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On the girth and diameter of generalized Johnson graphs

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Abstract

Let \(v > k > i\) be non-negative integers. The generalized Johnson graph, \(J(v, k, i)\), is the graph whose vertices are the \(k\)-subsets of a \(v\)-set, where vertices \(A\) and \(B\) are adjacent whenever \(|A \cap B| = i\). In this article, we derive general formulas for the girth and diameter of \(J(v, k, i)\). Additionally, we provide a formula for the distance between any two vertices \(A\) and \(B\) in terms of the cardinality of their intersection.

Keywords: girth; diameter; generalized Johnson graphs; uniform subset graphs

1. Introduction

Let \(v > k > i\) be non-negative integers. The generalized Johnson graph, \(X = J(v, k, i)\), is the graph whose vertices are the \(k\)-subsets of a \(v\)-set, where vertices \(A\) and \(B\) are adjacent whenever \(|A \cap B| = i\). Generalized Johnson graphs were introduced by Chen and Lih in [2]. Special cases include the Kneser graphs \(J(v, k, 0)\), the odd graphs \(J(2k+1, k, 0)\), and the Johnson graphs \(J(v, k, k-1)\). The Johnson graph \(J(v, k, k-1)\) is well known to have diameter \(\min\{k, v-k\}\), and formulas for the distance and diameter of Kneser graphs were proved in [5].

Generalized Johnson graphs have also been studied under the name uniform subset graphs, and a result in [3] offers a general formula for the diameter of \(J(v, k, i)\). However, that formula gives incorrect values when \(i > \frac{v}{2}k\), an important case that includes the Johnson graphs. In this paper we extend (and, in places, correct) those expressions for the diameter of generalized Johnson graphs and we additionally provide a formula for the girth.

Note that it is possible to extend the definition of \(X = J(v, k, i)\) to include \(v \geq k \geq i\). However, \(X\) is an empty graph when \(k = i\) or \(v = k\). If \(v = 2k\) and \(i = 0\), then \(X\) is isomorphic to the disjoint union of copies of \(K_2\). Furthermore, by taking complements, the graphs \(J(v, k, i)\) and \(J(v, v-k, v-2k+i)\) are easily seen to be isomorphic (see [4, p.11]). For the remainder of this article, we will be concerned with generalized Johnson graphs that are connected, so we make the following global definition.

Definition 1.1. Assume \(v > k > i\) are nonnegative integers, and let \(X = J(v, k, i)\) denote the corresponding generalized Johnson graph. To avoid trivialities, further assume that \(v \geq 2k\), and that \((v, k, i) \neq (2k, k, 0)\).

2. Girth

In this section we derive an expression for the girth \(g(X)\) of a generalized Johnson graph, \(X\). We begin with a lemma that characterizes when two vertices have a common neighbor.

Lemma 2.1. With reference to Definition 1.1, let \(A\) and \(B\) be vertices and let \(x = |A \cap B|\). Then \(A\) and \(B\) have a common neighbor if and only if \(x \geq \max\{-v + 3k - 2i, 2i - k\}\).

Proof. Vertices \(A\) and \(B\) have a common neighbor \(C\) if and only if there exists a nonnegative integer \(s\), such that every region in the following diagram (Figure 1) has nonnegative size.

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By simplifying the resulting inequalities, we find that $A$ and $B$ have a common neighbor if and only if there exists $s \in \mathbb{Z}$, such that
\[
\max\{0, i + x - k, 2i - k\} \leq s \leq \min\{x, i, v - 3k + 2i + x\}.
\]
Such an integer $s$ exists if and only if the expression on the left-hand side above does not exceed the expression on the right-hand side. Under our global assumptions, this is equivalent to $x \geq \max\{-v + 3k - 2i, 2i - k\}$.

\[\square\]

**Lemma 2.2.** With reference to Definition 1.1, the girth $g(X) = 3$ if and only if $v \geq 3(k - i)$.

**Proof.** The graph $X$ contains a 3-cycle if and only if there exist adjacent vertices $A$ and $B$ that have a common neighbor. By Lemma 2.1, this occurs if and only if $i \geq \max\{-v + 3k - 2i, 2i - k\}$. Since $i \geq 2i - k$ holds in all $J(v, k, i)$ graphs, this condition is equivalent to $v \geq 3(k - i)$.

