  $\ell = \ell_{n, \beta}(k) := \langle 3^n(\beta - kt3^{-n}) \rangle$  where  $\langle x \rangle$  is the integer characterized by the condition  $-1/2 < x - \langle x \rangle \leq 1/2$ . If we also introduce the notation  $k_n(x) = \langle 3^n x \rangle$ , then we can write

$$\begin{aligned} s(n, \beta) &= 3^n \sum_{k \in \mathbf{Z}} (\nu_n * \nu_n')(k3^{-n}, \ell_{n, \beta}(k)3^{-n}) = \\ &= 9^n \int_{-\infty}^{+\infty} (\nu_n * \nu_n')(k_n(x)3^{-n}, k_n(\beta - tk_n(x))3^{-n}) dx . \end{aligned}$$

Define  $b_i(u, v) = b(\langle 3^i u \rangle, \langle 3^i v \rangle)$ . Then we can write for  $-\frac{3^n-1}{2} \leq k, \ell \leq \frac{3^n-1}{2}$  that

$$(\nu_n * \nu'_n)(k3^{-n}, \ell3^{-n}) = 9^{-n} \prod_{i=1}^n b_i(k3^{-n}, \ell3^{-n}).$$

If  $k, \ell$  are outside that range, then  $(\nu_n * \nu'_n)(k3^{-n}, \ell3^{-n}) = 0$ . Hence we can write

$$s(n, \beta) = \int_{-1/2}^{1/2} \prod_{i=1}^n B_i(x, \beta - tk_n(x)) dx$$

where functions  $B_i : T^2 \rightarrow \mathbf{R}$  are defined below.

**Definition 1** *If  $(x, y) \in T^2$  and  $i > 0$ , then*

$$B_i(x, y) := b(k_i(x), k_i(y)).$$

## 2 Averaging Estimates

We will denote  $T^1 := (-1/2, 1/2]$  and  $T^2 := T^1 \times T^1$  and think of identifying pieces of the boundary so that tori are obtained. Let  $\pi(x) := x'$  where  $x' \in T^1$  and  $x - x' \in \mathbf{Z}$ . Recall that  $k_n(x) = \langle 3^n x \rangle$ .

### 2.1 Partitions related to base 3 expansions

It will be useful to think of the circle  $T^1$  with the Lebesgue measure as a probabilistic space.

**Definition 2** *Say that an interval  $I \subset T^1$  is a basic interval of order  $n$ ,  $n \geq 0$ , if the transformation  $x \rightarrow \pi(3^n x)$  maps  $I$  onto  $T^1$  with degree 1.*

For example, interval  $(-\frac{1}{6}, \frac{1}{6}]$  is basic of order 1. Let  $\mathcal{P}_n$  denote the partition of  $T^1$  into basic intervals of order  $n$ .

Now let  $\phi : \mathbf{R} \rightarrow \mathbf{R}$ . For a positive integer  $n$ , define  $\phi_n : T^1 \rightarrow T^1$  by

$$\phi_n(x) := \phi(3^{-n} k_n(x)). \tag{5}$$

So,  $\phi_n$  is a  $\mathcal{P}_n$ -measurable approximation of  $\phi$ .

**Lemma 1** *Let  $q \geq 0$  and  $n > 0$  and  $\phi(x) = tx + t_0$ . Then for every  $x \in T^1$*

$$\phi_n(\pi(3^q x)) + T(x) - 3^q \phi_{n+q}(x)$$

*is an integer, where  $T(x) = k_q(x)t + (3^q - 1)t_0$ .*

**Proof:**

The expression which is to be shown to yield an integer is measurable with respect to  $\mathcal{P}_{n+q}$ . It suffices to prove the claim for  $x = (3^n J + j)3^{-n-q}$  with integers  $J$  and  $j$  ranging over  $[-\frac{3^q-1}{2}, \frac{3^q-1}{2}]$  and  $[-\frac{3^n-1}{2}, \frac{3^n-1}{2}]$ , respectively. For  $x$  in such a form

$$3^q \phi_{n+q}(x) = 3^q(xt + t_0) = Jt + jt3^{-n} + 3^q t_0 .$$

On the other hand,

$$\phi_n(\pi(3^q x)) = \phi_n(\pi(J + j3^{-n})) = \phi_n(j3^{-n}) = \pi(jt3^{-n} + t_0) .$$

Finally,  $k_q(x) = J$  and

$$T(x) = Jt + (3^q - 1)t_0$$

which implies the claim.

