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## Ideals, Big Varieties, and Dynamic Networks

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# IDEALS, BIG VARIETIES, AND DYNAMIC NETWORKS

IAN H DINWOODIE

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ABSTRACT. The advantage of using algebraic geometry over enumeration for describing sets related to attractors in large dynamic networks from biology is advocated. Examples illustrate the gains.

Biological applications of dynamical systems can be in discrete spaces like  $\{0, 1\}^5$ , where “1” may code for “on” or “high” or “cancer cell terminated” and “0” may code for “off” or “low” or “cancer cell still going fine.” Then the subset

$$***01 := \{00001, 00101, 01001, 01101, 10001, 10101, 11001, 11101\}$$

can be enumerated easily, and the rule for membership is clear:  $x_4 = 0, x_5 = 1$ .

Nowadays, some dynamic networks on  $S := \{0, 1\}^d$  (called Boolean networks) are fitted or “learned” or modeled with large  $d$ ,  $d$  being the number of molecules or proteins or variables, using data and combined knowledge from experimentation and literature. The data involved for fitting may be Big Data, a term for large data sets often gathered mechanically ([21]), and the fitting procedures may include machine learning methods that can give complex models. One example is in [9], where a dynamical model of 15 nodes after variable selection was fitted with machine and statistical learning methods to data from motion sensors (see [13], [15], [22] for background on the data).

There are computational advantages in describing sets of interest as *varieties* – sets of roots of multivariate polynomials. Each coordinate gets an indeterminate, so for the set  $***01$  above we could describe it as the roots to two polynomials

$$\langle s_4, s_5 - 1 \rangle$$

in a ring of polynomials  $\mathbb{C}[s_1, s_2, s_3, s_4, s_5]$ . This set of two polynomials is the *ideal* for the variety  $***01$ . For large problems such as the  $d = 58$  node example of Section 4, it is useful to use finite field coefficients to speed computations. So with field coefficients in characteristic 2, simply  $\{0, 1\}$  with mod 2 operations, we have the polynomial ring  $F_2[s_1, s_2, s_3, s_4, s_5]$ . The elements of  $***01$  are the simultaneous roots of

$$\langle s_4, s_5 + 1 \rangle.$$

The entire state space  $\{0, 1\}^d$  has size  $2^d$ , yet it can be described as the roots of only  $d$  polynomials, the logarithm of its cardinality:

$$\langle s_1^2 + s_1, \dots, s_d^2 + s_d \rangle.$$

So useful subsets of a big state space  $S$  can sometimes be studied more efficiently with the polynomials that vanish on the subset, rather than by enumeration.

Traditional logical models for dynamics were small and conceptual ([30]), but dynamical models on discrete state spaces have become larger as more data has helped fit and verify more complex interactions (see [18], [19], [26], [27], [28], [32]). Commutative

algebra – the algebra of multivariate polynomials – has become one tool for doing calculations (see [24]). The methods and foundations for using commutative algebra are in [5], whose title this article plays on, and [16]. Software implementations include CoCoA [1], Macaulay2 [12] and Singular [7]. The essential operations for our applications are elimination and primary decomposition.

The new example here is an exact calculation on the network of 58 nodes as described in [33], which first appeared in [32].

## 1. BOOLEAN NETWORKS.

The traditional Boolean network with values in  $\{0, 1\}$  at each node is exemplified by the *lac* operon network formulated in [14] and described in [24]. In logical notation the dynamics are

$$\begin{aligned} f_1(x_1, x_2, x_3) &= !u_1 \& (x_3 \mid u_2) \\ f_2(x_1, x_2, x_3) &= x_1 \\ f_3(x_1, x_2, x_3) &= !u_1 \& ((x_2 \& u_2) \mid (x_3 \& !x_2)) \end{aligned}$$

where  $u_1, u_2 \in \{0, 1\}$  are fixed parameters for control that can be set at 0 or 1, “!” means “not” and “|” means “or.” Coding Boolean functions as polynomials is a standard procedure, with early foundations in [2], [34]. In practice it can usually be done easily without applying general theory.

