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FUZZINESS AND CATASTROPHE

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ABSTRACT

In a recent short note, Flondor has alluded to a possible linkage of fuzzy set theory and catastrophe theory. We consider several features of catastrophe theory, namely the properties of discontinuous jumps, hysteresis, and divergence in the "cusp catastrophe," and the role of the bias factor in the "butterfly catastrophe," which have affinities to and suggest possible extensions of fuzzy set ideas. Certain functions extensively considered in catastrophe theory lend themselves in some cases to interpretation as membership functions. The use of such functions may be of interest for the characterization of linguistic descriptions which are time-varying and encompass both discrete and fuzzy distinctions.

TEXT

In a recent short note [1], Flondor has alluded to a possible linkage of fuzzy set theory [2,3,4] and catastrophe theory [5,6,7]. In the present paper we make explicit an interpretation of this proposal, and consider other aspects of catastrophe-theoretic models which suggest possible extensions of and/or alterations to fuzzy set ideas. Specifically, certain functions which are extensively considered in catastrophe theory may sometimes be interpreted as membership functions. These functions suggest an interesting and mathematically deep way of introducing into fuzzy set theory both a temporal dimension and a topological linkage between discrete and continuous logics. (Other ways of incorporating these features in the theory are no doubt possible.)

Catastrophe theory analyzes systems governed by a particular class of potential functions (let us use the notation $V = f(x)$) in the neighborhood of certain points of interest (topological singularities). It is assumed that $dx/dt = -K\,\partial V/\partial x$ where K is large. The theory studies the surfaces described by values of x (the "behavioral variable") for which the system is at equilibrium, i.e., where $\partial V/\partial x = 0$. Consider the case where $V = x^4/4 + ax^2/2 + bx$ in the neighborhood of the singularity, S. For this case, known as the "cusp catastrophe," the equation $\partial V/\partial x = x^3 + ax + b = 0$ describes the equilibrium or "behavior" surface (Figure 1). For each combination of the "control parameters," a and b, the equilibrium value of x is specified by the point on the behavior surface directly above the "control point," (a,b). As the control point moves on the control surface, a "behavior point" follows above it on the behavior surface. This latter surface is an

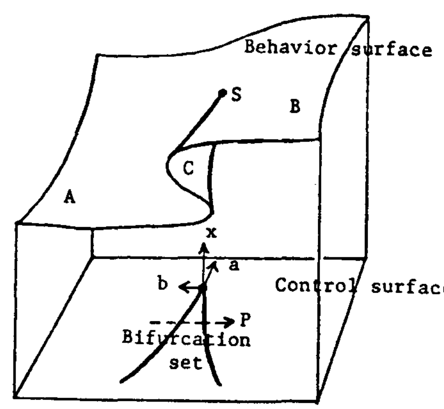

Figure-1

"attractor" in the sense that a displacement away from it (or an initial value of x not satisfying the equilibrium equation) results in rapid motion to the surface. Motion of the behavior point <u>on</u> the behavior surface is usually slower than such returns to equilibrium, and depends upon the motion of the control point.

For negative values of the control parameter, a, the behavior surface splits into overlapping upper and lower sheets. The resulting folds in the surface project onto a cusp-shaped set of points in the control surface known as the "bifurcation set," this term also being used more loosely to indicate the (internal) region bounded by these points. Within the bifurcation set, there is, as it were, a "struggle" between the two attractor surfaces, A and B. That is, if the behavior point is displaced or initially located away from the equilibrium surface, it moves rapidly to one or the other of these surfaces. Region C represents a condition of unstable equilibrium, i.e., an initial value of x on C will result in motion to either A or B. Trajectories of the control point which cross both boundaries of the bifurcation set, e.g., path P in the above figure, cause discontinuous jumps ("catastrophes") from one attractor surface to the other.

Without explicitly proposing this particular use of the cusp model, Flondor speaks of the struggle of attractors as the source of a kind of fuzziness, and suggests that "the fuzzy part of such a system consists of the 'way' to stability," stability here meaning, *"to be caught by an attractor." Fuzziness* is thus correlated with rapidly changing values of x, i.e., with conditions either of non-equilibrium or unstable equilibrium.

This is one possible connection which might be made between fuzzy-set and catastrophe-theory ideas. We propose a different approach: one in which

1) fuzziness is associated with the variety of values taken by x on the attractor surface(s); and,

2) fuzziness also may be superimposed on discrete (in this paper, 2- or 3-valued) characterizations.

