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Shift-Symmetric Configurations in Two-Dimensional Cellular Automata: Irreversibility, Insolubility, and Enumeration

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The search for symmetry as an unusual yet profoundly appealing phenomenon, and the origin of regular, repeating configuration patterns have been for a long time a central focus of complexity science, and physics.

Here, we introduce group-theoretic concepts to identify and enumerate the symmetric inputs, which result in irreversible system behaviors with undesired effects on many computational tasks. The concept of so-called *configuration shift-symmetry* is applied on two-dimensional cellular automata as an ideal model of computation. The results show the universal insolubility of “non-symmetric” tasks regardless of the transition function. By using a compact enumeration formula and bounding the number of shift-symmetric configurations for a given lattice size, we efficiently calculate how likely a configuration randomly generated from a uniform or density-uniform distribution turns shift-symmetric. Further, we devise an algorithm detecting the presence of shift-symmetry in a configuration.

The enumeration and probability formulas can directly help to lower the minimal expected error for many crucial (non-symmetric) distributed problems, such as leader election, edge detection, pattern recognition, convex hull/minimum bounding rectangle, and encryption. Besides cellular automata, the shift-symmetry analysis can be used to study the non-linear behavior in various synchronous rule-based systems that include inference engines, Boolean networks, neural networks, and systolic arrays.

Keywords: configuration shift-symmetry, two-dimensional cellular automata, insolubility, irreversibility, enumeration, symmetry detection, prime factorization, prime orbit, mutually-independent generators, leader election

I. INTRODUCTION

Symmetry is a synonym for beauty and rarity, and generally perceived as something desired. In this paper we investigate an opposing side of symmetry and show how it can irreversibly *corrupt* a computation and restrict the system’s dynamics.

The structure of the computational rules that result in regular, repeating system configurations has been studied by many, yet the question of how the natural and engineered system organize into symmetric structures is

not completely known. To understand the role of symmetry of the starting configurations (the inputs), how they are processed (the machine), and produce the final configurations with desired properties (the outputs) we use a cellular automata (CA) as a simple distributed model of computation. First introduced by John von Neumann, CAs allowed to explore logical requirements for machine self-replication and information processing in nature [38]. Despite having no central control and limited communication among the components, CAs are capable of universal computation and can exhibit various dynamical regimes [8, 38, 46, 52]. As one of the structurally simplest distributed systems, CAs have become a fundamental model for studying complexity in its purist form [16, 53]. Subsequently, CAs have been successfully employed in numerous research fields and applications, such as modeling artificial life [29], physical equations [19, 47], and social and biological simulations [18, 25, 42, 43].

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The CA input configurations define a language that is processed by the machine. Exploring the structural symmetries of the input language not only translates to an efficient machine implementation, but allows us to present a theoretical argument of a problem insolvability and the irreversibility of computation.

In this paper, we introduce the concept of shift-symmetry and show that any standard CA maintains a configuration shift-symmetry due to uniformity and synchronicity of cells. We prove that once a system reaches a symmetric, i.e., spatially regular configuration, the computation will never revert from this attractor and will fail to solve all problems that require the asymmetric solutions. As a result, the number of symmetries of the system is non-decreasing.

Using group theory, we prove that the CA's configuration space irreversibly folds causing a permanent regime "shift" when a configuration slips to a symmetric, repeating pattern. Consequently, a non-symmetric solution cannot be reached from a shift-symmetric configuration. A more general implication is that a configuration is unreachable (even if symmetric) if a source configuration has a symmetry not contained in a target one. Non-symmetric tasks, such as leader election or pattern recognition, i.e., tasks expecting a final configuration to be non-symmetric, are therefore principally insolvable, since for any lattice size there always exist input configurations that are symmetric.

We develop three progressively more efficient enumeration techniques based on mutually independent generators to answer the question of how many potential shift-symmetric configurations there are in any given two-dimensional CA lattice. As a side product, we demonstrate that the shift-symmetry is closely linked to prime factorization. We introduce and prove lower and upper bounds for the number of shift-symmetric configurations, where the lower bound (local minima) is tight and reached only for prime lattice sizes. We enumerate shift-symmetric configurations for a given lattice size and number of active cells.

Finally, we derive a formula and bounds for the probability of selecting shift-symmetric configuration randomly generated from a uniform or density-uniform distribution. We develop a shift-symmetry detection algorithm and derive its worst and average-case time complexities.

A. Applications

Since all the formulas and proofs presented in this paper assume two-dimensional CAs with any number of states, and arbitrary uniform transition and neighborhood functions, our results are widely applicable.

Knowing the number of shift-symmetric configurations, we can directly determine the probability of randomly selecting a shift-symmetric configuration. This probability then equals an error lower bound or expected insolvability for any non-symmetric task. As we show the

shift-symmetry-caused insolvability is rapidly asymptotically decreasing with the lattice size for a uniform distribution. For instance, the probability is 0.5 for a 2×2 lattice, but it drops to around 2.7×10^{-15} for a 10×10 lattice. Since the number of shift-symmetric configurations heavily depends on the prime factorization of a lattice size, the probability function is non-monotonously decreasing. To minimize the occurrence of shift-symmetries for uniform distribution, our general recommendation is to use prime lattices, or at least avoid even ones. On the other side, the probability for a density-uniform distribution is quite high, regardless of primes: it is around 10^{-3} even for a 45×45 lattice.

The distribution error-size constraints have important consequences for designing robust and efficient computational procedures for non-symmetric tasks. A class of non-symmetric tasks covers many crucial distributed problems, such as leader election [5, 46], pattern recognition [41], edge detection [45], convex hull/minimum bounding rectangle [11], and encryption [49]. For these tasks an expected final configuration, e.g., reproduction of a certain two-dimensional image, is in a general case non-shift-symmetric, and therefore unreachable from a symmetric configuration.

Practical implications of these properties includes performance degradation of systolic CPU arrays and nanoscale multicore systems [56]. Our results span to the hardware implementations of synchronous CAs with FPGAs, used, e.g., for traffic signals control [26], random number generation [44], and reaction-diffusion model [24]; and spintronics, where computation is achieved by coupled oscillators [9, 48]. Also, current efforts of implementing two or three-dimensional cellular automata using DNA tiles [20, 50] and/or gel-separated compartments in so-called *gellular automata* [21, 28] might face problems related to configuration shift-symmetry if a synchronous update is considered.

B. Related Work

In his seminal work, Packard [39] identified the importance of symmetry and showed that CA's global properties emerge as a function of transition function's reflective and rotational symmetries. The fundamental algebraic properties of additive and non-additive CAs were studied by Martin *et al.* [34], who demonstrated on simple cases that there is a connection between the global behavior and the structure of the configuration transitions. Wolz and deOliveira [54] exploited the structure and symmetry in the transition table to design an efficient evolutionary algorithm that found the best results for the density classification and parity problems. Marquez-Pita *et al.* [32, 33] used a brute-force approach to find similar input configurations that produce the same outputs. Their results are a compact transition function re-description schema that used wild-cards to represent the many-to-one computation rules on a majority problem. Bagnoli

et al. [4] explored different methods of master-slave synchronization and control of totalistic cellular automata. CA computation-theoretic results were summarized by Culik II *et al.* [17], who investigated CAs through the eyes of set theory and topology. The concept of symmetry in number theory has been applied to so-called tapestry design and periodic forests [3, 35], which relates to CA configurations. However, the triangular topology and geometric branching differs from a discrete toroidal Cartesian topology typically used for CAs.

Despite a substantial focus on the symmetry of *transition functions*, and the design of transition functions resulting in regular or synchronized patterns, the theoretical CA research did not address the general structure and implications of the shift-symmetric *configurations* without assuming anything specific about a transition function, as we do here.

One of our main motivations is the pioneering work of Angluin [2], who noticed that a ring containing anonymous components (processors), which are all in the same state, will never break its homogeneous configuration and elect a leader. This intuitive observation is, in fact, a special case of our concept of configuration shift-symmetry for CAs. We will show that Angluin's homogeneous state, which corresponds to a configuration of all zeros or all ones in a binary CA, is the most symmetric configuration for a given lattice size.

The concept of shift-symmetry is related to the notion of regular domains in computational mechanics [15, 22]. A shift-symmetric configuration is essentially a (global) regular domain spread to a full lattice. Although we cannot apply the results directly to regular domains at the level of sub-configurations because we pay no attention to local symmetries and non-cyclic and non-regular borders, the number of possible shift-symmetric configurations gives at least an upper bound on the number of possible regular domains.

In our previous work [7] we proved that configuration shift-symmetry along with loose-coupling of active cells prevents a leader to be elected in a one-dimensional CA [5]. The *leader election* problem, first introduced by Smith [46], requires processors to reach a final configuration where exactly one processor is in a leader state (one) and all others are followers (zero). Leader election is a representative of a problem class where a solution is an asymmetric, non-homogeneous, transitionally and rotationally invariant system configuration. A final fixed-point configuration is asymmetric, since it contains only one processor in a leader state. Clearly, leader election and symmetry are *enemies*, and, in fact, leader election is often called symmetry-breaking.

To enumerate shift-symmetric configurations for a one-dimensional case [7] we employed only basic combinatorics. Here, in order to span to two dimensions, we extended our enumeration machinery to group theory and independent generators. We show that the insolubility caused by configuration symmetry extends beyond leader election to a whole class of non-symmetric problems.

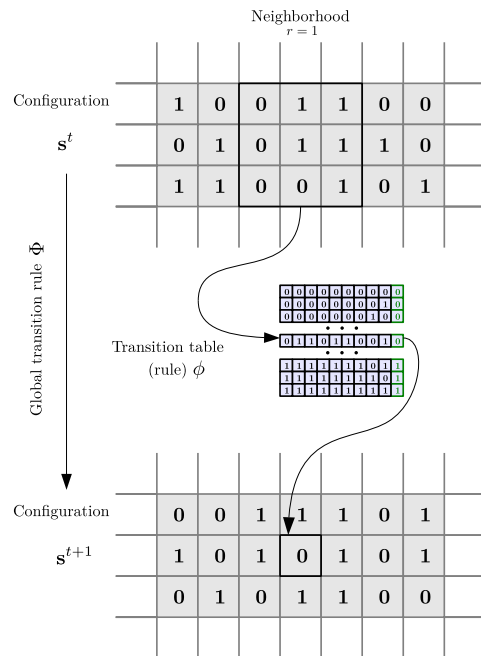


FIG. 1. Schematic of the configuration update for a binary two-dimensional cellular automaton.

C. Model

By definition, a CA [13] consists of a lattice of N components, called *cells*, and a *state set* Σ . A state of the cell with index i is denoted $s_i \in \Sigma$. A *configuration* is then a sequence of cell states:

$$\mathbf{s} = (s_0, s_1, \dots, s_{N-1}).$$

Given a topology for the lattice and the number of neighbors b , a *neighborhood* function $\eta : \mathbb{N} \times \Sigma^N \rightarrow \Sigma^b$ maps any pair (i, \mathbf{s}) to the b -tuple $\eta_i(\mathbf{s})$ of cells' states that are accessible (visible) to cell i in configuration \mathbf{s} . Note that each cell is usually its own neighbor.

The transition rule $\phi : \Sigma^b \rightarrow \Sigma$ is applied in parallel to each cell's neighborhood, resulting in the synchronous update of all of the cells' states $s_i^{t+1} = \phi(\eta_i(\mathbf{s}^t))$. The transition rule is represented either by a transition table, also called a look-up table, or a finite state transducer [23]. Here we focus exclusively on *uniform* CAs, where all cells share the same transition function. The *global transition rule* $\Phi : \Sigma^N \rightarrow \Sigma^N$ is defined as the transition rule with the scope over all configurations

$$\mathbf{s}^{t+1} = \Phi(\mathbf{s}^t).$$

In this paper we analyze two-dimensional CAs, where cells are topologically organized on a two-dimensional grid with cyclic boundaries, i.e., we treat them as tori. The true power of our analysis is that it applies to two-dimensional CAs with arbitrary neighborhood and transition functions. We rely only on their uniformity:

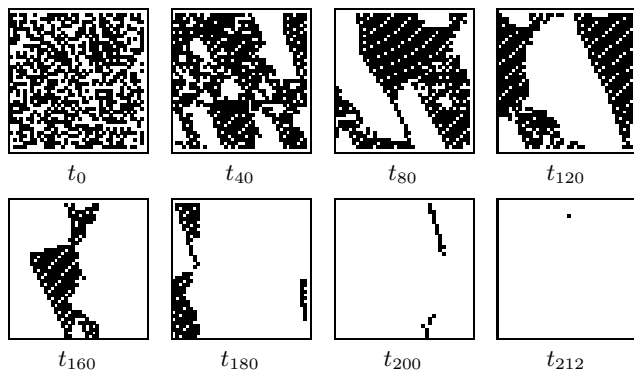


FIG. 2. Example space-time diagrams of a leader-electing CA on lattice size $N = 40^2$ [6]. Figures show a CA computation starting with a random initial configuration (time t_0), followed by 7 configuration snapshots. The CA reaches a final configuration with a single active cell (leader) at time t_{212} .

each cell has the same neighborhood and transition function; and synchronous update, the attributes typically assumed for a standard CA.

Figure 1 shows the update mechanism for a two-dimensional binary CA with a Moore neighborhood, a square neighborhood with radius $r = 1$ containing 9 cells. The dynamics of two-dimensional CAs are illustrated as a series of configuration snapshots, where an active cell is black and an inactive cell white (Figure 2).

