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CONVERGENCE ANALYSIS OF SOME TENT-BASED SCHEMES FOR LINEAR HYPERBOLIC SYSTEMS

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Abstract. Finite element methods for symmetric linear hyperbolic systems using unstructured advancing fronts (satisfying a causality condition) are considered in this work. Convergence results and error bounds are obtained for mapped tent pitching schemes made with standard discontinuous Galerkin discretizations for spatial approximation on mapped tents. Techniques to study semidiscretization on mapped tents, design fully discrete schemes, prove local error bounds, prove stability on spacetime fronts, and bound error propagated through unstructured layers are developed.

1. Introduction

Tent-based numerical methods for hyperbolic equations stratify a spacetime simulation region in an unstructured manner, by tent-shaped subregions, and advance solutions across them progressively in time. The partitioning into tents provides a rational design for local time stepping, maintaining high order accuracy in both space and time, and without any ad hoc projection or extrapolation steps, an advantage that has been and continues to be effectively leveraged by many researchers [1, 2, 10, 17, 18, 20, 22]. Nevertheless, a drawback of tents is that they are not tensor products of a spatial domain with a time interval, necessitating development of new tent (spacetime) discretizations coupling too many spatiotemporal unknowns. In [13], we overcame this drawback by using a mapping technique that transforms spacetime tents to tensor product spacetime cylinders. This opened avenues to use standard techniques like the discontinuous Galerkin (DG) discretizations for spatial discretization, together with efficient matrix-free time stepping schemes, on the mapped tents. Such methods, referred to as *Mapped Tent Pitching* (MTP) schemes, have been applied to solve a variety of linear and nonlinear hyperbolic systems [12, 13, 14].

This is the first paper to provide convergence theorems for MTP schemes. Although the scope of the analysis here is limited to linear hyperbolic systems, we identify what we consider to be the basic ingredients for error analysis of MTP schemes, such as a norm in which stability on spacetime fronts can be obtained. We use techniques to bound the propagation of error through layers of tents similar

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to those in [10, 17]. However, a number of new tools are needed to overcome difficulties arising from a time-dependent mass matrix generated due to the mapping. As we already noted in [12], the use of classical explicit Runge-Kutta time stepping on these mapped systems leads to loss of higher orders of convergence due to the complications created by the map. We outlined an algorithmic solution in [12], namely the *Structure-Aware Taylor* (SAT) time stepping scheme, that accounts for the specific structure of the time-dependent mass matrix. Here we shall provide *a priori* error bounds for these as well as a few other schemes.

Our analysis is divided into the next three sections. First, we borrow a spatial DG discretization framework from [3, 9] which permits the treatment of many important examples of symmetric hyperbolic systems and many boundary condition choices, all at once. The application of this framework to the mapped equation (the pull back of the hyperbolic system from the physical tent to the spacetime cylinder), is detailed in Section 2. We then combine it with a semidiscrete analysis in Section 3. While the analysis in that section ignores errors due to time discretization that are undoubtedly present in practice, it immediately clarifies in what norms one may expect stability on spacetime advancing fronts, and what error bounds might be provable after time discretization. The main result of this section is that under the conditions spelled out later, we may expect the error in the numerical solution at the final time to be $O(h^{p+1/2})$ where h represents a spatial mesh size parameter and p denotes the spatial polynomial degree (used in the spatial DG discretization). In this form, the result is comparable to [17, Theorem 5.1] that provides the same rate for their spacetime DG method using spacetime polynomials of degree p on tents.

In Section 4, we discuss several fully discrete schemes that combine the spatial DG discretization on a mapped tent with SAT or other time stepping. We find that proving stability of the fully discrete schemes requires some trickery. Ever since the classical work of [16], we know that stability regions and "naive spectral stability analysis based on scalar eigenvalues arguments may be misleading." Many researchers have since pursued energy-type arguments to prove stability of time stepping schemes with spatial DG discretizations [3, 5, 21, 23]. However, we are not able to directly apply existing techniques due to the nonstandard nature of the system we obtain after mapping the hyperbolic equation. Therefore, we start afresh, beginning with the most basic scheme and proceeding to more complicated cases. Namely, in §4.2 we prove unconditional strong stability for a lowest order tent-implicit scheme. Then, we proceed to analyze a lowest order "iterated" explicit scheme in §4.3, constructed as an iterative solver for the implicit scheme, for which we prove a nonstandard conditional stability. Then, in §4.4, we proceed to an s-stage SAT scheme and show that its local error in a tent is $O(h^s)$, which is comparable to the sth power of the time step since the amount of local time advance in tents is tied to its spatial mesh size. We are able to prove, in one case, stability under a traditional Courant-Friedrichs-Levy (CFL) condition, by which we mean that the amount of *local* time advance within a tent is limited by a constant multiple of the *local* spatial mesh size. In another case, we prove stability under a "3/2 CFL" condition. In some non-tent-based DG methods, others have encountered a similar (4/3 CFL) limitation in stability analyses [3]. We offer the above-mentioned cases not as the last word on stability, but rather to spur further research into this interesting topic.

In the remainder of this section, we establish notation and the lingua franca of tents that we use throughout. Consider a cylindrical domain $\Omega \times (0, T)$ in the physical spacetime, where the spatial domain Ω is an open bounded subset of $\mathbb{R}^N$. We assume that Ω is subdivided by a simplicial mesh Ω_h. The subscript h denotes the maximal element diameter of the spatial mesh Ω_h. Spacetime tents are built atop this spatial mesh, using the algorithms in [13] or [8]. We start by viewing the spatial mesh Ω_h at time $t = 0$ as the initial advancing front. When one mesh vertex $\mathtt{v}$ is moved forward in time, while keeping all other vertices fixed, the advancing front is updated to the piecewise planar surface formed by connecting the raised $\mathtt{v}$ to its neighboring vertices. The new front differs from the old by a tent-shaped region, which we denote by $T^{\mathtt{v}}$. Its projection onto Ω gives the vertex patch $\Omega^{\mathtt{v}}$ of all spatial simplices connected to the vertex $\mathtt{v}$. We shall refer to this process as *pitching* the tent $T^{\mathtt{v}}$. For concurrency, one pitches multiple tents simultaneously at vertices whose vertex patches do not have a mesh element in their pairwise intersections, as in Figure 1b. The canopies of these spacetime tents can be represented as the graph of $\varphi_1(x)$, a continuous function that is piecewise linear with respect to the mesh Ω_h (whose value is zero in locations where tents are not yet erected). These canopies together form the next advancing front: $C_1 = \{(x, \varphi_1(x)) : x \in \Omega\}$. Note that the time coordinate of a point in C_1 is never less than that of the corresponding point in the first front $C_0 = \Omega \times \{0\}$. This process is repeated by pitching tents atop C_1, and later atop the subsequent advancing fronts that result from each step (as illustrated in Figures 1c–1f).