Consider sections of the behavior surface with the control parameter, a, being held constant at some positive value. These sections appear as continuous, single-valued, and typically, but not necessarily, monotonically increasing curves of the same form as the membership function of some fuzzy sets. For an example, see Figure-2.

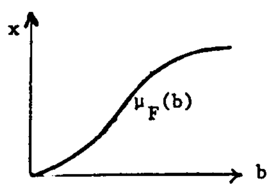

Figure-2

Here b is some base variable, e.g., height in inches, F is some fuzzy set, e.g., "tall," and x is the value of the membership function, $\mu_F(b)$, which indicates the degree of belonging of b to F. If one wishes, one can have $\mu_F(b)$ approach 0 and 1 at low and high values of b, respectively. Membership functions different from the one shown in the above figure can also be represented, either by selecting different sections of the behavior surface, or by transforming the entire surface through some appropriate bending and/or stretching operation.

Now consider what happens when $a < 0$, i.e., forward (in Figure-1) of the topological singularity, S. The sections now bifurcate into two domains (e.g., A and B in Figure-3). This could model a phenomenon in which the existence of a clear distinction suddenly emerges, but within which some fuzziness continues to exist.

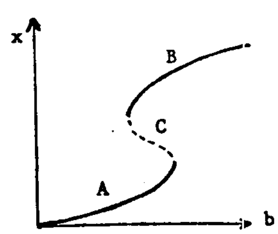

Figure-3

An example which illustrates the above is embryological differentiation.* Zeeman [6] has suggested that the cusp can be used to model the differentiation of cells into distinct types, as shown in Figure-4.

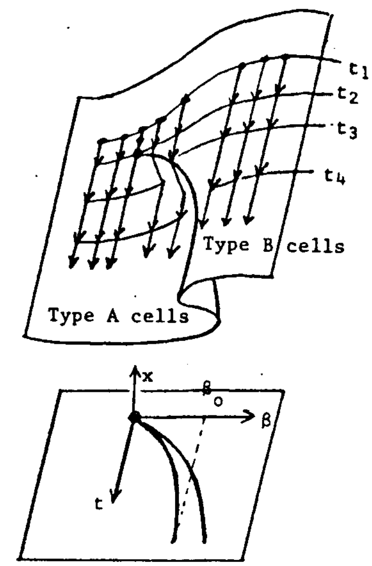

Figure-4

Here t represents time, and β is a spatial coordinate along which there exists some physical or chem ical gradient. This gradient is the basis for the eventual bifurcation of a population of cells into two tissue types separated one from another by a definite boundary (at β_0). The cell population is represented by the several control points whose tra jectories are plotted. Initially at time t_1, we have a continuous spectrum ranging between the two ultimate cell types. A plot of x versus β resemble the fuzzy set membership function shown in Figure-2 At t_2, the spectrum undergoes a sudden splitting, and as t continues, the distinction between what no may be called type A and type B cells becomes more clear+. But even after this distinction occurs, a fuzziness is still embodied in the range of values which x assumes on the upper or lower equilibrium surfaces. That is, cells are type A or type B, but they are <u>also</u> more or less "A-like" or "B-like." I this example, x represents something like the relative degree of "A-likeness" versus "B-likeness." W have thus in this case a description which is time-dependent and which can encompass <u>both</u> 2-valued and fuzzy distinctions.

This example is only illustrative, but it suggests the possibility that fuzzy set theory might b enriched by the use of an integrated logic of conti nuity and discontinuity based on catastrophe-theoretic concepts. Fuzzy set theory is a response to the fact that ordinary 2-valued logic (something is either a member of a set or not a member of it)

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*The above figure is an approximation to Zeeman's drawing; his and Thom's application of catastrophe theory to embryological differentiation is also much more extensively developed than this. These analyses are not, however, settled matters. An active controversy currently rages about the validity of many applications of catastrophe theory.

+Indeed, the boundary between the two cell types stabilizes at time t_4.

does not properly describe many phenomena. Continuous membership functions (such as the one shown in Figure-2), whose use corresponds to multi-valued schemes of logic [8], are a possible alternative, but one would like to retain the possibility also of the classical sharp dichotomies. Within fuzzy set theory, classical 2-valued logic can be obtained, of course, if values of the membership function are restricted to either 0 or 1. But a more interesting kind of dichotomy, which retains the possibility of membership values over the whole [0,1] interval, is represented in the way in which the linguistic modifier "not" is interpreted. For example, if, in Figure-2, b is the height of persons whose membership in the fuzzy set "tall" is given as

$$x = \mu_{tall}(b)$$

then "not tall" is defined by the complement of the set "tall" as follows:

$$\mu_{not\ tall}(b) = 1 - \mu_{tall}(b).$$

We can also define the fuzzy set "short," to be identical to "not tall." We have then the result shown in Figure-5.