II. SHIFT-SYMMETRIC CONFIGURATIONS

As stated by Angluin [2], homogeneous configurations are insolvable by any anonymous deterministic algorithm (including CAs). The CA uniformity can be embedded in its transition function, the deterministic update, synchronicity, topology, configuration, and cells' anonymity. Intuitively, a fully uniform system in terms of its structure, configuration, and computational mechanisms cannot produce any reasonable or complex dynamics.

We show that Angluin's homogeneous configurations of 0^N and 1^N belong to a much larger class of so-called shift-symmetric configurations. In this section we formalize the concept of configuration shift-symmetry by employing vector translations and group theory. Figure 3 shows a CA computation on a two-dimensional shift-symmetric configuration. Compared to the one-dimensional case [7], two dimensions are more symmetry potent.

It is important to mention that we deal with *square* configurations only. Nevertheless, we suggest most of the lemmas and theorems could be extended to incorporate arbitrary rectangular shapes. Also, the formulas and methodology to enumerate two-dimensional shift-symmetric configurations could be generalized to arbitrarily many dimensions. For consistency, however, we leave the rectangular as well as n -dimensional extensions for future consideration.

First, we formally define a shift-symmetric (square) configuration by a given vector as shown in Figure 4.

Definition II.1 For a non-zero vector (pattern shift) $\mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$ we denote by

$$S_{n \times n}(\mathbf{v}) = \{\mathbf{s} \in \Sigma^{n \times n} \mid \forall \mathbf{u} \in \mathbb{Z}_n \times \mathbb{Z}_n : s_{\mathbf{u}} = s_{\mathbf{u} \oplus \mathbf{v}}\}$$

the set of all **shift-symmetric square configurations** of size $N = n^2$ relative to \mathbf{v} over the alphabet Σ , where \oplus denotes coordinate-wise addition on $\mathbb{Z}_n \times \mathbb{Z}_n$. \square

Note that as opposed to our previous work [6], we renamed *symmetric* configurations to *shift-symmetric* configurations to avoid confusion with reflective or rotational symmetries. These two symmetry types, unlike shift-symmetry, are not generally preserved by a transition function unless we impose certain "symmetric" properties on the transitions.

Since any translation by a non-zero vector \mathbf{v} defines a configuration symmetry, we can study shift-symmetric configurations with the techniques of group theory. From now on, we will call such a vector \mathbf{v} that we use for state translation a *generator*.

Lemma II.1 For any non-zero vector (generator) $\mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$,

$$S_{n \times n}(\mathbf{v}) = \{\mathbf{s} \in \Sigma^{n \times n} \mid \forall \mathbf{u} \in \mathbb{Z}_n \times \mathbb{Z}_n \forall \mathbf{w} \in \langle \mathbf{v} \rangle : s_{\mathbf{u}} = s_{\mathbf{u} \oplus \mathbf{w}}\},$$

where $\langle \mathbf{v} \rangle$ is the cyclic subgroup of $\mathbb{Z}_n \times \mathbb{Z}_n$ generated by \mathbf{v} . \square

Lemma II.2 For any non-zero $\mathbf{v} = (l_1, l_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$, the following hold:

- (i). $|S_{n \times n}(\mathbf{v})| = |\Sigma|^{\frac{n^2}{|\langle \mathbf{v} \rangle|}}$.
- (ii). $|\langle \mathbf{v} \rangle| = \frac{n}{\gcd(l_1, l_2, n)}$,
- (iii). $|S_{n \times n}(\mathbf{v})| = |\Sigma|^{n \gcd(l_1, l_2, n)}$.

Proof (i). When $\mathbf{v} = (l_1, l_2)$ is repeatedly applied to any cell in the lattice, an orbit is generated, consisting of $|\langle \mathbf{v} \rangle|$ cells that must share a common state for any configuration in $S_{n \times n}(\mathbf{v})$. The number of distinct orbits

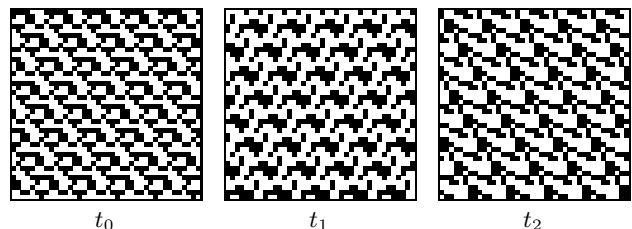


FIG. 3. Space-time diagrams of CA computation on a two-dimensional binary shift-symmetric configuration showing a lattice at three consecutive time steps. Once reached, a shift-symmetry cannot be broken.

of cells in the lattice is simply $\frac{n^2}{|\langle \mathbf{v} \rangle|}$. Any configuration in $S_{n \times n}(\mathbf{v})$ is thus uniquely determined by choosing a state from Σ for each orbit of cells, so (i) follows.

(ii). For $l \in \mathbb{Z}_n$ it is easily shown that $|\langle l \rangle| = \frac{n}{\gcd(l, n)}$, so

$$|\langle \mathbf{v} \rangle| = \text{lcm} \left(\frac{n}{\gcd(l_1, n)}, \frac{n}{\gcd(l_2, n)} \right) = \frac{n}{\gcd(l_1, l_2, n)},$$

where lcm denotes the least common multiple.

(iii). By (ii), the exponent in (i) becomes

$$\frac{n^2}{|\langle \mathbf{v} \rangle|} = \frac{n^2}{\frac{n}{\gcd(l_1, l_2, n)}} = n \gcd(l_1, l_2, n)$$

as desired. \square

Lemma II.3 Fix any non-zero vector $\mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$ and any shift-symmetric square configuration $\mathbf{s} \in S_{n \times n}(\mathbf{v})$. Then for any $\mathbf{w} \in \mathbb{Z}_n \times \mathbb{Z}_n$, the neighborhoods satisfy

$$\eta_{\mathbf{w}}(\mathbf{s}) = \eta_{\mathbf{w} \oplus \mathbf{v}}(\mathbf{s}).$$

Proof Suppose the neighborhood function, which is uniformly shared by all cells, is defined by (relative) vectors $\mathbf{u}_1, \dots, \mathbf{u}_m$, i.e., $\eta_{\mathbf{w}}(\mathbf{s}) = (s_{\mathbf{w} \oplus \mathbf{u}_1}, \dots, s_{\mathbf{w} \oplus \mathbf{u}_m})$ and assume the lemma does not hold, i.e., there exists \mathbf{w} for which $\eta_{\mathbf{w}}(\mathbf{s}) \neq \eta_{\mathbf{w} \oplus \mathbf{v}}(\mathbf{s})$. Then

$$(s_{\mathbf{w} \oplus \mathbf{u}_1}, \dots, s_{\mathbf{w} \oplus \mathbf{u}_m}) \neq (s_{(\mathbf{w} \oplus \mathbf{v}) \oplus \mathbf{u}_1}, \dots, s_{(\mathbf{w} \oplus \mathbf{v}) \oplus \mathbf{u}_m})$$

and so there exists some \mathbf{u}_j such that $s_{\mathbf{w} \oplus \mathbf{u}_j} \neq s_{(\mathbf{w} \oplus \mathbf{v}) \oplus \mathbf{u}_j}$, i.e., $s_{\mathbf{w} \oplus \mathbf{u}_j} \neq s_{(\mathbf{w} \oplus \mathbf{u}_j) \oplus \mathbf{v}}$, which contradicts the assumption that $\mathbf{s} \in S_{n \times n}(\mathbf{v})$. \square

Theorem II.4 If $\mathbf{s} \in S_{n \times n}(\mathbf{v})$ then $\Phi(\mathbf{s}) \in S_{n \times n}(\mathbf{v})$ for any uniform global transition rule Φ .

Proof Suppose $\mathbf{q} = \Phi(\mathbf{s})$ is not symmetric by \mathbf{v} . Then, there exists $\mathbf{u} \in \mathbb{Z}^n \times \mathbb{Z}^n$, such that $q_{\mathbf{u}} \neq q_{\mathbf{u} \oplus \mathbf{v}}$. By Lemma II.3, $\eta_{\mathbf{u}}(\mathbf{s}) = \eta_{\mathbf{u} \oplus \mathbf{v}}(\mathbf{s})$, and so

$$q_{\mathbf{u}} = \phi(\eta_{\mathbf{u}}(\mathbf{s})) = \phi(\eta_{\mathbf{u} \oplus \mathbf{v}}(\mathbf{s})) = q_{\mathbf{u} \oplus \mathbf{v}},$$

which is a contradiction. \square

Corollary II.5 If $\mathbf{s} \in S_{n \times n}(\mathbf{v})$ and $\mathbf{q} \notin S_{n \times n}(\mathbf{v})$ then a non-symmetric configuration \mathbf{q} is unreachable from a shift-symmetric configuration \mathbf{s} for any uniform global transition rule Φ , i.e., for $\forall i \in \mathbb{N}$, $\Phi^i(\mathbf{s}) \neq \mathbf{q}$, where $\Phi^i(\mathbf{s})$ denotes i applications of Φ on \mathbf{s} .

Proof By induction $\Phi^i(\mathbf{s}) \in S_{n \times n}(\mathbf{v}) \not\equiv \mathbf{q}$. \square

Corollary II.6 Leader election from a symmetric square configuration \mathbf{s} is impossible for $n > 1$.

Proof A target configuration \mathbf{q} for leader election contains exactly one cell in the leader state $a \in \Sigma$. This configuration is asymmetric for $n > 1$, and therefore unreachable from a shift-symmetric configuration \mathbf{s} defined by any vector \mathbf{v} . \square

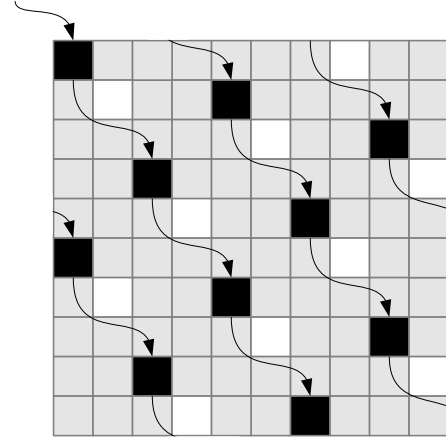


FIG. 4. Schematic of a shift-symmetric two-dimensional configuration generated by the vector $v = (3, 4)$ on $Z_{10 \times 10}$.

III. ENUMERATING SHIFT-SYMMETRIC CONFIGURATIONS

In this section we will further investigate shift-symmetric two-dimensional configurations and ask how many there are in a square lattice of size $N = n^2$. First, to generalize shift-symmetry and lay a solid ground for group-centric analysis we define the symmetric configurations over several generators. Then, we construct effective generators using prime factors of n and prove they are mutually independent. Finally, we enumerate shift-symmetric configurations using inclusion-exclusion principle as stated in Lemma III.10 and III.11, and conclude in the final formula in Theorem III.12.

Definition III.1 Let $\mathbb{L} \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$. We define the set of \mathbb{L} -symmetric configurations to be the set

$$S_{n \times n}(\mathbb{L}) = \{\mathbf{s} \in \Sigma^{n \times n} \mid \forall \mathbf{u} \in \mathbb{Z}_n \times \mathbb{Z}_n, \forall \mathbf{v} \in \langle \mathbb{L} \rangle : s_{\mathbf{u}} = s_{\mathbf{u} \oplus \mathbf{v}}\},$$

where $\langle \mathbb{L} \rangle = \{c_1 \mathbf{v}_1 \oplus \dots \oplus c_{|\mathbb{L}|} \mathbf{v}_{|\mathbb{L}|} \mid c_i \in \mathbb{Z}_n\}$. In other words, $S_{n \times n}(\mathbb{L})$ denotes the set of all shift-symmetric square configurations of size $N = n^2$ over the alphabet Σ with generator set \mathbb{L} . \square

Corollary III.1 For any subset $\mathbb{L} \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$,

$$|S_{n \times n}(\mathbb{L})| = |\Sigma|^{\frac{n^2}{|\langle \mathbb{L} \rangle|}}.$$

Proof Similar to Lemma II.2(i). \square

Corollary III.2 For any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$

$$S_{n \times n}(\mathbf{u}) \cap S_{n \times n}(\mathbf{v}) = S_{n \times n}(\{\mathbf{u}, \mathbf{v}\}).$$

Proof Immediate from Definition III.1. \square

Lemma III.3 For any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$, the following hold:

- (i). $|\langle \mathbf{u}, \mathbf{v} \rangle| = \frac{|\langle \mathbf{u} \rangle| |\langle \mathbf{v} \rangle|}{|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle|}$.
- (ii). $|S_{n \times n}(\mathbf{u}) \cap S_{n \times n}(\mathbf{v})| = |\Sigma| \frac{n^2 |\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle|}{|\langle \mathbf{u} \rangle| |\langle \mathbf{v} \rangle|}$.
- (iii). $|S_{n \times n}(\mathbf{u}) \cup S_{n \times n}(\mathbf{v})| = |\Sigma| n^2 \left(\frac{1}{|\langle \mathbf{u} \rangle|} + \frac{1}{|\langle \mathbf{v} \rangle|} - \frac{|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle|}{|\langle \mathbf{u} \rangle| |\langle \mathbf{v} \rangle|} \right)$.

