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The Decay and Formation of One-Dimensional Nonconservative Shocks

Marek Elźanowski
Portland State University, elzanowskim@pdx.edu

Marcelo Epstein
University of Calgary

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The decay and formation of one-dimensional nonconservative shocks

Marek Elżanowski

Department of Mechanical Engineering and Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada.

and Marcelo Epstein

Department of Mechanical Engineering, The University of Calgary, Calgary, Alberta, Canada.
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The method of singular surfaces is used to obtain explicit conditions under which a one-dimensional acceleration wave develops into a shock when some dissipation mechanism is present. The conditions which secure the initial growth of the strong shock wave propagating into an undeformed nonlinear and dissipative medium are also derived. The analysis is presented for a single balance law, one-dimensional elasticity, and the nonlinear Maxwellian continuum.

Keywords: nonconservative hyperbolic systems, balance law, existence of global solutions, shock formation, wave propagation, nonlinear elasticity, nonlinear partial differential equations

Introduction

Weak waves (acceleration waves) that can be modelled as solutions of a quasilinear hyperbolic system of conservation laws generally form singularities (shock waves) in finite time. When a hyperbolic system of balance, rather than conservation, laws is considered, the problem of predicting the formation of such singularities ("breaking" or "blowing-up" of waves) is rather involved. Physically, this situation corresponds to the presence of a dissipative mechanism, such as external viscous friction or internal viscoplastic flow, which in some cases may prevent the breaking of waves of relatively weak strength. Moreover, even if the shock does form, the damping mechanism will have a strong influence on its growth. Indeed, it may attenuate it to such an extent that if the amplitude of the "acceleration wave" accompanying the shock is too weak the shock will decay, whereas otherwise it would grow.

The problem of existence of global solutions to quasilinear hyperbolic systems of balance laws has been studied by basically two methods. In the first method (e.g., Refs. 1–4) an estimate of the growth of spatial derivatives of the Reimann invariants along characteristics shows that in most cases waves break if these derivatives are initially large enough. In the second approach[5,6] more general results concerning the existence of global solutions are obtained via energy estimates.

In this paper a different technique is used, based on the concept of a singular surface,[8] to derive, in a one-dimensional situation, explicit conditions, rather than estimates, under which the acceleration wave (Lipschitz continuous solution of a hyperbolic system) blows up into a shock when some dissipation mechanism is present. We also show conditions which secure the initial growth of the strong shock wave propagating into an undeformed nonlinear medium. The singular surface method used here proves to be very effective because it yields explicit results, yet it may not work when the equations are more complicated.

We start our analysis with the single balance law and then proceed to the so-called p-system[7] and the nonlinear Maxwellian one-dimensional continuum.

Single balance law

Let γ be a C^1-smooth curve $y = \xi(t)$ on the y-t plane, and let $u(x,t)$ be a solution of

$$u_{,t} + f(u)_{,x} + \psi(u) = 0 \tag{1}$$

defined and Lipschitz continuous everywhere except possibly on $\xi(t)$ and such that its first spatial derivative as well as u itself attains limits $u^+_{,x}(\xi,t)$, $u^-_{,x}(\xi,t)$, $u^+(\xi,t)$, $u^-(\xi,t)$, on γ^+ and γ^-, respectively, subject to the Rankine-Hugoniot condition on γ,

$$\dot{\xi}(u^+(\xi,t) - u^-(\xi,t)) = f(u^+(\xi,t)) - f(u^-(\xi,t)) \tag{2}$$

Moreover, $t \to u\ (\xi(t),t)$ is assumed to be at least C^1-smooth. Here f and ψ are given smooth functions of u such that

(A_0) $f_{,u}(u) > 0$, $u > 0$ and $f(0) = 0$, $f_{,u}(0) = 0$.
(A_1) $f_{,uu}(u) > 0$, $-\infty < u < \infty$.
(A_2) $f_{,u}(u) > f(u)/u$, $u > 0$.
(A_3) $\psi(u) > 0$ and $\psi_{,u}(u) > 0$, $u > 0$.