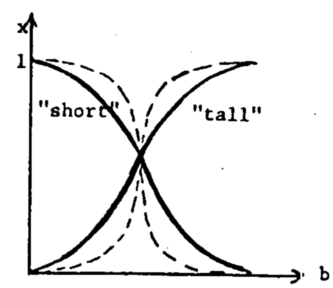

Figure-5

If we make these curves steeper, they will approach, in the limit, a conventional dyadic description. (Or the reverse would illustrate a transition from classical to fuzzy logic.)

The catastrophe-theoretic curve of Figure-3 is a different - and, we suggest, more interesting - solution to the problem of uniting 2-valued and multi-valued logics. Here we have a 2-valued distinction in the separate regions A and B, and multi-valued distinctions within these regions. A problem arises, however, over what these regions should be called in our present linguistic example. B and A are obviously correlated with the attributes of "tall" and "short," respectively, but this usage is not appropriate, since we have spoken of x as indicating the degree of membership in the fuzzy set "tall." "Short" is the name of a different fuzzy set which has its own membership function. Yet, it seems quite natural to give these names to the two attractor surfaces. The problem arises from the fact that the very possibility of bifurcation in the equilibrium surface indicates that the behavior variable is "bipolar," that is, refers not to a single behavioral attribute, but to a complementary pair of such attributes.* (This is clear in the example of differentiation.)

The difficulty is not serious. It is certainly legitimate to give linguistic "labels" to separate parts of catastrophe-theoretic surfaces. One must take care only not to confuse these separate labels with separate fuzzy sets. Regions A and B could alternatively be labelled as "weakly tall" and "strongly tall," but there is a natural tendency to prefer positive linguistic characterizations over negative ones, e.g., "short" over "weakly tall."

A more serious complication arises from the fact that Figure-3 does not describe a single-valued function inside the bifurcation set. In practice, this ambiguity is resolved by the fact that x depends not simply on the present location of the control point, but also on its history. Figure-4 shows control point trajectories which illustrate the cusp properties of divergence and discontinuous jumps ("catastrophes"). For a value of b inside the bifurcation set,

$$x = \mu_A(b) \quad \text{or} \quad \mu_B(b)$$

depending on which side of the singularity the trajectory occurs.

A trajectory such as path P of Figure-1, which results in the curve of Figure-3, shows how the property of jumps can be combined with that of hysteresis. The system is in state A as the control point moves across the bifurcation set, and jumps to state B when the second (right-most) boundary of the bifurcation set is crossed. For motion in the reverse direction, the catastrophe occurs at the other (leftmost) boundary of the bifurcation set. In terms of our linguistic example, Figure-3 could be regarded as a composite, as it were, of two (discontinuous) membership functions. One of these (μ^1_{tall}) specifies membership in "tall" for a sequence of observed subjects with gradually increasing heights, i.e., corresponds to motion of the control point towards increasing values of b; the other (μ^2_{tall}) specifies membership in "tall" for a sequence of subjects with gradually decreasing heights, i.e., corresponds to motion of the control point towards decreasing values of b. This is summarized in Figure-6. (This interpretation does not, however, account for the occurrence of the unstable equilibrium region, C.) The hysteresis effect might be said to model the conditioning by past experience of linguistic judgments involving dyadic distinctions.

It should be understood that the trajectories which have been discussed are only illustrative. Thom's theory does not dictate how the control point moves, but only supplies the necessary topological relationship between the control and behavior variables in the neighborhood of the singularity. The main point here is that the overlap of the behavior surface presents no technical difficulty, because the value of x is determined by the control point trajectory.

Of course, whether linguistic descriptions of some phenomena require such features as divergence, sudden jumps, and/or hysteresis effects, is an empirical question. But if fuzzy set theory is to be applied to phenomena for which the cusp catastrophe is an appropriate model, it would seem plausible that some of the above features must be capable of being accomodated within the theory.

