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# A Note on Lattice Chains and Delannoy Numbers

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# A note on lattice chains and Delannoy numbers


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**Abstract.** Fix nonnegative integers $n_1, \ldots, n_d$ and let $L$ denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. Let $L$ be partially ordered by the usual dominance ordering. In this paper we offer combinatorial derivations of a number of results concerning chains in $L$. In particular, the results obtained are established without recourse to generating functions or recurrence relations. We begin with an elementary derivation of the number of chains in $L$ of a given size, from which one can deduce the classical expression for the total number of chains in $L$. Then we derive a second, alternative, expression for the total number of chains in $L$ when $d = 2$. Setting $n_1 = n_2$ in this expression yields a new proof of a result of Stanley [7] relating the total number of chains to the central Delannoy numbers. We also conjecture a generalization of Stanley's result to higher dimensions.


## 1 Introduction

Fix nonnegative integers $n_1, \ldots, n_d$ and let $L$ denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. Partially order $L$ by setting $(a_1, \ldots, a_d) \preceq (b_1, \ldots, b_d)$ whenever $a_i \leq b_i$ for each $i$ $(1 \leq i \leq d)$. In various contexts [2, 3, 5], the number of chains in $L$ of a given size has been computed using either recurrence relations or generating functions. Summing this expression over all possible sizes, one obtains an expression for the total number of chains $L$. In the case when the dimension $d = 2$ and the lattice $L$ is a square (so that $n_1, n_2$ share a common value $n$), an alternative expression for this quantity was given by Stanley [7]. In particular, he used generating functions to establish that the total number of chains in $L$ equals $2^{n+1} d_n$, where $d_n$ denotes the $n^{th}$ Delannoy number. In [8], a bijective proof of Stanley's result is given. The bijection given there is the composition of five combinatorially defined bijections, perhaps a testament to its subtlety.

In this paper we begin with an elementary derivation of the number of chains in $L$ of a given size using inclusion/exclusion. We then derive a formula for the total number of chains in $L$ when $d = 2$. Setting $n_1 = n_2$ in this expression yields a new proof of Stanley's result. We conclude with a few remarks on the hypergeometric form of the expressions derived, and finally, we conjecture a generalization of Stanley's result to higher dimensions.

## 2 Lattice chains

Fix nonnegative integers $n_1, \ldots, n_d$. Let $L = L(n_1, \ldots, n_d)$ denote the lattice of integer points $(a_1, \ldots, a_d) \in \mathbb{Z}^d$ satisfying $0 \leq a_i \leq n_i$ for $1 \leq i \leq d$. Recall $L$ is partially ordered by the dominance relation, defined as follows. Given $\mathbf{a}, \mathbf{b} \in L$ with $\mathbf{a} = (a_1, \ldots, a_d)$ and $\mathbf{b} = (b_1, \ldots, b_d)$, we say $\mathbf{a} \preceq \mathbf{b}$ whenever $a_i \leq b_i$ for $1 \leq i \leq d$.

By a *chain* in $L$ we mean a subset of $L$ that is totally ordered by $\preceq$. A *k-chain* is a chain with $k$ elements. Define $k_{max} = n_1 + \cdots + n_d + 1$ and observe that $k_{max}$ is the maximum number of elements of a chain in $L$.

Let $C = C(n_1, \dots n_d)$ denote the set of chains in $L$, and for each integer $k$, let $C_k = C_k(n_1, \dots n_d)$ denote the set of $k$-chains in $L$. These sets have been studied in the contexts of subsets of multi-sets and partitions of a set [2, 3, 5]. In the next two sections we study expressions for $|C_k|$ and $|C|$.

## 2.1 Counting $k$-chains

One obvious way to obtain an expression for $|C|$ is to sum $|C_k|$ over all $k$. This requires us to first find an expression for $|C_k|$. And indeed, a simple expression for $|C_k|$ is not difficult to derive, and has been computed in several places [3, 5] for the special case $n_i = 1$ for all $i$, and, in [2], for the general case. Each of these derivations proceeds either by solving an appropriate recurrence or through the use of generating functions. In this section we offer a direct counting argument for $|C_k|$ using the principle of inclusion/exclusion.