**QED**

**Lemmas about circle rotations.** Let  $R_t : T^1 \rightarrow T^1$  be defined by  $R_t(x) = \pi(x + t)$ .

**Definition 3** *Define the set  $U(t, K) \subset \mathbf{N}$  by the following requirement:  $m \in U(t, K)$  if and only if for every  $x \in T^1$  and every  $J$  which is a sub-arc of  $T^1$  with length  $3^{-m}$ , the set*

$$\{R_t^p(x) : p = 0, 1, \dots, 3^m - 1\} \cap J$$

*has no more than  $K$  elements.*

Thus, for  $m \in U(t, K)$  the first  $3^m$  points of any orbit are uniformly spread out, in the sense that no ‘‘lumps’’ are formed.

**Lemma 2** *For every  $t$  irrational the set  $U(t, 6)$  is infinite.*



**Proof:**

Let  $q$  be a closest return time for the rotation  $x \rightarrow x+t \pmod 1$ . Then the orbit  $x, \dots, R_t^{q-1}(x)$  cuts the circle into pieces of two sizes and the shorter ones are never adjacent. Hence, any arc of length not exceeding  $1/q$  may contain at most two points of the orbit. If  $m$  is chosen so that  $3^{m-1} < q \leq 3^m$ , the the orbit  $x, \dots, R_t^{3^m-1}(x)$  can be covered by three orbits of length  $q$ . Thus, no arc of length  $3^{-m}$  contains more than 6 points.

QED

## 2.2 Averages along graphs

**Definition 4** *Suppose that  $F : T^2 \rightarrow \mathbf{R}$  and  $g : T^1 \rightarrow \mathbf{R}$  are given. Then we can form a function  $F_g : T^1 \rightarrow \mathbf{R}$  by the following formula:  $F_g(x) = F(x, g(x))$ .*

The general type of the problem we will consider is as follows. We wish to average  $F_g$  along basic intervals, which corresponds to taking conditional expectations with respect to partitions  $\mathcal{P}_n$ . The problem is under what assumptions these averages can be estimated in terms of the average of  $F$  over  $T^2$ .

**Proposition 1** *Consider  $\phi(x) = tx + t_0$  and choose  $N > 0$  and  $K$  so that  $N \in U(t, K)$ .*

*For every  $n \geq 0$  and every set  $A \subset T^2$  which is measurable with respect to  $\mathcal{P}_N \times \mathcal{P}_N$  we consider the function  $F : T^2 \rightarrow \mathbf{R}$  given by  $F^{(n)}(x, y) = \chi_A(\pi(3^{n+N}x), \pi(3^{n+N}y))$ .*

*Then,*

$$E(F_{\phi_{n+2N}}^{(n)} | \mathcal{P}_n)(x) \leq K \int_{T^2} \chi_A d\lambda_2$$

*for every  $x \in T^1$ , using the notation of Definition 4.*

**Proof:**

Choose an interval  $I \in \mathcal{P}_n$ . Observe first that without loss of generality  $n = 0$ . Indeed, for  $n \geq 0$  the interval  $I$  can be parameterized by a variable  $x' = \pi(3^n x)$  which runs over  $T^1$ . We get  $\pi(3^N 3^n x) = \pi(3^N x')$  and, by Lemma 1,

$$\pi(3^{n+N} \phi_{n+2N}(x)) = \pi(3^N(\phi + T(x))_{2N}(x'))$$

with  $T(x)$  constant and equal to  $T(I)$  on  $I$ . Thus for every  $x \in I$

$$E(F_{\phi_{n+2N}}^{(n)} | \mathcal{P}_n)(x) = E(F_{\phi_{2N}+T(I)}^{(0)})$$

which leads to the initial problem with  $n = 0$  and  $t_0$  increased by  $T(I)$ . Since the claim is supposed to be valid for every  $t_0$ , the reduction is complete.