Reiterating, the *advancing front* at step i is the graph of a lowest-order Lagrange finite element function $\varphi_i(x)$:

$$C_i = \{(x, \varphi_i(x)) : x \in \Omega\}. \tag{1.1}$$

We shall refer to the region between two successive advancing fronts as a *layer*, i.e.,

$$L_i = \{(x, t) \in \Omega \times (0, T) : \ \varphi_{i-1}(x) \leqslant t \leqslant \varphi_i(x)\} \tag{1.2}$$

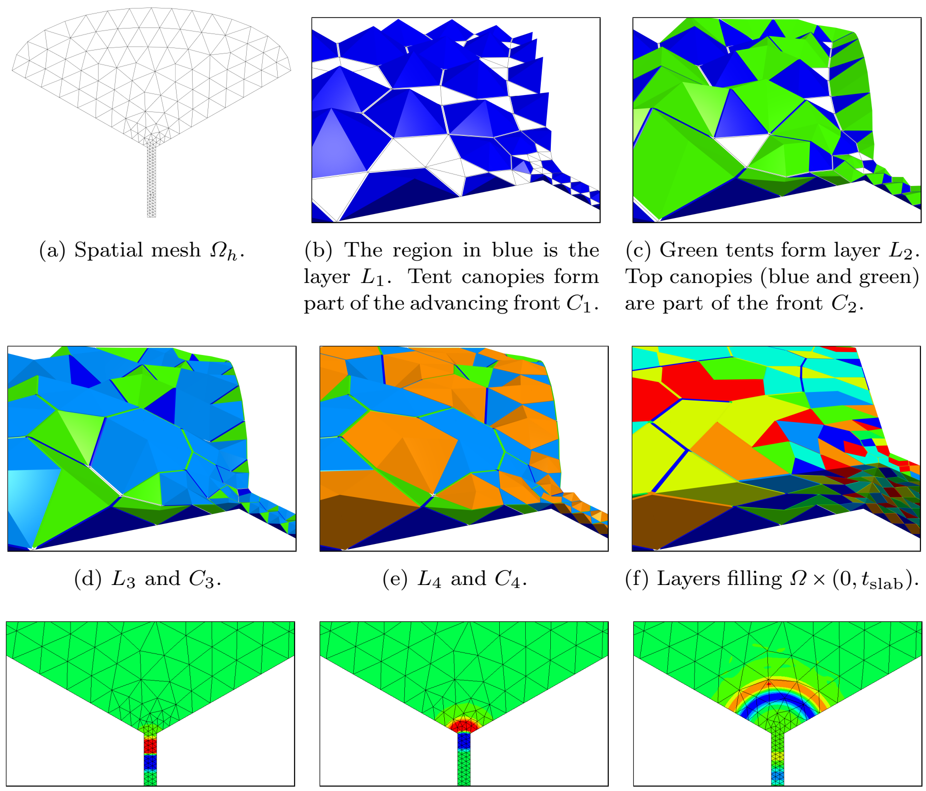
denotes the ith layer for $i = 1, 2, \ldots, m$. The layer L_i (see Figure 1) is made above C_{i-1} by pitching tents atop vertex patches associated with a subset of mesh vertices of Ω_h, which form the *pitch locations* at that stage. Let V_i denote the collection of such vertices identifying the pitch locations on C_{i-1}. Then $L_i = \bigcup_{\mathtt{v} \in V_i} T^{\mathtt{v}}$. The spacetime will generally contain multiple tents pitched at the same vertex $\mathtt{v}$ at different time coordinates. Although referring to a tent by its spatial pitch location alone, as in $T^{\mathtt{v}}$ above, is generally ambiguous, it will not confuse us since we will usually be occupied with analyzing one tent at a time.

A tent can be expressed as

$$T^{\mathtt{v}} = \{(x, t) : x \in \Omega^{\mathtt{v}}, \ \varphi^{\mathtt{v}}_{\mathrm{bot}}(x) \leqslant t \leqslant \varphi^{\mathtt{v}}_{\mathrm{top}}(x)\} \tag{1.3}$$

where $\varphi^{\mathtt{v}}_{\mathrm{bot}}$ and $\varphi^{\mathtt{v}}_{\mathrm{top}}$ are continuous functions on $\Omega^{\mathtt{v}}$ that are piecewise linear with respect to the mesh elements forming the vertex patch $\Omega^{\mathtt{v}}$. The function $\delta^{\mathtt{v}}(x) = \varphi^{\mathtt{v}}_{\mathrm{top}}(x) - \varphi^{\mathtt{v}}_{\mathrm{bot}}(x)$ on $\Omega^{\mathtt{v}}$ will feature often in the sequel. It arises as a weight in transformed integrals and is degenerate at the points where tent top meets tent bottom.

One reason for working with tents is the ease by which *causality* can be imposed, simply by adjusting the height of the *tent pole*, the line segment connecting $(\mathtt{v}, \varphi^{\mathtt{v}}_{\mathrm{bot}}(\mathtt{v}))$ to $(\mathtt{v}, \varphi^{\mathtt{v}}_{\mathrm{top}}(\mathtt{v}))$. By the definition of hyperbolicity, the maximal wave



(a) Spatial mesh Ω_h. (b) The region in blue is the layer L_1. Tent canopies form part of the advancing front C_1. (c) Green tents form layer L_2. Top canopies (blue and green) are part of the front C_2.

(d) L_3 and C_3. (e) L_4 and C_4. (f) Layers filling $\Omega \times (0, t_{\text{slab}})$.

(g) An initial pulse. (h) Solution at $t = t_{\text{slab}}$. (i) Solution at $t = 3t_{\text{slab}}$.

Figure 1. Tents, layers, advancing fronts, and solution snapshots from an acoustic wave simulation using the MTP scheme of §4.4: Figures 1b–1e show successive layers of tents, viewed at an angle to show where small and large features meet (and time t is in the vertical direction). Figure 1f shows how tents asynchronously enable both large and small time advances within a spacetime slab $\Omega \times (0, t_{\text{slab}})$. Plots of a time evolving wave solution at $t = 0, t_{\text{slab}}$, and $3t_{\text{slab}}$, computed using the tents in Figure 1f, are shown in Figures 1g, 1h, and 1i, respectively.

speed c is finite. When each spacetime tent encloses the domain of dependence of all its points, causality holds. In other words, if

$$\|(\operatorname{grad}_x \varphi_i)(x)\|_2 < \frac{1}{c}, \qquad x \in \Omega,$$

on all advancing fronts, then causality holds. In this paper, in place of the strict inequality, we assume we are given a strict upper bound $\hat{c}$ for the maximal wave speed c and that our mesh of spacetime tents is constructed so that

$$\|(\operatorname{grad}_x \varphi_i)(x)\|_2 \leqslant \frac{1}{\hat{c}} < \frac{1}{c} \qquad x \in \Omega. \tag{1.4}$$

We will refer to (1.4) as the *causality condition*. Algorithms for constructing tent meshes satisfying (1.4) can be found in [8, 13]. They terminate filling the spacetime $\Omega \times (0, T)$ with tents so that the first and the last advancing fronts, C_0 and C_m, are flat, i.e., $\varphi_0(x) \equiv 0$ and $\varphi_m(x) \equiv T$. In practice, we often select (as in Figure 1) a $t_{\rm slab} < T$ such that $T = n_{\rm slabs} t_{\rm slab}$, run the tent meshing algorithm to fill the subregion $\Omega \times (0, t_{\rm slab})$, and then translate the same mesh to reuse it $n_{\rm slabs} - 1$ times to cover the full spacetime domain $\Omega \times (0,T)$ without further meshing overhead. However, for the ensuing analysis, we ignore this extra subdivision into smaller spacetime slabs (so $t_{\rm slab} = T$ henceforth). With this background in mind, we proceed to show how to map tents and construct an MTP scheme after fixing a model problem.

2. A model problem for analysis

In this section, we describe a model symmetric linear hyperbolic system and its MTP discretization that we shall be occupied with. Although MTP schemes can be applied much more generally (as shown in [13, 14]), we restrict to this model for transparently presenting the essential new ideas needed for a convergence analysis.