Proof (i). By definition, $\langle \mathbf{u}, \mathbf{v} \rangle = \{c_1 \mathbf{u} + c_2 \mathbf{v} \mid c_1, c_2 \in \mathbb{Z}_n\}$, hence there are $|\langle \mathbf{u} \rangle| |\langle \mathbf{v} \rangle|$ selections of vectors from $\langle \mathbf{u} \rangle$ and $\langle \mathbf{v} \rangle$. However, each vector from $\langle \mathbf{u}, \mathbf{v} \rangle$ is included $|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle|$ times.

- (ii). Immediate from (i) and Corollary III.2.
- (iii). Inclusion-exclusion using (ii). \square

Lemma III.4 For any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$

$$S_{n \times n}(\mathbf{u}) \subseteq S_{n \times n}(\mathbf{v}) \iff \langle \mathbf{v} \rangle \leq \langle \mathbf{u} \rangle.$$

Proof (\Rightarrow). Suppose $S_{n \times n}(\mathbf{u}) \subseteq S_{n \times n}(\mathbf{v})$. Then

$$S_{n \times n}(\mathbf{u}) = S_{n \times n}(\mathbf{u}) \cap S_{n \times n}(\mathbf{v}) = S_{n \times n}(\{\mathbf{u}, \mathbf{v}\})$$

by Corollary III.2. But then $|\langle \mathbf{u} \rangle| = |\langle \mathbf{u}, \mathbf{v} \rangle|$ by Corollary III.1, which forces $\langle \mathbf{u} \rangle = \langle \mathbf{u}, \mathbf{v} \rangle$, so that $\mathbf{v} \in \langle \mathbf{u} \rangle$ and $\langle \mathbf{v} \rangle \leq \langle \mathbf{u} \rangle$ as desired.

(\Leftarrow). By way of contradiction, suppose that $S_{n \times n}(\mathbf{u}) \not\subseteq S_{n \times n}(\mathbf{v})$ and $\langle \mathbf{v} \rangle \leq \langle \mathbf{u} \rangle$. Let $\mathbf{s} \in S_{n \times n}(\mathbf{u})$ such that $\mathbf{s} \notin S_{n \times n}(\mathbf{v})$. Then \mathbf{s} is symmetric under \mathbf{u} but not under \mathbf{v} . Consequently, there exists $\mathbf{w} \in \mathbb{Z}_n \times \mathbb{Z}_n$ such that $s_{\mathbf{w}} \neq s_{\mathbf{w} \oplus \mathbf{v}}$. But $\mathbf{s} \in S_{n \times n}(\mathbf{u})$ and $\mathbf{v} \in \langle \mathbf{u} \rangle$ by assumption, so Lemma II.1 implies that $s_{\mathbf{w}} = s_{\mathbf{w} \oplus \mathbf{v}}$, which is a contradiction. \square

Definition III.2 We denote by $S_{n \times n}$ the set of all square shift-symmetric configurations of length $N = n^2$ over the alphabet Σ , so that

$$S_{n \times n} = \bigcup_{\mathbf{0} \neq \mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n} S_{n \times n}(\mathbf{v}).$$

\square

Lemma III.5 For any prime p that divides n and any i ($0 \leq i < n$), the cyclic group $\langle (\frac{n}{p}, i \frac{n}{p}) \rangle$ is simple, i.e., it has no nontrivial proper subgroups.

Proof By Lemma II.2(ii), we see that $\langle (\frac{n}{p}, i \frac{n}{p}) \rangle$ has order p , and by Lagrange's Theorem, any group with prime order is simple. \square

Remark: By swapping the coordinates, the proof applies also to each subgroup of the form $\langle (i \frac{n}{p}, \frac{n}{p}) \rangle$.

Definition III.3 Fix any natural number n and let $n = \prod_{j=1}^{\omega(n)} p_j^{\alpha_j}$ be the prime factorization of n , where $\omega(n)$ denotes the number of distinct prime factors. For each prime divisor p_j we define

$$G_n(p_j) = \left\{ \left(0, \frac{n}{p_j} \right) \right\} \cup \left\{ \left(\frac{n}{p_j}, i \frac{n}{p_j} \right) : 0 \leq i \leq p_j - 1 \right\},$$

and we define $G_n = \bigcup_{j=1}^{\omega(n)} G_n(p_j)$. \square

Corollary III.6 For any natural number n ,

$$|G_n| = \omega(n) + \sum_{i=1}^{\omega(n)} p_i.$$

Proof Immediate from Definition III.3. \square

Lemma III.7 Fix any natural number n and let $n = \prod_{j=1}^{\omega(n)} p_j^{\alpha_j}$ be the prime factorization of n , where $\omega(n)$ denotes the number of distinct prime factors. Then

$$S_{n \times n} = \bigcup_{\mathbf{w} \in G_n} S_{n \times n}(\mathbf{w}),$$

where G_n is defined as in Definition III.3.

Proof See Appendix A. \square

Lemma III.8 Fix any $n \in \mathbb{N}$. For any distinct $\mathbf{u}, \mathbf{v} \in G_n$,

$$|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle| = 1.$$

Proof See Appendix A. \square

Lemma III.9 Fix any $n \in \mathbb{N}$ and any prime divisor p of n . Let $\hat{n} = n/p$. Then for any distinct $\mathbf{u}, \mathbf{v} \in G_n(p)$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (\hat{n}, 0), (0, \hat{n}) \rangle.$$

In particular, $|\langle \mathbf{u}, \mathbf{v} \rangle| = p^2$.

Proof See Appendix A. \square

Definition III.4 Given any $\mathbf{v}, \mathbf{w} \in \mathbb{Z}^k$, we write $\mathbf{v} \trianglelefteq \mathbf{w}$ whenever the coordinates satisfy $v_i \leq w_i$ for every i ($1 \leq i \leq k$). We write $\mathbf{v} \triangleleft \mathbf{w}$ if $\mathbf{v} \trianglelefteq \mathbf{w}$ and $\mathbf{v} \neq \mathbf{w}$. We denote the sum of the coordinates by $|\mathbf{v}| = \sum_{i=1}^k v_i$, and for any $m \in \mathbb{Z}$, we write \mathbf{m} for the k -tuple whose coordinates all equal m . \square

Lemma III.10 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| = \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{p}+1} (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k \binom{p_i + 1}{v_i} |\Sigma|^{f(\mathbf{v})},$$

where $\mathbf{p} = (p_1, \dots, p_k)$ and $f(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-\min(v_i, 2)}$.

Proof See Appendix A. \square

Lemma III.11 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then an alternative counting of $|S_{n \times n}|$ is

$$|S_{n \times n}| = \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} |\Sigma|^{g(\mathbf{v})} \left(\sum_{\mathbf{v} \triangleleft \mathbf{u} \triangleleft \text{top}(\mathbf{v})} (-1)^{1+|\mathbf{u}|} \prod_{i=1}^k \binom{p_i + 1}{u_i} \right)$$

where $g(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-v_i}$ and $\text{top}(\mathbf{v}) \in \mathbb{Z}^k$ has i th coordinate

$$\text{top}(i) = \begin{cases} v_i & \text{if } v_i < 2 \\ p_i + 1 & \text{if } v_i = 2. \end{cases}$$

Proof See Appendix A. \square

Theorem III.12 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} \prod_{i=1}^k r(i)$$

where $g(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-v_i}$ and

$$r(i) = \begin{cases} 1 & \text{if } v_i = 0 \\ p_i + 1 & \text{if } v_i = 1 \\ p_i & \text{if } v_i = 2. \end{cases}$$

Proof See Appendix A. \square

Corollary III.13 Let $n = p$, where p is a prime. Then

$$|S_{n \times n}| = |\Sigma|^n (n+1) - |\Sigma|n$$

Proof For $\mathbf{v} = (1)$, $g(\mathbf{v}) = n$ and $b(\mathbf{v}) = (n+1)$, and for $\mathbf{v} = (2)$, $g(\mathbf{v}) = 1$ and $b(\mathbf{v}) = -n$. \square

The alternative counting method presented in Lemma III.11 is more efficient than the method from Lemma III.10 because of the grouping of the exponential elements, which are costly to calculate. The final formula presented in Theorem III.12 is the most efficient because, besides having the exponential elements grouped, it also reduces the inner binomial sum to a simple expression $r(i)$. This is illustrated for $n = 2^{\alpha_1} 3^{\alpha_2}$ in Appendix B.

Also, note that a one-by-one lattice offers no symmetries since there exists no non-zero shift in $\mathbb{Z}_1 \times \mathbb{Z}_1$.

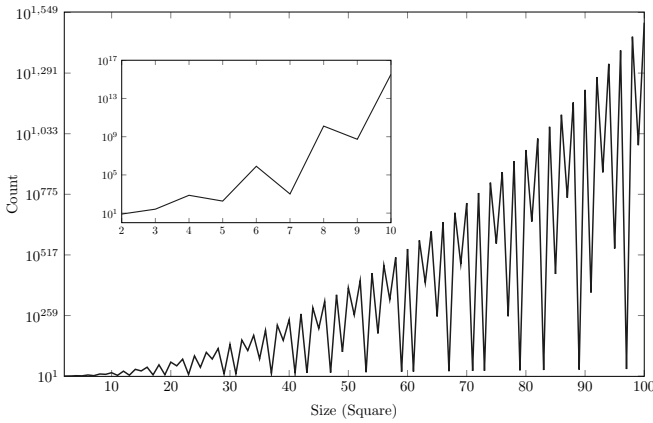


FIG. 5. Number of shift-symmetric two-dimensional binary ($|\Sigma| = 2$) configurations for the lattice sizes 2^2 to 100^2 with an inset focused on the area 2^2 to 10^2 . Note the local minima for prime and local maxima for even sizes.

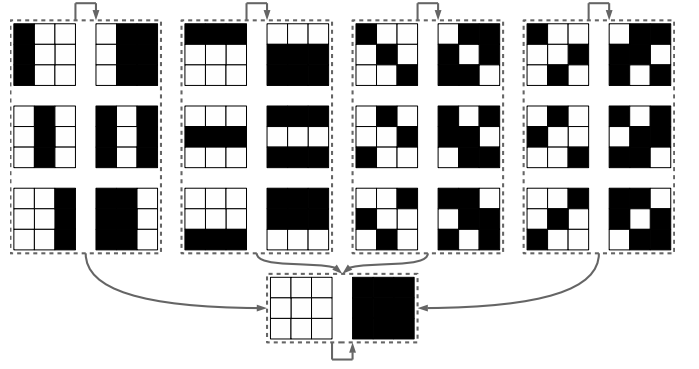


FIG. 6. All 26 binary shift-symmetric configurations for the lattice size 3^2 grouped into 5 classes based on the generating vector(s). The vectors are from left to right: $(0,1)$, $(1,0)$, $(1,1)$, $(1,2)$ and the one at the bottom containing all of them. The arrows show allowed transitions. Note that for prime-size binary lattices $|S_{n \times n}| = 2^n (n+1) - 2n$.

A. Bounding the Number of Shift-Symmetric Configurations

In the previous section we derived a closed and efficient formula for counting the number of shift-symmetric configurations in a square lattice $N = n^2$. To get a deeper and more qualitative insight we now bound this number from the top and the bottom. We prove that the lower bound is tight and reached only on prime lattices.

Lemma III.14 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n , and for each m ($1 \leq m < k$), let

$$q_{n \times n}^m = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} \prod_{i=1}^m r(i),$$

where $\mathbf{v} \in \mathbb{Z}^m$ and $g(\mathbf{v})$ and $r(i)$ are defined as before. Then

$$q_{n \times n}^m \leq q_{n \times n}^{m+1}.$$

Note that $|S_{n \times n}| = q_{n \times n}^k$.

Proof See Appendix A. \square

Lemma III.15

$$|\Sigma|^n (n+1) - |\Sigma|n \leq |S_{n \times n}|,$$

where equality holds if and only if n is a prime.