We will write  $F$  for  $F^{(0)}$  and  $\phi$  for  $\phi + T(I)$ .

$$\begin{aligned} F_{\phi_{2N}}(x) &= \chi_A(\pi(3^N x), \pi(3^N \phi_{2N}(x))) = \\ &= \chi_A[\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x))] \end{aligned}$$

by Lemma 1 used with  $n = q = N$ . If we write  $x = J3^{-N} + j3^{-2N}$  with  $J, j$  both integers from the range  $[-\frac{3^N-1}{2}, \frac{3^N-1}{2}]$ , we get  $\pi(3^N x) = j3^{-N}$  and  $T(x) = Jt + T_0$  where  $T_0$  is a constant. We can then write

$$\begin{aligned} E(F_{\phi_{2N}}) &= E[\chi_A(\pi(3^N x), \pi(\phi_N(\pi(3^N x)) + T(x)))] = \\ &= 3^{-2N} \sum_{j=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \sum_{J=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \chi_A(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)). \end{aligned}$$

For  $j$  fixed, points  $\pi(Jt + \phi(j3^{-n}) + T_0)$  form an orbit of the rotation  $R_t$  of length  $3^N$ . By the hypothesis of the Lemma, each square of the partition  $\mathcal{P}_N \times \mathcal{P}_N$  contains no more than  $K$  points in the form  $(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0))$ . Hence,

$$3^{-2N} \sum_{J=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \sum_{j=-\frac{3^N-1}{2}}^{\frac{3^N-1}{2}} \chi_A(j3^{-N}, \pi(Jt + \phi(j3^{-n}) + T_0)) \leq K \int_{T^2} \chi_A d\lambda_2.$$

**QED**

**Lemma 3** *For every  $t \in \mathbf{R}$ ,  $m \geq n \geq 0$ , if  $\phi(x) = tx$ ,  $t_0 \in \mathbf{R}$ , and  $F : T^2 \rightarrow [0, \infty)$  is measurable with respect to  $\mathcal{P}_n \times \mathcal{P}_n$ , then*

$$F(\pi(x), \pi(\phi_m(x) + t_0)) \leq \sum_{2|i| < |t|+2} F(\pi(x), \pi(\phi_n(x) + t_0 + i3^{-n}))$$

where  $i$  runs through integer values only.

**Proof:**

Estimate  $|\phi_m(x) - \phi_n(x)| \leq |t| \frac{3^{-n}}{2}$ . Thus, for every  $x$  we can choose  $\tau(x)$  in the form  $i3^{-n}$ , where  $i$  is an integer and  $-|t|/2 - 1 < i < |t|/2 + 1$  so that  $\pi(\phi_n(x) + t_0 + \tau(x))$  and  $\pi(\phi_m(x) + t_0)$  belong to the same element of  $\mathcal{P}_n$ , and so

$$F(\pi(x), \pi(\phi_m(x) + t_0)) = F(\pi(x), \pi(\phi_n(x) + t_0 + \tau(x))) .$$

Now  $\tau(x)$  only takes values in the set  $i3^{-n}$  with  $|i| < |t|/2 + 1$  and so the lemma follows.

**QED**

**Proposition 2** *Let  $t \in \mathbf{R}$ ,  $K > 0$ ,  $n \geq 0$  and  $N \in U(t, K)$ , see Definition 3. Denote  $\phi(x) = tx$ . Let  $F : T^2 \rightarrow [0, \infty)$  be measurable with respect to  $\mathcal{P}_n \times \mathcal{P}_n$ . Suppose that for a fixed  $I$  and every  $t_1 \in \mathbf{R}$ ,*

$$\int_{T^1} F_{\phi_n+t_1}(x) dx \leq I ,$$

*see Definition 4 for the explanation of the notation.*

*Now  $G : T^2 \rightarrow [0, \infty)$  is measurable with respect to  $\mathcal{P}_N \times \mathcal{P}_N$ ,  $N > 0$ . Define  $\tilde{G}(x, y) = G(\pi(3^{n+N}x), \pi(3^{n+N}y))$ .*