**Theorem 1** *[2] Fix $n_1, \dots, n_d \in \mathbb{Z}^{\geq 0}$ and set $k_{max} = 1 + \sum_1^d n_i$. Then for any integer $k$ $(0 \leq k \leq k_{max})$,*

$$|C_k(n_1, \dots, n_d)| = \sum_{r=0}^{k-1} (-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

□

Our proof begins with Lemma 1, which counts the number of sequences $\langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle$ in $L$ satisfying $\mathbf{a}_1 \preceq \cdots \preceq \mathbf{a}_k$. Since such sequences allow duplicate entries, while chains do not, Lemma 1 does not directly compute $|C_k|$.

**Lemma 1** *With the notation of Theorem 1, fix any integer $k$ $(0 \leq k \leq k_{max})$. Let $S_k$ denote the set of all sequences $\langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle$ in $L$ satisfying $\mathbf{a}_1 \preceq \cdots \preceq \mathbf{a}_k$. Then*

$$|S_k| = \prod_{i=1}^{d} \binom{n_i + k}{n_i}. \tag{1}$$

**Proof.** Consider a sequence $\langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle$ in $L$, where $\mathbf{a}_j = (a_{j1}, \dots, a_{jd})$ for $(1 \leq j \leq k)$. This sequence belongs to $S_k$ if and only if for each $i$ $(1 \leq i \leq d)$

$$0 \leq a_{1i} \leq \cdots \leq a_{ki} \leq n_i. \tag{2}$$

The number of integer sequences $a_{1i}, \dots, a_{ki}$ satisfying (2) is given by $\binom{n_i+k}{n_i}$. Multiplying these factors together as $i$ ranges from 1 to $d$, we obtain the result. □

As discussed above, $S_k$ includes sequences with repeated elements. So we will apply inclusion/exclusion to obtain $|C_k|$. The next lemma considers the sets to be excluded.

**Lemma 2** *With the notation of Theorem 1, fix any integer $k$ $(0 \leq k \leq k_{max})$, and let $S_k$ be as in Lemma 1. For each $1 \leq i \leq k-1$, let*

$$S_k(i) = \{\langle \mathbf{a}_1, \dots, \mathbf{a}_k \rangle \in S_k \mid \mathbf{a}_i = \mathbf{a}_{i+1}\}.$$

*Then for any integers $i_i, \dots, i_r$ such that $1 \leq i_i < \cdots < i_r \leq k-1$, we have*

$$|S_k(i_1) \cap S_k(i_2) \cdots \cap S_k(i_r)| = \prod_{i=1}^{d} \binom{n_i + k - r}{n_i}.$$

**Proof.** Fix integers $i_i, \dots, i_r$ such that $1 \leq i_i < \cdots < i_r \leq k-1$. Each sequence $\mathbf{a} \in S_k(i_1) \cap S_k(i_2) \cdots \cap S_k(i_r)$ satisfies $\mathbf{a}_{i_j} = \mathbf{a}_{i_j+1}$ for $1 \leq j \leq r$. Such a sequence corresponds naturally to a sequence in $S_{k-r}$ by deleting the $r$ terms $\mathbf{a}_{i_j}$ for $1 \leq j \leq r$. Replacing $k$ by $k-r$ in Lemma 1 counts $|S_{k-r}|$. The result follows. □

We now prove the theorem.

**Proof of Theorem 1.** Observe $|C_k| = \left|S_k \setminus \bigcup_{i=1}^{k-1} S_k(i)\right|$. By the principle of inclusion/exclusion,

$$\begin{aligned}|C_k| &= \sum_{r=0}^{k-1}(-1)^r \sum_{i_1<\cdots<i_r} |S_k(i_1)\cap\cdots\cap S_k(i_r)| \\ &= \sum_{r=0}^{k-1}(-1)^r \binom{k-1}{r} \prod_{i=1}^{d} \binom{n_i+k-r}{n_i}. \qquad \square\end{aligned}$$

## 2.2 Counting the total number of chains

In this section we compute $|C|$, the total number of chains in $L$. One expression is easily obtained using the result of the previous section. Indeed, recall that a chain in $L$ has at most $k_{max} = n_1 + \cdots + n_d + 1$ elements. It follows that

$$|C(n_1,\ldots,n_d)| = \sum_{k=0}^{k_{max}} |C_k(n_1,\ldots,n_d)|. \tag{3}$$

In the special case when $d = 2$ and the lattice $L$ is a square (so that $n_1 = n_2$), Stanley [7, Section 6.3] used generating functions to find an alternative expression for this quantity, which we will obtain as Corollary 4 below. In this section, however, we begin with a combinatorial derivation of a slight generalization of Stanley's result; in particular, we count the total number of chains for the case $d = 2$ without the assumption that $n_1 = n_2$. Central to our proof is the notion of a *$y$-strict* chain: a chain in which no two elements have the same $y$-coordinate (ie., 2nd coordinate). We obtain the following theorem.

