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Abstract

Let $v > k > i$ be non-negative integers. The generalized Johnson graph, $J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, where vertices A and B are adjacent whenever $|A \cap B| = i$. In this article, we derive general formulas for the girth and diameter of $J(v, k, i)$. Additionally, we provide a formula for the distance between any two vertices A and B in terms of the cardinality of their intersection.

Keywords: girth; diameter; generalized Johnson graphs; uniform subset graphs

1. Introduction

Let $v > k > i$ be non-negative integers. The *generalized Johnson graph*, $X = J(v, k, i)$, is the graph whose vertices are the k-subsets of a v-set, where vertices A and B are adjacent whenever $|A \cap B| = i$. Generalized Johnson graphs were introduced by Chen and Lih in [2]. Special cases include the Kneser graphs $J(v, k, 0)$, the odd graphs $J(2k+1, k, 0)$, and the Johnson graphs $J(v, k, k-1)$. The Johnson graph $J(v, k, k-1)$ is well known to have diameter $\min\{k, v-k\}$, and formulas for the distance and diameter of Kneser graphs were proved in [5].

Generalized Johnson graphs have also been studied under the name *uniform subset graphs*, and a result in [3] offers a general formula for the diameter of $J(v, k, i)$. However, that formula gives incorrect values when $i > \frac{2}{3}k$, an important case that includes the Johnson graphs. In this paper we extend (and, in places, correct) those expressions for the diameter of generalized Johnson graphs and we additionally provide a formula for the girth.

Note that it is possible to extend the definition of $X = J(v, k, i)$ to include $v \geq k \geq i$. However, X is an empty graph when $k = i$ or $v = k$. If $v = 2k$ and $i = 0$, then X is isomorphic to the disjoint union of copies of K_2. Furthermore, by taking complements, the graphs $J(v, k, i)$ and $J(v, v-k, v-2k+i)$ are easily seen to be isomorphic (see [4, p.11]). For the remainder of this article, we will be concerned with generalized Johnson graphs that are connected, so we make the following global definition.

Definition 1.1. Assume $v > k > i$ are nonnegative integers, and let $X = J(v, k, i)$ denote the corresponding generalized Johnson graph. To avoid trivialities, further assume that $v \geq 2k$, and that $(v, k, i) \neq (2k, k, 0)$.

2. Girth

In this section we derive an expression for the girth $g(X)$ of a generalized Johnson graph, X. We begin with a lemma that characterizes when two vertices have a common neighbor.

Lemma 2.1. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Then A and B have a common neighbor if and only if $x \geq \max\{-v + 3k - 2i, 2i - k\}$.*

Proof. Vertices A and B have a common neighbor C if and only if there exists a nonnegative integer s, such that every region in the following diagram (Figure 1) has nonnegative size.

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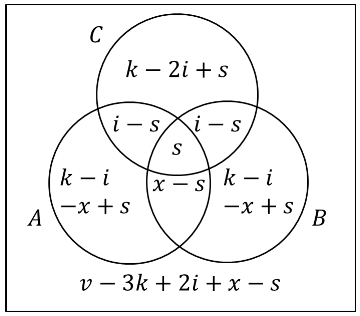
Figure 1: Diagram for proof of Lemma 2.1



By simplifying the resulting inequalities, we find that A and B have a common neighbor if and only if there exists $s \in \mathbb{Z}$, such that

$$\max\{0,\ i+x-k,\ 2i-k\} \leq s \leq \min\{x,\ i,\ v-3k+2i+x\}.$$

Such an integer s exists if and only if the expression on the left-hand side above does not exceed the expression on the right-hand side. Under our global assumptions, this is equivalent to $x \geq \max\{-v+3k-2i, 2i-k\}$. □

Lemma 2.2. *With reference to Definition 1.1, the girth $g(X) = 3$ if and only if $v \geq 3(k-i)$.*

Proof. The graph X contains a 3-cycle if and only if there exist adjacent vertices A and B that have a common neighbor. By Lemma 2.1, this occurs if and only if $i \geq \max\{-v+3k-2i, 2i-k\}$. Since $i \geq 2i-k$ holds in all $J(v,k,i)$ graphs, this condition is equivalent to $v \geq 3(k-i)$. □

A sufficient condition for the girth to be at most 4 is the existence of a 4-cycle.