2.1. **A symmetric linear hyperbolic system.**

Let L be a positive integer (and let N, as before, denote the spatial dimension). Suppose that $\mathcal{L}^{(j)} : \Omega \to \mathbb{R}^{L\times L}$, for $j = 1,\ldots,N$ and $\mathcal{G} : \Omega \to \mathbb{R}^{L\times L}$ are symmetric bounded matrix-valued functions and suppose $\mathcal{G}$ is uniformly positive definite in $\bar{\Omega}$. Our model problem is the following linear hyperbolic system of L equations, in L unknowns, denoted by $u(x,t)$, or in terms of scalar components, by $u_k(x,t)$:

$$\partial_t\, g(u) + \mathrm{div}_x\, f(u) = 0, \tag{2.1}$$

with

$$[g(u)]_l = \sum_{k=1}^{L} \mathcal{G}_{lk} u_k, \qquad [f(u)]_{lj} = \sum_{k=1}^{L} \mathcal{L}^{(j)}_{lk} u_k.$$

We have restricted ourselves to time-independent coefficients, and shall do so also for boundary conditions, which are expressed using a matrix field $\mathcal{B} : \partial\Omega \to \mathbb{R}^{L\times L}$. Boundary conditions are considered in the form studied by Friedrichs [11], namely,

$$(\mathcal{D} - \mathcal{B})u = 0 \qquad \text{on } \partial\Omega \tag{2.2}$$

where

$$\mathcal{D} = \sum_{j=1}^{N} n_j \mathcal{L}^{(j)}. \tag{2.3}$$

Note that $\mathcal{D}$, in general, depends on the point $x \in \partial\Omega$ as well as on $n(x)$, the spatial unit outward normal at x, and when we wish to emphasize these dependencies, we shall denote it by $\mathcal{D}^{(n)}$, $\mathcal{D}(x)$, or $\mathcal{D}(x)^{(n)}$. Note that later in the sequel n (or $n(x)$) will also be used to generically denote the outward unit normal on boundaries of other domains (such as mesh elements). Of course, the hyperbolic system (2.1) must also be supplemented with an initial condition,

$$u(x,0) = u^0(x) \qquad x \in \Omega \tag{2.4}$$

at time $t = 0$ for some given initial data u^0.

Friedrichs [11] identified conditions on the operator $\mathcal{B}$ for obtaining well-posed boundary value problems. We shall borrow the same conditions and impose them pointwise on $\partial\Omega \times (0,T)$. In particular, at each $x \in \partial\Omega$ and $t \in (0,T)$, we assume $\ker(\mathcal{D} - \mathcal{B}) + \ker(\mathcal{D} + \mathcal{B}) = \mathbb{R}^L$ and $\mathcal{B} + \mathcal{B}^t \geqslant 0$. (The latter is to be interpreted as $(\mathcal{B}(x) + \mathcal{B}(x)^t)y \cdot y = 2\mathcal{B}(x)y \cdot y \geqslant 0$ for all vectors $y \in \mathbb{R}^L$ at any $x \in \partial\Omega$.) To simplify the analysis, we shall also assume that

$$\sum_{j=1}^{N} \partial_j \mathcal{L}^{(j)} = 0 \tag{2.5}$$

in the sense of distributions (so jumps in $\mathcal{L}^{(j)}(x)$ are allowed so long as (2.5) holds) and

$$\mathcal{G} \text{ and } \mathcal{L}^{(j)} \text{ are constant on each mesh element } K \in \Omega_h. \tag{2.6}$$

Across a mesh facet F of normal n_F, it is easy to see that (2.5) implies the continuity of $\mathcal{D}^{(n_F)}$. As long as we obtain this normal continuity (which is needed in our analysis), assumption (2.5) can be relaxed to other forms (such as what is suggested in [9, equation (A.4)]), at the expense of a few additional technicalities. Assumption (2.6) allows us to zero out some projection error terms instead of tracking such small terms in error estimates.

2.2. **Mapping tents.** Consider a tent expressed as in (1.3). We map to a tent $T^{\mathtt{v}}$ from a cylindrical tensor product domain $\hat{T}^{\mathtt{v}} = \Omega^{\mathtt{v}} \times (0, 1)$ using the map $\Phi^{\mathtt{v}}(x, \hat{t}) = (x, \varphi^{\mathtt{v}}(x, \hat{t}))$ where

$$\varphi^{\mathtt{v}}(x, \hat{t}) = (1 - \hat{t})\varphi^{\mathtt{v}}_{\mathrm{bot}}(x) + \hat{t}\varphi^{\mathtt{v}}_{\mathrm{top}}(x) = \varphi^{\mathtt{v}}_{\mathrm{bot}}(x) + \hat{t}\,\delta^{\mathtt{v}}(x)$$

(see [13, Fig. 2] for an illustration of this map). We will drop the superscript $\mathtt{v}$ when it is obvious from context. Clearly, $\Phi(\hat{T}^{\mathtt{v}}) = T^{\mathtt{v}}$ and the interior of $\hat{T}^{\mathtt{v}}$ is mapped one-to-one onto the interior of $T^{\mathtt{v}}$ (but the map is not one-to-one from the boundary of $\hat{T}^{\mathtt{v}}$ to the boundary of $T^{\mathtt{v}}$). The coordinate $\hat{t}$ in $\hat{T}^{\mathtt{v}}$ will be referred to as the *pseudotime* coordinate. Note that the time coordinate t in the physical spacetime twists space and pseudotime together since it is given by $t = \varphi(x, \hat{t})$.

The Jacobian matrix of the map Φ is easily computed:

$$\operatorname{grad}_{x\hat{t}} \Phi = \begin{bmatrix} I & 0 \\ (\operatorname{grad}_x \varphi)^T & \delta \end{bmatrix} \tag{2.7}$$

Using it in a Piola transformation, it can be shown [13, Theorem 2] that the mapped hyperbolic solution $\hat{u} = u \circ \Phi$ satisfies

$$\partial_{\hat{t}} \left[g(\hat{u}) - f(\hat{u}) \operatorname{grad}_x \varphi \right] + \operatorname{div}_x \left[\delta f(\hat{u}) \right] = 0, \qquad \text{in } \hat{T}^{\mathtt{v}}, \tag{2.8}$$

whenever u solves (2.1). MTP schemes proceed by solving (2.8) by various discretization strategies (particularly those that leverage the tensor product nature of space and pseudotime in $\hat{T}^{\mathtt{v}}$) and then pulling back the computed solution to the physical spacetime. We now proceed to discuss a discretization strategy that uses a DG spatial discretization on $\Omega^{\mathtt{v}}$. We will combine it with pseudotime discretizations later (in Section 4).

2.3. **Spatial discretization on mapped tents.** The mapped equation (2.8) can be approximated by any standard scheme that allows for a time dependent mass matrix. Our focus is on discontinuous Galerkin (DG) discretizations. Since there are numerous flavors of DG schemes and numerical fluxes, for efficiently covering various choices, we use the framework of [9] (see also [3]), a simplified version of which, adapted to our purposes, is described next. Their framework is motivated by the previously mentioned early work of Friedrichs [11], and other previous authors have also been similarly motivated while considering boundary conditions, notably [10] and [17] in the context of tents and spacetime methods.