A control problem for the *lac* operon is to find values of  $u_1, u_2$  that lead to steady state (fixed point)  $x_1 x_2 x_3 = 111$  (which means the operon is “on”). The problem has a geometric interpretation as follows. Expand the dynamics to  $F : \{0, 1\}^5 \rightarrow \{0, 1\}^5$  with  $F = (f_1, f_2, f_3, f_4, f_5)$  where  $f_4 = u_1$  and  $f_5 = u_2$  fix  $u_1$  and  $u_2$  respectively. Now we are looking for subsets of the form  $***u_1u_2$  inside the basin of attraction of steady state  $111u_1u_2$ , where “\*” is a wild card place holder and  $u_1, u_2$  can be any values. Such a subset will have the property that setting  $u_1, u_2$  at the specified values will drive the system to  $(1, 1, 1, u_1, u_2)$  regardless of the initial values of  $x_1, x_2, x_3$ . The map  $F$  has four steady states, including one useful one  $11101$ . The basin of attraction of this steady state is all eight points  $***01$ , so any initial values for  $x_1 x_2 x_3$  with  $u_1 = 0, u_2 = 1$  will lead to the “on” steady state. This is a small version of the general control problem for large networks in biology like our main example in Section 4.

Standard software for Boolean networks identifies basins of attraction by complete enumeration, and will solve this problem easily. However, in standard software the number of nodes or coordinate values is limited (typically to around 30 for basin analysis), their values must be 0-1, and the output as a list of elements does not explain what conditions imply membership. For large problems, a geometric approach can be better for computations and for comprehension.

## 2. BASINS OF ATTRACTION.

A Boolean network with  $d$  coordinates or nodes has a state space  $\{0, 1\}^d$ , a  $d$ -fold product of logical or off/on or low/high symbols 0 and 1. Let  $F = (f_1, \dots, f_d)$  be an update function on  $\{0, 1\}^d$ , where  $f_j : \{0, 1\}^d \rightarrow \{0, 1\}$  is the rule for node  $j$ . This map is deterministic, called the synchronous update. For a state  $\mathbf{x}$ , define the limiting set

$$A_{\mathbf{x}} = \bigcap_{k=1}^{\infty} \bigcup_{n=k}^{\infty} \{F^n(\mathbf{x})\}$$

where  $F^n$  is the  $n$ -fold composition of  $F$ . The resulting sets, as  $\mathbf{x}$  varies in the state space  $\{0, 1\}^d$ , are the limiting sets or attractor sets of the system. These are fixed points or cycles.

Let  $B_A$  be the usual deterministic basin of attraction for an attractor  $A$ , the points that eventually hit  $A$ :

$$B_A := \bigcup_{k=1}^{\infty} \{\mathbf{x} : F^k(\mathbf{x}) \in A\} = \lim_{k \rightarrow \infty} F^{-k}(A).$$

The main control problem is to find simple conditions for membership of an initial state in the attractor basin  $B_A$ , as in the *lac* operon opening example where membership in  $B_{\{11101\}}$  was determined by simple conditions  $u_1 = 0, u_2 = 1$ . These membership rules give the set  $***01$  which we call a *cylinder set*, from the probability terminology for product spaces where a few of the coordinates are restricted and others are free.

But biological applications need a randomized version of the map  $F$  ([4], [25], [32]), since the timing of the coordinate maps may not be synchronized perfectly in the real world. One type of asynchronous updates, the one we use, proceeds by choosing a node or coordinate  $j$  with the uniform probability distribution, then changing current state  $\mathbf{x}$  to  $\mathbf{x}'$  like:

$$\mathbf{x} = (x_1, \dots, x_{j-1}, x_j, x_{j+1}, \dots, x_d) \rightarrow (x_1, \dots, x_{j-1}, f_j(\mathbf{x}), x_{j+1}, \dots, x_d) = \mathbf{x}'.$$

The resulting process  $\mathbf{x}_0, X_1, X_2, X_3, \dots$  is a Markov chain in state space  $\{0, 1\}^d$ , starting at nonrandom initial configuration  $\mathbf{x}_0$  and with random variables  $X_n$  for random states or configurations. The probability that  $X_n$  is in a set  $S$  will be written:

$$P_{\mathbf{x}_0}(X_n \in S).$$

The process has a probability transition matrix of size  $2^d \times 2^d$ . A 60 node example would be impossible to study with the transition matrix because of its size. Control methods that use a transition matrix ([17]) are not realistic for large Boolean networks.