Lemma 2.3. *With reference to Definition 1.1, if $(v,k,i) \neq (2k+1,k,0)$ then $g(X) \leq 4$.*

Proof. We proceed in two cases.

Case 1: $i \geq 2$ or $v > 2k+1$. In this case we get that $v \geq 2k-i+2$. So we can find disjoint sets, A_1, A_2, A_3, A_4, and B_1, B_2, and C such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$, and $|B_1| = |B_2| = k-i-1$, and $|C| = i$. Then

$$A_1 \cup B_1 \cup C, \quad A_2 \cup B_2 \cup C, \quad A_3 \cup B_1 \cup C, \quad A_4 \cup B_2 \cup C$$

is a 4-cycle in X.

Case 2: $i = 1$. In this case, since $v \geq 2k$, we can find disjoint sets A_1, A_2, A_3, A_4 and B_1, B_2 such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$ and $|B_1| = |B_2| = k-2$. Then

$$A_1 \cup A_2 \cup B_1, \quad A_2 \cup A_3 \cup B_2, \quad A_3 \cup A_4 \cup B_1, \quad A_4 \cup A_1 \cup B_2$$

is a 4-cycle in X. □

Combining the above lemmas, we obtain a general expression for the girth.

Theorem 2.4. *With reference to Definition 1.1, the girth of X is given by*

$$g(X) = \begin{cases} 3 & \text{if } v \geq 3(k-i); \\ 4 & \text{if } v < 3(k-i) \text{ and } (v,k,i) \neq (2k+1,k,0); \\ 5 & \text{if } (v,k,i) = (5,2,0); \\ 6 & \text{if } (v,k,i) = (2k+1,k,0) \text{ and } k > 2. \end{cases}$$

Proof. The first two cases follow from Lemmas 2.2 and 2.3. The remaining cases are odd graphs, for which the girth is well-known. (See, for example, [1, p.58].) $\square$

3. Distance

In this section we derive a general expression for the distance between two vertices in terms of their intersection.

Lemma 3.1. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x < -v + 3k - 2i$, then*

$$\text{dist}(A, B) = 3.$$

Proof. Since $x < i$, $\text{dist}(A, B) \geq 2$. By Lemma 2.1, $\text{dist}(A, B) > 2$. Let $A' \subseteq A \setminus B$, such that $|A'| = i - x$. Let $B' \subseteq B \setminus A$, such that $|B'| = k - i$. Let $C = A \cap B$, and let $D = C \cup A' \cup B'$. Then $|D| = x + (i - x) + (k - i) = k$, and $|A \cap D| = x + (i - x) = i$, so D is a vertex adjacent to A. Note that $|D \cap B| = k - i + x \geq -v + 3k - 2i$. Also, since $x < -v + 3k - 2i$, we have $2i - k < -(v - 2k) - x \leq 0$, so $|D \cap B| \geq 2i - k$. Hence by Lemma 2.1, $\text{dist}(D, B) \leq 2$. Hence $\text{dist}(A, B) = 3$. $\square$

Together with the previous lemma, the next result characterizes the distance between vertices whose intersection is less than i.

Lemma 3.2. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x \geq -v + 3k - 2i$, then*

$$\text{dist}(A, B) = \left\lceil \frac{k - x}{k - i} \right\rceil.$$

Proof. We proceed in two cases.

Case 1: $x \geq 2i - k$. Since $x < i$, we know $\text{dist}(A, B) \geq 2$. Since $x \geq 2i - k$, Lemma 2.1 implies that $\text{dist}(A, B) = 2$. Note that the above inequalities imply $k - i < k - x \leq 2(k - i)$. Hence $\lceil \frac{k-x}{k-i} \rceil = 2$.