We assume that Ω_h is a shape regular conforming simplicial mesh of domain Ω. We use $a \lesssim b$ to indicate that there is a constant $C > 0$ such that $a \leqslant Cb$ and that the value of C may be chosen independently of any spatial mesh chosen from the shape regular family. The value of the generic constant C in "$\lesssim$" may differ at different occurrences and is allowed to depend on the wave speed, material coefficients, spatial polynomial degree, etc. Let $P_p(K)$ denote the space of polynomials of degree at most p restricted to the domain K and let $V_h = \{v : v|_K \in P_p(K)^L$ for any mesh element $K \in \Omega_h\}$. Let Ω_h^{v} denote the collection of elements in the vertex patch Ω^{v} of a mesh vertex v. Let V_h^{v} denote the restriction of the spatial DG space on Ω^{v} and let $\psi_j(x)$ denote a basis for V_h^{v}. The semidiscrete approximation of $\hat{u}$ is of the form

$$\hat{u}_h(x, \hat{t}) = \sum_j U_j(\hat{t})\psi_j(x).$$

We consider a DG semidiscretization of (2.8) of the form displayed next in (2.9). In the spatial integrals there and throughout, we do not explicitly indicate the measure (volume or boundary measure) as it will be understood from context. For each fixed $0 < \hat{t} \leqslant 1$, the function $\hat{u}_h(\cdot, \hat{t})$ satisfies

$$\int_{\Omega^{\mathrm{v}}} \partial_{\hat{t}} \left[g(\hat{u}_h) - f(\hat{u}_h) \operatorname{grad}_x \varphi \right] \cdot v = \sum_{K \in \Omega_h^{\mathrm{v}}} \left[\int_K \delta f(\hat{u}_h) : \operatorname{grad}_x v - \int_{\partial K} \delta \hat{F}^n_{\hat{u}_h} \cdot v \right], \tag{2.9}$$

for all $v \in V_h^{\mathrm{v}}$, where the numerical flux $\hat{F}^n_{\hat{u}_h}$ on an element boundary ∂K is defined using the values of $\hat{u}_h$ from the element K as well as from the neighboring element K_o, as follows. For any $w \in V_h^{\mathrm{v}}$, at a point $x \in \partial K \cap \partial K_o$, letting $w_o = w|_{K_o}$, define

$$\{w\}(x) = \frac{1}{2}(w + w_o), \qquad [\![w]\!](x) = w - w_o.$$

Then, $\hat{F}^n_w$ is assumed to take the form

$$\hat{F}^n_w = \begin{cases} \mathcal{D}\{w\} + S[\![w]\!] & \text{on } \partial K \backslash \partial \Omega, \\ \frac{1}{2}(\mathcal{D} + B)w & \text{on } \partial K \cap \partial \Omega. \end{cases} \tag{2.10}$$

Here, $\mathcal{D}$ is defined by (2.3), with the vector n now denoting the outward unit normal on ∂K, $S : \mathcal{F}_i \to \mathbb{R}^{L \times L}$ denotes a stabilization matrix on interior facets of $\mathcal{F}_i = \bigcup \{\partial K \backslash \partial \Omega : \ K \in \Omega_h^{\mathrm{v}}\}$, and $B : \partial \Omega \to \mathbb{R}^{L \times L}$, is used to model the exact boundary condition $\mathcal{B}$ with any needed extra stabilization on boundary facets. Note that S is single-valued on $\mathcal{F}_i$ (while $\mathcal{D}$ is multivalued and depends on the sign of the normal on an element boundary). Let $\| \cdot \|_2$ denote the Euclidean norm (of a vector, or the induced norm for a matrix) and let $|y|_S = (Sy \cdot y)^{1/2}$ and $|y|_B = (By \cdot y)^{1/2}$. We assume that

$$\ker(\mathcal{D}(x) - \mathcal{B}(x)) \subseteq \ker(\mathcal{D}(x) - B(x)) \qquad x \in \partial\Omega, \tag{2.11a}$$

$$B(x) + B(x)^t \geqslant 0, \qquad \|B(x)\|_2 \lesssim 1, \qquad x \in \partial\Omega, \tag{2.11b}$$

$$(\mathcal{D}(x) + B(x))\, y \cdot z \lesssim \|y\|_2 \, |z|_B, \qquad x \in \partial\Omega, \ y, z \in \mathbb{R}^L, \tag{2.11c}$$

$$S(x) + S(x)^t \geqslant 0, \qquad \|S(x)\|_2 \lesssim 1, \qquad x \in \mathcal{F}_i, \tag{2.11d}$$

$$S(x) y \cdot z \lesssim |y|_S \, |z|_S, \qquad x \in \mathcal{F}_i, \ y, z \in \mathbb{R}^L. \tag{2.11e}$$

$$\mathcal{D}(x)y \cdot z \;\lesssim\; \|y\|_2\, |z|_S, \qquad x \in \mathcal{F}_i,\; y, z \in \mathbb{R}^L. \tag{2.11f}$$

These form a subset of the "design conditions for DG methods" in [9] that we shall need for our analysis in the next section.

2.4. **Examples.** The following examples show a variety of equations, boundary conditions, and their well-known discretizations that conform to the framework introduced above. We will work out the first example in some detail and describe the rest telegraphically since similar examples can be found in the literature [3, 9, 10, 17].

Example 2.1 (Maxwell equations with impedance boundary conditions)**.** Suppose we are given electric permittivity $\varepsilon(x)$ and magnetic permeability $\mu(x)$ as positive functions on Ω and let $Z > 0$. The Maxwell system for the electric field $E(x,t)$ and magnetic field $H(x,t)$, with impedance boundary conditions, consists of

$$\varepsilon\, \partial_t\, E - \operatorname{curl} H = 0, \qquad \mu\, \partial_t\, H + \operatorname{curl} E = 0, \qquad \text{in } \Omega \times (0,T), \tag{2.12a}$$

$$n \times E - Zn \times (H \times n) = 0, \qquad \text{on } \partial\Omega \times (0,T). \tag{2.12b}$$

To fit this into the prior setting, we put $N = 3$, $L = 6$, and

$$u = \begin{bmatrix} E \\ H \end{bmatrix}, \quad \mathcal{L}^{(j)} = \begin{bmatrix} 0 & [\epsilon^j] \\ [\epsilon^j]^t & 0 \end{bmatrix}, \quad \mathcal{G} = \begin{bmatrix} \varepsilon I & 0 \\ 0 & \mu I \end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix} 0 & \mathcal{N} \\ \mathcal{N} & -2Z\mathcal{N}^t\mathcal{N} \end{bmatrix},$$

where ϵ^j is the 3×3 matrix whose (l,m)th entry equals the value of the Levi-Civita alternator ϵ_{jlm} and $\mathcal{N} = \sum_{j=1}^3 n_j \epsilon^j \in \mathbb{R}^{3\times 3}$. Noting that $\mathcal{N}^t = -\mathcal{N}$, $\mathcal{N}E = E \times n$, and $\mathcal{N}^t\mathcal{N}H = n \times (H \times n)$, it is easy to see that

$$\mathcal{D} = \begin{bmatrix} 0 & \mathcal{N} \\ \mathcal{N}^t & 0 \end{bmatrix}, \quad (\mathcal{D} - \mathcal{B}) \begin{bmatrix} E \\ H \end{bmatrix} = 2 \begin{bmatrix} 0 \\ n \times E - Zn \times (H \times n) \end{bmatrix}, \tag{2.13}$$

so that (2.2) indeed imposes the impedance boundary condition (2.12b).