A sufficient condition for the girth to be at most 4 is the existence of a 4-cycle.

**Lemma 2.3.** With reference to Definition 1.1, if $(v, k, i) \neq (2k + 1, k, 0)$ then $g(X) \leq 4$.

**Proof.** We proceed in two cases.

**Case 1:** $i \geq 2$ or $v > 2k + 1$. In this case we get that $v \geq 2k - i + 2$. So we can find disjoint sets, $A_1, A_2, A_3, A_4$, and $B_1, B_2$, and $C$ such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$, and $|B_1| = |B_2| = k - i - 1$, and $|C| = i$. Then
\[
A_1 \cup B_1 \cup C,\ A_2 \cup B_2 \cup C,\ A_3 \cup B_1 \cup C,\ A_4 \cup B_2 \cup C
\]
is a 4-cycle in $X$.

**Case 2:** $i = 1$. In this case, since $v \geq 2k$, we can find disjoint sets $A_1, A_2, A_3, A_4$ and $B_1, B_2$ such that $|A_1| = |A_2| = |A_3| = |A_4| = 1$ and $|B_1| = |B_2| = k - 2$. Then
\[
A_1 \cup A_2 \cup B_1,\ A_2 \cup A_3 \cup B_2,\ A_3 \cup A_4 \cup B_1,\ A_4 \cup A_1 \cup B_2
\]
is a 4-cycle in $X$.

Combining the above lemmas, we obtain a general expression for the girth.

**Theorem 2.4.** With reference to Definition 1.1, the girth of $X$ is given by
\[
g(X) = \begin{cases} 
3 & \text{if } v \geq 3(k - i); \\
4 & \text{if } v < 3(k - i) \text{ and } (v, k, i) \neq (2k + 1, k, 0); \\
5 & \text{if } (v, k, i) = (5, 2, 0); \\
6 & \text{if } (v, k, i) = (2k + 1, k, 0) \text{ and } k > 2.
\end{cases}
\]
Lemma 3.3. With reference to Definition 1.1, let $A$ and $B$ be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x < -v + 3k - 2i$, then
\[ \text{dist}(A, B) = 3. \]

Proof. Since $x < i$, $\text{dist}(A, B) \geq 2$. By Lemma 2.1, $\text{dist}(A, B) > 2$. Let $A' \subseteq A \setminus B$, such that $|A'| = i - x$. Let $B' \subseteq B \setminus A$, such that $|B'| = k - i$. Let $C = A \cap B$, and let $D = C \cup A' \cup B'$. Then $|D| = x + (i - x) + (k - i) = k$, and $|A \cap D| = x + (i - x) = i$, so $D$ is a vertex adjacent to $A$. Note that $|D \cap B| = k - i + x \geq -v + 3k - 2i$. Also, since $x < -v + 3k - 2i$, we have $2i - k < -(v - 2k) - x \leq 0$, so $|D \cap B| \geq 2i - k$. Hence by Lemma 2.1, $\text{dist}(D, B) \leq 2$. Hence $\text{dist}(A, B) = 3$. \[\square\]

Together with the previous lemma, the next result characterizes the distance between vertices whose intersection is less than $i$.

Lemma 3.2. With reference to Definition 1.1, let $A$ and $B$ be vertices and let $x = |A \cap B|$. Suppose $x < i$. If $x \geq -v + 3k - 2i$, then
\[ \text{dist}(A, B) = \left\lceil \frac{k - x}{k - i} \right\rceil. \]

Proof. We proceed in two cases.

Case 1: $x \geq 2i - k$. Since $x < i$, we know $\text{dist}(A, B) \geq 2$. Since $x \geq 2i - k$, Lemma 2.1 implies that $\text{dist}(A, B) = 2$. Note that the above inequalities imply $k - i < k - x \leq 2(k - i)$. Hence $\left\lceil \frac{k - x}{k - i} \right\rceil = 2$.

