Portland State University

PDXScholar

Electrical and Computer Engineering Faculty
Publications and Presentations

Electrical and Computer Engineering

5-2011

Quantum Phase Estimation Using Multivalued Logic

Marek Perkowski

Portland State University, marek.perkowski@pdx.edu

Vamsi Parasa

Portland State University

Follow this and additional works at: https://pdxscholar.library.pdx.edu/ece_fac

Part of the [Electrical and Computer Engineering Commons](#)

Let us know how access to this document benefits you.

Citation Details

Perkowski, Marek and Parasa, Vamsi, "Quantum Phase Estimation Using Multivalued Logic" (2011).
Electrical and Computer Engineering Faculty Publications and Presentations. 202.
https://pdxscholar.library.pdx.edu/ece_fac/202

This Conference Proceeding is brought to you for free and open access. It has been accepted for inclusion in Electrical and Computer Engineering Faculty Publications and Presentations by an authorized administrator of PDXScholar. Please contact us if we can make this document more accessible: pdxscholar@pdx.edu.

Quantum Phase Estimation using Multivalued Logic

Vamsi Parasa
Marek Perkowski

Department of Electrical and Computer Engineering,
Portland State University

ISMVL 2011, 23-25 May 2011, Tuusula, Finland

Agenda

- Importance of Quantum Phase Estimation (QPE)
- QPE using binary logic
- QPE using MVL
- Performance Requirements
- Salient features
- Conclusion

Abstract

- We generalize the *Quantum Phase Estimation algorithm* to MVL logic.
- We show the quantum circuits for QPE using **qudits**.
- We derive the performance requirements of the QPE to achieve high probability of success.
- We show how this leads to logarithmic decrease in the number of qudits and exponential decrease in error probability of the QPE algorithm as the value of the radix d increases.

Introduction

- QPE – one of the most important quantum subroutines, used in :
 - 1) Shor's Algorithm
 - 2) Abrams and Lloyd Algorithm
 - (Simulating quantum systems)
 - Calculation of Molecular Ground State Energies
 - 3) Quantum Counting (for Grover Search)
 - 4) Fourier Transform on arbitrary Z_p

Quantum phase estimation (QPE)

Let $|u\rangle$ be the eigenstate of a unitary operator U with an eigenvalue $e^{2\pi i\varphi_u}$, where the value of the phase φ_u is unknown. The goal of the QPE algorithm is to determine the best approximation to the phase φ_u .

QPE – Formal Definition

phase

- More formally, if we have $U|u\rangle = e^{2\pi i\varphi_u} |u\rangle$
- Let $\varphi_u \approx \frac{\tilde{\varphi}_u}{d^t} = 0.\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3\dots\tilde{\varphi}_{t-1}\tilde{\varphi}_t$ be the best t ‘dit’ approximation to φ_u
- This implies,
$$\frac{\tilde{\varphi}_u}{d^t} = \tilde{\varphi}_1 d^{-1} + \tilde{\varphi}_2 d^{-2} + \dots \tilde{\varphi}_{t-1} d^{-(t-1)} + \tilde{\varphi}_t d^{-t}$$
for each $\tilde{\varphi}_i \in [0, d-1]$.
- Thus the goal of the QPE algorithm is to find the value of $\tilde{\varphi}_u$ which gives the best estimate of the original phase φ_u

QPE Algorithm– Binary logic case

We first review the binary case:

Let $\varphi_u \approx \frac{\tilde{\varphi}_u}{2^t} = \tilde{\varphi}_1 2^{-1} + \tilde{\varphi}_2 2^{-2} + \dots \tilde{\varphi}_{t-1} 2^{-(t-1)} + \tilde{\varphi}_t 2^{-t}$ be the best t bit approximation to φ_u . In short, it is represented as $\varphi_u \approx 0.\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3\dots\tilde{\varphi}_{t-1}\tilde{\varphi}_t$