Next, for the DG discretization, set

$$B = \frac{1}{1+Z} \begin{bmatrix} 2\mathcal{N}\mathcal{N}^t & (1-Z)\mathcal{N} \\ (1-Z)\mathcal{N} & 2Z\mathcal{N}\mathcal{N}^t \end{bmatrix}, \quad S = \begin{bmatrix} \mathcal{N}^t\mathcal{N} & 0 \\ 0 & \mathcal{N}^t\mathcal{N} \end{bmatrix}.$$

Since $\|n \times (E \times n)\|_2^2 = \|\mathcal{N}E\|_2^2$, setting $b = n \times E - Zn \times (H \times n)$, it is easy to see that

$$(\mathcal{D} - B) \begin{bmatrix} E \\ H \end{bmatrix} = \frac{2}{1+Z} \begin{bmatrix} n \times b \\ b \end{bmatrix}, \quad \left| \begin{bmatrix} E \\ H \end{bmatrix} \right|_B^2 = \frac{2}{1+Z} \left(\|n \times E\|_2^2 + Z\|n \times H\|_2^2 \right), \tag{2.14}$$

and the latter shows (2.11b) since $Z > 0$. The first identity of (2.14), together with (2.13) implies (2.11a). Similar computations establish (2.11c). Finally, noting that $|[\begin{smallmatrix} E \\ H \end{smallmatrix}]|_S^2 = \|n \times E\|_2^2 + \|n \times H\|_2^2$, the remaining properties (2.11d) and (2.11f), are easily established. Note that if the above S is scaled by $1/2$, then the conditions of (2.11) continue to hold and we get the "classic upwind" flux [15, p. 434] for Maxwell equations.

Example 2.2 (Maxwell equations with perfect electric boundary conditions)**.** Reconsider Example 2.1 with $Z = 0$. This yields a Dirichlet boundary condition modeling the electrical isolation by a perfect electric conductor. Substituting $Z = 0$ in previous choices of $\mathcal{B}, B$, and S, one can show that all conditions of (2.11) continue to hold even though the $|\cdot|_B$ seminorm is now weaker: $|[\begin{smallmatrix} E \\ H \end{smallmatrix}]|_B^2 = 2\|n \times E\|_2^2$ from (2.14).

Example 2.3 (Advection)**.** The advection problem with inflow boundary conditions,

$$\partial_t u + \operatorname{div}_x(bu) = 0 \quad \text{in } \Omega \times (0,T), \qquad u = 0 \quad \text{on } \partial_{\text{in}}\Omega \times (0,T),$$

where $b : \Omega \to \mathbb{R}^N$, is some given vector field, fits the above setting with $L = 1$ (keeping the spatial dimension N arbitrary), $\mathcal{L}^{(j)} = b_j \in \mathbb{R}^{1\times 1}$, $\mathcal{G} = 1$ and $\mathcal{D} = b \cdot n$. Examples of b that satisfy (2.5) and (2.6) are offered by divergence-free functions in the lowest order Raviart-Thomas finite element space. The inflow boundary condition is recovered by setting $\mathcal{B} = |b \cdot n|$. The choices

$$B = |b \cdot n|, \qquad S = \frac{1}{2}|b \cdot n|,$$

are easily seen to verify (2.11) and yield the classical upwind DG discretization.

Example 2.4 (Wave equation with Dirichlet boundary conditions)**.** Rewriting

$$\partial_{tt}\phi - \Delta\phi = 0, \quad \text{in } \Omega \times (0,T), \qquad \phi = 0 \quad \text{on } \partial\Omega \times (0,T), \tag{2.15}$$

as a first order hyperbolic system for $L = N + 1$ variables using the flux $q = -\operatorname{grad}_x \phi$ and $\mu = \partial_t \phi$, we match the prior framework. Put $u = \left[\begin{smallmatrix} q \\ \mu \end{smallmatrix}\right]$, $\mathcal{G}$ to identity, and $\mathcal{L}^{(j)} = e_j e_{N+1}^t + e_{N+1} e_j^t$ (using the standard unit basis vectors e_j of $\mathbb{R}^{N+1}$). The Dirichlet boundary conditions on ϕ can be imposed by requiring that $\mu = 0$ on $\partial\Omega \times (0,T)$, which is what (2.2) yields with

$$\mathcal{B} = \sum_{j=1}^{N} n_j (e_{N+1} e_j^t - e_j e_{N+1}^t) + 2 e_{N+1} e_{N+1}^t = \begin{bmatrix} 0 & -n \\ n^t & 2 \end{bmatrix}.$$

All conditions of (2.11) are satisfied by setting

$$B = \mathcal{B}, \qquad S = \begin{bmatrix} nn^t & 0 \\ 0 & 1 \end{bmatrix}.$$

These choices yield the DG discretization with upwind-like fluxes for the wave equation.

Example 2.5 (Wave equation with Robin boundary conditions)**.** We reconsider Example 2.4 after replacing the boundary condition in (2.15) by $\partial\phi/\partial n + \rho\partial_t\phi = 0$ for some $\rho > 0$, or equivalently, in terms of the variables q, μ introduced there,

$$n \cdot q - \rho\mu = 0, \quad \text{on } \partial\Omega \times (0,T). \tag{2.16}$$

Keeping the same S and changing

$$\mathcal{B} = \begin{bmatrix} 0 & n \\ -n^t & 2\rho \end{bmatrix}, \quad B = \begin{bmatrix} \rho^{-1} nn^t & 0 \\ 0 & \rho \end{bmatrix},$$

all conditions of (2.11) are satisfied.

Example 2.6 (Wave equation with Neumann boundary conditions)**.** This is the boundary condition obtained when $\rho = 0$ in (2.16). The boundary condition as well as the conditions of (2.11) are verified with

$$\mathcal{B} = \begin{bmatrix} 0 & n \\ -n^t & 0 \end{bmatrix}, \quad B = \begin{bmatrix} nn^t & n \\ -n^t & 0 \end{bmatrix},$$

keeping S unchanged.

3. Analysis of semidiscretization

In this section, we prove stability of the DG semidiscretization (2.9) on the advancing fronts. When combined with standard finite element approximation estimates, this leads to the main result of this section, namely the error estimate of Theorem 3.12 below.

3.1. Preparatory observations.

Let $H^1(\mathit{\Omega}_h^{\mathtt{v}})$ denote the broken Sobolev space isomorphic to $\Pi_{K\in\mathit{\Omega}_h^{\mathtt{v}}} H^1(K)$. Since our variables have L unknown components, we will need L copies of this space. To ease notation, we abbreviate $H_h^{\mathtt{v}} = H^1(\mathit{\Omega}_h^{\mathtt{v}})^L$, $H^{\mathtt{v}} = H^1(\mathit{\Omega}^{\mathtt{v}})^L$, and $L^{\mathtt{v}} = L^2(\mathit{\Omega}^{\mathtt{v}})^L$. Since the traces of a $w \in H_h^{\mathtt{v}}$ on element boundaries are square integrable, the following definition of the bilinear form $a : H_h^{\mathtt{v}} \times H_h^{\mathtt{v}} \to \mathbb{R}$, with the numerical fluxes $\hat{F}_w^n$ from (2.10), makes sense:

$$a(w,v) = \sum_{K\in\mathit{\Omega}_h^{\mathtt{v}}} \bigg[\int_K \delta f(w) : \operatorname{grad}_x v - \int_{\partial K} \delta \hat{F}_w^n \cdot v \bigg].$$

Let $(w,v)_D$ denote the inner product in $L^2(D)$, or its Cartesian products, for any domain D, and let $\|w\|_D = (w,w)_D^{1/2}$. Using this notation, we may alternately write $a(\cdot,\cdot)$ as

$$a(w,v) = \sum_{K\in\mathit{\Omega}_h^{\mathtt{v}}} \left[\sum_{j=1}^N (\delta\mathcal{L}^{(j)} w, \partial_j v)_K - (\delta\hat{F}_w^n, v)_{\partial K} \right]. \tag{3.1}$$