Consider now a solution $u(x,t)$ of (1), (2) such that u and $u_{,x}$ experience jumps across γ; i.e., $[u](\xi(t)) = u^+(\xi(t),t) - u^-(\xi(t),t)$ and $[u_{,x}](\xi(t)) = u^+_{,x}(\xi,t) - u^-_{,x}(\xi,t)$, and such that $u^+(x,t) \equiv u^+_{,x}(x,t) \equiv 0$. According to Hadamard's lemma,[6] the chain rule can be used on either side of γ for evaluating derivatives of u and $u_{,x}$ on γ itself. Thus,

$$\begin{aligned}[u]_{,t} &= [u_{,x}]\dot{\xi}(t) + [u_{,t}] \quad \text{and}\\ [u_{,x}] &= [u_{,xx}]\dot{\xi}(t) + [u_{,xt}]\end{aligned} \tag{3}$$

It follows from the Rankine-Hugoniot condition (2) and condition (A_0) that

$$\begin{aligned}\dot{\xi}(t)[u](\xi(t)) &= f(u^+(\xi,t)) - f(u^-(\xi,t))\\ &= -f(u^-(\xi,t)) = -f(-[u](\xi(t)))\end{aligned} \tag{4}$$

Since the function $u(x,t)$ satisfies (1) on either side of γ, u and $u_{,x}$ attain their limits on γ, and f and ψ are smooth functions, one can obtain on $\xi(t)$ an analog of (1); that is,

$$[u_{,t}](\xi(t)) + [f_{,u}u_{,x}](\xi(t)) + [\psi](\xi(t)) = 0 \tag{5}$$

Combining (3), (4), and (A_0), we can write equation (5) as

$$\begin{aligned}[u]_{,t} + [u_{,x}](t)\{f(-[u](t))/[u](t)\\ + f_{,u}(-[u](t))\} + [\psi](t) = 0\end{aligned} \tag{6}$$

since

$$[f_{,u}u_{,x}](t) = -[f_{,u}](t)[u_{,x}](t) \qquad \text{for } u^+ \equiv u^+_{,x} \equiv 0$$

Let us consider now a solution of (1) and (2) that is a shock wave propagating into an undeformed medium ($u^+ \equiv 0$) such that $u^-(\xi(t),t) > 0$ and $u^-_{,x}(\xi(t),t) < 0$ for any $t \geqq 0$. It is elementary to observe from the evolution equation (6) that the amplitude of the shock wave $|[u](\xi(t))|$ will grow if the accompanying acceleration wave $[u_{,x}](\xi(t))$ is strong enough; specifically

$$[u_{,x}](t) > \frac{\psi(-[u](t))}{f_{,u}(-[u](t)) + f(-[u](t))/[u](t)} \qquad t \geqq 0 \tag{7}$$

In particular, if one would like to secure the initial growth of a shock of initial amplitude $[u](0) = u_0 < 0$, the solution behind the wave must be such that

$$u^+_{,x}(\xi(0)) < \frac{\psi(-u_0)}{f_{,u}(-u_0) + f(-u_0)/u_0} \tag{8}$$

Even if a shock starts with a decay it may later recover* its growth, as was observed in Ref. 12.

The damping mechanism which so much influences the propagation of a shock wave has a strong impact on its own formation. In most cases it prevents some weak waves from blowing up, but it has essentially no influence on strong waves. There are, however, some mechanisms which dominate the wave evolution to such an extent that any wave stays bounded at finite time.[5] It is usually the relation between the nonlinearity of the divergence term in (1) ($f_{,uu}$) and the form of the dissipation $\psi(u)$ which decides for one or the other. Consider a C^1-solution $u(x,t)$ of (1) defined on $(-\infty,+\infty) \times [0,T)$. Suppose that (1) is such that (A_0), (A_1), and (A_3) are satisfied. Then