Proof If $k = 1$, i.e., n is a prime, the equality holds as shown in Corollary III.13. If $k > 1$ using Lemma III.14 and $p_1 < n$, $p_1 \leq \frac{n}{2}$

$$\begin{aligned}
|S_{n \times n}| &= q_{n \times n}^k \geq q_{n \times n}^{k-1} \geq \dots \geq q_{n \times n}^1 \\
&= |\Sigma|^{n^2 p_1^{-1}} (p_1 + 1) - |\Sigma|^{n^2 p_1^{-2}} p_1 \\
&> |\Sigma|^{n^2 n^{-1}} (n + 1) - |\Sigma|^{n^2 n^{-2}} n
\end{aligned}
\quad \square$$

Lemma III.16 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| \leq 2 \sum_{i=1}^k |\Sigma|^{n^2 p_i^{-1}} (p_i + 1).$$

Proof See Appendix A. \square

Lemma III.17 Let p be a prime divisor of n . Then

$$|\Sigma|^{n^2 p^{-1}} (p + 1) \leq |\Sigma|^{n^2 (p-1)^{-1}} p.$$

Proof Let $B = |\Sigma|^{n^2 p^{-1}}$. Then

$$\begin{aligned}
B^{\frac{p}{p-1}} p - B(p + 1) &= B(B^{\frac{1}{p-1}} p - (p + 1)) \\
&\geq B(|\Sigma|^{\frac{n}{p-1}} p - (p + 1)) \\
&\geq B(2^{\frac{n}{p-1}} p - (p + 1)) & |\Sigma| \geq 2 \\
&\geq B(2p - (p + 1)) & \frac{n}{p-1} \geq 1 \\
&\geq 0
\end{aligned}
\quad \square$$

Corollary III.18 Let p be a prime of n . Then

$$|\Sigma|^{n^2 p^{-1}} (p + 1) \leq 3 |\Sigma|^{\frac{n^2}{2}}.$$

\square

Lemma III.19 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| \leq 6 \log_2(n) |\Sigma|^{\frac{n^2}{2}}.$$

Proof By Lemma III.16 and Corollary III.18

$$|S_{n \times n}| \leq 2 \sum_{i=1}^k |\Sigma|^{n^2 p_i^{-1}} (p_i + 1) \leq 6k |\Sigma|^{\frac{n^2}{2}} \leq 6 \log_2(n) |\Sigma|^{\frac{n^2}{2}}$$

\square

Corollary III.20 The number of shift-symmetric configurations $|S_{n \times n}|$ satisfies

$$|\Sigma|^{n(n+1)} - |\Sigma|^n \leq |S_{n \times n}| \leq 6 \log_2(n) |\Sigma|^{\frac{n^2}{2}}.$$

Proof By Lemma III.15 and Lemma III.19. \square

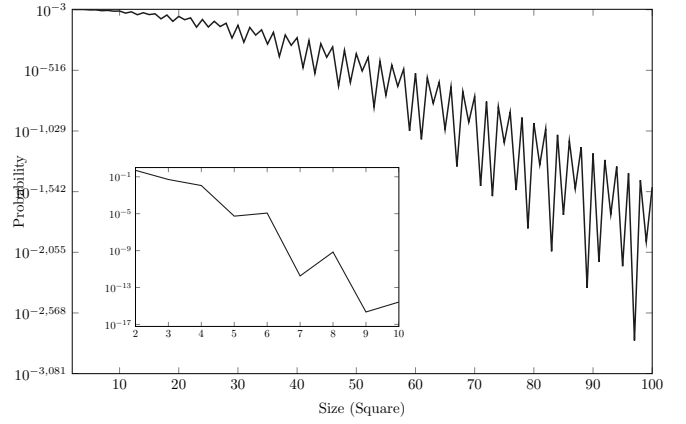


FIG. 7. Probability of selecting a shift-symmetric two-dimensional binary ($|\Sigma| = 2$) configuration using uniform distribution for the lattice sizes 2^2 to 100^2 with an inset focused on the area 2^2 to 10^2 . Note the local minima for prime and local maxima for even sizes ($n > 4$).

B. Probability of Selecting Shift-Symmetric Configuration over Uniform Distribution

To calculate the probability that a randomly drawn configuration is shift-symmetric, we use a uniform distribution, where the probability of selecting each symbol from Σ for s_i in configuration \mathbf{s} is the same. For non-symmetric tasks, this probability directly equals a least expected insolvability (or error lower bound).

Lemma III.21 The probability of selecting a shift-symmetric configuration in a square lattice of size $N = n^2$ over uniform distribution is

$$P_{n \times n}^{\text{unif}} = \frac{|S_{n \times n}|}{|\Sigma|^{n^2}}.$$

Proof Overall, there exist $|\Sigma|^{n^2}$ configurations and each configuration is equally likely. \square

Lemma III.22 The probability of selecting a shift-symmetric configuration in a square lattice of size $N = n^2$ over uniform distribution satisfies

$$n |\Sigma|^{-n^2+n} \leq P_{n \times n}^{\text{unif}} \leq 6 \log_2(n) |\Sigma|^{-\frac{n^2}{2}}.$$

Proof By Corollary III.20 and $n |\Sigma|^n \leq |\Sigma|^{n(n+1)} - |\Sigma|^n$. \square

As shown in Figure 7 and proved by Lemma III.22, the probability $P_{n \times n}^{\text{unif}}$ decreases rapidly: square-exponentially by n or exponentially by the lattice size $N = n^2$. Since $|S_{n \times n}|$ depends on the prime factorization of n the probability is non-monotonous. Similarly to $|S_{n \times n}|$ the probability $P_{n \times n}^{\text{unif}}$ reaches local minima for prime and local maxima for even lattices ($n > 4$).

IV. ENUMERATING SHIFT-SYMMETRIC CONFIGURATIONS FOR k ACTIVE CELLS

Having enumerated *all* shift-symmetric configurations we now tackle a subproblem of enumerating configurations with a specific number of cells in a given state, such as the state *active*. The motivation behind this endeavour is to calculate the probability of selecting a shift-symmetric configuration for a density-uniform distribution. Similarly to Section III we present three progressively more efficient counting techniques based on mutually-independent generators of prime factors of n in Lemma IV.3, Lemma IV.5, and Theorem IV.6.

Definition IV.1 For any state $a \in \Sigma$ and $n, k \in \mathbb{N}$, define $D_{n \times n, k}^a$ to be the set of all square configurations with exactly k sites in state a :

$$D_{n \times n, k}^a = \{\mathbf{s} \in \Sigma^{n \times n} \mid \#_a \mathbf{s} = k\}.$$

Accordingly, let $S_{n \times n, k}^a$ be the set of such configurations that are symmetric:

$$S_{n \times n, k}^a = S_{n \times n} \cap D_{n \times n, k}^a.$$

And for any $\mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$, let $S_{n \times n, k}^a(\mathbf{v})$ denote the set of configurations in $S_{n \times n, k}^a$ that are generated by \mathbf{v} , so that

$$S_{n \times n, k}^a(\mathbf{v}) = S_{n \times n}(\mathbf{v}) \cap D_{n \times n, k}^a.$$

□

Corollary IV.1 For any $a \in \Sigma$, any $n, k \in \mathbb{N}$, and $\mathbf{v} = (l_1, l_2) \in \mathbb{Z}_n \times \mathbb{Z}_n$,

$$S_{n \times n, k}^a(\mathbf{v}) \neq \emptyset$$

iff $|\langle \mathbf{v} \rangle| = \frac{n}{\gcd(l_1, l_2, n)}$ is an integer that divides k . □

Lemma IV.2 For any $a \in \Sigma$, any $k \in \mathbb{N}$, and $\mathbf{v} \in \mathbb{Z}_n \times \mathbb{Z}_n$ such that $|\langle \mathbf{v} \rangle|$ divides k

$$|S_{n \times n, k}^a(\mathbf{v})| = \binom{\frac{n^2}{|\langle \mathbf{v} \rangle|}}{\frac{k}{|\langle \mathbf{v} \rangle|}} (|\Sigma| - 1)^{\frac{n^2 - k}{|\langle \mathbf{v} \rangle|}}.$$

Proof Let $\mathbf{s} \in S_{n \times n, k}^a(\mathbf{v})$. Then the number of selections of state in \mathbf{s} , i.e., the pattern size, is $n^2/|\langle \mathbf{v} \rangle|$. To enumerate the number of such configurations, we first have to choose $k/|\langle \mathbf{v} \rangle|$ out of $n^2/|\langle \mathbf{v} \rangle|$ sites to be in state a , and then fill the remaining $n^2/|\langle \mathbf{v} \rangle| - k/|\langle \mathbf{v} \rangle|$ sites with states from $\Sigma \setminus \{a\}$. □

Lemma IV.3 Pick $n, k \in \mathbb{N}$ with $k \leq n$ and let $d = \gcd(k, n)$. Let $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, $k = \prod_{i=1}^{\omega(k)} q_i^{\beta_i}$, and $d = \prod_{i=1}^{\omega(d)} r_i^{\gamma_i}$ be the prime factorizations of n, k, d , respectively. Then for any $a \in \Sigma$,

$$|S_{n \times n, k}^a| = \sum_{\mathbf{0} < \mathbf{u} \leq \mathbf{r} + \mathbf{1}} (-1)^{1+|\mathbf{u}|} \left(\prod_{i=1}^{\omega(d)} \binom{r_i + 1}{u_i} \right) \binom{\frac{n^2}{h(\mathbf{u})}}{\frac{k}{h(\mathbf{u})}} (|\Sigma| - 1)^{\frac{n^2 - k}{h(\mathbf{u})}},$$

where $\mathbf{r} = (r_1, \dots, r_{\omega(d)})$ and $h(\mathbf{u}) = \prod_{i=1}^{\omega(d)} r_i^{\min(u_i, 2)}$.

Proof See Appendix A. □

Corollary IV.4 For any state set Σ and state $a \in \Sigma$, the set $S_{n \times n, 0}^a$ equals the set $S_{n \times n}$ for the state set $\Sigma \setminus \{a\}$. □

Lemma IV.5 Pick $n, k \in \mathbb{N}$ with $k \leq n$ and let $d = \gcd(k, n)$. Let $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, $k = \prod_{i=1}^{\omega(k)} q_i^{\beta_i}$, and $d = \prod_{i=1}^{\omega(d)} r_i^{\gamma_i}$ be the prime factorizations of n, k, d , respectively. Then for any $a \in \Sigma$,

$$|S_{n \times n, k}^a| = \sum_{\substack{\mathbf{0} < \mathbf{v} \leq \mathbf{2} \\ \mathbf{v} \leq \mathbf{u} \leq \text{top}(\mathbf{v})}} (-1)^{1+|\mathbf{u}|} \binom{\frac{n^2}{h(\mathbf{v})}}{\frac{k}{h(\mathbf{v})}} (|\Sigma| - 1)^{\frac{n^2 - k}{h(\mathbf{v})}} \prod_{i=1}^{\omega(d)} \binom{r_i + 1}{u_i},$$

where $h(\mathbf{v}) = \prod_{i=1}^{\omega(d)} r_i^{\min(v_i, 2)}$ and $\text{top}(\mathbf{v}) \in \mathbb{Z}^{\omega(d)}$ has i th coordinate

$$\text{top}(i) = \begin{cases} v_i & \text{if } v_i < 2 \\ r_i + 1 & \text{if } v_i = 2. \end{cases}$$

Proof Similar to the proof of Lemma III.11. □

Theorem IV.6 Pick $n, k \in \mathbb{N}$ with $k \leq n$ and let $d = \gcd(k, n)$. Let $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, $k = \prod_{i=1}^{\omega(k)} q_i^{\beta_i}$, and $d = \prod_{i=1}^{\omega(d)} r_i^{\gamma_i}$ be the prime factorizations of n, k, d , respectively. Then for any $a \in \Sigma$,

$$|S_{n \times n, k}^a| = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} (-1)^{1+|\mathbf{v}|} \binom{\frac{n^2}{h(\mathbf{v})}}{\frac{k}{h(\mathbf{v})}} (|\Sigma| - 1)^{\frac{n^2 - k}{h(\mathbf{v})}} \prod_{i=1}^{\omega(d)} r(i),$$

where $h(\mathbf{v}) = \prod_{i=1}^{\omega(d)} r_i^{\min(v_i, 2)}$ and

$$r(i) = \begin{cases} 1 & \text{if } v_i = 0 \\ p_i + 1 & \text{if } v_i = 1 \\ p_i & \text{if } v_i = 2. \end{cases}$$

Proof Similar to the proof of Theorem III.12. □

Corollary IV.7 The number of binary symmetric configurations ($|\Sigma| = 2$) with k sites in state a is given by

$$|S_{n \times n, k}^a| = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} (-1)^{1+|\mathbf{v}|} \binom{\frac{n^2}{h(\mathbf{v})}}{\frac{k}{h(\mathbf{v})}} \prod_{i=1}^{\omega(d)} r(i).$$

□

As in Section III, the alternative counting method presented in Lemma IV.5 is more efficient than the method from Lemma IV.3 due to the grouping of the exponential elements. The final counting Theorem IV.6 is the most efficient, since it further simplifies the formula by collapsing the inner binomial sum to a simple expression $r(i)$. This evolution is illustrated for $n = 2^{\alpha_1} 3^{\alpha_2}$ and $k = 2^{\beta_1} 3^{\beta_2}$, $\beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2$ in Appendix B.

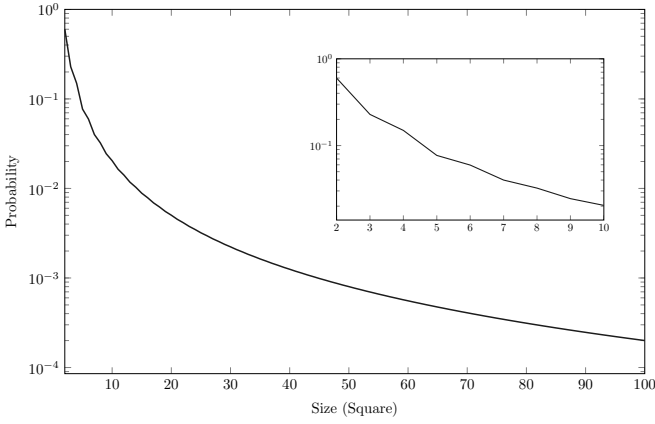


FIG. 8. Probability of selecting a shift-symmetric two-dimensional binary ($|\Sigma| = 2$) configuration using density-uniform distribution for the lattice sizes 2^2 to 100^2 with an inset focused on the area 2^2 to 10^2 .

A. Probability of Selecting Shift-Symmetric Configuration over Density-Uniform Distribution

Besides a uniform distribution, a CA's performance is commonly evaluated using a so-called density-uniform distribution. In a density-uniform distribution the probability of selecting k active cells ($\#_a \mathbf{s} = k$) is uniformly distributed, therefore, each *density* is equally likely.