Case 2: $x < 2i - k$. In this case, $k - x > 2(k - i)$. Therefore, there exist positive integers $q, m$ such that $k - x = (q + 1)(k - i) + m$ with $0 < m \leq k - i$. Let $C = A \cap B$. Then we can write $A$ and $B$ as disjoint unions
\[ A = A_1 \cup \cdots \cup A_{q+2} \cup C \quad \text{and} \quad B = B_1 \cup \cdots \cup B_{q+2} \cup C, \]
where $|A_j| = |B_j| = k - i$ for $j \in \{1, \ldots, q + 1\}$ and $|A_{q+2}| = |B_{q+2}| = m$. Define
\[ X_j = (B_1 \cup \cdots \cup B_j) \cup (A_{j+1} \cup \cdots \cup A_{q+2}) \cup C \]
for each $j \in \{1, \ldots, q\}$. Then $A, X_1, \ldots, X_q$ is a path of length $q$. Note that $|X_q \cap B| = x + q(k - i) = i - m$, so $2i - k \leq |X_q \cap B| < i$ and therefore Case 1 applies. Thus, $\text{dist}(X_q, B) = 2$ and so $\text{dist}(A, B) \leq q + 2 = \left\lceil \frac{k - x}{k - i} \right\rceil$. On the other hand, since adjacent vertices differ by $k - i$ elements, $\text{dist}(A, B) \geq \left\lceil \frac{k - x}{k - i} \right\rceil$. \[\square\]

We now address the case where the intersection between $A$ and $B$ is greater than $i$. The following lemma adapts Lemmas 1 and 2 in [6] to generalized Johnson graphs.

Lemma 3.3. With reference to Definition 1.1, let $A$ and $B$ be vertices and let $x = |A \cap B|$. Suppose $x > i$ and assume there is an $AB$-path of length $d$.

(i) If $d = 2p$, then
\[ p \geq \left\lceil \frac{k - x}{v - 2k + 2i} \right\rceil. \]
The inductive hypothesis, with reference to Definition 1.1, let bound is sharp.

Proof. For brevity, let $\Delta = v - 2k + 2i$. If $d = 0$, then $A = B$ so, $x = k$ and $p = 0 \geq \left\lfloor \frac{k-x}{\Delta} \right\rfloor$. If $d = 1$, then $x = i$, so $p = 0 \geq \left\lfloor \frac{x}{\Delta} \right\rfloor$. If $d = 2$, then by Lemma 2.1, $x \geq -v + 3k - 2i$, which implies $k - x \leq \Delta$. Hence, $p = 1 \geq \left\lfloor \frac{k-x}{\Delta} \right\rfloor$. Assume $d \geq 3$ and that the claim holds for all paths of length less than $d$. We proceed in two cases.

Case 1: $d = 2p$. We can find a vertex $C$ such that $\text{dist}(A, C) = 2(p - 1)$ and $\text{dist}(C, B) = 2$. By the inductive hypothesis, $k - |A \cap C| \leq (p - 1)\Delta$ and $k - |C \cap B| \leq \Delta$. Therefore, $k - x = |A \setminus B| \leq |A \setminus C| + |C \setminus B| = (k - |A \cap C|) + (k - |C \cap B|) \leq p\Delta$. Hence $p \geq \left\lfloor \frac{k-x}{\Delta} \right\rfloor$.

Case 2: $d = 2p + 1$. We can find a vertex $C$ adjacent to $B$ and such that $\text{dist}(A, C) = 2p$. By the inductive hypothesis, $|A \setminus C| \leq p\Delta$. Therefore, $x - i = |A \cap B| - i \leq |A \setminus C| + |B \cap C| - i \leq p\Delta$. Hence $p \geq \left\lfloor \frac{k-x}{\Delta} \right\rfloor$.

The previous lemma implies a lower bound on the distance. The next result will show that this bound is sharp.

Lemma 3.4. With reference to Definition 1.1, let $A$ and $B$ be vertices and let $x = |A \cap B|$. Suppose $x > i$. Then

$$\text{dist}(A, B) = \min \left\{ 2 \left\lfloor \frac{k-x}{v - 2k + 2i} \right\rfloor, 2 \left\lfloor \frac{x-i}{v - 2k + 2i} \right\rfloor + 1 \right\}.$$ 