When the domain is the often used vertex patch $\mathit{\Omega}^{\mathtt{v}}$, we use the abbreviated notation $(w,v)_{\mathtt{v}} = (w,v)_{\mathit{\Omega}^{\mathtt{v}}} = \int_{\mathit{\Omega}^{\mathtt{v}}} w\cdot v$. Using it, we define $M_0 : L^{\mathtt{v}} \to L^{\mathtt{v}}$, and $M_1 : L^{\mathtt{v}} \to L^{\mathtt{v}}$ by

$$(M_0 w, v)_{\mathtt{v}} = (\mathcal{G}w, v)_{\mathtt{v}} - \sum_{j=1}^N ((\partial_j \varphi_{\mathrm{bot}})\, \mathcal{L}^{(j)} w, v)_{\mathtt{v}}, \tag{3.2}$$

$$(M_1 w, v)_{\mathtt{v}} = \sum_{j=1}^N ((\partial_j \delta)\, \mathcal{L}^{(j)} w, v)_{\mathtt{v}} \tag{3.3}$$

for all $w, v \in L^{\mathtt{v}}$. Let $M(\tau) = M_0 - \tau M_1$. (We will often abbreviate $M(\tau)$ to simply M.) Using these definitions, we may now rewrite (2.9) succinctly as $(\partial_{\hat{t}}(M(\hat{t})\hat{u}_h), v)_{\mathtt{v}} = a(\hat{u}_h, v)$. Let $\|v\|_{\mathtt{v}} = (v,v)_{\mathtt{v}}^{1/2}$, and for any operator $\mathcal{O}$ on $L^{\mathtt{v}}$, let

$$\|\mathcal{O}\|_{\mathtt{v}} = \sup_{v,w\in L^{\mathtt{v}}} \frac{(\mathcal{O}v, w)_{\mathtt{v}}}{\|v\|_{\mathtt{v}} \|w\|_{\mathtt{v}}}. \tag{3.4}$$

Lemma 3.1. *The causality condition implies that $M(\tau)$ is a selfadjoint positive definite operator on $L^{\mathtt{v}}$ for any $0 \leqslant \tau \leqslant 1$ and that there is a mesh-independent constant $C_{\mathcal{L},c}$ (depending on $\mathcal{L}^{(j)}$ and c) such that*

$$\left(1 - \frac{c}{\hat{c}}\right)(\mathcal{G}w, w)_{\mathtt{v}} \leqslant (Mw, w)_{\mathtt{v}} \leqslant C_{\mathcal{L},c}(\mathcal{G}w, w)_{\mathtt{v}} \tag{3.5}$$

holds for all $w \in L^{\mathtt{v}}$. Moreover,

$$\max\left(\|M_0\|_{\mathtt{v}}, \|M_0^{-1}\|_{\mathtt{v}}, \|M_1\|_{\mathtt{v}}, \|M\|_{\mathtt{v}}, \|M^{-1}\|_{\mathtt{v}}\right) \lesssim 1.$$

Proof. Let $\varphi = \tau\varphi_{\text{top}} + (1-\tau)\varphi_{\text{bot}}$. Since

$$(M(\tau)v, w)_{\mathtt{v}} = (\mathcal{G}v, w)_{\mathtt{v}} - \sum_{j=1}^{N} ((\partial_j\varphi)\,\mathcal{L}^{(j)}v, w)_{\mathtt{v}} \tag{3.6}$$

and $\partial_j\varphi$ is constant on each element, the selfadjointness is immediate from the symmetry of $\mathcal{L}^{(j)}$ and $\mathcal{G}$. It remains to prove the stated positive definiteness. In accordance with (2.3), let $\mathcal{D}^{(\nu)} = \sum_{j=1}^{N} \nu_j \mathcal{L}^{(j)}$. Recall [6, p. 53] that hyperbolicity implies the existence of real eigenvalues λ_i and accompanying eigenvectors e_i (forming a complete set) satisfying $\mathcal{D}^{(\nu)} e_i = \lambda_i \mathcal{G} e_i$ for any unit vector $\nu \in \mathbb{R}^N$. Since $\mathcal{D}^{(\nu)}$ and $\mathcal{G}$ are symmetric, it is easy to see that the eigenvectors e_i must be orthogonal in the $\langle x, y\rangle_{\mathcal{G}} = \mathcal{G}x \cdot y$ inner product. Expanding any vector $v \in \mathbb{R}^L$ in the eigenbasis e_i as follows,

$$v = \sum_{j=1}^{L} v_i e_i \qquad \text{with } v_i = \langle v, e_i\rangle_{\mathcal{G}},$$

and recalling that the maximal wave speed c is the maximum of all such $|\lambda_i|$,

$$\mathcal{D}^{(\nu)} v \cdot v = \sum_{i=1}^{L} v_i \lambda_i \mathcal{G} e_i \cdot v = \sum_{i=1}^{L} \lambda_i\, |v_i|^2 \leqslant c \sum_{i=1}^{L} |v_i|^2 = c\,\langle v, v\rangle_{\mathcal{G}}.$$

Using this inequality with $\nu = (\operatorname{grad}_x \varphi)/\|\operatorname{grad}_x \varphi\|_2$, we have

$$\begin{aligned} \mathcal{G}v \cdot v - \sum_{j=1}^{N} (\partial_j\varphi)\mathcal{L}^{(j)} v \cdot v &= \mathcal{G}v \cdot v - \|\operatorname{grad}_x \varphi\|_2\, \mathcal{D}^{(\nu)} v \cdot v \\ &\geqslant (1 - \|\operatorname{grad}_x \varphi\|_2\, c)\ \mathcal{G}v \cdot v. \end{aligned} \tag{3.7}$$

Since φ is a convex combination of φ_{bot} and φ_{top}, both of which satisfy the causality condition (1.4), we have $\|\operatorname{grad}_x \varphi\|_2 \leqslant 1/\hat{c}$. Applying this, after using (3.7) in (3.6), the proof of the lower bound of (3.5) is finished. The upper bound is a consequence of the boundedness of the $\mathcal{L}^{(j)}$ and $\mathcal{G}$. Finally, the stated operator norm bounds on $M(\tau)$, $M_0 = M(0)$, and their inverses follow immediately from (3.5). The estimate for the operator norm of M_1 also follows easily since $|\partial_j \delta| \lesssim 1$. □

Let $\mathcal{F}^{\mathtt{v}}$ denote the set of facets (i.e., $(N-1)$-subsimplices) of the simplicial mesh $\varOmega_h^{\mathtt{v}}$ of the vertex patch $\varOmega^{\mathtt{v}}$. This set is partitioned into the collection of facets on the boundary $\partial\varOmega^{\mathtt{v}}$ of the vertex patch, denoted by $\mathcal{F}_b^{\mathtt{v}}$, and the remainder, denoted by $\mathcal{F}_i^{\mathtt{v}}$, the set of interior facets of $\varOmega_h^{\mathtt{v}}$. We assume that each facet F of the entire spatial mesh $\varOmega_h$ is endowed with a unit normal n_F whose orientation is arbitrarily fixed, unless if F is contained in the global boundary $\partial\varOmega$, in which case it points outward. Then, for any $x \in F$, set $[\![u]\!]_F(x) = \lim_{\varepsilon\to 0} u(x + \varepsilon n_F) - u(x - \varepsilon n_F)$. Note that $[\![u]\!]_F$ agrees with the previously defined jump $[\![u]\!]$ on element boundaries, except possibly for a sign. Let

$$d(w, v) = -\left[a(w, v) + a(v, w) + (M_1 w, v)_{\mathtt{v}}\right]$$

for $w, v \in H_h^{\mathtt{v}}$. The first identity of the next lemma shows that $d(w, w) \geqslant 0$ due to (2.11b) and (2.11d).