Lemma IV.8 *The probability of selecting a shift-symmetric configuration in a square lattice of size $N = n^2$ over a density-uniform distribution is*

$$P_{n \times n}^{\text{dens}} = \frac{1}{n^2 + 1} \sum_{k=0}^{n^2} \frac{|S_{n \times n, k}^a|}{\binom{n^2}{k} (|\Sigma| - 1)^{n^2 - k}}.$$

Proof For a density $k \in \{0, \dots, n^2\}$ there exist $\binom{n^2}{k} (|\Sigma| - 1)^{n^2 - k}$ configurations and each density (the number of active cells) is equally likely. \square

As presented in Figure 8, the probability for density-uniform distribution decreases a magnitude slower than for the uniform one and reaches 0.001 even for $N = 45^2$. That is due to the fact that density-uniform distribution selects configurations with a few or many active cells, which are combinatorially more symmetric, more often.

V. DETECTING A SHIFT-SYMMETRIC CONFIGURATION

For practical reasons, e.g., to find whether a current system's configuration is shift-symmetric and take an action (restart), we provide an algorithm to effectively detect an occurrence of shift-symmetry.

First, to find out whether a configuration is shift-symmetric by a shift \mathbf{v} we start at a cell $\mathbf{w} = (0, 0)$

and check if all the cells at the orbit $\mathbf{w} \oplus i\mathbf{v}$ are in the same state. If yes, we repeat this process for the next orbit and so on, moving in arbitrary but fixed order (e.g., left-right up-down), until we check all the cells. If a cell has been visited before we skip it and move on until we find an unvisited cell, which marks a start of the next orbit. Also, if the test fails at any point, a configuration is non-shift-symmetric (by \mathbf{v}), and the process can be terminated. Otherwise, the property holds for all the cells and a configuration is shift-symmetric.

To determine whether a configuration is shift-symmetric globally, a naive way would be to try all possible non-zero vectors \mathbf{v} and check if any of them passes the aforementioned procedure. Luckily, as we discovered in Section II each configuration shift-symmetry “overlaps” with mutually-independent generators from G_n . Recall that these generators are defined by prime factors

$$G_n = \bigcup_{j=1}^{\omega(n)} G_n(p_j).$$

The number of these generators $|G_n| = \omega(n) + \sum_{i=1}^{\omega(n)} p_i$ is significantly smaller than n^2 . Now, we calculate the worst and average-case time complexity of the shift-symmetry test using the generators from G_n .

Theorem V.1 *The worst-case time complexity of the shift-symmetry detection algorithm for a square configuration of size $N = n^2$ is*

$$O(n^3).$$

Proof In a worst-case scenario, when a configuration is non-shift-symmetric and there is only one cell breaking symmetry, each test requires to visit potentially all n^2 cells. The overall worst-case time complexity is therefore $O(|G_n|n^2)$. We know that the sum of distinct prime factors $\text{sopf}(n) = \sum_{i=1}^{\omega(n)} p_i$ also known as the integer logarithm is at most n (if n is prime), which gives us

$$\begin{aligned} O(|G_n|n^2) &= O\left(\left(\omega(n) + \sum_{i=1}^{\omega(n)} p_i\right)n^2\right) \\ &= O((\log_2(n) + \text{sopf}(n))n^2) \\ &= O(n^3). \end{aligned} \quad \square$$

Theorem V.2 *The average-case time complexity of the shift-symmetry detection algorithm for a square configuration of size $N = n^2$ generated from a uniform distribution is*

$$O(n^2).$$

Proof See Appendix A. \square

We proved the worst and average-case time complexity of $O(n^3)$ and $O(n^2)$ respectively, which translate to $O(\sqrt{NN})$ and linear $O(N)$ when interpreted by the optics of the number of cells $N = n^2$. The function $\text{sopf}(n)$, which plays a crucial role in both O formulas, is of a logarithmic nature in “most of the cases,” but n for primes. Since the number of primes is infinite we could not use any tighter asymptote than n . However, for randomly chosen n we expect the time complexities to be just $O(\log(n)n^2)$ and $O(\log^2(n))$ respectively.

It is worth mentioning that the presented algorithm detects *if* a configuration is shift-symmetric but does not count the number of shift-symmetries in a configuration. The validity of the detection holds because we know that *any* shift-symmetric configuration must obey at least one of the prime generators from G_n . Nevertheless, to determine the number of shift-symmetries, i.e., the number of vectors with distinct vector spaces in $\mathbb{Z}_n \times \mathbb{Z}_n$ for which the cells at a same orbit share the same state, we would need to consider also sub-vectors, whose satisfiability cannot be generally inferred from the prime generators. Construction of a *counting* algorithm is addressable but goes beyond the scope of this paper.

VI. DISCUSSION AND CONCLUSION

We showed that shift-symmetry decreases the system’s computational capabilities and expressivity, and is generally good to be avoided. For each shift-symmetry, a system falls into, a configuration folds by the order of symmetry and “independent” computation shrinks to a smaller, prime fraction of the system. The rest of the system is mirrored and lacks any intrinsic computational value or novelty. The number of reachable configurations shrinks proportionally as well.

One of the key aspects of shift-symmetry is that it is maintained (irreversible) for any number of states, and any uniform transition and neighborhood functions. It means that the occurrence of shift-symmetry is rooted in the CA model itself, specifically, in the cells’ uniformity, synchronous update, and toroidal topology. Shift-symmetry is preserved as long as a transition function is uniform (shared among the cells), even if non-deterministic. In other words, during each step a transition function can be discarded and regenerated at random. However, within the same synchronous update it must be consistent, i.e., two cells whose neighborhood’s sub-configurations are the same must be transitioned to the same state.

We showed that a non-symmetric solution is unreachable from a shift-symmetric configuration, which renders the non-symmetric tasks, such as leader election [5, 46], several image processing routines including pattern recognition [41], and encryption [49], insolvable in a general sense. These non-symmetric procedures are fundamental parts of many distributed protocols and algorithms. Additionally, leader election contributes to deci-

sion making of biological societies [14, 31], and is a key driver of cell differentiation [30, 37] responsible for structural heterogeneity and the specialization of cells.

To determine how likely a configuration randomly generated from a uniform distribution is shift-symmetric, hence insolvable, we efficiently enumerated and bounded the number of shift-symmetric configurations using mutually independent generators. We also introduced a lower, tight prime-size bound, and an upper bound, and showed that even-size lattices are locally most likely shift-symmetric. Overall, shift-symmetry is not as rare as one would think, especially for small or non-prime lattices, or when a configuration is generated using density-uniform distribution. Asymptotically, the probability for uniform distribution drops exponentially with the lattice size but a magnitude slower for a density-uniform distribution. For instance the probability for a 100^2 square lattice is around 10^{-1505} using uniform and 2×10^{-4} using density-uniform distribution.

To detect whether a configuration is shift-symmetric we constructed an algorithm, which, by using the base prime generators, can effectively determine a presence of shift-symmetry in linear $O(N)$ time for prime and just $O((\frac{1}{2} \log(N))^2)$ for randomly chosen N on average.

Further, we need to emphasize that shift-symmetry does not necessarily have to be harmful for all the tasks. For instance, the density classification [12, 16, 36, 40], which is widely used as a CA benchmark problem, requires a final configuration to be either 1^N if the majority of cells are initially in the state 1, and 0^N otherwise. Since the expected homogeneous configurations are fully shift-symmetric, they can be reached potentially from any configuration. Naturally, that depends on the structure of a transition function but shift-symmetry does not impose any strong restrictions here. The ability of reaching a *valid* answer does not mean reaching a *correct* answer. However, for the density classification, shift-symmetry tolerates the latter as well. It is because a shift-symmetric configuration consists purely of repeated sub-configurations, and so the density (ratio of ones) in a sub-configuration is the same as in the whole.

By moving from one to two dimensions we generalized our machinery to vector translations, which can be extended to the n -dimensional case [10]. It is expected that the number of shift-symmetric configurations will grow with the dimensionality of lattice. It will be interesting to investigate this relation from the perspective of prime-exponent divisors.

An important implication of shift-symmetry is that cyclic behavior must occur only within the same symmetry class defined by a set of prime shifts (vectors) as illustrated in Figure 6. Note that we count no-symmetry as a class as well. This leads to the realization that once a CA gains a symmetry, i.e., a configuration crosses symmetry classes, it cannot be injective and reversible, and there must exist a configuration without a predecessor, a so-called “Garden of Eden” configuration [1, 27]. It means that the only way for the CA to stay injective

is to decompose all the configurations into cycles, each fully residing in a certain shift-symmetry class. Again one large class would contain all the non-shift-symmetric configurations. Open question is for which lattices, i.e., for how many shift-symmetric configurations, CAs are non-injective, thus irreversible, on average. As opposed to our shift-symmetric endeavour, which applies to any transition function, investigating injectivity would require to assume something about the transition function, e.g., that is generated randomly. Trivially, for any lattice there always exists an injective transition function. An example is an identity function.

As shown, the number of symmetries in any synchronous toroidal CA is non-decreasing but could it be increasing in the “average” case for a random transition function? We know that the expected behavior of randomly generated CA is most likely chaotic and the attractor length is exponential to the lattice size N , as opposed to ordered or complex CAs with linear or quadratic attractors [55]. Would the length of attractor be sufficient to discover a shift-symmetry if we keep a random CA running long enough, potentially $|\Sigma|^N$ time steps? As seen in Figure 7, the ratio of shift-symmetric configurations assuming a uniform distribution is exponentially decreasing with the lattice size, and prime lattices could produce “only” around $n|\Sigma|^n$ symmetric configurations. For a randomly chosen lattice size, dimensions, and cell connectivity, we expect the number of reachable symmetries to be significantly smaller than the total number of symmetries available. However, for symmetry-rich lattices, we speculate that toroidal synchronous uniform systems, such as CAs, could undergo *spontaneous symmetrization* contracting an initial configuration to a fully homogeneous state (reverse Big Bang). If proven, it would directly imply the system’s non-injectivity and irreversibility, and would bind symmetrization with Gibbs entropy. This hypothesis will be addressed in our future work.

We suggest that several phenomena observed in CA dynamics, such as irreversibility, emergence of structured “patterns”, and self-organization could be explained or contributed to shift-symmetry. As demonstrated by Wolfram [51] on 256 elementary one-dimensional CAs, when run long enough, most of these CAs condensate to ordered structures: homogeneous configurations and self-similar patterns, which are in fact shift-symmetric.

A straightforward way to fight symmetry would be to introduce noise, i.e., to break the uniformity of cells and/or to use an asynchronous update. Based on the amount of noise, this could, however, disrupt the consistency of local, particle-based, interactions, which give rise to a global computation. Clearly, asynchronicity makes a system more robust but sacrifices the information processing by *algebraic* structures, which could exist only due to synchronous update. Using our enumeration formulas and probability calculations we could in principle minimize a desired shift-symmetry insolvability and the number of resources needed based on distributed application requirements.

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Appendix A: Proofs

Lemma III.7 Fix any natural number n and let $n = \prod_{j=1}^{\omega(n)} p_j^{\alpha_j}$ be the prime factorization of n , where $\omega(n)$ denotes the number of distinct prime factors. Then

$$S_{n \times n} = \bigcup_{\mathbf{w} \in G_n} S_{n \times n}(\mathbf{w}),$$

where G_n is defined as in Definition III.3.

Proof (\subseteq). Let $\mathbf{s} \in S_{n \times n}$, so that $\mathbf{s} \in S_{n \times n}(\mathbf{v})$ for some nonzero $\mathbf{v} = (a, b) \in \mathbb{Z}_n \times \mathbb{Z}_n$. It suffices to show that $\langle \mathbf{w} \rangle \leq \langle \mathbf{v} \rangle$ for some $\mathbf{w} \in G_n$, since this fact, by Lemma III.4, implies $S_{n \times n}(\mathbf{v}) \subseteq S_{n \times n}(\mathbf{w})$ and therefore $\mathbf{s} \in S_{n \times n}(\mathbf{w})$.

Without loss of generality, we may assume $\gcd(a, b, n) = 1$. Otherwise, we simply divide everything by $d = \gcd(a, b, n)$ to obtain $\hat{\mathbf{v}} = (\hat{a}, \hat{b})$, and \hat{n} , respectively. Once we show that $\langle \hat{\mathbf{w}} \rangle \leq \langle \hat{\mathbf{v}} \rangle$ for some $\hat{\mathbf{w}} \in G_{\hat{n}}$, we multiply throughout by d to obtain the desired result.

Case 1. Suppose $\gcd(a, n) = 1$. Then $ai \equiv_n b$ and $aj \equiv_n 1$ for some $i, j \in \mathbb{Z}$. Also, $n\mathbf{v} \equiv_n (0, 0)$, so $|\langle \mathbf{v} \rangle|$ divides n . Let p be any prime divisor of $|\langle \mathbf{v} \rangle|$ and write $n = pm$ for some $m \in \mathbb{Z}$. Let $\mathbf{w} = (m, im)$ and note that $\mathbf{w} \in G_n(p)$. Also observe $\mathbf{v} = a(1, i)$ and $\mathbf{w} = m(1, i)$, so that

$$mj\mathbf{v} = mja(1, i) = m(1, i) = \mathbf{w}.$$

Therefore $\mathbf{w} \in \langle \mathbf{v} \rangle$ and thus $\langle \mathbf{w} \rangle \leq \langle \mathbf{v} \rangle$ as desired.