Schematic for the QPE Algorithm

QPE : step 1 – Initialization, Binary logic case

$|0\rangle^t$ be the tensor product of t qubits each in the state $|0\rangle$

QPE : step 2 – Apply the operator U

$$\frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle |u\rangle \xrightarrow{U^j}$$

$$\frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} |j\rangle (U^j |u\rangle)$$

$$= \frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} \left(e^{2\pi i j \frac{\tilde{\phi}_u}{2^t}} \right) |j\rangle |u\rangle$$

Binary logic case

U^j denotes the controlled unitary operator U controlled on the state vector $|j\rangle$.

$$(U^j |u\rangle) = (e^{2\pi i \phi_u})^j |u\rangle = \left(e^{2\pi i \frac{\tilde{\phi}_u}{2^t}} \right)^j |u\rangle$$

QPE : step 3 – Apply Inverse QFT

$$\frac{1}{\sqrt{2^t}} \sum_{j=0}^{2^t-1} \left(e^{2\pi i j \frac{\tilde{\varphi}_u}{2^t}} \right) |j\rangle |u\rangle \xrightarrow{QFT^\dagger} |\tilde{\varphi}_u\rangle |u\rangle$$

Binary logic case

QFT Definition: $|j\rangle \rightarrow \frac{1}{\sqrt{2^n}} \sum_{k=0}^{2^n-1} e^{\frac{2\pi i j k}{2^n}} |k\rangle$

QPE : step 4 - Measurement

Binary logic case

- If the phase φ_u is an **exact binary fraction**, we measure the estimate of the phase φ_u with probability of 1.
- If not, then we measure the estimate of the phase with a very high probability close to 1.

Quantum circuit for u^j

If $j = j_1 2^{t-1} + j_2 2^{t-2} + \dots j_k 2^{t-k} \dots + j_{t-1} 2^1 + j_t 2^0$.

$$U^j = U^{j_1 2^{t-1}} \times U^{j_2 2^{t-2}} \times \dots U^{j_k 2^{t-k}} \dots \times U^{j_{t-1} 2^1} \times U^{j_t 2^0}$$

The circuit for QFT is well known and hence not discussed.

QPE – Generalization to MVL

MV logic case

- We represent the phase φ_u as a *d*-ary fraction given by $\varphi_u \approx \frac{\tilde{\varphi}_u}{d^t} = 0.\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3\dots\tilde{\varphi}_{t-1}\tilde{\varphi}_t$

$$\frac{\tilde{\varphi}_u}{d^t} = \tilde{\varphi}_1 d^{-1} + \tilde{\varphi}_2 d^{-2} + \dots \tilde{\varphi}_{t-1} d^{-(t-1)} + \tilde{\varphi}_t d^{-t}$$

Now we have qudits not qubits

We have t qudits for phase

Now we have arbitrary Chrestenson instead Hadamard

Now we Inverse QFT on base d , not base 2

Schematic for QPE using Qudits

Some definitions

QFT and Chrestenson

MV logic case

- The Multivalued logic **QFT** on **n qudits** is defined as :

$$|j\rangle \xrightarrow{QFT} \frac{1}{\sqrt{d^n}} \sum_{k=0}^{d^n-1} e^{\frac{2\pi i jk}{d^n}} |k\rangle$$

- The action of a **Chrestenson (CH)** gate on a **single qudit** is defined as:

$$CH|x\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{y \in [0, d-1]} e^{\frac{2\pi i xy}{d}} |y\rangle$$

For $d = 2$, the CH gate reduces to Hadamard gate

MVL QPE : Step 1 - Initialization

MV logic case

$$|0\rangle^t |u\rangle \xrightarrow{CH^{\otimes t}} \frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} |j\rangle |u\rangle$$

$$\left(CH|0\rangle \rightarrow \frac{1}{\sqrt{d}} \sum_{j \in [0, d-1]} |j\rangle \right)$$

MVL QPE: step 2 – Apply Controlled u gate

MV logic case

$$\frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} |j\rangle |u\rangle \xrightarrow{U^j} \frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} |j\rangle (U^j |u\rangle)$$

$$= \frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} |j\rangle \left(e^{2\pi i \varphi_u} \right)^j |u\rangle = \frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} \left(e^{2\pi i j \frac{\tilde{\varphi}_u}{d^t}} \right) |j\rangle |u\rangle$$

MVL QPE: Step 3 – apply iqft

MV logic case

We apply inverse
Quantum Fourier
Transform

MVL QPE: Step 4 -- measurement

MV logic case

- After making a measurement on the first qudit register, we now get $\tilde{\varphi}_u$ which is an estimate of the phase φ_u
- If φ_u is not an exact *d-ary* fraction then we can only measure the phase with a high success probability close to 1 but not exactly 1.