*Then, for every choice of  $t$  and  $K$  and every  $t_0 \in \mathbf{R}$*

$$\int_{T^1} F_{\phi_{n+2N+t_0}}(x) \tilde{G}_{\phi_{n+2N+t_0}}(x) dx \leq K(|t| + 3)I \int_{T^2} G d\lambda_2 .$$

**Proof:**

Fix some  $t_1 \in \mathbf{R}$ . Function  $F_{\phi_n+t_1}$  is measurable with respect to  $\mathcal{P}_n$ . By Proposition 1,

$$E(\tilde{G}_{\phi_{n+2N+t_0}} | \mathcal{P}_n)(x) \leq KI_G$$

for every  $x$  where  $I_G := \int_{T^2} G d\lambda_2$ . Now,

$$\int_{T^1} F_{\phi_n+t_1}(x) \tilde{G}_{\phi_{n+2N+t_0}}(x) dx = \int_{T^1} E(F_{\phi_n+t_1} \tilde{G}_{\phi_{n+2N+t_0}} | \mathcal{P}_n)(x) dx \leq \tag{6}$$

$$\leq KI_G \int_{T^1} F_{\phi_n+t_1}(x) dx \leq KI_G I$$

by the hypothesis of Proposition 2.

By Lemma 3

$$\begin{aligned} F_{\phi_{n+2N}+t_0}(x) &= F(x, \pi(\phi_{n+2N}(x) + t_0)) \leq \\ &\leq \sum_{j \in (-1-|t|/2, |t|/2+1)} F_{\phi_n+t_0+j3^{-n}}(x). \end{aligned}$$

If we use estimate (6) for all  $t_1 = t_0 + j3^{-n}$ , we get

$$\begin{aligned} \int_{T^1} F_{\phi_{n+2N}+t_0}(x) \tilde{G}_{\phi_{n+2N}+t_0}(x) dx &\leq \\ &\leq KI_G(|t| + 3)I. \end{aligned}$$

**QED**

### 2.3 Averages of products

**Theorem 2** *Fix  $t$  irrational and let  $\phi(x) = tx + t_0$ . Then for every  $\lambda > \sqrt{5/3}$  and  $t_0 \in \mathbf{R}$  we have*

$$\lim_{m \rightarrow \infty} \left[ \lambda^{-m} \int_{T^1} \prod_{i=1}^m B_i(x, \phi_m(x)) dx \right] = 0.$$

Recall that functions  $B_i$  are given by Definition 1. Since

$$s(m, \beta) = \int_{T^1} \prod_{i=1}^m B_i(x, \phi_m(x))$$

with  $\phi(x) = \beta - tx$ , Theorem 2 implies that  $\lambda^{-m} S_m \rightarrow 0$ . By estimate (3),  $I_\alpha(\mu^t) < \infty$  provided that  $3^{1-\alpha} > \sqrt{5/3}$  and Theorem 1 follows.

**Hölder estimate.** From Lemma 2, see that  $U(t, 6)$  is infinite and choose  $N \in U(t, 6)$ . Let  $J_{k,0}$  denote the set of integers  $i$  which belong to  $(2(j-1)N, (2j-1)N]$  for some  $j = 1, \dots, k$  and  $J_{k,1}$  be the complement of  $J_{k,0}$  in the set  $1, 2, \dots, 2kN$ . Define  $P_{k,0}(x, y) = \prod_{i \in J_{k,0}} B_i(x, y)$  and  $P_{k,1}(x, y) = \prod_{i \in J_{k,1}} B_i(x, y)$ . Then

$$\prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x)) = P_{k,0}(x, \phi_{2kN}(x)) P_{k,1}(x, \phi_{2kN}(x)) .$$