Case 2. Suppose $\gcd(a, n) \neq 1$. Let p be any prime divisor of both a and n , so that $a = pa'$ and $n = pn'$ for some $a', n' \in \mathbb{Z}$. Let $\mathbf{w} = (0, n')$ and note that $\mathbf{w} \in G_n(p)$. Observe that $an' = a'pn' = a'n \equiv_n 0$, so

$$n'\mathbf{v} = n'(a, b) = (an', bn') = (0, bn') = b\mathbf{w}.$$

Therefore $n'\mathbf{v} \in \langle \mathbf{w} \rangle$. But by Lemma II.2(ii), $|\langle \mathbf{w} \rangle| = p$, a prime. So if $n'\mathbf{v}$ is nonzero, then it generates $\langle \mathbf{w} \rangle$. But $n'\mathbf{v}$ is indeed nonzero, since its second coordinate is bn' , and if $bn' \equiv_n 0$, then $n|bn'$. Dividing by n' , we see $p|b$. But recall that p divides a and n , and we assumed at the beginning (without loss of generality) that $\gcd(a, b, n) = 1$. So p cannot divide b . This contradiction shows $n'\mathbf{v}$ is nonzero and so $n'\mathbf{v}$ generates $\langle \mathbf{w} \rangle$. Thus

$$\mathbf{w} \in \langle \mathbf{w} \rangle = \langle n'\mathbf{v} \rangle \leq \langle \mathbf{v} \rangle.$$

So $\langle \mathbf{w} \rangle \leq \langle \mathbf{v} \rangle$ as desired.

(\supseteq). Immediate by Definition III.2. \square

Lemma III.8 Fix any $n \in \mathbb{N}$. For any distinct $\mathbf{u}, \mathbf{v} \in G_n$,

$$|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle| = 1. \quad (\text{A1})$$

Proof First, suppose that $\mathbf{u} \in G_n(p)$ and $\mathbf{v} \in G_n(q)$, where $p \neq q$. By Lemma II.2(i), $|\langle \mathbf{u} \rangle| = p$ and $|\langle \mathbf{v} \rangle| = q$.

Since $|\langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle|$ must divide both of these primes, line (A1) must hold as claimed.

Next, suppose $\mathbf{u}, \mathbf{v} \in G_n(p)$ and write $n = \hat{n}p$ for some $\hat{n} \in \mathbb{Z}$. Suppose $\mathbf{u} = (\hat{n}, i\hat{n})$ and $\mathbf{v} = (\hat{n}, j\hat{n})$ for some $0 \leq i < j < p$. If $\mathbf{x} \in \langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle$ then $\exists k, l$ ($0 \leq k, l < p$) such that $\mathbf{x} = k\mathbf{u} = l\mathbf{v}$. But then $(k\hat{n}, ki\hat{n}) = (l\hat{n}, lj\hat{n})$, so $k\hat{n} \equiv_n l\hat{n}$ and thus $k \equiv_p l$. But also, $ki\hat{n} \equiv_n lj\hat{n}$, so that $ki \equiv_p lj$. Since $i \not\equiv_p j$, this forces $k \equiv_p 0$, so that $\mathbf{x} = 0$ and (A1) must hold as claimed.

Finally, suppose $\mathbf{u}, \mathbf{v} \in G_n(p)$ and suppose $\mathbf{u} = (0, \hat{n})$ and $\mathbf{v} = (\hat{n}, i\hat{n})$ for some $0 \leq i < p$. If $\mathbf{x} \in \langle \mathbf{u} \rangle \cap \langle \mathbf{v} \rangle$ then $\exists k, l$ ($0 \leq k, l < p$) such that $\mathbf{x} = k\mathbf{u} = l\mathbf{v}$. But then $(0, k\hat{n}) = (l\hat{n}, li\hat{n})$, so $0 \equiv_n l\hat{n}$ and thus $0 \equiv_p l$. But also, $k\hat{n} \equiv_n li\hat{n}$, so that $k \equiv_p li$ and therefore $k \equiv_p 0$. Now $\mathbf{x} = 0$ and (A1) must hold as claimed. \square

Lemma III.9 Fix any $n \in \mathbb{N}$ and any prime divisor p of n . Let $\hat{n} = n/p$. Then for any distinct $\mathbf{u}, \mathbf{v} \in G_n(p)$,

$$\langle \mathbf{u}, \mathbf{v} \rangle = \langle (\hat{n}, 0), (0, \hat{n}) \rangle.$$

In particular, $|\langle \mathbf{u}, \mathbf{v} \rangle| = p^2$.

Proof (\subseteq). First suppose $\mathbf{u} = (\hat{n}, i\hat{n})$ and $\mathbf{v} = (\hat{n}, j\hat{n})$ for some $0 \leq i < j < p$. Then $\mathbf{u} = (\hat{n}, 0) + i(0, \hat{n})$ and $\mathbf{v} = (\hat{n}, 0) + j(0, \hat{n})$. So $\langle \mathbf{u}, \mathbf{v} \rangle \subseteq \langle (\hat{n}, 0), (0, \hat{n}) \rangle$ as desired. A similar argument holds when $\mathbf{u} = (\hat{n}, i\hat{n})$ and $\mathbf{v} = (0, \hat{n})$.

(\supseteq). Again suppose $\mathbf{u} = (\hat{n}, i\hat{n})$ and $\mathbf{v} = (\hat{n}, j\hat{n})$ for some $0 \leq i < j < p$. Then $\mathbf{u} - \mathbf{v} \in \langle (0, \hat{n}) \rangle$. But $\mathbf{u} - \mathbf{v} \neq 0$ and $|\langle (0, \hat{n}) \rangle| = p$, so $\mathbf{u} - \mathbf{v}$ generates $\langle (0, \hat{n}) \rangle$. Thus $(0, \hat{n}) \in \langle \mathbf{u} - \mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$. Likewise, $(\hat{n}, 0) \in \langle j\mathbf{u} - i\mathbf{v} \rangle \subseteq \langle \mathbf{u}, \mathbf{v} \rangle$, so the desired containment holds. A similar argument can be made when $\mathbf{u} = (\hat{n}, i\hat{n})$ and $\mathbf{v} = (0, \hat{n})$, showing that $(\hat{n}, 0) \in \langle \mathbf{u} - i\mathbf{v} \rangle$, which implies the desired result. \square

Lemma III.10 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| = \sum_{0 < \mathbf{v} \leq \mathbf{p}+1} (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k \binom{p_i + 1}{v_i} |\Sigma|^{f(\mathbf{v})},$$

where $\mathbf{p} = (p_1, \dots, p_k)$ and $f(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-\min(v_i, 2)}$.

Proof By Lemma III.7, inclusion-exclusion, and Corollary III.2,

$$|S_{n \times n}| = \left| \bigcup_{\mathbf{w} \in G_n} S_{n \times n}(\mathbf{w}) \right| = \sum_{\emptyset \neq J \subseteq G_n} (-1)^{|J|+1} |S_{n \times n}(J)|.$$

Since $G_n = \bigcup_{j=1}^k G_n(p_j)$, we have k sets from which to choose the elements of J , so

$$|S_{n \times n}| = \sum_{\substack{J_1 \subseteq G_n(p_1) \\ J_k \subseteq G_n(p_k)}} (-1)^{1+\sum_{i=1}^k |J_i|} \left| S_{n \times n} \left(\bigcup_{i=1}^k J_i \right) \right|,$$

where the sum excludes the case when $J_i = \emptyset$ for all i . It follows from Corollary III.2 that $S_{n \times n}(\bigcup J_i) = \bigcap S_{n \times n}(J_i)$ and so Corollary III.1 gives

$$\left| S_{n \times n} \left(\bigcup_{i=1}^k J_i \right) \right| = |\Sigma|^{\frac{n^2}{|\bigcup_{i=1}^k J_i|}}.$$

But by Lemma III.8 we know $|\langle J_i \rangle \cap \langle J_j \rangle| = 1$ when $i \neq j$, so

$$\left| \left\langle \bigcup_{i=1}^k J_i \right\rangle \right| = \prod_{i=1}^k |\langle J_i \rangle|.$$

Since $J_i \subseteq G_n(p_i)$, recall that $\langle J_i \rangle = \langle (\frac{n}{p_i}, 0), (0, \frac{n}{p_i}) \rangle$ when $|J_i| \geq 2$ by Lemma III.9. So $|\langle J_i \rangle| = 1, p_i,$ and p_i^2 when $|J_i| = 0, 1,$ and $\geq 2,$ respectively. Therefore

$$\prod_{i=1}^k |\langle J_i \rangle| = \prod_{i=1}^k p_i^{\min(|J_i|, 2)}$$

Substituting all this into the expression for $|S_{n \times n}|$, we obtain

$$|S_{n \times n}| = \sum_{\substack{J_1 \subseteq G_n(p_1) \\ J_k \subseteq \ddot{G}_n(p_k)}} (-1)^{1 + \sum_{i=1}^k |J_i|} \left(|\Sigma|^{\frac{n^2}{\prod_{i=1}^k p_i^{\min(|J_i|, 2)}}} \right)$$

Now, because the content of J_i is irrelevant and we care only about the cardinality $|J_i|$, for each size $v_i = |J_i|$ we have $\binom{|G_n(p_i)|}{v_i} = \binom{p_i+1}{v_i}$ ways of choosing v_i elements from $G_n(p_i)$, which produces the final formula as required. \square

Lemma III.11 *Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then an alternative counting of $|S_{n \times n}|$ is*

$$|S_{n \times n}| = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} |\Sigma|^{g(\mathbf{v})} \left(\sum_{\mathbf{v} \leq \mathbf{u} \leq \text{top}(\mathbf{v})} (-1)^{1+|\mathbf{u}|} \prod_{i=1}^k \binom{p_i+1}{u_i} \right)$$

where $g(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-v_i}$ and $\text{top}(\mathbf{v}) \in \mathbb{Z}^k$ has i th coordinate

$$\text{top}(i) = \begin{cases} v_i & \text{if } v_i < 2 \\ p_i + 1 & \text{if } v_i = 2. \end{cases}$$

Proof We know that the exponent of each p_i in $S_{n \times n}$ from Lemma III.10 is at most 2. Therefore for given $v_1, \dots, v_k \in \{0, 1, 2\}$ we can combine all binomial expressions associated with $|\Sigma|^{\frac{n^2}{\prod_{i=1}^k p_i^{v_i}}}$. If $v_i \leq 1$ then we have $\binom{p_i+1}{v_i}$ selections from $G_n(p_i)$, and $\bigcup_{u_i=2}^{p_i+1} \binom{p_i+1}{u_i}$ for $v_i = 2$. These two expressions could be generalized as $\bigcup_{u_i=v_i}^{\text{top}(i)} \binom{p_i+1}{u_i}$ using the *top* function defined above.

Therefore the total coefficient of $|\Sigma|^{\frac{n^2}{\prod_{i=1}^k p_i^{v_i}}}$ is

$$\sum_{\substack{v_1 \leq u_1 \leq \text{top}(1) \\ v_k \leq u_k \leq \text{top}(k)}} (-1)^{1+\sum_{i=1}^k u_i} \prod_{i=1}^k \binom{p_i+1}{u_i}$$

as required. \square

Theorem III.12 *Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then*

$$|S_{n \times n}| = \sum_{\mathbf{0} < \mathbf{v} \leq \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} \prod_{i=1}^k r(i)$$

where $g(\mathbf{v}) = n^2 \prod_{i=1}^k p_i^{-v_i}$ and

$$r(i) = \begin{cases} 1 & \text{if } v_i = 0 \\ p_i + 1 & \text{if } v_i = 1 \\ p_i & \text{if } v_i = 2. \end{cases}$$

Proof For a given vector \mathbf{v} with $v_1, \dots, v_k \in \{0, 1, 2\}$ we define

$$b(\mathbf{v}) = \sum_{\mathbf{v} \leq \mathbf{u} \leq \text{top}(\mathbf{v})} (-1)^{1+|\mathbf{u}|} \prod_{i=1}^k \binom{p_i+1}{u_i},$$

where $\text{top}(i) = v_i$ if $v_i < 2$ and $p_i + 1$ if $v_i = 2$.

Using Lemma III.11, we are left to show that

$$b(\mathbf{v}) = (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k r(i).$$

We prove it by induction on k . As the induction basis we choose $k = 1$, and so $n = p$, where p is a prime. Since $b(\mathbf{v}) = r(0)$ we need to confirm it equals $-1, p + 1,$ or $-p$ for three different cases of \mathbf{v} defined by the function r .