Quantum circuit for U^j

$$\text{Let } |j\rangle = \bigotimes_{k=1}^t |J_k\rangle \Rightarrow U^j = U^{\left(\sum_{k=1}^t J_k d^{t-k}\right)} = \prod_{k=1}^t U^{J_k d^{t-k}}$$

MV logic case

$$U_{d^k} \triangleq \text{diag}\left(I, U^{(1d^k)}, U^{(2d^k)}, \dots, U^{((d-2)d^k)}, U^{((d-1)d^k)}\right)$$

D-valued quantum multiplexers

Case $d=3$

control

Target (date)

QUANTUM Circuit

MV logic case

- IQFT can be implemented with a complexity of $O(n \log n)$ but the expensive part is implementing higher order powers of U .
- This determines the complexity of the circuit.

QPE - Performance

MV logic case

$$\frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} \left(e^{2\pi i j \frac{\tilde{\varphi}_u}{d^t}} \right) |j\rangle |u\rangle \xrightarrow{QFT^\dagger} |\tilde{\varphi}_u\rangle |u\rangle$$

If the phase φ_u is an exact d-ary fraction i.e.

$\varphi_u = \frac{\tilde{\varphi}_u}{d^t}$ then after QFT ... QPE algorithm gives correct answer with probability of 1.

What if it is not an exact fraction ?

$$i.e. \quad \varphi_u = \frac{\tilde{\varphi}_u}{d^t} + \delta \quad (0 \leq \delta < d^{-t})$$

QPE Performance

- It can be shown that, in the general case, when the phase is not an exact fraction, **QPE** ^{binary} succeeds with minimum probability of $\frac{8}{\pi^2} = 81.5\%$
- What can we do to increase this success probability very close to 1 ?
- Will MVL help in this aspect? YES

QPE Performance: PHASE $\neq$ d-ary fraction

- Let's analyze what happens if the phase is not an exact fraction. Applying the QFT gives a new superposition state

MV logic case

$$\frac{1}{\sqrt{d^t}} \sum_{j=0}^{d^t-1} \left(e^{2\pi i j \frac{\tilde{\varphi}_u}{d^t}} \right) |j\rangle \xrightarrow{QFT^\dagger} \sum_{l=0}^{d^t-1} \alpha_l |l\rangle$$

It is not hard to show that, the probability of measuring an l given by $P(l)$ is

$$P(l) = |\alpha_l|^2 = \left| \frac{1}{d^t} \sum_{j=0}^{d^t-1} \exp \left(2\pi i \frac{(\varphi_u - j)}{d^t} l \right) \right|^2$$

QPE Performance

- Using the fact that $\varphi_u = \frac{\tilde{\varphi}_u}{d^t} + \delta \quad (0 \leq \delta < d^{-t})$

We get $P(l) = |\alpha_l|^2 = \left| \frac{1}{d^t} \left(\frac{1 - e^{2\pi i(d^t \delta + (\tilde{\varphi}_u - l))}}{1 - e^{2\pi i(\delta + \frac{\tilde{\varphi}_u - l}{d^t})}} \right) \right|^2$ MV logic case

- Thus after measurement, we get some value l with the probability given above. i.e. this implies $\varphi_u = \frac{l}{d^t}$
- If l is close to $\tilde{\varphi}_u$ then we can say QPE succeeded else QPE fails.
- How close is close ?