Proposition 1

The Cauchy problem for (1) with initial condition $u_0(x) \in C^1(R)$, with bounded derivatives, and such that $u_0(x) > 0$ and $u_{0,x}(x) < 0$ has only a local C^1-smooth solution which blows up to infinity at some $T_c < \infty$ provided that

$$(f_{,uu}\psi(u))_{,u} < 0 \qquad \text{for any } u > 0 \tag{9}$$

Proof. We set $r(x,t) = u_{,x}(x,t)$ and differentiate (1) with respect to x to obtain

$$\bar{r} + f_{,uu}r^2 + \psi_{,u}r = 0 \tag{10}$$

where $\bar{r} = r_{,t} + f_{,u}r_{,x}$ denotes differentiation in the characteristic direction of $f_{,u}$. Let h be a function such that $\bar{h} = \psi_{,u}$. In place of (1) we now obtain on a characteristic $\mathrm{d}\xi/\mathrm{d}t = f_{,u}(\xi(t),t)$ that

$$\bar{q} + kq^2 = 0 \tag{11}$$

where $q(t) = r(\xi(t),t)e^h$ and $k(t) = f_{,uu}(\xi(t),t)e^{-h}$. Any initial-value problem of (11) has the solution

$$q(t) = q(0)/(1 + \dot{q}(0)K(t)) \tag{12}$$

where $K(t) = \int_0^t k(s)\,\mathrm{d}s$, and the integration is per-

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* When $\psi(u) \equiv 0$, the amplitude of the shock grows as long as $u^-_{,x} < 0$.

formed along a characteristic. On the other hand, according to (1) $\ddot{u} = \psi_u$. Thus,

$$K(t) = \int_0^t f_{uu} e^{-h}\,ds = \int_0^t f_{uu}\frac{\ddot{u}(s)}{\dot{u}(0)}\,ds = -\frac{1}{\dot{u}(0)}\int_0^t f_{uu}(-\ddot{u}(s))\,du = -\frac{1}{\dot{u}(0)}(f_u(u(\xi(t),t)) - f_u(u(\xi(0),0))) = \frac{1}{\psi(u(\xi(0),0))}(f_u(u(\xi(0),0)) - f_u(u(\xi(t),t))) \tag{13}$$

It is clear now that if (9) holds then $f_{uu}(u(\xi(t),t))\psi(u(\xi(t),t))$ increases along characteristic $\xi(t)$, so

$$K(t) \geqq f_{uu}(u(\xi(0),0))\psi(u(\xi(0),0))$$

Consequently, as long as $q(0) = u_x(\xi(0),0) < 0$, the solution (12) exists only locally and at $T_c < -1/q(0)f_{uu}(u_0)\psi(u_0)$ forms a singularity.

In every case the critical time T_c can be determined from (12) and (13). If equation (1) is such that condition (9) is not satisfied, the Cauchy problem will still have a local solution only, as we showed in Ref. 10, provided the amplitude of the weak wave $[u_{,x}(x,t)]$ is large enough. For instance, if $f(u) = \frac{1}{3}u^3$, $\psi(u) = \alpha u$, and $\alpha > 0$, the C^1-solution of (1) blows up at finite time as long as at some x $u_{0,x}(x) < -\alpha/u_0(x)$.

The p-system

Consider an evolution of the Lipschitz continuous initial data $u_0(x)$ under

$$\rho u_{,tt} - f(u_{,x})_{,x} + \psi(u_{,t}) = 0 \tag{14}$$

where ρ = const and $f(u_{,x}(x,t))$ and $\psi(u_{,t}(x,t))$ are smooth given functions satisfying (A_0)–(A_4). Equation (14) represents the equation of motion of one-dimensional nonlinear elasticity with external viscous friction $\psi(u_{,t})$. It is equivalent to the genuinely nonlinear system of hyperbolic balance laws

$$\begin{aligned} &v_{,t} - \frac{1}{\rho}f(\epsilon)_{,x} + \frac{1}{\rho}\psi(v) = 0 \\ &v_{,x} - \epsilon_{,t} = 0 \end{aligned} \tag{15}$$

if one sets $v = u_{,t}$ and $\epsilon = u_{,x}$. System (15) is sometimes called the p-system.[7]