Proof. For brevity, let $\Delta = v - 2k + 2i$. When $x = k$ the result is trivial, so assume $x < k$. Let $C = A \cap B$ and $D = \overline{A} \cup \overline{B}$; it follows that $|C| = x$ and $|D| = v - 2k + x$. There exist non-negative integers $q, m$ such that $k - x = q\Delta + m$, with $0 < m \leq \Delta$. We can write $A$ and $B$ as disjoint unions $A = C \cup \{a_1, \ldots, a_{k-x}\}$ and $B = C \cup \{b_1, \ldots, b_{k-x}\}$. If $q = 0$, then $k - x = m \leq \Delta$, which implies $x \geq -v + 3k - 2i$. Since $x > i$, we also have $x > 2i - k$. Hence, by Lemma 2.1, $\text{dist}(A, B) = 2$ as needed. Now, assume $q \geq 1$. For $j \in \{1, \ldots, q\}$, let

$$A_j = \{a_1, \ldots, a_{(j-1)\Delta+i}\} \quad \text{and} \quad A_j' = \{a_{j\Delta+1}, \ldots, a_{k-x}\},$$

$$B_j = \{b_1, \ldots, b_{\Delta}\} \quad \text{and} \quad B_j' = \{b_{\Delta+i+1}, \ldots, b_{k-x}\},$$

and define

$$X_{2j-1} = D \cup A_j \cup B_j' \quad \text{and} \quad X_{2j} = C \cup B_j \cup A_j'.$$

Then $A, X_1, \ldots, X_{2q}$ is a path of length $2q$. Note that $|X_{2q} \cap B| = k - m \geq k - \Delta = -v + 3k - 2i$. Also, since $m \leq k - x$, we have $|X_{2q} \cap B| = k - m \geq x > i \geq 2i - k$. Hence $\text{dist}(X_{2q}, B) = 2$, by Lemma 2.1. Thus, there is an $AB$-path of length $2(q+1) = 2\left\lfloor (k-x)/\Delta \right\rfloor$ from $A$ to $B$.

Now, let $D' \subseteq D$, $C' \subseteq C$ be such that $|D'| = |C'| = x - i$. Let $A' = (B \setminus C') \cup D'$. Then $A'$ is a vertex adjacent to $A$. Further, $|A' \cap B| = k - x + i > i$. By applying the previous argument to $A'$ and $B$, there is an $A'B$-path of length $2\left\lfloor k - (k-x+1)/\Delta \right\rfloor = 2\left\lfloor \frac{x-i}{\Delta} \right\rfloor$. By Lemma 3.3, $\text{dist}(A, B) = \min\{2\left\lfloor \frac{k-x}{v - 2k + 2i} \right\rfloor, 2\left\lfloor \frac{x-i}{v - 2k + 2i} \right\rfloor + 1\}$. □

From the above results, we obtain a general formula for the distance between two vertices.

Theorem 3.5. With reference to Definition 1.1, let $A$ and $B$ be vertices and let $x = |A \cap B|$. Then

$$\text{dist}(A, B) = \begin{cases} 
\frac{3}{2} \left[ \frac{k-x}{v - 2k + 2i} \right] & \text{if } x < \min\{i, -v + 3k - 2i\}; \\
\min\{2\left[ \frac{k-x}{v - 2k + 2i} \right], 2\left[ \frac{x-i}{v - 2k + 2i} \right] + 1\} & \text{if } x \geq i. 
\end{cases}$$
Proof. Apply Lemmas 3.1, 3.2, and 3.4. Note that when $x = i$, we have $\text{dist}(A, B) = 1 = \min\{2\left\lceil \frac{k-x}{v-2k+2i} \right\rceil, 2\left\lceil \frac{x-i}{v-2k+2i} \right\rceil + 1\}$. \hfill \square

4. Diameter

In this section, we will use Theorem 3.5 to derive a general expression for the diameter of generalized Johnson graphs. The following lemma determines the maximum value of the expression in Lemma 3.4.

Lemma 4.1. Assume $k > i + 1$ and let $f(x) = \min\{2\left\lceil \frac{k-x}{v-2k+2i} \right\rceil, 2\left\lceil \frac{x-i}{v-2k+2i} \right\rceil + 1\}$. Then

$$\max_{x \in \mathcal{I}} f(x) = \left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1,$$

where $\mathcal{I} = \{i + 1, \ldots, k\}$.