Our approach is to apply the Hölder inequality to this product. It is easier to estimate the second norm of  $P_{k,1}(x, (\phi + t_1)_{2kN})$  with  $t_1 \in \mathbf{R}$ . Using Proposition 2 with  $n = 2(k-1)N + N$ ,  $F := P_{k-1,1}^2$  and  $G(x, y) = \prod_{i=1}^N B_i^2(x, y)$ , we get

$$\|P_{k,1}(x, (\phi + t_1)_{2kN})\| \leq 6(|t| + 3)I \int_{T^2} G d\lambda_2$$

where  $I$  is an upper estimate for  $\|P_{k-1,1}(x, (\phi + t_1)_{2(k-1)N})\|$  for any  $t_1 \in \mathbf{R}$ . Note that  $\int_{T^2} G d\lambda_2 = (5/3)^N$  and hence one gets by induction starting with  $P_{0,1} \equiv 1$  that

$$\|P_{k,1}(x, \phi_{2kN}(x))\|_2^2 \leq K_1^k (5/3)^{kN} .$$

The same method is used to estimate the second norm of

$$P_{k,0}(x, (\phi + t_1)_{(2k-1)N}(x)) .$$

This time, the induction starts with  $\|P_{1,0}(x, (\phi + t_1)_N(x))\|_2^2 \leq 3^N$  since  $3^N$  is the maximum. Thus,

$$\|P_{k,0}(x, (\phi + t_1)_{(2k-1)N}(x))\|_2^2 \leq 3^N K_1^{k-1} (5/3)^{(k-1)N} .$$

Using Lemma 3 and applying the previous estimate for

$$t_1 = j3^{-2(k-1)N} , \quad 2|j| < |t| ,$$

we get

$$\begin{aligned} \|P_{k,0}(x, \phi_{2kN}(x))\|_2^2 &\leq (|t| + 3) \|P_{k,0}(x, \phi_{(2k-1)N}(x))\|_2^2 \leq \\ &\leq (|t| + 1) 3^N K_1^{k-1} (5/3)^{2(k-1)N} . \end{aligned}$$

By Hölder's inequality,

$$\lambda^{-2kN} \int_{T^1} \prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x)) dx \leq \sqrt{3^N(|t| + 3)} \left[ \frac{5}{3} \frac{\sqrt[N]{K_1}}{\lambda^2} \right]^{kN}.$$

If  $\lambda > \sqrt{5/3}$  then  $N$  can be chosen so large that

$$\frac{5}{3} \frac{\sqrt[N]{K_1}}{\lambda^2} < 1.$$

Then

$$\lim_{k \rightarrow \infty} \left[ \lambda^{-2kN} \int_{T^1} \prod_{i=1}^{2kN} B_i(x, \phi_{2kN}(x)) dx \right] = 0. \quad (7)$$

Any  $j > 0$  can be represented as  $2k_jN + j_0$  with  $j_0 < 2N$ . Then

$$\prod_{i=1}^j B_i(x, \phi_j(x)) \leq 3^{j_0} \prod_{i=1}^{2k_jN} B_i(x, \phi_j(x)).$$

Again using Lemma 3 and the fact that we estimate

$$\int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, \phi_j(x)) dx \leq (|t| + 1) \int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, (\phi + t_1)_{2k_jN}(x)) dx$$

where  $t_1$  was chosen to attain the supremum of the integral on the right-hand side. Hence, for any  $j > 0$ ,

$$\int_{T^1} \prod_{i=1}^j B_i(x, \phi_j(x)) dx \leq 9^N (|t| + 1) \int_{T^1} \prod_{i=1}^{2k_jN} B_i(x, (\phi + t_1)_{2k_jN}(x)) dx$$

and Theorem 2 follows from this together with assertion (7). Notice that (7) holds for any  $t_0$ , in particular one can set  $t_0 := t_0 + t_1$  in that estimate.

## References

- [1] Hutchinson, J.E.: *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), 713-747

- [2] Jonker, L.B. & Veerman, J.J.P.: *Semicontinuity of Dimension and Measure of Locally Scaling Fractals*, preprint, Stony Brook IMS Series, (1997) #2
- [3] Kenyon, R.: *Projecting the one-dimensional Sierpinski gasket*, Israel J. Math. **97** (1997), 221-238
- [4] Lagarias, J.C. & Wang, Y.: *Tiling the line with translates of one tile*, Invent. Math. **124** (1996), 341-365
- [5] Mattila, P.: *Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability.*, Cambridge Studies in Advanced Mathematics vol. 44, Cambridge University Press (1995)
- [6] Veerman, J.J.P. & Stošić, B.D.: *On the Dimensions of Certain Incommensurably Constructed Sets*, Experim. Math. **9** (2000), 413-425