QPE : Success Probability lower bound

- It is easy to show that the probability that QPE returns either l or $l+1$ such that $l \leq \tilde{\varphi}_u \leq l+1$ is

$$\frac{8}{\pi^2} = 81.5\% \text{ as}$$

$$P(l) = |\alpha_l|^2 = \left| \frac{1}{d^t} \frac{1 - e^{2\pi i(d^t \varphi_u - l)}}{1 - e^{2\pi i\left(\varphi_u - \frac{l}{d^t}\right)}} \right|^2 \geq \frac{4}{\pi^2}$$

MV logic case

- Although encouraging, the lower bound is not good enough.
- We need SUCCESS PROBABILITY close to 1.
- How to define SUCCESS PROBABILITY ?

Success probability = 1 – failure probability

MV logic case

- Suppose we have a t dit approximation to the phase

$$\varphi_u \approx \frac{\tilde{\varphi}_u}{d^t} = 0.\tilde{\varphi}_1\tilde{\varphi}_2\tilde{\varphi}_3\ldots\tilde{\varphi}_{t-1}\tilde{\varphi}_t$$

- If we are only interested in a precision of only upto n dits

$$i.e. \quad \varphi_u = \frac{\tilde{\varphi}_u}{d^t} + \delta \quad (0 \leq \delta < d^{-n})$$

- then as long as QPE returns some l i.e. $\varphi_u = \frac{l}{d^t}$ such that the above condition is satisfied, we have a success.

$$e = l - \tilde{\varphi}_u = \delta d^t \quad \text{but} \quad (0 \leq \delta < d^{-n})$$

- The error e is $\Rightarrow e < d^{t-n}$

Failure probability

- We define the failure probability as

$$\varepsilon = p(|l - \tilde{\varphi}_u| > e) = \sum_{l=0, l \notin [\tilde{\varphi}_u - e, \tilde{\varphi}_u + e]}^{d^t - 1} |\alpha_l|^2 \leq \frac{1}{2(e-1)}$$

- The failure probability has a lower bound and hence the success probability has an upper bound.

$$p(\text{Success}) = 1 - \varepsilon > 1 - \frac{1}{2(e-1)}$$

$$\text{where } e = d^{t-n} - 1$$

Success probability: REQUIREMENTS

- Thus to achieve phase estimation with a success probability of $1 - \varepsilon$ with precision/accuracy up to n dits, we need to use a system with t dits. The value of t is given by

$$t = n + p = n + \log_d \left(2 + \frac{1}{2\varepsilon} \right)$$

- Now we show quantitative results in some graphs

How MVL HELPS

- **Failure probability** decreases exponentially with increase in radix d of the logic used

Less number of qudits for a given precision

These are the requirements for a real world problem of calculating molecular energies

*Number of qudits required (t) vs dimension of qudits (d)
to obtain a precision of upto 5 decimal digits with a success probability of 98%*

More RESULTS

NUMBER OF QUDITS REQUIRED FOR QPE ALGORITHM

<i>Precision in decimal digits</i>	<i>Success probability</i>	$d=2$	$d=3$	$d=4$	$d=5$	$d=6$
5	99.5%	24	15	12	11	10
4	98 %	19	12	10	9	8
4	95 %	18	12	9	8	7
4	85 %	17	11	9	8	7
3	98 %	17	11	9	8	7
2	99 %	14	9	7	6	6
2	90 %	11	7	6	5	5

Conclusions

- Quantum Phase Estimation has **many applications** in Quantum Computing
- **MVL is very helpful** for Quantum Phase Estimation
- Using MVL causes **exponential decrease** in the **failure probability** for a given precision of phase required.
- Using MVL results in **signification reduction in the number of qudits** required as radix d increases

Conclusions 2

- The method creates **high power unitary matrices** U^k of the original Matrix U for which eigenvector $|u\rangle$ we want to find phase.
- We cannot design these matrices as powers. This would be extremely **wasteful**
- We have to calculate these matrices and **decompose** them to gates
- **New type** of quantum logic synthesis problem: not permutative U , not arbitrary U , there are **other problems** like that, we found
- This **research problem** has been not solved in literature even in case of binary unitary matrices U