Case 2: $x < 2i - k$. In this case, $k - x > 2(k - i)$. Therefore, there exist positive integers q, m such that $k - x = (q + 1)(k - i) + m$ with $0 < m \leq k - i$. Let $C = A \cap B$. Then we can write A and B as disjoint unions

$$A = A_1 \cup \cdots \cup A_{q+2} \cup C \quad \text{and} \quad B = B_1 \cup \cdots \cup B_{q+2} \cup C,$$

where $|A_j| = |B_j| = k - i$ for $j \in \{1, \ldots, q + 1\}$ and $|A_{q+2}| = |B_{q+2}| = m$. Define

$$X_j = (B_1 \cup \cdots \cup B_j) \cup (A_{j+1} \cup \cdots \cup A_{q+2}) \cup C$$

for each $j \in \{1, \ldots, q\}$. Then $A, X_1, \ldots, X_q$ is a path of length q. Note that $|X_q \cap B| = x + q(k - i) = i - m$, so $2i - k \leq |X_q \cap B| < i$ and therefore Case 1 applies. Thus, $\text{dist}(X_q, B) = 2$ and so $\text{dist}(A, B) \leq q + 2 = \lceil \frac{k-x}{k-i} \rceil$. On the other hand, since adjacent vertices differ by $k - i$ elements, $\text{dist}(A, B) \geq \lceil \frac{k-x}{k-i} \rceil$. $\square$

We now address the case where the intersection between A and B is greater than i. The following lemma adapts Lemmas 1 and 2 in [6] to generalized Johnson graphs.

Lemma 3.3. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x > i$ and assume there is an AB-path of length d.*

(i) If $d = 2p$, then

$$p \geq \left\lceil \frac{k - x}{v - 2k + 2i} \right\rceil.$$

(ii) If $d = 2p+1$, then

$$p \geq \left\lceil \frac{x-i}{v-2k+2i} \right\rceil.$$

Proof. For brevity, let $\Delta = v - 2k + 2i$. If $d = 0$, then $A = B$ so, $x = k$ and $p = 0 \geq \lceil \frac{k-x}{\Delta} \rceil$. If $d = 1$, then $x = i$, so $p = 0 \geq \lceil \frac{x-i}{\Delta} \rceil$. If $d = 2$, then by Lemma 2.1, $x \geq -v + 3k - 2i$, which implies $k - x \leq \Delta$. Hence, $p = 1 \geq \lceil \frac{k-x}{\Delta} \rceil$. Assume $d \geq 3$ and that the claim holds for all paths of length less than d. We proceed in two cases.

Case 1: $d = 2p$. We can find a vertex C such that $\mathrm{dist}(A, C) = 2(p-1)$ and $\mathrm{dist}(C, B) = 2$. By the inductive hypothesis, $k - |A \cap C| \leq (p-1)\Delta$ and $k - |C \cap B| \leq \Delta$. Therefore, $k - x = |A \setminus B| \leq |A \setminus C| + |C \setminus B| = (k - |A \cap C|) + (k - |C \cap B|) \leq p\Delta$. Hence $p \geq \lceil \frac{k-x}{\Delta} \rceil$.

Case 2: $d = 2p+1$. We can find a vertex C adjacent to B and such that $\mathrm{dist}(A, C) = 2p$. By the inductive hypothesis, $|A \setminus C| \leq p\Delta$. Therefore, $x - i = |A \cap B| - i \leq |A \setminus C| + |B \cap C| - i \leq p\Delta$. Hence $p \geq \lceil \frac{x-i}{\Delta} \rceil$. □

The previous lemma implies a lower bound on the distance. The next result will show that this bound is sharp.

Lemma 3.4. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Suppose $x > i$. Then*

$$\mathrm{dist}(A, B) = \min\left\{ 2\left\lceil \frac{k-x}{v-2k+2i} \right\rceil, 2\left\lceil \frac{x-i}{v-2k+2i} \right\rceil + 1 \right\}.$$

Proof. For brevity, let $\Delta = v - 2k + 2i$. When $x = k$ the result is trivial, so assume $x < k$. Let $C = A \cap B$ and $D = \overline{A \cup B}$; it follows that $|C| = x$ and $|D| = v - 2k + x$. There exist non-negative integers q, m such that $k - x = q\Delta + m$, with $0 < m \leq \Delta$. We can write A and B as disjoint unions $A = C \cup \{a_1, \ldots, a_{k-x}\}$ and $B = C \cup \{b_1, \ldots, b_{k-x}\}$. If $q = 0$, then $k - x = m \leq \Delta$, which implies $x \geq -v + 3k - 2i$. Since $x > i$, we also have $x > 2i - k$. Hence, by Lemma 2.1, $\mathrm{dist}(A, B) = 2$ as needed. Now, assume $q \geq 1$. For $j \in \{1, \ldots, q\}$, let

$$A_j = \{a_1, \ldots, a_{(j-1)\Delta+i}\} \quad \text{and} \quad A'_j = \{a_{j\Delta+1}, \ldots, a_{k-x}\},$$

$$B_j = \{b_1, \ldots, b_{j\Delta}\} \quad \text{and} \quad B'_j = \{b_{j\Delta-i+1}, \ldots, b_{k-x}\},$$

and define

$$X_{2j-1} = D \cup A_j \cup B'_j \quad \text{and} \quad X_{2j} = C \cup B_j \cup A'_j.$$