If $\mathbf{v} = (0)$, $\text{top}(\mathbf{v}) = (0)$ and the only \mathbf{u} is $\mathbf{u} = (0)$, which gives $b(\mathbf{v}) = -\binom{p+1}{0} = -1$. If $\mathbf{v} = (1)$, $\text{top}(\mathbf{v}) = (1)$ and the only \mathbf{u} is $\mathbf{u} = (1)$, and so $b(\mathbf{v}) = \binom{p+1}{1} = p + 1$. If $\mathbf{v} = (2)$, $\text{top}(\mathbf{v}) = (p + 1)$ and \mathbf{u} ranges from (2) to $(p + 1)$. Therefore

$$\begin{aligned} b(\mathbf{v}) &= \sum_{u_1=2}^{p+1} (-1)^{1+u_1} \binom{p+1}{u_1} \\ &= \underbrace{\sum_{u_1=0}^{p+1} (-1)^{1+u_1} \binom{p+1}{u_1}}_0 + \binom{p+1}{0} - \binom{p+1}{1} \\ &= -p \end{aligned}$$

For the induction step we prove:

$$b(\mathbf{v}) = (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k r(i) \Rightarrow b(\mathbf{w}) = (-1)^{1+|\mathbf{w}|} \prod_{i=1}^{k+1} r(i),$$

where $\mathbf{w} = (v_1, \dots, v_k, v_{k+1})$. Similarly to the induction basis we need to consider three cases for v_{k+1} :

If $v_{k+1} = 0$

$$\begin{aligned} b(\mathbf{w}) &= \binom{p_{k+1} + 1}{0} \underbrace{\sum_{\mathbf{v} \leq \mathbf{u} \leq \text{top}(\mathbf{v})} (-1)^{1+|\mathbf{u}|} \prod_{i=1}^k \binom{p_i + 1}{u_i}}_{b(\mathbf{v})} \\ &= (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k r(i) \quad \text{by induction step} \\ &= (-1)^{1+|\mathbf{w}|} \prod_{i=1}^{k+1} r(i) \end{aligned}$$

If $v_{k+1} = 1$

$$\begin{aligned} b(\mathbf{w}) &= -\binom{p_{k+1} + 1}{1} \underbrace{\sum_{\mathbf{v} \leq \mathbf{u} \leq \text{top}(\mathbf{v})} (-1)^{1+|\mathbf{u}|} \prod_{i=1}^k \binom{p_i + 1}{u_i}}_{b(\mathbf{v})} \\ &= -(p_{k+1} + 1)(-1)^{1+|\mathbf{v}|} \prod_{i=1}^k r(i) \quad \text{by induction step} \\ &= (-1)^{1+|\mathbf{w}|} \prod_{i=1}^{k+1} r(i) \end{aligned}$$

If $v_{k+1} = 2$

$$\begin{aligned} b(\mathbf{w}) &= \underbrace{\sum_{u_{k+1}=2}^{p_{k+1}+1} (-1)^{u_{k+1}} \binom{p_{k+1} + 1}{u_{k+1}}}_{p_{k+1}} b(\mathbf{v}) \\ &= p_{k+1} (-1)^{1+|\mathbf{v}|} \prod_{i=1}^k r(i) \quad \text{by induction step} \\ &= (-1)^{1+|\mathbf{w}|} \prod_{i=1}^{k+1} r(i) \quad \square \end{aligned}$$

Lemma III.14 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n , and for each m ($1 \leq m < k$), let

$$q_{n \times n}^m = \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} \prod_{i=1}^m r(i),$$

where $\mathbf{v} \in \mathbb{Z}^m$ and $g(\mathbf{v})$ and $r(i)$ are defined as before. Then

$$q_{n \times n}^m \leq q_{n \times n}^{m+1}.$$

Note that $|S_{n \times n}| = q_{n \times n}^k$.

Proof Let $\mathbf{v} = (v_1, \dots, v_m)$ and $\mathbf{w} = (v_1, \dots, v_m, v_{m+1})$. Then

$$\begin{aligned} q_{n \times n}^{m+1} &= \sum_{\mathbf{0} \triangleleft \mathbf{w} \triangleleft \mathbf{2}} (-1)^{1+|\mathbf{w}|} |\Sigma|^{g(\mathbf{w})} \prod_{i=1}^{m+1} r(i) \\ &= \sum_{v_{m+1}=0}^2 \left((-1)^{v_{m+1}} r(m+1) \right. \\ &\quad \left. \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} p_{m+1}^{-v_{m+1}} \prod_{i=1}^m r(i) \right) \\ &\quad + \sum_{v_{m+1}=1}^2 (-1)^{1+v_{m+1}} |\Sigma|^{n^2} p_{m+1}^{-v_{m+1}} r(m+1) \end{aligned}$$

We split the expression into five parts:

$$q_{n \times n}^{m+1} = x_0 + x_1 + x_2 + y_1 + y_2$$

and define, for any $c \in \mathbb{R}$

$$q_{n \times n}^m(c) = \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} (-1)^{1+|\mathbf{v}|} |\Sigma|^{g(\mathbf{v})} c \prod_{i=1}^m r(i),$$

i.e., $q_{n \times n}^m = q_{n \times n}^m(1)$. Then

$$\begin{aligned} x_0 &= q_{n \times n}^m \\ x_1 &= -(p_{m+1} + 1) q_{n \times n}^m(p_{m+1}^{-1}) \\ x_2 &= p_{m+1} q_{n \times n}^m(p_{m+1}^{-2}) \\ y_1 &= |\Sigma|^{n^2} p_{m+1}^{-1} (p_{m+1} + 1) \\ y_2 &= -|\Sigma|^{n^2} p_{m+1}^{-2} p_{m+1} \end{aligned}$$

Now we show that

$$y_1 + x_1 + y_2 \geq 0$$

Let $A = n^2 p_{m+1}^{-1}$. Then $(y_1 + x_1 + y_2)(p_{m+1} + 1)^{-1}$

$$\begin{aligned} &\geq |\Sigma|^A - q_{n \times n}^m(p_{m+1}^{-1}) - |\Sigma|^{\frac{A}{p_{m+1}}} \\ &\geq |\Sigma|^A - \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} |\Sigma|^{g(\mathbf{v})} p_{m+1}^{-1} \prod_{i=1}^m r(i) - |\Sigma|^{\frac{A}{p_{m+1}}} \\ &\geq |\Sigma|^A - \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} |\Sigma|^{\frac{A}{p_l}} - |\Sigma|^{\frac{A}{p_{m+1}}} \quad p_l = \min\{p_1, \dots, p_m\} \\ &\geq |\Sigma|^A - 3^m |\Sigma|^{\frac{A}{p_l}} - |\Sigma|^{\frac{A}{p_{m+1}}} \quad |\mathbf{v}| = m \\ &\geq |\Sigma|^A - |\Sigma|^{2m} |\Sigma|^{\frac{A}{p_l}} - |\Sigma|^{\frac{A}{p_{m+1}}} \quad |\Sigma| \geq 2 \\ &\geq |\Sigma|^A - |\Sigma|^{2m + \frac{A}{p_l} + \frac{A}{p_{m+1}}} \\ &\geq |\Sigma|^A - |\Sigma|^{2 \log_2(n) + \frac{A}{p_l} + \frac{A}{p_{m+1}}} \quad m \leq k-1 < \log_2(n) \\ &\geq |\Sigma|^A - |\Sigma|^{2 \log_2(n) + \frac{A}{2} + \frac{A}{3}} \quad p_l, p_{m+1} \geq 2, p_l \neq p_{m+1} \\ &\geq 0 \end{aligned}$$

Since x_2 is non-negative we can conclude that

$$q_{n \times n}^{m+1} = \underbrace{x_0}_{q_{n \times n}^m} + \underbrace{x_1 + y_1 + y_2}_{\geq 0} + \underbrace{x_2}_{\geq 0} \geq q_{n \times n}^m \quad \square$$

Lemma III.16 Let $n = \prod_{i=1}^k p_i^{\alpha_i}$ be the prime factorization of n , where $k = \omega(n)$, the number of distinct prime factors of n . Then

$$|S_{n \times n}| \leq 2 \sum_{i=1}^k |\Sigma|^{n^2 p_i^{-1}} (p_i + 1).$$

Proof As in the proof of Lemma III.14 we employ the function $q_{n \times n}$, which can be decomposed into five parts as defined earlier

$$q_{n \times n}^{m+1} = x_0 + x_1 + x_2 + y_1 + y_2$$

Now we show that

$$y_1 \geq x_1 + x_2 + y_2$$

Let $A = n^2 p_{m+1}^{-1}$. Then $(y_1 - x_1 - x_2 - y_2)(p_{m+1} + 1)^{-1}$

$$\begin{aligned} &\geq |\Sigma|^A + \underbrace{q_{n \times n}^m(p_{m+1}^{-1})}_{\geq 0} - \underbrace{q_{n \times n}^m(p_{m+1}^{-2})}_{\geq 0} + \underbrace{|\Sigma|^{\frac{A}{p_{m+1}}}}_{\geq 0} \\ &\geq |\Sigma|^A - q_{n \times n}^m(p_{m+1}^{-2}) \\ &\geq |\Sigma|^A - \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} |\Sigma|^{g(\mathbf{v}) p_{m+1}^{-2}} \prod_{i=1}^m r(i) \\ &\geq |\Sigma|^A - \sum_{\mathbf{0} \triangleleft \mathbf{v} \triangleleft \mathbf{2}} |\Sigma|^{\frac{A}{p_1 p_{m+1}}} \quad p_l = \min\{p_1, \dots, p_m\} \\ &\geq |\Sigma|^A - 3^m |\Sigma|^{\frac{A}{p_1 p_{m+1}}} \quad |\mathbf{v}| = m \\ &\geq |\Sigma|^A - |\Sigma|^{2m} |\Sigma|^{\frac{A}{p_1 p_{m+1}}} \quad |\Sigma| \geq 2 \\ &\geq |\Sigma|^A - |\Sigma|^{2m + \frac{A}{p_1 p_{m+1}}} \\ &\geq |\Sigma|^A - |\Sigma|^{2 \log_2(n) + \frac{A}{p_1 p_{m+1}}} \quad m \leq k - 1 < \log_2(n) \\ &\geq |\Sigma|^A - |\Sigma|^{2 \log_2(n) + \frac{A}{6}} \quad p_l, p_{m+1} \geq 2, p_l \neq p_{m+1} \\ &\geq 0 \end{aligned}$$

Since $y_1 \geq x_1 + x_2 + y_2$

$$q_{n \times n}^{m+1} = x_0 + x_1 + x_2 + y_1 + y_2 \leq x_0 + 2y_1.$$

By substituting x_0 and y_1 we obtain a recursive inequality

$$\begin{aligned} q_{n \times n}^{m+1} &\leq q_{n \times n}^m + 2|\Sigma|^{n^2 p_{m+1}^{-1}} (p_{m+1} + 1) \\ &\leq q_{n \times n}^{m-1} + 2|\Sigma|^{n^2 p_m^{-1}} (p_m + 1) + 2|\Sigma|^{n^2 p_{m+1}^{-1}} (p_{m+1} + 1) \\ &\dots \\ &\leq 2 \sum_{i=1}^{m+1} |\Sigma|^{n^2 p_i^{-1}} (p_i + 1) \end{aligned}$$

To finalize the proof we use $|S_{n \times n}| = q_{n \times n}^k$. \square

Lemma IV.3 Pick $n, k \in \mathbb{N}$ with $k \leq n$ and let $d = \gcd(k, n)$. Let $n = \prod_{i=1}^{\omega(n)} p_i^{\alpha_i}$, $k = \prod_{i=1}^{\omega(k)} q_i^{\beta_i}$, and $d = \prod_{i=1}^{\omega(d)} r_i^{\gamma_i}$ be the prime factorizations of n, k, d , respectively. Then for any $a \in \Sigma$,

$$|S_{n \times n, k}^a| = \sum_{\mathbf{0} \triangleleft \mathbf{u} \triangleleft \mathbf{r} + \mathbf{1}} (-1)^{1+|\mathbf{u}|} \left(\prod_{i=1}^{\omega(d)} \binom{r_i + 1}{u_i} \right) \binom{\frac{n^2}{h(\mathbf{u})}}{\frac{k}{h(\mathbf{u})}} (|\Sigma| - 1)^{\frac{n^2 - k}{h(\mathbf{u})}},$$

where $\mathbf{r} = (r_1, \dots, r_{\omega(d)})$ and $h(\mathbf{u}) = \prod_{i=1}^{\omega(d)} r_i^{\min(u_i, 2)}$.

Proof Using Definition IV.1, Lemma III.7, and Corollary IV.1,

$$\begin{aligned} S_{n \times n, k}^a &= \left(\bigcup_{\mathbf{w} \in G_n} S_{n \times n}(\mathbf{w}) \right) \cap D_{n \times n, k}^a \\ &= \bigcup_{i=1}^{\omega(n)} \bigcup_{\mathbf{w} \in G_n(p_i)} S_{n \times n, k}^a(\mathbf{w}) \\ &= \bigcup_{i=1}^{\omega(d)} \bigcup_{\mathbf{w} \in G_n(r_i)} S_{n \times n, k}^a(\mathbf{w}). \end{aligned}$$

By the inclusion-exclusion principle

$$|S_{n \times n, k}^a| = \sum_{\substack{J_1 \subseteq G_n(r_1) \\ J_{\omega(d)} \subseteq \ddot{G}_n(r_{\omega(d)})}} (-1)^{1+\sum_{i=1}^{\omega(d)} |J_i|} \left| \bigcap_{\mathbf{w} \in \cup_i J_i} S_{n \times n}(\mathbf{w}) \right|.$$

Now, by Corollary III.2

$$\begin{aligned} \bigcap_{\mathbf{w} \in \cup_{i=1}^{\omega(d)} J_i} S_{n \times n, k}^a(\mathbf{w}) &= \left(\bigcap_{\mathbf{w} \in \cup_{i=1}^{\omega(d)} J_i} S_{n \times n}(\mathbf{w}) \right) \cap D_{n \times n, k}^a \\ &= S_{n \times n}(\cup_{i=1}^{\omega(d)} J_i) \cap D_{n \times n, k}^a \\ &= S_{n \times n, k}^a(\cup_{i=1}^{\omega(d)} J_i) \end{aligned}$$

Finally let $m = |\langle \cup_{i=1}^{\omega(d)} J_i \rangle|$, then using Lemma IV.2,

$$|S_{n \times n, k}^a(\cup_{i=1}^{\omega(d)} J_i)| = \left(\frac{n^2}{m} \right) (|\Sigma| - 1)^{\frac{n^2 - k}{m}},$$

where $m = |\langle \cup_{i=1}^{\omega(d)} J_i \rangle| = \prod_{i=1}^{\omega(d)} r_i^{\min(|J_i|, 2)}$. \square

Theorem V.2 *The average-case time complexity of the shift-symmetry detection algorithm for a square configuration of size $N = n^2$ generated from a uniform distribution is*

$$O(n^2).$$

Proof Let $m = n^2 p^{-1}$ be the number of orbits for a prime p . Assuming a uniform distribution the probability of passing an orbit is $Q = |\Sigma|^{1-p}$. If successful we move to a next orbit, otherwise we terminate with the probability $1 - Q$. The probability of terminating at i^{th} orbit can be therefore generalized as

$$P_i = \begin{cases} (1 - Q)Q^{i-1} & \text{if } i < m \\ Q^{m-1} & \text{if } i = m. \end{cases}$$

It is easy to show that these probabilities sum to 1, i.e., we must terminate at one of m orbits. Further, the probability of successfully passing the test for all the orbits—the probability that a configuration generated from a uniform distribution is shift-symmetric by a vector with an order p —equals $|\Sigma|^{n^2(p^{-1}-1)}$.