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*This calls to mind the classical Hegelian principle of dialectics, "the mutual interpretation of opposites." For an exposition of this principle, which shows it to be a philosophical precursor of fuzzy set theory, see [9]. For a discussion of the relations of dialectics and catastrophe theory, see [10]

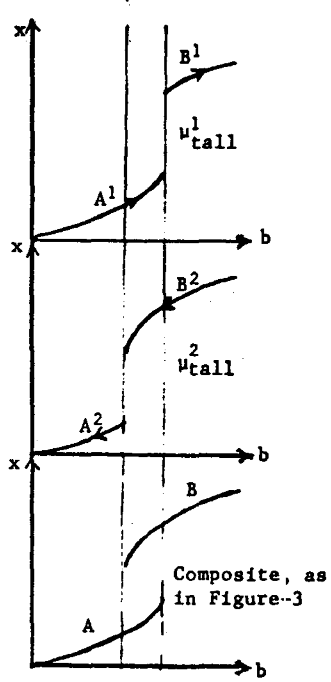

Figure-6

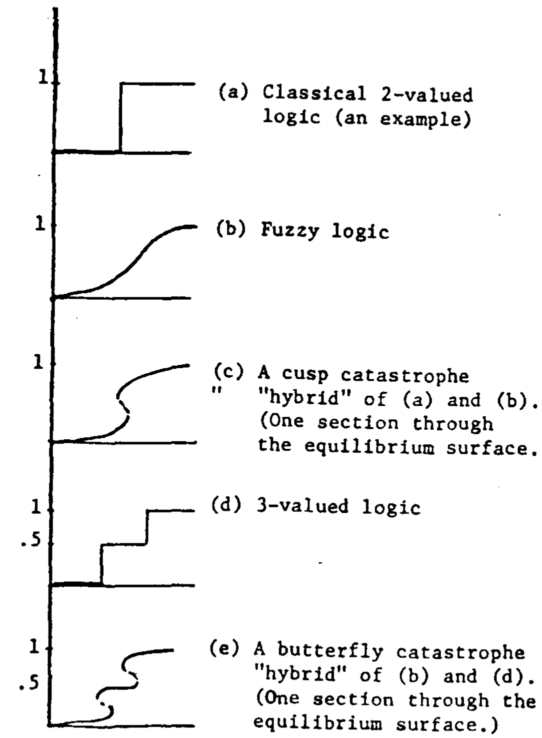

Figure-7

We can also make use of a second of Thom's archetypes, the "butterfly catastrophe," to model more complicated unions of discrete and continuous logics as well as the effect of bias on the judgment of observers. The butterfly has the potential function, $V = x^6/_6 + dx^4/_4 + cx^3/_3 + bx^2/_2 + ax$, and a corresponding equilibrium surface which can have three attractor surfaces. If the cusp can be considered a kind of "hybrid" of classical 2-valued logic and fuzzy logic, the butterfly can be considered a "hybrid" of some 3-valued logic and fuzzy logic. The three attractor surfaces can be regarded as fuzzy versions of the logical values 0, 1, and ½, just as attractors A and B are fuzzy versions of 0 and 1. Figure-7 illustrates possible examples of these various logics.

When the parameter d in the potential function is positive, the butterfly "reduces" to a two-surfaced cusp, but one in which the values of the parameter, c, the "bias factor" can cause shifts in the dependence of the behavioral variable on b, as shown in Figure-8.

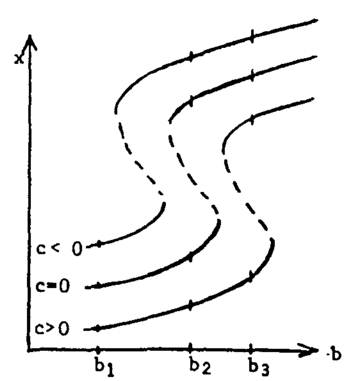

Figure-8

To continue the previous example, assume that membership in "tall" is given by x; b is the height of the "subject," i.e., the person being observed. Positive, zero, and negative values of the bias factor, c, correspond to tall, intermediate-height, and short observers, respectively. When the person being observed has height b_1, i.e., is fairly short, all three observers agree to label the subject short, but the short observer ($c < 0$) actually means by this a greater value of membership in "tall" than does the tall observer ($c > 0$).* A subject of height b_2 is considered by the short observer to be tall, and by the tall observer to be short; the intermediate-height observer ($c = 0$) may regard the subject either as short or tall (probably depending upon the height of previously observed subjects). A slightly taller subject, b_3, is labelled unambiguously as tall by both the short and intermediate-height observers, but might be considered either tall or short by the tall observer. And so on.

In summary, fuzzy-set theory and catastrophe theory have some affinities. Their linkage might suggest some novel ways to join together continuous and discrete logics to model aspects of natural language. The present note, like Flondor's which

*In this discussion, we speak of the fuzzy set "tall," but also use "short" and "tall" to label parts of the membership functions. It will be clear from the context which meaning of "tall" is intended.

inspired it, is highly preliminary, but may help point the way to more rigorous developments.

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