What was a Lipschitz continuous solution of (14) is now a piecewise Lipschitz continuous solution vector $\mathbf{u} = (v,\epsilon)(x,t)$ of (15). We assume that $\mathbf{u}$ as well as $\mathbf{u}_{,x}$ attains side limits on a C^1-smooth curve $y = \xi(t)$ and that these are subject to Rankine-Hugoniot conditions on $\xi(t)$; i.e.,

$$\begin{aligned} &\rho\dot{\xi}(v^+(\xi(t),t) - v^-(\xi(t),t)) \\ &\qquad = -(T(\epsilon^+(\xi(t),t)) - T(\epsilon^-(\xi(t,t)) \\ &\dot{\xi}(\epsilon^+(\xi(t),t) - \epsilon^-(\xi(t),t)) \\ &\qquad = -(v^+(\xi(t),t) - v^-(\xi(t),t)) \end{aligned} \tag{16}$$

As before, $t \to u(\xi(t),t)$ is assumed to be C^1-smooth. Such a solution of (15) and (16) is called a shock wave of amplitude $[\epsilon](t) = \epsilon^+(\xi(t),t) - \epsilon^-(\xi(t),t)$. For simplicity, we consider a shock wave propagating into an undeformed medium; i.e., $\mathbf{u}^+(x,t) \equiv 0$.

The analog of system (15) on the shock curve $\xi(t)$ is

$$\begin{aligned} &[v_{,t}] - \frac{1}{\rho}[f(\epsilon)_{,x}] + \frac{1}{\rho}[\psi(v)] = 0 \\ &[v_{,x}] = [\epsilon_{,t}] \end{aligned} \tag{17}$$

Using (3), the properties of the functions f and ψ, and the fact that $[\mathbf{u}] \equiv 0$, one can easily reduce (17) to

$$\rho\{-V[\epsilon]^2\}_{,t}f(-[\epsilon])[\epsilon_{,x}] - f_\epsilon(-[\epsilon])[\epsilon][\epsilon_{,x}] - [\epsilon]f(V[\epsilon]) = 0 \tag{18}$$

where* $V \equiv \dot{\xi} = ([f(\epsilon)]/\rho[\epsilon])^{1/2}$, as is clear from (16). Assuming that $[\epsilon](0) < 0$, which is consistent with the Lax (entropy) condition,[11] ($[\epsilon](0) > 0$ would not propagate as a front-shock), we observe that the shock wave will grow only if

$$[\epsilon_{,x}](t) > \frac{\psi(V[\epsilon])(t)[\epsilon](t)}{f_\epsilon(-[\epsilon](t))(-[\epsilon](t) - f(-[\epsilon](t)))} \qquad t \geqq 0 \tag{19}$$

which is similar to (7) for a single equation (1). In particular, if $f(\epsilon) = A\epsilon + B\epsilon^2$ and $\psi(v) = \gamma v$, then to secure the initial growth of the shock in the presence of the external viscous friction $\psi(v)$, where $B > 0$, $\gamma > 0$, and the initial amplitude $[\epsilon_{,x}](0) > 0$, of the acceleration wave associated with the shock, we must require

$$\gamma < \frac{B[\epsilon_{,x}](0)}{((A - B[\epsilon](0))/\rho)^{1/2}} \tag{20}$$

This observation, obtained by means of some numerical examples in Ref. 12, prompted us to look at the problem of the so-called breaking of waves of the hyperbolic p-system of balance laws (15). Our intention was to obtain by standard analysis precise conditions for the local existence of smooth enough solutions.