Then $A, X_1, \ldots, X_{2q}$ is a path of length $2q$. Note that $|X_{2q} \cap B| = k - m \geq k - \Delta = -v + 3k - 2i$. Also, since $m \leq k - x$, we have $|X_{2q} \cap B| = k - m \geq x > i \geq 2i - k$. Hence $\mathrm{dist}(X_{2q}, B) = 2$, by Lemma 2.1. Thus, there is an AB-path of length $2(q+1) = 2\lceil (k-x)/\Delta \rceil$ from A to B.

Now, let $D' \subseteq D$, $C' \subseteq C$ be such that $|D'| = |C'| = x - i$. Let $A' = (B \setminus C') \cup D'$. Then A' is a vertex adjacent to A. Further, $|A' \cap B| = k - x + i > i$. By applying the previous argument to A' and B, there is an $A'B$-path of length $2\lceil \frac{k-(k-x+i)}{\Delta} \rceil = 2\lceil \frac{x-i}{\Delta} \rceil$. By Lemma 3.3, $\mathrm{dist}(A, B) = \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\}$. □

From the above results, we obtain a general formula for the distance between two vertices.

Theorem 3.5. *With reference to Definition 1.1, let A and B be vertices and let $x = |A \cap B|$. Then*

$$dist(A, B) = \begin{cases} 3 & \text{if } \; x < \min\{i, -v + 3k - 2i\}; \\ \lceil \frac{k-x}{k-i} \rceil & \text{if } \; -v + 3k - 2i \leq x < i; \\ \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\} & \text{if } \; x \geq i. \end{cases}$$

Proof. Apply Lemmas 3.1, 3.2, and 3.4. Note that when $x = i$, we have $\mathrm{dist}(A, B) = 1 = \min\{2\lceil \frac{k-x}{v-2k+2i} \rceil, 2\lceil \frac{x-i}{v-2k+2i} \rceil + 1\}$. $\square$

4. Diameter

In this section, we will use Theorem 3.5 to derive a general expression for the diameter of generalized Johnson graphs. The following lemma determines the maximum value of the expression in Lemma 3.4.

Lemma 4.1. *Assume $k > i + 1$ and let $f(x) = \min\left\{2\left\lceil \frac{k-x}{v-2k+2i} \right\rceil, 2\left\lceil \frac{x-i}{v-2k+2i} \right\rceil + 1\right\}$. Then*

$$\max_{x \in \mathcal{I}} f(x) = \left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1,$$

where $\mathcal{I} = \{i+1, \dots, k\}$.

Proof. For brevity, let $\Delta = v - 2k + 2i$ and let $x \in \mathcal{I}$. There exist $\epsilon \in \{0, 1\}$ and non-negative integers q, m such that $k - i - 1 = (2q + \epsilon)\Delta + m$ and $0 < m \leq \Delta$. We prove $\max_{x \in \mathcal{I}} f(x) = 2q + \epsilon + 2$.

Let $x_0 = (q + \epsilon)\Delta + i$. If $x > x_0$, then $2\lceil \frac{k-x}{\Delta} \rceil \leq 2\lceil \frac{k-(x_0+1)}{\Delta} \rceil = 2(q+1) \leq 2q + \epsilon + 2$. If $x \leq x_0$, then $2\lceil \frac{x-i}{\Delta} \rceil + 1 \leq 2\lceil \frac{x_0-i}{\Delta} \rceil + 1 = 2(q+\epsilon) + 1 \leq 2q + \epsilon + 2$. Hence, $f(x) \leq 2q + \epsilon + 2$.

