Analytically Continued Hypergeometric Expression of the Incomplete Beta Function

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Abstract The Incomplete Beta Function is rewritten as a Hypergeometric Function that is the analytic continuation of the conventional form, a generalization of the finite series, which simpifies the Stieltjes transform of powers of a monomial divided by powers of a binomial.

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The finite hypergeometric series expression for the Incomplete Beta Function, [1]

$$ {}_2F_1(-n,1;c;z) = (1-c)z^{1-c}(z-1)^{n+c-1}B_{1-1/z}(1-c-n,n+1)\ , \tag{1}$$

may be generalized to

Theorem

$$ {}_2F_1(-\nu,1;\gamma;z) = (1-\gamma)z^{1-\gamma}(z-1)^{\nu+\gamma-1}\quad \Big[B_{1-1/z}(1-\gamma-\nu,\nu+1) - B(1-\gamma-\nu,\nu+1)\left(1-\frac{(-1)^{-\nu}\sin[\pi(\gamma+\nu)]}{\sin(\pi\gamma)}\right)\Big]\ . \tag{2}$$

The Incomplete Beta Function [2] is conventionally defined [3] with real parameters for statistical problems,

$$ B_x(p,q) = \int_0^x t^{p-1}(1-t)^{q-1}\ dt \qquad (0\le x\le 1, \quad p,q>0)\ , \tag{3}$$

but is a smooth function of p, q or x when any or all are taken off the real axis (though it diverges as x takes on large, real values). Its hypergeometric expression [4] is likewise well-behaved for complex parameters, so we rewrite this expression in its more general form

$$\begin{aligned} {}_2F_1(\alpha,\beta;\beta+1;w) &= \beta w^{-\beta}B_w(\beta,1-\alpha) = \beta w^{-\beta}B(\beta,1-\alpha)\left(1-I_{1-w}(1-\alpha,\beta)\right) \\ &= \beta w^{-\beta}\left[B(1-\alpha,\beta) - B_{1-w}(1-\alpha,\beta)\right]\ . \end{aligned}\tag{4}$$

One may analytically continue the left-hand side to [5]

$$\begin{aligned} {}_2F_1(\alpha,\beta;\beta+1;w) &= (-1)^{-\alpha}(w)^{-\alpha}\frac{\Gamma(\beta+1)\Gamma(\beta-\alpha)}{\Gamma(\beta)\Gamma(\beta+1-\alpha)}{}_2F_1(\alpha,\alpha-\beta;\alpha+1-\beta;1/w) \\ &+ (-1)^{-\beta}(w)^{-\beta}\frac{\Gamma(\beta+1)\Gamma(\alpha-\beta)}{\Gamma(\alpha)\Gamma(1)}{}_2F_1(\beta,0;\beta+1-\alpha;1/w)\ , \end{aligned}\tag{5}$$

Then equating right-hand sides of (4) and (5) and transforming the nontrivial hypergeometric function again [6] gives

$$(B(1-\alpha,\beta)-B_{1-w}(1-\alpha,\beta)) = (-1)^{-\alpha}w^{-\alpha+\beta}\frac{1}{(\beta-\alpha)}\left(1-\frac{1}{w}\right)^{1-\alpha}{}_2F_1(1-\beta,1;\alpha+1-\beta;1/w)$$
$$+(-1)^{-\beta}B(1-\alpha,\beta)\frac{\Gamma[1-(\alpha-\beta)]\Gamma(\alpha-\beta)}{\Gamma(1-\alpha)\Gamma(\alpha)}\ , \tag{6}$$

Letting $z=1/w$ this simplifies [7] to

$$B_{1-1/z}(1-\alpha,\beta) = z^{\alpha-\beta}\frac{1}{(\beta-\alpha)}(z-1)^{1-\alpha}{}_2F_1(1-\beta,1;\alpha+1-\beta;z)$$
$$+B(1-\alpha,\beta)\left(1+(-1)^{1-\beta}\frac{\sin[\pi\alpha)]}{\sin[\pi(\alpha-\beta)]}\right)\ , \tag{7}$$

Finally one substitutes $\beta=\nu+1$ and $\alpha=\gamma+\beta-1$ and rearranges sides to obtain Eq. (2).

In addition, if one substitutes $\beta=1-\nu$, $\alpha=2-\mu$, and $z=\frac{\beta}{\gamma}$ and analytically continues the Gauss function, [8] one may obtain a more useful form for the known [9] Stieltjes transform [10] of powers of a monomial divided by powers of a binomial,

Corollary

$$\int_0^\infty \frac{x^{\nu-1}(\beta+x)^{1-\mu}}{\gamma+x}dx = 2\int_0^\infty \frac{x^{\nu-1/2}(\beta+x^2)^{1-\mu}}{\gamma+x^2}dx = \pi\gamma^{\nu-1}(\beta-\gamma)^{1-\mu}\csc(\nu\pi)I_{1-\frac{\gamma}{\beta}}(\mu-1,1-\nu)$$
$$=\pi\gamma^{\nu-1}(\beta-\gamma)^{1-\mu}\csc(\nu\pi)\left(1+(-1)^{\nu}\frac{\sin[\pi(2-\mu)]}{\sin[\pi(1+\nu-\mu)]}\right)$$
$$-\frac{\pi\csc(\nu\pi)\beta^{\nu+1-\mu}}{(\mu-1-\nu)(\beta-\gamma)B(\mu-1,1-\nu)}\ {}_2F_1(2-\mu,1;2-\mu+\nu;\frac{\beta}{\beta-\gamma})\ , \tag{8}$$

$(|arg\gamma|<\pi|,|arg\beta|<\pi|,0<Re\ \nu<Re\ \mu)$ which is a finite series for integer $\mu>1$.

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[6] *ibid.*, p. 1043, No. 9.131c.

[7] *ibid.*, p. 937, No. 8334.3.

[8] *ibid.*, p. 1043, No. 9.131b.

[9] *ibid.*, p. 290, No. 3.227.2; A. Erdélyi *et al., Tables of Integral Transforms,* Vol. II (McGraw Hill, New York, 1954), p. 217, No. 10.

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