Consider an acceleration wave $(v,\epsilon)(x,t)$ of (15) propagating into an undeformed medium with a positive speed along the characteristic $y = \xi(t)$. The vector $(v,\epsilon)(x,t)$ is assumed to be C^1-smooth, except possibly on $(\epsilon(t),t)$ for $t \in [0,T)$ where it is continuous and its first and higher derivatives attain side limits. Differentiating $(15)_1$ with respect to x and taking jumps of all the terms, we obtain, on $y = \xi(t)$,

$$\rho[\epsilon_{,tt}] - f_{\epsilon\epsilon}[\epsilon_{,x}^2] - f_\epsilon[\epsilon_{,xx}] + \psi_v[v_{,x}] = 0 \tag{21}$$

* We have chosen to follow the front shock, that is, the one with $\dot{\xi} > 0$.

since (15) is satisfied on either side of $\xi(t)$. Using (3), where $\dot{\xi} = (f_\epsilon(\epsilon)/\rho)^{1/2}$, and the fact that ϵ is continuous across the characteristic $\xi(t)$, we finally arrive at

$$-2\rho\dot{\xi}[\epsilon_{,x}]_{,t} + f_{\epsilon\epsilon}[\epsilon_{,x}]^2 - \psi_v\dot{\xi}[\epsilon_{,x}] = 0 \qquad (22)$$

Taking into account that $\epsilon(x,t)$ and $v(x,t)$ vanish ahead of the acceleration wave, (22) yields on the wavefront ξ a scalar Riccati equation with constant coefficients; that is,

$$[\epsilon_{,x}]_{,t} - \frac{\frac{1}{2}f_{\epsilon\epsilon}[\epsilon_{,x}]^2}{(\rho f_\epsilon)^{1/2}} + \frac{1}{2\rho}\psi_v[\epsilon_{,x}] = 0 \qquad (23)$$

Equation (23) is the evolution equation for the amplitude of the single acceleration wave of (15) propagating into an undeformed medium.

Proposition 2

A single acceleration wave $u(x,t)$ of (15) such that* $\epsilon(x,t) > 0$ and $\epsilon_{,x}^-(\xi(t),t) < 0$ propagating into an undeformed medium breaks at finite time provided that initially

$$[\epsilon_{,x}](0) > \frac{\psi_v(0)(f_\epsilon(0)/\rho)^{1/2}}{f_{\epsilon\epsilon}(0)} \qquad (24)$$

Proof. Take the evolution equation (23) for the amplitude $[\epsilon_{,x}](t)$. A standard substitution $q = [\epsilon_{,x}]e^h$ and $k = -f_{\epsilon\epsilon}/2\rho(f_\epsilon/\rho)^{1/2}e^{-h}$, where $h(t) = (1/2\rho)\psi_v t$, reduces (23) to (11), the solution of which has the well-known form (12).

The function $K(t) = \int_0^t k(s)\,ds$ can now easily be evaluated as

$$K(t) = \frac{2\rho\lambda_\epsilon}{\psi_v}(e^{-\psi_v t/2\rho} - 1) \qquad (25)$$

where $\lambda_\epsilon = f_{\epsilon\epsilon}/2\rho(f_\epsilon/\rho)^{1/2}$. Consequently, the solution of (11) blows up in finite time if and only if

$$q(0) = [\epsilon_{,x}](0) > \psi_v(0)/2\rho\lambda_\epsilon(0) \qquad (26)$$

For example if $\psi(v) = \gamma v$, where $\gamma > 0$, and $f(\epsilon) = A\epsilon + \frac{1}{2}B\epsilon^2$, where $B > 0$, condition (26) tells us that the acceleration wave breaks into a shock if its initial amplitude is greater than $(\gamma/B)(A/\rho)^{1/2}$.

Inelastic continuum

In this section we analyze the formation of the stress waves (shocks) in a one-dimensional inelastic (Maxwellian) continuum. Our objective is to show the power and the simplicity of the singular surface method in yielding the threshold amplitude beyond which acceleration waves break.

* If $\epsilon_{,x}(x,t)$ jumps across the characteristic $\xi(t)$, the jump of $v_{,x}(x,t)$ on $\xi(t)$ is uniquely determined by the so-called compatibility conditions (3).