By using the formula for a geometric sum we can prove that

$$\sum_{i=0}^{n-1} (i+1)r^i = \frac{1 - r^n(1 + n(1-r))}{(1-r)^2}.$$

We apply this to calculate the expected number of visited orbits as

$$\begin{aligned} E_p[\#\text{orbits}] &= \sum_{i=1}^m iP_i \\ &= (1-Q) \sum_{i=0}^{m-2} (i+1)Q^i + mQ^{m-1} \\ &= \frac{1 - Q^{m-1}(m - Qm + Q) + (1-Q)mQ^{m-1}}{1-Q} \\ &= \frac{1 - Q^m}{1-Q}. \end{aligned}$$

Owing to $Q < 1$ we can bound the expected (average) number of visited orbits for a prime p as

$$E_p[\#\text{orbits}] \leq (1-Q)^{-1} = (1 - |\Sigma|^{1-p})^{-1}.$$

Each p -orbit contains p cells and so the expected number of visited cells is simply

$$E_p[\#\text{cells}] \leq 2p(1 - |\Sigma|^{1-p})^{-1}.$$

Note that while moving from one orbit to a next one we can potentially revisit some cells, however, because the order is fixed we can visit each cell at most twice.

The overall expected number of visited cells, i.e., the average-case time complexity in O-notation is

$$\sum_{i=1}^{\omega(n)} (p_i + 1)p_i(1 - |\Sigma|^{1-p_i})^{-1}.$$

Since the expression $(1 - |\Sigma|^{1-p_i})^{-1}$ is at most 2 ($p_i \geq 2$) and the integer logarithm $\text{sopf}(n)$ is at most n , the average-case time complexity of the shift-symmetry test is

$$\begin{aligned} O\left(\sum_{i=1}^{\omega(n)} p_i^2 + \sum_{i=1}^{\omega(n)} p_i\right) &= O(\text{sopf}^2(n) + \text{sopf}(n)) \\ &= O(n^2). \end{aligned} \quad \square$$

Appendix B: Examples

Example: Let $n = 2^{\alpha_1}3^{\alpha_2}$, then using counting from Lemma III.10, $|S_{n \times n}| =$

$$\begin{aligned}
& \binom{3}{1} |\Sigma|^{\frac{n^2}{2}} + \binom{4}{1} |\Sigma|^{\frac{n^2}{3}} \\
& - \binom{3}{2} |\Sigma|^{\frac{n^2}{2^2}} - \binom{3}{1} \binom{4}{1} |\Sigma|^{\frac{n^2}{2 \cdot 3}} - \binom{4}{2} |\Sigma|^{\frac{n^2}{3^2}} \\
& + \binom{3}{3} |\Sigma|^{\frac{n^2}{2^2}} + \binom{3}{2} \binom{4}{1} |\Sigma|^{\frac{n^2}{2^2 \cdot 3}} + \binom{3}{1} \binom{4}{2} |\Sigma|^{\frac{n^2}{2 \cdot 3^2}} \\
& \quad + \binom{4}{3} |\Sigma|^{\frac{n^2}{3^2}} \\
& - \binom{3}{3} \binom{4}{1} |\Sigma|^{\frac{n^2}{2^2 \cdot 3}} - \binom{3}{2} \binom{4}{2} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} - \binom{3}{1} \binom{4}{3} |\Sigma|^{\frac{n^2}{2 \cdot 3^2}} \\
& \quad - \binom{4}{4} |\Sigma|^{\frac{n^2}{3^2}} \\
& + \binom{3}{3} \binom{4}{2} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} + \binom{3}{2} \binom{4}{3} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} + \binom{3}{1} \binom{4}{4} |\Sigma|^{\frac{n^2}{2 \cdot 3^2}} \\
& - \binom{3}{3} \binom{4}{3} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} - \binom{3}{2} \binom{4}{4} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} \\
& + \binom{3}{3} \binom{4}{4} |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}}
\end{aligned}$$

by Lemma III.11, $|S_{n \times n}| =$

$$\begin{aligned}
& |\Sigma|^{\frac{n^2}{2}} \left[+ \binom{3}{1} \right] + \\
& |\Sigma|^{\frac{n^2}{3}} \left[+ \binom{4}{1} \right] + \\
& |\Sigma|^{\frac{n^2}{2 \cdot 3}} \left[- \binom{3}{1} \binom{4}{1} \right] + \\
& |\Sigma|^{\frac{n^2}{2^2}} \left[- \binom{3}{2} + \binom{3}{3} \right] + \\
& |\Sigma|^{\frac{n^2}{3^2}} \left[- \binom{4}{2} + \binom{4}{3} - \binom{4}{4} \right] + \\
& |\Sigma|^{\frac{n^2}{2^2 \cdot 3}} \left[+ \binom{3}{2} \binom{4}{1} - \binom{3}{3} \binom{4}{1} \right] + \\
& |\Sigma|^{\frac{n^2}{2 \cdot 3^2}} \left[+ \binom{3}{1} \binom{4}{2} - \binom{3}{1} \binom{4}{3} + \binom{3}{1} \binom{4}{4} \right] + \\
& |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} \left[- \binom{3}{2} \binom{4}{2} + \binom{3}{3} \binom{4}{2} + \binom{3}{2} \binom{4}{3} \right. \\
& \quad \left. - \binom{3}{3} \binom{4}{3} - \binom{3}{2} \binom{4}{4} + \binom{3}{3} \binom{4}{4} \right]
\end{aligned}$$

and finally by Theorem III.12, $|S_{n \times n}| =$

$$\begin{aligned}
& |\Sigma|^{\frac{n^2}{2}} 3 + |\Sigma|^{\frac{n^2}{3}} 4 - |\Sigma|^{\frac{n^2}{2 \cdot 3}} 3 \cdot 4 - |\Sigma|^{\frac{n^2}{2^2}} 2 - |\Sigma|^{\frac{n^2}{3^2}} 3 + \\
& |\Sigma|^{\frac{n^2}{2^2 \cdot 3}} 2 \cdot 4 + |\Sigma|^{\frac{n^2}{2 \cdot 3^2}} 3 \cdot 3 - |\Sigma|^{\frac{n^2}{2^2 \cdot 3^2}} 2 \cdot 3
\end{aligned}$$

Example: Let $n = 2^{\alpha_1}3^{\alpha_2}$, $a \in \Sigma$, and $k = 2^{\beta_1}3^{\beta_2}$, where $\beta_1 \leq \alpha_1, \beta_2 \leq \alpha_2$, and $\sigma = |\Sigma| - 1$. Then using counting from Lemma IV.3 $|S_{n \times n, k}^a| =$

$$\begin{aligned}
& \binom{3}{1} \binom{\frac{n^2}{2}}{\frac{k}{2}} \sigma^{\frac{n^2-k}{2}} + \binom{4}{1} \binom{\frac{n^2}{3}}{\frac{k}{3}} \sigma^{\frac{n^2-k}{3}} \\
& - \binom{3}{2} \binom{\frac{n^2}{2^2}}{\frac{k}{2^2}} \sigma^{\frac{n^2-k}{2^2}} - \binom{3}{1} \binom{4}{1} \binom{\frac{n^2}{2 \cdot 3}}{\frac{k}{2 \cdot 3}} \sigma^{\frac{n^2-k}{2 \cdot 3}} - \binom{4}{2} \binom{\frac{n^2}{3^2}}{\frac{k}{3^2}} \sigma^{\frac{n^2-k}{3^2}} \\
& + \binom{3}{3} \binom{\frac{n^2}{2^2}}{\frac{k}{2^2}} \sigma^{\frac{n^2-k}{2^2}} + \binom{3}{2} \binom{4}{1} \binom{\frac{n^2}{2^2 \cdot 3}}{\frac{k}{2^2 \cdot 3}} \sigma^{\frac{n^2-k}{2^2 \cdot 3}} \\
& \quad + \binom{3}{1} \binom{4}{2} \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} + \binom{4}{3} \binom{\frac{n^2}{3^2}}{\frac{k}{3^2}} \sigma^{\frac{n^2-k}{3^2}} \\
& - \binom{3}{3} \binom{4}{1} \binom{\frac{n^2}{2^2 \cdot 3}}{\frac{k}{2^2 \cdot 3}} \sigma^{\frac{n^2-k}{2^2 \cdot 3}} - \binom{3}{2} \binom{4}{2} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} \\
& \quad - \binom{3}{1} \binom{4}{3} \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} - \binom{4}{4} \binom{\frac{n^2}{3^2}}{\frac{k}{3^2}} \sigma^{\frac{n^2-k}{3^2}} \\
& + \binom{3}{3} \binom{4}{2} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} + \binom{3}{2} \binom{4}{3} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} \\
& \quad + \binom{3}{1} \binom{4}{4} \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} \\
& - \binom{3}{3} \binom{4}{3} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} - \binom{3}{2} \binom{4}{4} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} \\
& \quad + \binom{3}{3} \binom{4}{4} \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}}
\end{aligned}$$

by Lemma IV.5 $|S_{n \times n, k}^a| =$

$$\begin{aligned}
& \binom{\frac{n^2}{2}}{\frac{k}{2}} \sigma^{\frac{n^2-k}{2}} \left[+ \binom{3}{1} \right] + \\
& \binom{\frac{n^2}{3}}{\frac{k}{3}} \sigma^{\frac{n^2-k}{3}} \left[+ \binom{4}{1} \right] + \\
& \binom{\frac{n^2}{2 \cdot 3}}{\frac{k}{2 \cdot 3}} \sigma^{\frac{n^2-k}{2 \cdot 3}} \left[- \binom{3}{1} \binom{4}{1} \right] + \\
& \binom{\frac{n^2}{2^2}}{\frac{k}{2^2}} \sigma^{\frac{n^2-k}{2^2}} \left[- \binom{3}{2} + \binom{3}{3} \right] + \\
& \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} \left[- \binom{4}{2} + \binom{4}{3} - \binom{4}{4} \right] + \\
& \binom{\frac{n^2}{2^2 \cdot 3}}{\frac{k}{2^2 \cdot 3}} \sigma^{\frac{n^2-k}{2^2 \cdot 3}} \left[+ \binom{3}{2} \binom{4}{1} - \binom{3}{3} \binom{4}{1} \right] + \\
& \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} \left[+ \binom{3}{1} \binom{4}{2} - \binom{3}{1} \binom{4}{3} + \binom{3}{1} \binom{4}{4} \right] + \\
& \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} \left[- \binom{3}{2} \binom{4}{2} + \binom{3}{3} \binom{4}{2} + \binom{3}{2} \binom{4}{3} \right. \\
& \quad \left. - \binom{3}{3} \binom{4}{3} - \binom{3}{2} \binom{4}{4} + \binom{3}{3} \binom{4}{4} \right]
\end{aligned}$$

and finally by Theorem IV.6, $|S_{n \times n, k}^a| =$

$$\begin{aligned} & \binom{\frac{n^2}{2}}{\frac{k}{2}} \sigma^{\frac{n^2-k}{2}} 3 + \binom{\frac{n^2}{3}}{\frac{k}{3}} \sigma^{\frac{n^2-k}{3}} 4 - \binom{\frac{n^2}{2 \cdot 3}}{\frac{k}{2 \cdot 3}} \sigma^{\frac{n^2-k}{2 \cdot 3}} 3 \cdot 4 \\ & - \binom{\frac{n^2}{2^2}}{\frac{k}{2^2}} \sigma^{\frac{n^2-k}{2^2}} 2 - \binom{\frac{n^2}{3^2}}{\frac{k}{3^2}} \sigma^{\frac{n^2-k}{3^2}} 3 + \binom{\frac{n^2}{2^2 \cdot 3}}{\frac{k}{2^2 \cdot 3}} \sigma^{\frac{n^2-k}{2^2 \cdot 3}} 2 \cdot 4 \\ & + \binom{\frac{n^2}{2 \cdot 3^2}}{\frac{k}{2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2 \cdot 3^2}} 3 \cdot 3 - \binom{\frac{n^2}{2^2 \cdot 3^2}}{\frac{k}{2^2 \cdot 3^2}} \sigma^{\frac{n^2-k}{2^2 \cdot 3^2}} 2 \cdot 3 \end{aligned}$$

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