Define the single coordinate updates on a state  $\mathbf{x} = (x_1, \dots, x_d)$  by

$$F_j(\mathbf{x}) = (x_1, \dots, x_{j-1}, f_j(\mathbf{x}), x_{j+1}, \dots, x_d), \quad j = 1, 2, \dots, d.$$

Now with  $B_0 = B_A$ ,  $F_j^{-1}$  the inverse of  $F_j$ , and  $B^c$  the complement of a set  $B$ , define the exclusive asynchronous basin of the attractor  $A$  by

$$B_i = B_{i-1} \setminus \bigcup_{j=1}^d F_j^{-1}(B_{i-1}^c), \quad i = 1, 2, 3, \dots$$

$$B_{ex,A} := \lim_{i \rightarrow \infty} B_i.$$

Probabilistically, this can be described as

$$B_{ex,A} = \{\mathbf{x} : P_{\mathbf{x}}(X_n \in B_A, n = 0, 1, 2, 3, \dots) = 1\}.$$

In words, the points in  $B_{ex,A}$  are the states that can never leave the basin of attraction  $B_A$  over time under perturbations in the form of asynchronous updates.

The exclusive asynchronous basin  $B_{ex,A}$  gives a robust basin for the purpose of targeting useful attractors, and the control problem is characterizing membership in  $B_{ex,A}$  with settings on a few nodes.

### 3. POLYNOMIAL DYNAMICS FOR BASIN IDEALS.

An effective way to compute basins of attraction for large networks uses commutative algebra on dynamics written as polynomials, and basins are represented first with their ideals. Then the associated primes of the ideal of the basin lead to simple cylinder sets (sometimes an associated prime will give more than one cylinder), and these give simple conditions for basin membership and clear control strategies by setting values on nodes that appear in the cylinder sets. Although the notion of exclusive asynchronous basin involves

randomization, the polynomial calculations for basins are exact. This differs from methods based on computer simulation of randomized dynamics.

Now we describe an algorithm for finding  $B_A$ , using finite field coefficients for speed and memory efficiency. Let  $A$  be an attractor of interest, with basin of attraction  $B_A$ . We will use twice as many indeterminates as the number of coordinates  $d$ . Define the ring of polynomials  $R := F_2[s_1, \dots, s_d, t_1, \dots, t_d] = F_2[\mathbf{s}, \mathbf{t}]$ , and define ideals

$$\begin{aligned} I_{01} &= \langle s_1^2 + s_1, \dots, s_d^2 + s_d, t_1^2 + t_1, \dots, t_d^2 + t_d \rangle \\ F_{st} &= \langle f_1(\mathbf{s}) - t_1, f_2(\mathbf{s}) - t_2, \dots, f_d(\mathbf{s}) - t_d \rangle \\ F_{ts} &= \langle f_1(\mathbf{t}) - s_1, f_2(\mathbf{t}) - s_2, \dots, f_d(\mathbf{t}) - s_d \rangle \\ I_A &= \bigcap_{\mathbf{x} \in A} \langle t_1 - x_1, \dots, t_d - x_d \rangle. \end{aligned}$$

Define the ideal  $I_1$  by

$$I_1 = (F_{st} + I_A + I_{01}) \cap F_2[\mathbf{s}],$$

and define recursively a sequence of ideals  $I_2, I_3, I_4, \dots$  by

$$\begin{aligned} J &= (F_{ts} + I_i + I_{01}) \cap F_2[\mathbf{t}] \\ I_{i+1} &= (F_{st} + J + I_{01}) \cap F_2[\mathbf{s}], \quad i = 1, 2, 3, \dots \end{aligned}$$