**Theorem 2** *Let $n_1$ and $n_2$ be nonnegative integers. Then*

$$|C(n_1,n_2)| = 2^{n_1+1} \sum_{i=0}^{n_2} \binom{n_1+i}{i}\binom{n_2}{i}. \tag{4}$$

**Proof.** To each chain $\xi$ in $L(n_1,n_2)$ we associate a pair $(A_\xi,\xi')$ where $A_\xi \subseteq \{0,1,\ldots,n_1\}$ and $\xi'$ is a $y$-strict chain in $L(n_1,n_2-1)$. If $\xi$ denotes the chain $(x_1,y_1) \preceq (x_2,y_2) \preceq \cdots \preceq (x_k,y_k)$, then we define $A_\xi = \{x_i | y_i = y_{i+1} \text{ or } y_i = n_2\}$ and $\xi' = \xi \backslash \{(x_i,y_i) | x_i \in A_\xi\}$. See Figure 1.

This is a bijective correspondence, and we now exhibit the inverse map which associates a chain in $L(n_1,n_2)$ with each pair $(A,\xi')$ consisting of a subset $A \subseteq \{0,1,\ldots,n_1\}$ and a $y$-strict chain $\xi'$ in $L(n_1,n_2-1)$. Let $(A,\xi')$ be such a pair. For each $i$ $(0 \le i \le n_1)$ let $\mathrm{top}_{\xi'}(i)$ denote the maximum $y$ such that $\xi' \cup \{(i,y)\}$ is a chain in $L(n_1,n_2)$. Then the chain associated with the pair $(A,\xi')$ is the chain $\xi = \xi' \cup \{(i,\mathrm{top}_{\xi'}(i)) | i \in A\}$.

Given this correspondence, it remains only to count the number of pairs $(A,\xi')$. Let $\xi'$ be an $i$-element $y$-strict chain in $L(n_1,n_2-1)$. Then there are $\binom{n_2}{i}$ choices for the $y$-coordinates that appear in $\xi'$. Once the $y$-coordinates have been chosen, we may choose the $x$-coordinates freely, as long as the resulting choice maintains the chain condition for $\xi'$. That is, we must choose $i$ $x$-coordinates such that $0 \le x_1 \le x_2 \le \cdots \le x_i \le n_1$. The number of such choices for the $x$-coordinates is given by $\binom{n_1+i}{i}$. Thus, we have

$$\sum_{i=0}^{n_2} \binom{n_1+i}{i}\binom{n_2}{i}$$

$y$-strict chains in $L(n_1,n_2-1)$. For each such chain $\xi'$, we have $2^{n_1+1}$ choices for a subset $A$. It follows that the number of pairs $(A,\xi')$ is given by the right side of (4). $\square$

**Remark.** A recursive proof of (4), albeit somewhat less illuminating, can also be given as follows. A simple inclusion/exclusion argument shows that for positive integers $n_1$ and $n_2$

$$|C(n_1,n_2)| = 2|C(n_1,n_2-1)| + 2|C(n_1-1,n_2)| - 2|C(n_1-1,n_2-1)|. \tag{5}$$

It is readily verified that the expression on the right side of (4) also satisfies this recurrence, along with appropriate initial conditions.

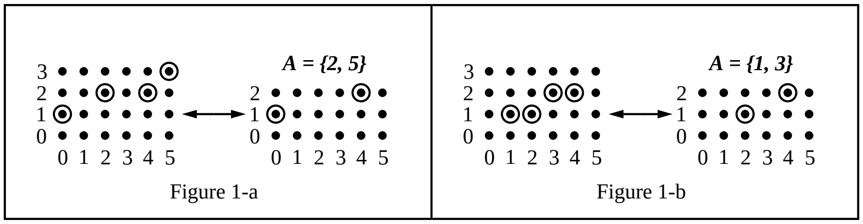


Figure 1: Mapping Chains

We close this section with a few comments on symmetry. Observe that although the quantity $|C(n_1, n_2)|$ is, by its definition, symmetric in $n_1$ and $n_2$, the expression in (4) is less obviously so. It is perhaps also worth noting, then, that our expression for $|C(n_1, n_2)|$ is quite conveniently stated using the notation of hypergeometric series. Recall that for any complex number $a$ and any natural number $n$, we define $(a)_n := (a)(a+1)\cdots(a+n-1)$. Using this notation, the ${}_2F_1$ hypergeometric series is defined as follows:

$$ {}_2F_1(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!} $$

for complex $a, b, c, z$ with $c \neq 0$. There are many identities involving such series. For example, one of the so-called Euler transformations (see [6, p.33]) gives that

$$ {}_2F_1(a, b; c; 1/2) = 2^a {}_2F_1(a, c-b; c; -1). \tag{6} $$

By Theorem 2, we see that

$$ C(n_1, n_2) = 2^{n_2+1} {}_2F_1(-n_1, n_2+1; 1; -1). $$

Applying (6), we obtain the more symmetric expression

$$ C(n_1, n_2) = 2^{n_1+n_2+1} {}_2F_1(-n_1, -n_2; 1; 1/2). $$

# 3 Delannoy numbers

The Delannoy numbers count the number of lattice paths in $L(n_1, n_2)$ from $(0, 0)$ to $(n_1, n_2)$ in which only vertical $\mathfrak{v} = (0, 1)$, horizontal $\mathfrak{h} = (1, 0)$, and diagonal $\mathfrak{d} = (1, 1)$ steps are allowed. Such a path is sometimes referred to as a (restricted) king's walk. When $n_1$ and $n_2$ share a common value $n$, we refer to $d_n = D(n, n)$ as the *central Delannoy numbers*. For arbitrary dimension $d$, we set $D(n_1, ..., n_d)$ equal to the number of lattice paths in $L(n_1, ..., n_d)$ that begin from the origin and terminate at $(n_1, ..., n_d)$, in which only positive steps from the $d$-dimensional unit hypercube are allowed. This follows [4]. Indeed, for more about generalizations of the Delannoy numbers, we refer the reader to [4, 1].

Although the Delannoy numbers are typically derived recursively or with generating functions, we can count the number of restricted king's walks as follows. Observe that $D(n_1, n_2) = D(n_2, n_1)$ so we may assume without loss of generality that $n_1 \leq n_2$.

**Theorem 3** *Let $n_1, n_2$ be nonnegative integers such that $n_1 \leq n_2$. Then*

$$ D(n_1, n_2) = \sum_{i=0}^{n_1} \binom{n_2+i}{i} \binom{n_2}{n_1-i}. \tag{7} $$

**Proof.** A walk is a sequence of $\mathfrak{h}, \mathfrak{v}$, and $\mathfrak{d}$ steps. To reach $(n_1, n_2)$, the number of $\mathfrak{h}, \mathfrak{d}$ steps must sum to $n_1$ and the number of $\mathfrak{v}, \mathfrak{d}$ steps must sum to $n_2$. Let $i$ denote the number of diagonal $\mathfrak{d}$ steps in a given walk. Clearly, $0 \leq i \leq n_1$. Also, the total number of steps of such a walk is $n_1 + n_2 - i$. The number of $\mathfrak{h}$

steps is given by $n_1 - i$. The number of $\mathfrak{v}$ steps is $n_2 - i$. So the total number of such walks is given by the trinomial coefficient $\binom{n_1+n_2-i}{i, n_1-i, n_2-i} = \binom{n_1+n_2-i}{n_1-i}\binom{n_2}{i}$. Thus, the total number of walks is

$$\sum_{i=0}^{n_1} \binom{n_1+n_2-i}{n_1-i}\binom{n_2}{i}.$$

Reindexing the sum (replacing $i$ by $n_1 - i$) we obtain the desired result. □

Comparing lines (4) and (7), we have the following.

**Corollary 4** *[7, 8] For any nonnegative integer $n$, $C(n,n) = 2^{n+1}D(n,n)$.* □

**Remark.** We note that the expression in (7) can also be established recursively. Indeed, a simple inclusion/exclusion argument shows that

$$D(n_1, n_2) = D(n_1, n_2 - 1) + D(n_1 - 1, n_2) + D(n_1 - 1, n_2 - 1).$$

It is then readily verified that the expression on the right side of (7) also satisfies this recurrence, along with appropriate initial conditions.

We conclude this paper with a conjecture, analogous to Corollary 4, that appears to be supported by numerical evidence.

**Conjecture 1** *If $n_1 = n_2 = \cdots = n_d$, then*

$$C(n_1, n_2, \ldots, n_d) = 2^{n_1+1}D(n_1, n_2, \ldots, n_d).$$

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