Lemma 3.2. *For all $v, w \in H_h^{\mathtt{v}}$,*

$$d(w, w) = \sum_{F \in \mathcal{F}_i^{\mathtt{v}}} 2\big(\delta\, S[\![w]\!]_F, [\![w]\!]_F\big)_F + \sum_{F \in \mathcal{F}_b^{\mathtt{v}}} (\delta\, Bw, w)_F, \tag{3.8}$$

$$
\begin{aligned}
(3.9) \qquad -a(v,w) = & \sum_{K\in\Omega_h^{\mathrm{v}}} (\mathrm{div}_x(\delta f(v)), w)_K \\
& + \sum_{F\in\mathcal{F}_i^{\mathrm{v}}} \Big[(\delta \mathcal{D}^{(n_F)} [\![v]\!]_F, \{w\})_F + (\delta S [\![v]\!]_F, [\![w]\!]_F)_F \Big] \\
& - \sum_{F\in\mathcal{F}_b^{\mathrm{v}}} \frac{1}{2} (\delta(\mathcal{D}-B)v, w)_F.
\end{aligned}
$$

Proof. Integrating by parts on an element $K \in \Omega_h^{\mathrm{v}}$,

$$
\sum_{j=1}^N (\delta\, \mathcal{L}^{(j)} w, \partial_j w)_K = (\delta \mathcal{D} w, w)_{\partial K} - \sum_{j=1}^N (\partial_j(\delta \mathcal{L}^{(j)} w), w)_K.
$$

Applying the product rule to expand the derivative in the last term, using (2.5), and the symmetry of $\mathcal{L}^{(j)}$, we obtain

$$
(3.10) \qquad 2\sum_{j=1}^N (\delta \mathcal{L}^{(j)} w, \partial_j w)_K = -\sum_{j=1}^N (w, (\partial_j \delta) \mathcal{L}^{(j)} w)_K + (\delta w, \mathcal{D} w)_{\partial K}.
$$

Using this in the first term of the definition of $a(w,w)$, we have

$$
\begin{aligned}
d(w,w) &= -2\, a(w,w) - \sum_{j=1}^N (\partial_j \delta\ \mathcal{L}^{(j)} w, w)_{\mathrm{v}} && \text{by (3.3)}, \\
&= \sum_{K\in\Omega_h^{\mathrm{v}}} \big[-(\delta\, w, \mathcal{D} w)_{\partial K} + 2(\delta\, w, \hat{F}_w^n)_{\partial K} \big] && \text{by (3.1) and (3.10)}, \\
&= \sum_{K\in\Omega_h^{\mathrm{v}}} \big[-(\delta\, w, \mathcal{D} w)_{\partial K\cap\partial\Omega} + (\delta\, w, (\mathcal{D}+B) w)_{\partial K\cap\partial\Omega} \big] \\
&+ \sum_{K\in\Omega_h^{\mathrm{v}}} \big[-(\delta\, w, \mathcal{D} w)_{\partial K\setminus\partial\Omega} + (\delta\, w, 2\mathcal{D}\{w\} + 2S[\![w]\!])_{\partial K\setminus\partial\Omega} \big],
\end{aligned}
$$

where we used the definition of the numerical flux in (2.10), splitting the right hand side sum into two to accomodate the two cases in (2.10). The first sum, when rewritten using boundary facets, immediately yields the last term of (3.8) since $\delta = 0$ on $\partial\Omega^{\mathrm{v}} \setminus \partial\Omega$. The second sum can be rearranged to a sum over interior facets $F \in \mathcal{F}_i^{\mathrm{v}}$, where $[\![\mathcal{D}^{(n_F)}]\!]_F = 0$ due to (2.5), which allows for further simplifications, eventually yielding the other term on the right hand side of (3.8).

The proof of (3.9) involves a similar integration by parts starting from (3.1) and a similar rearrangement of sums over element boundaries to sums over facets. □

We use $C^s(0,\tau,X)$, for some Banach space X, to denote the X-valued function $w : [0,\tau] \to X$ that is s times continuously differentiable. For any $v, w \in C^1(0,1,H_h^{\mathrm{v}})$, define

$$
b_\tau(v,w) = \int_0^\tau \big(\partial_{\hat{t}} [M(\hat{t}) v(\hat{t})], w(\hat{t}) \big)_{\mathrm{v}}\, d\hat{t}\ - \int_0^\tau a(v(\hat{t}), w(\hat{t}))\, d\hat{t}.
$$

Note that the temporal snapshots $w(\hat{t})$ and $v(\hat{t})$ used above, being in H_h^{v}, are admissible as arguments of the form a. For any $z \in L^{\mathrm{v}}$, define $\|z\|_{M(\tau)} = (M(\tau)z, z)_{\mathrm{v}}^{1/2}$. This is a norm due to Lemma 3.1.

Lemma 3.3. *For all* $w \in C^1(0,1,H_h^{\mathsf{v}})$,

$$2\, b_\tau(w,w) = \|w(\tau)\|^2_{M(\tau)} - \|w(0)\|^2_{M(0)} + \int_0^\tau d(w(\hat t), w(\hat t))\; d\hat t.$$

Proof. The proof relies on a simple but key identity, which is best expressed writing M for $M(\hat t) = M_0 - \hat t M_1$, as follows:

$$\frac{d}{d\hat t}\int_{\Omega^{\mathsf{v}}} Mw\cdot w = \int_{\Omega^{\mathsf{v}}} 2\frac{\partial}{\partial \hat t}(Mw)\cdot w - \int_{\Omega^{\mathsf{v}}} \frac{dM}{d\hat t} w\cdot w.$$

It implies $2\left(\partial_{\hat t}[M(\hat t)w], w\right)_{\mathsf{v}} = \partial_{\hat t}(M(\hat t)w,w)_{\mathsf{v}} - (M_1 w, w)_{\mathsf{v}}$. Using this in the definition of b_τ, we obtain

$$\begin{aligned}2b_\tau(w,w) &= \int_0^\tau \left[\frac{d}{d\hat t}\left(M(\hat t)w(\hat t), w(\hat t)\right)_{\mathsf{v}} - \left(M_1 w(\hat t), w(\hat t)\right)_{\mathsf{v}} - 2\, a(w(\hat t), w(\hat t))\right] d\hat t\\ &= (M(\tau)w,w)_{\mathsf{v}} - (M(0)w,w)_{\mathsf{v}} - \int_0^\tau \left[2\, a(w(\hat t), w(\hat t)) + (M_1 w(\hat t), w(\hat t))\right] d\hat t\end{aligned}$$

so the result follows from the definition of $d(\cdot,\cdot)$. ☐

3.2. **Stability on spacetime surfaces.** We will first establish a bound on the exact solution on spacetime tents (Proposition 3.4), which will then serve as motivation for our approach to proving stability (Lemma 3.6). Let $\partial_{\mathrm{top}} T^{\mathsf{v}}, \partial_{\mathrm{bot}} T^{\mathsf{v}}$ and $\partial_{\mathrm{bdr}} T^{\mathsf{v}}$ denote the top, bottom, and boundary parts, respectively, of the boundary of a tent T^{v}, i.e.,

$$\partial_{\mathrm{top}} T^{\mathsf{v}} = \{(x,t)\in \partial T^{\mathsf{v}} :\ t = \varphi_{\mathrm{top}}(x)\}, \qquad \partial_{\mathrm{bot}} T^{\mathsf{v}} = \{(x,t)\in \partial T^{\mathsf{v}} :\ t = \varphi_{\mathrm{bot}}(x)\}$$
$$\partial_{\mathrm{bdr}} T^{\mathsf{v}} = \{(x,t)\in \partial T^{\mathsf{v}} :\ (x,t) \text{ is neither in } \partial_{\mathrm{top}} T^{\mathsf{v}} \text{ nor in } \partial_{\mathrm{bot}} T^{\mathsf{v}}\}.$$