Stop the iteration when  $I_i \subset I_{i+1}$ , an ideal containment condition that shows the reverse containment of varieties  $\mathbf{V}(I_{i+1}) \subset \mathbf{V}(I_i)$  – when this happens the basin computation has reached its fixed “point” or fixed and final set. Note that this terminating condition can be established using the division algorithm on polynomials, the varieties themselves need not be examined. This comes from the “ideal membership” property of a Groebner basis – membership can be determined by reducing a polynomial using a Groebner basis ([5]). This is a place where algebraic geometry helps with big varieties.

**Theorem 1.** *At a positive integer  $i^*$  where  $I_{i^*} = I_{i^*+1}$ , the variety for  $I_{i^*}$  is the basin  $B_A$ .*

*Proof.* We sketch the ideas in a proof. The first elimination step

$$I_1 = (F_{st} + I_A + I_{01}) \cap F_2[\mathbf{s}]$$

gives the polynomials in  $\mathbf{s}$  indeterminates on the set of points in  $\{0, 1\}^d$  that hit  $A$  in one step,  $F^{-1}(A)$ . Seidenberg’s Lemma ([16]) shows that the ideal is radical. Adding in the state space set  $I_{01}$  guarantees no unwanted points are included after elimination (that is, there are no extra unwanted roots of the polynomials in  $\mathbf{s}$  that actually do not map under  $F_{st}$  to points in  $I_A$ ), by the Extension Theorem ([5]). The Extension Theorem requires a small extension to  $F_2$  coefficients (not an algebraically closed field) but adding the  $I_{01}$  ideal makes it valid ([8]).

Then  $I_i$  is the ideal for points  $\{\mathbf{x} \in \{0, 1\}^d : F^{2i-1}(\mathbf{x}) \in A\}$ , the  $(2i-1)^{st}$  inverse image. When the inverse images become fixed, the set is fixed at the basin of attraction.  $\square$

A similar algorithm can compute the exclusive asynchronous basin  $B_{ex,A}$ . It is also based on elimination, and terminates when an ideal containment condition is satisfied.

#### 4. EXAMPLE.

A 60-node Boolean model for leukemia was formulated in [32], and a version was studied in [25] and [33]. The model is given in Tables 1 and 2 on 58 nodes, two receptors that play no role in Apoptosis being removed. Apoptosis is an outcome where the cancer cell is terminated, and its onset is coded with a node Apoptosis hitting value 1. The goal is

to find out how to make that happen by setting values on other nodes that may be subject to intervention or control.

The model has 7 attractors, which are all cycles of length 4. Three of them have Apoptosis fixed at 1, so these are the ones to study first. Only one has a nonempty exclusive asynchronous basin of attraction, and it is the 4-cycle below, with the Apoptosis coordinate 48 underlined:

```
{0011101110111111011001011101011110111011101100010010101010,
 0111101110111111011001011101011110111011101100010010101010,
 1111101110111111011001011101011110111011101100010010101010,
 1011101110111111011001011101011110111011101100010010101010}.
```

The exclusive asynchronous basin is given by the union of two cylinders, found with the associated primes of its ideal. The membership conditions are

S1P<sup>52</sup>=0, Ceramide<sup>39</sup>=1, sFas<sup>38</sup>=0, Fas<sup>37</sup>=1, FasT<sup>36</sup>=1, FasL<sup>35</sup>=1, TBET<sup>26</sup>=1, IL2RB<sup>19</sup>=1, IL2RBT<sup>18</sup>=1, NFKB<sup>14</sup>=1, PI3K<sup>13</sup>=1, ERK<sup>12</sup>=1, MEK<sup>11</sup>=1, GAP<sup>10</sup>=0, RAS<sup>9</sup>=1, GRB2<sup>7</sup>=1

S1P<sup>52</sup>=0, Ceramide<sup>39</sup>=1, sFas<sup>38</sup>=0, Fas<sup>37</sup>=1, FasT<sup>36</sup>=1, FasL<sup>35</sup>=1, NFKB<sup>14</sup>=1, PI3K<sup>13</sup>=1, PDGFR<sup>3</sup>=1.

in addition to the fixed values 101010 on the last 6 signaling nodes 53-58 specified in Table 2.