Proof. For brevity, let $\Delta = v - 2k + 2i$ and let $x \in \mathcal{I}$. There exist $\epsilon \in \{0, 1\}$ and non-negative integers $q, m$ such that $k - i - 1 = (2q + \epsilon) \Delta + m$ and $0 < m \leq \Delta$. We prove $\max_{x \in \mathcal{I}} f(x) = 2q + \epsilon + 2$.

Let $x_0 = (q + \epsilon) \Delta + i$. If $x > x_0$, then $2\left\lceil \frac{k-x}{\Delta} \right\rceil \leq 2\left\lceil \frac{k-(x_0+1)}{\Delta} \right\rceil = 2(q+1) \leq 2q + \epsilon + 2$. If $x < x_0$, then $2\left\lceil \frac{x-i}{\Delta} \right\rceil + 1 \leq 2\left\lceil \frac{x-i}{\Delta} \right\rceil + 1 = 2(q+\epsilon) + 1 \leq 2q + \epsilon + 2$. Hence, $f(x) \leq 2q + \epsilon + 2$.

Let $x_1 = q\Delta + i + 1 + \epsilon(m-1) \in \mathcal{I}$. It follows that $\left\lceil \frac{k-x_1}{\Delta} \right\rceil = q + \epsilon + 1$ and $\left\lceil \frac{x_1-i}{\Delta} \right\rceil = q + 1$. Therefore, $f(x_1) = \min\{2(q+\epsilon+1), 2q+3\} = 2q + \epsilon + 2$. It follows that $\max_{x \in \mathcal{I}} f(x) = 2q + \epsilon + 2$. \hfill \square

We now present our main result, which extends and corrects that in [3].

Theorem 4.2. With reference to Definition 1.1, we have

$$\text{diam}(X) = \begin{cases} 
\left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1 & \text{if } v < 3(k-i) - 1 \text{ or } i = 0; \\
3 & \text{if } 3(k-i) - 1 \leq v < 3k - 2i \text{ and } i \neq 0; \\
\left\lceil \frac{k}{v-k-1} \right\rceil & \text{if } v \geq 3k - 2i \text{ and } i \neq 0.
\end{cases}$$

Proof. We will use the distance expression from Theorem 3.5. We proceed in three cases.

Case 1: $v < 3(k-i) - 1$ or $i = 0$. If $i = 0$, the result is proved in [5]. Assume $v < 3(k-i) - 1$. In this case $\left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1 \geq 3$. Also, $2k \leq v < 3(k-i)$, so $\left\lceil \frac{k}{v-k-1} \right\rceil \leq \left\lceil \frac{3}{2} \right\rceil = 2$. Hence, $\left\lceil \frac{k}{v-k-1} \right\rceil \leq \left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1$. Since $0 \leq i < k < v < 3(k-i) - 1$ by Definition 1.1, it follows that $k > i + 1$. By Lemma 4.1, there exist vertices $A$ and $B$ such that $\text{dist}(A, B) = \left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1$. From Theorem 3.5, it follows that $\text{diam}(X) = \left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1$.

Case 2: $3(k-i) - 1 \leq v < 3k - 2i$ and $i \neq 0$. Since $v \geq 3(k-i)$, we have $\left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1 \leq 2$. Since $2k \leq v < 3k - 2i$, we have $\left\lceil \frac{k}{v-k-1} \right\rceil \leq 2$. By Theorem 3.5, if $A$ and $B$ are disjoint vertices, $\text{dist}(A, B) = 3$; hence $\text{diam}(X) = 3$.

Case 3: $v \geq 3k - 2i$ and $i \neq 0$. In this case $\left\lceil \frac{k-i-1}{v-2k+2i} \right\rceil + 1 \leq 2$. Since $v \geq 3k - 2i$, we have $-v + 3k - 2i \leq 0$, so the first case in Theorem 3.5 does not occur. Since $i \neq 0$, we have $\left\lceil \frac{k}{v-k-1} \right\rceil \geq 2$. If $A$ and $B$ are disjoint vertices, $\text{dist}(A, B) = \left\lceil \frac{k}{v-k-1} \right\rceil$, by Theorem 3.5. Hence $\text{diam}(X) = \left\lceil \frac{k}{v-k-1} \right\rceil$. \hfill \square

References