Consider a Maxwellian one-dimensional body whose material obeys the constitutive relation

$$\sigma_{,t} = g(\sigma)v_{,x} - \phi(\sigma) \qquad (27)$$

where σ denotes the stress and v the particle velocity. Here, $g(\sigma)$ and $\phi(\sigma)$ are given material functions which are positive and have positive derivatives for $\sigma > 0$.

The law of motion in the case of the one-dimensional theory has the form

$$\sigma_{,x} + \rho\mathbf{b} = \rho v_{,t} \qquad (28)$$

where ρ is the density in the reference configuration and $\mathbf{b}$ are the body forces.

Let $\mathbf{u} = (\sigma,v)$ denote a single acceleration wave propagating into a constant stress state, i.e., a Lipschitz continuous solution u on $(-\infty, +\infty) \times [0,T)$ of the system (27), (28) whose first, and possibly higher, derivatives observe jumps across the characteristic curve γ: $dy/dt = (\rho^{-1}g(\sigma(y(t),t)))^{1/2}$, and such that ahead of γ, $\sigma(x,t) \equiv \sigma_0$.

Equations (27) and (28) are satisfied on either side of γ. Thus, on γ one gets

$$[\sigma_{,t}] - g(\sigma)[v_{,x}] = 0 \qquad (29)$$

$$[v_{,t}] - 1/\rho[\sigma_{,x}] = 0 \qquad (30)$$

Using (3) and the fact that $[\sigma] = [v] = 0$, we obtain

$$g(\sigma)[v_{,x}] = -\dot{\gamma}[\sigma_{,x}] \quad \text{and}$$
$$g(\sigma)[v_{,x}] = -\rho\dot{\gamma}[v_{,t}] \qquad (31)$$

respectively.

Differentiating (27) and (28) with respect to x and taking the jumps across γ, we can easily obtain the following system of ordinary differential equations in the characteristic direction:

$$[\sigma_{,x}]_{,t} - \dot{\gamma}[\sigma_{,xx}] + \rho g'(\sigma)[v_{,t}][v_{,x}] - g(\sigma)[v_{,xx}] + \rho\phi'(\sigma)[v_{,t}] = 0 \qquad (32)$$
$$[\sigma_{,xx}] = \rho\{[v_{,x}]_{,t} - \dot{\gamma}[v_{,xx}]\}$$

Utilizing once again the compatibility conditions (3) for $[v_{,t}]$, we can reduce equation $(32)_1$ with the help of $(31)_1$ to the familiar form

$$[v_{,x}]_{,t} + \tfrac{1}{2}g'(\sigma)[v_{,x}]^2 + \tfrac{1}{2}\phi'(\sigma)[v_{,x}] = 0 \qquad (33)$$

If, as before, we now set $q(t) = [v_{,x}]e^h$, $k(t) = \frac{1}{2}g'(\sigma)e^{-h}$, and $h(t) = \frac{1}{2}\phi'(\sigma)t$, (33) takes the form of (11) with a solution given by (12). Elementary calculations show now that

$$K(t) = -\frac{g'(\sigma)}{\phi'(\sigma)}e^{-\phi'(\sigma)t/2} - 1 \qquad (34)$$

which enables us to conclude that the acceleration wave $\mathbf{u}$ of amplitude $[v_{,x}] < 0$ will form a shock if

$$[v_{,x}](0) < -\phi'(\sigma_0)/g'(\sigma_0) \qquad (35)$$

or, equivalently, if

$$[\sigma_{,x}](0) > (\rho g'(\sigma_0))^{1/2}\phi'(\sigma_0)/g'(\sigma_0) \qquad (36)$$

If, as in Ref. 1, we choose $\phi(\sigma) = k_1\sigma + k_0$ and

$g(\sigma) = c^2\phi^2(\sigma)$, then condition (36) yields

$$[\sigma_{,x}](0) > (k_1\rho/2c^2(k_1\sigma_0 + k_0))^{1/2} \quad (37)$$

which can be compared with the restrictions obtained therein.

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