In the wild card notation, they can be written:

```
1-12      13-25      26-34      35-52
*****1*1011  11***11*****  1*****  11101*****0
**1*****  1*****  *****  11101*****0
```

So the two cylinder sets above specify the nodes and their initial settings to guarantee a robust evolution towards Apoptosis.

The size of the asynchronous exclusive basin is easy, from inclusion-exclusion:  $2^{58-22} + 2^{58-15} - 2^{58-23} = 8, 830, 452, 760, 576$ . Yet finding and describing the basin did not require enumeration.

There are two out of seven attractors that cycle in the Apoptosis value, and both have empty exclusive asynchronous basins. (Any appearance of Apoptosis in the original model [32] would shut the cell down totally, so such cycles do not appear in that model.)

### 5. GENERALIZED STATE SPACES.

It is not uncommon for network nodes to have values more general than  $\{0, 1\}$ , and the algebraic methods generalize easily. The only changes to the Boolean methodology are to get the transition functions in polynomial form, and to write the state space of the system as the roots of an ideal.

The ‘‘ABC’’ dynamic model for flowering of the plant species *Arabidopsis thaliana* is explained in [29] and transition functions are given in Table 3.

Consider the following indicator polynomials:

$$\begin{aligned}
 l_i &= (x_i^2 - 3x_i + 2)/2 \\
 m_i &= -x_i^2 + 2x_i \\
 h_i &= (x_i^2 - x_i)/2, \quad i = 1, 2, 3.
 \end{aligned}$$

nodes 1-29	logical rule
1	CTLA4
2	TCR
3	Stimuli & !CTLA4
4	PDGFR
5	S1P   PDGF
6	TCR   IL2RB
7	LCK & !FYN
8	CD45   ((TCR   IL2RB) & !ZAP70)
9	LCK & !FYN
10	IL2RB   ZAP70
11	GRB2   PDGFR
12	GRB2   PLCG1 & !GAP
13	RAS
14	(GRB2   PLCG1) & !GAP
15	GAP
16	(RAS   (PDGFR & GAP)) & !(IL15   IL2)
17	MEK
18	RAS
19	ERK
20	MEK & PI3K
21	PI3K
22	PDGFR   RAS
23	NFKB
24	(TPL2   PI3K)   (FLIP & TRADD & IAP)
25	NFAT
26	PI3K
27	RANTES
28	NFKB
29	IL2
30	(NFKB   STAT3   NFAT) & !TBET
31	IL2RBT
32	ERK & TBET
33	IL2RB
34	IL2RBT & (IL2   IL15)
35	IL2RAT
36	IL2 & (STAT3   NFKB)
37	IL2RA
38	IL2 & IL2RAT & !IL2RA
39	JAK
40	(IL2RA   IL2RB   RANTES   IFNG) & !(SOCS   CD45)
41	SOCS
42	JAK & !(IL2   IL15)
43	STAT3
44	JAK
45	P27
46	STAT3
47	TBET
48	JAK   TBET
49	CREB
50	ERK & IFNG
51	IFNGT
52	TBET   STAT3   NFAT
53	IFNG
54	((IL2   IL15   Stimuli) & IFNGT) & !(SMAD   P2)

TABLE 1. Boolean model of T-LGL dynamics from [33].