Let $x_1 = q\Delta + i + 1 + \epsilon(m-1) \in \mathcal{I}$. It follows that $\lceil \frac{k-x_1}{\Delta} \rceil = q + \epsilon + 1$ and $\lceil \frac{x_1-i}{\Delta} \rceil = q + 1$. Therefore, $f(x_1) = \min\{2(q+\epsilon+1), 2q+3\} = 2q + \epsilon + 2$. It follows that $\max_{x \in \mathcal{I}} f(x) = 2q + \epsilon + 2$. $\square$

We now present our main result, which extends and corrects that in [3].

Theorem 4.2. *With reference to Definition 1.1, we have*

$$\mathrm{diam}(X) = \begin{cases} \lceil \frac{k-i-1}{v-2k+2i} \rceil + 1 & \text{if } v < 3(k-i) - 1 \text{ or } i = 0; \\ 3 & \text{if } 3(k-i) - 1 \leq v < 3k - 2i \text{ and } i \neq 0; \\ \lceil \frac{k}{k-i} \rceil & \text{if } v \geq 3k - 2i \text{ and } i \neq 0. \end{cases}$$

Proof. We will use the distance expression from Theorem 3.5. We proceed in three cases.

Case 1: $v < 3(k-i) - 1$ or $i = 0$. If $i = 0$, the result is proved in [5]. Assume $v < 3(k-i) - 1$. In this case $\lceil \frac{k-i-1}{v-2k+2i} \rceil + 1 \geq 3$. Also, $2k \leq v < 3(k-i)$, so $\lceil \frac{k}{k-i} \rceil \leq \lceil \frac{3}{2} \rceil = 2$. Hence, $\lceil \frac{k}{k-i} \rceil \leq 3 \leq \lceil \frac{k-i-1}{v-2k+2i} \rceil + 1$. Since $0 \leq i < k < v < 3(k-i) - 1$ by Definition 1.1, it follows that $k > i + 1$. By Lemma 4.1, there exist vertices A and B such that $\mathrm{dist}(A, B) = \lceil \frac{k-i-1}{v-2k+2i} \rceil + 1$. From Theorem 3.5, it follows that $\mathrm{diam}(X) = \lceil \frac{k-i-1}{v-2k+2i} \rceil + 1$

Case 2: $3(k-i) - 1 \leq v < 3k - 2i$ and $i \neq 0$. Since $v \geq 3(k-i)$, we have $\lceil \frac{k-i-1}{v-2k+2i} \rceil + 1 \leq 2$. Since $2k \leq v < 3k - 2i$, we have $\lceil \frac{k}{k-i} \rceil \leq 2$. By Theorem 3.5, if A and B are disjoint vertices, $\mathrm{dist}(A, B) = 3$; hence $\mathrm{diam}(X) = 3$.

Case 3: $v \geq 3k - 2i$ and $i \neq 0$. In this case $\lceil \frac{k-i-1}{v-2k+2i} \rceil + 1 \leq 2$. Since $v \geq 3k - 2i$, we have $-v + 3k - 2i \leq 0$, so the first case in Theorem 3.5 does not occur. Since $i \neq 0$, we have $\lceil \frac{k}{k-i} \rceil \geq 2$. If A and B are disjoint vertices, $\mathrm{dist}(A, B) = \lceil \frac{k}{k-i} \rceil$, by Theorem 3.5. Hence $\mathrm{diam}(X) = \lceil \frac{k}{k-i} \rceil$. $\square$

References

[1] N. Biggs. *Algebraic Graph Theory.* Cambridge University Press, Cambridge, 1974.

[2] Bor-Liang Chen and Ko-Wei Lih. Hamiltonian uniform subset graphs. *Journal of Combinatorial Theory B*, 42:257–263, 1987.

[3] Yongzhu Chen and Weifan Wang. Diameters of uniform subset graphs. *Discrete Mathematics*, 308:6645–6649, 2008.

[4] C. D. Godsil and G. Royle. *Algebraic Graph Theory*. Springer-Verlag, New York, 2001.

[5] Mario Valencia-Pabon and Juan-Carlos Vera. On the diameter of Kneser graphs. *Discrete Mathematics.*, 305:383–385, 2005.

[6] Yingqian Wang Yongzhu Chen. On the diameter of generalized Kneser graphs. *Discrete Mathematics*, 308:4276–4279, 2008.