nodes 30-58	logical rule
30	P2
31	(IFNG   P2) & !Stimuli2
32	GZMB
33	(CREB & IFNG)   TBET
34	TPL2
35	TAX   (PI3K & TNF)
36	TNF
37	NFKB
38	TRADD
39	TNF & !(IAP   A20)
40	FasL
41	STAT3   NFKB   NFAT   ERK
42	FasT
43	NFKB
44	Fas
45	FasT & FasL & !sFas
46	sFas
47	FasT & S1P
48	Ceramide
49	Fas & !S1P
50	DISC
51	FasT & ((Fas & IL2)   Ceramide   (Fas & !FLIP))
52	Caspase
53	((TRADD   GZMB) & BID) & !IAP   DISC
54	FLIP
55	(NFKB   (CREB & IFNG)) & !DISC
56	A20
57	NFKB
58	BID
59	(Caspase   GZMB) & !(BclxL   MCL1)
60	IAP
61	NFKB & !BID
62	BclxL
63	(NFKB   STAT3) & !(BID   GZMB   DISC)
64	MCL1
65	(IL2RB & STAT3 & NFKB & PI3K) & !DISC
66	Apoptosis
67	Caspase
68	GPCR
69	S1P
70	SMAD
71	GPCR
72	SPHK1
73	PDGFR
74	S1P
75	SPHK1 & !Ceramide
76	Stimuli
77	1
78	Stimuli2
79	0
80	PDGF
81	1
82	CD45
83	0
84	IL15
85	1
86	TAX
87	0

TABLE 2. Boolean model of T-LGL dynamics from [33].

One can check that for  $\mathbf{x} \in \{0, 1, 2\}^3$ ,  $l_i(\mathbf{x}) = I_{\{0\}}(x_i)$ , where  $I_A(x)$  is the usual indicator function notation for the set  $A$  with value 1 on the set and 0 off the set. That is,  $l_i$  serves

a b c	$f_1$ $f_2$ $f_3$
0 0 0	0 0 0
0 0 1	0 0 1
0 0 2	0 1 1
0 1 0	1 0 1
0 1 1	0 0 2
0 1 2	0 1 2
1 0 0	1 0 0
1 0 1	0 0 0
1 0 2	0 1 0
1 1 0	2 0 0
1 1 1	0 0 0
1 1 2	0 1 0
2 0 0	1 1 0
2 0 1	0 1 0
2 0 2	0 1 0
2 1 0	2 1 0
2 1 1	0 1 0
2 1 2	0 1 0

TABLE 3. Table of update functions  $\bar{F} = (f_1, f_2, f_3)$  from Table 2 of [29].

as an indicator function of the low value 0 for variable  $i$ . Similarly,  $m_i$  and  $h_i$  are indicator functions for medium and high values on the domain  $S$ . These functions are ternary analogs of “ $x_i$ ” which works as an indicator function  $I_{\{1\}}(x_i)$  in the 0-1 valued Boolean case.

We can represent  $f_1$  from Table 3 in polynomial form as

$$f_1 = 2 \cdot (m_1 \cdot m_2 \cdot l_3 + h_1 \cdot m_2 \cdot l_3) + 1 \cdot (l_1 \cdot m_2 \cdot l_3 + m_1 \cdot l_2 \cdot l_3 + h_1 \cdot l_2 \cdot l_3)$$

and similarly for  $f_2, f_3$ . The use of polynomial indicator functions for defining subsets of domains is developed in [23] for similar applications in experimental design. The 3-level state space is the variety of the radical ideal

$$I_{012} = \langle s_1^3 - 3s_1^2 + 2s_1, s_2^3 - 3s_2^2 + 2s_2, \dots, s_d^3 - 3s_d^2 + 2s_d \rangle$$

in the ring  $\mathbb{C}[s_1, \dots, s_d]$ .

In the polynomial framework, one can quickly compute big biology examples with node values in  $\{0, 1, 2\}$  like the 13-node flowering example of [3] (and further clarified in [10]). We have also done the four-level cancer example in [11].

In conclusion, sets related to attractors in discrete dynamics can be big and yet still have manageable ideals. Checking containment of sets can be done with the ideals. Probabilistic dynamics can be done with ideals. Sets can be represented efficiently and intuitively by means of associated prime ideals. And general node values beyond  $\{0, 1\}$  are not a problem.



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