Note that $\partial_{\mathrm{bdr}} T^{\mathsf{v}}$ is empty whenever $\partial\Omega^{\mathsf{v}}$ does not intersect $\partial\Omega$. The next result shows that the solution on $\partial_{\mathrm{top}} T^{\mathsf{v}}$ can be bounded, in a tent-specific norm, by that on $\partial_{\mathrm{bot}} T^{\mathsf{v}}$. Specifically, defining

$$\|w\|^2_{\partial_{\mathfrak{b}} T^{\mathsf{v}}} = \int_{\Omega^{\mathsf{v}}} \left[g(w(x,\varphi_{\mathfrak{b}}(x))) - f(w(x,\varphi_{\mathfrak{b}}(x)))\, \mathrm{grad}_x\, \varphi_{\mathfrak{b}}\right]\cdot w(x,\varphi_{\mathfrak{b}}(x)), \tag{3.11}$$

for $\mathfrak{b} \in \{\mathrm{top}, \mathrm{bot}\}$, it follows from the next result that $\|u\|_{\partial_{\mathrm{top}} T^{\mathsf{v}}} \leqslant \|u\|_{\partial_{\mathrm{bot}} T^{\mathsf{v}}}$ (because $\|u\|_{\partial_{\mathrm{top}} T^{\mathsf{v}}}$ and $\|u\|_{\partial_{\mathrm{bot}} T^{\mathsf{v}}}$ coincide with $\|\hat u\|_{M(1)}$ and $\|\hat u\|_{M(0)}$, respectively).

Proposition 3.4. *On a spacetime tent* T^{v} *satisfying causality, suppose a solution* u *of*

$$\partial_t\, g(u) + \mathrm{div}_x\, f(u) = 0 \qquad \text{in } T^{\mathsf{v}}, \tag{3.12a}$$
$$(\mathcal{D} - \mathcal{B})u = 0 \qquad \text{on } \partial_{\mathrm{bdr}} T^{\mathsf{v}}, \tag{3.12b}$$

is smooth enough for $\hat u = u\circ\Phi$ *to be in* $C^1(0,1,H^{\mathsf{v}})$. *Then* $\hat u = u\circ\Phi$ *at pseudotime* τ, *for any* $0 \leqslant \tau \leqslant 1$, *satisfies*

$$\|\hat u(\tau)\|_{M(\tau)} \leqslant \|\hat u(0)\|_{M(0)}.$$

Proof. Since $\hat u$ satisfies the mapped equation $\partial_{\hat t}(g(\hat u) - f(\hat u)\,\mathrm{grad}_x\,\varphi) + \mathrm{div}_x\left[\delta f(\hat u)\right] = 0$, we have $(\partial_{\hat t}[M(\hat t)\hat u], v)_{\mathsf{v}} + (\mathrm{div}_x(\delta f(\hat u)), v)_{\mathsf{v}} = 0$ for all $v \in H^{\mathsf{v}}$. Now, observe that

$$-a(\hat u(\hat t), v) = \left(\,\mathrm{div}_x\left[\delta f(\hat u(\hat t))\right], v\right)_{\mathsf{v}}$$

by Lemma 3.2: indeed, the jumps in (3.9) of the lemma vanish when applied to $\hat{u}(\hat{t})$ since $\hat{u}(\hat{t}) \in H^{\mathrm{v}}$, and moreover, the last term of (3.9) also vanishes due to (3.12b) and (2.11a). Thus,

$$(\partial_{\hat{t}}[M(\hat{t})\hat{u}], v)_{\mathrm{v}} - a(\hat{u}(\hat{t}), v) = 0$$

for all $v \in H^{\mathrm{v}}$ and each $0 \leqslant \hat{t} \leqslant 1$. Integrating over $\hat{t}$ from 0 to τ, we obtain

$$b_\tau(\hat{u}, w) = 0, \tag{3.13}$$

for all $w \in C^1(0, 1, H^{\mathrm{v}})$. Choosing $w = \hat{u}$ and applying Lemma 3.3, we have

$$\|\hat{u}(\tau)\|^2_{M(\tau)} - \|\hat{u}(0)\|^2_{M(0)} + \int_0^\tau d(\hat{u}(\hat{t}), \hat{u}(\hat{t}))\, d\hat{t} = 0.$$

Finally, we apply (3.8) of Lemma 3.2. Noting that $[\![\hat{u}(\hat{t})]\!]_F = 0$ on all interior facets F and recalling the positivity assumption (2.11b) on B, we complete the proof. □

Definition 3.5 (Semidiscrete flow: $R_h^{\mathrm{sem}}(\tau)$)**.** For any $0 \leqslant \tau \leqslant 1$, define $R_h^{\mathrm{sem}}(\tau) : V_h^{\mathrm{v}} \to V_h^{\mathrm{v}}$ as follows. Given a $v_h^0 \in V_h^{\mathrm{v}}$, let $v_h \in C^1(0, 1, V_h^{\mathrm{v}})$ solve

$$\begin{aligned} \big(\partial_{\hat{t}}\big[g(v_h) - f(v_h)\operatorname{grad}_x \varphi\big], w\big)_{\mathrm{v}} &= a(v_h(\hat{t}), w), && 0 < \hat{t} \leqslant 1, \\ v_h(0) &= v_h^0, && \hat{t} = 0, \end{aligned} \tag{3.14}$$

for all $w \in V_h^{\mathrm{v}}$. Set $R_h^{\mathrm{sem}}(\tau)v_h^0$ to $v_h(\tau)$. (In particular $R_h^{\mathrm{sem}}(0)v_h^0 = v_h(0) = v_h^0$.)

Lemma 3.6 (Stability of semidiscretization)**.** *For any $0 \leqslant \tau \leqslant 1$, and any $v \in V_h^{\mathrm{v}}$,*

$$\|R_h^{\mathrm{sem}}(\tau)v\|_{M(\tau)} \leqslant \|v\|_{M(0)}.$$

Proof. The argument is similar to the proof of Proposition 3.4. Let $v_h(\tau) = R_h^{\mathrm{sem}}(\tau)v$. We need to bound $v_h(\tau)$ by $v_h(0) = v$. Replacing w in (3.14) by a time-dependent test function $\tilde{v} \in C^1(0, 1, V_h^{\mathrm{v}})$ and integrating over $\hat{t}$ from 0 to τ, we have

$$\int_0^\tau \Big[(\partial_{\hat{t}}[M(\hat{t})v_h], \tilde{v})_{\mathrm{v}} - a(v_h(\hat{t}), \tilde{v}(\hat{t}))\Big]\, d\hat{t} = 0,$$

or equivalently, $b_\tau(v_h, \tilde{v}) = 0$, for all $\tilde{v} \in C^1(0, 1, V_h^{\mathrm{v}})$. Now, choosing $\tilde{v} = v_h$ and applying Lemma 3.3, we find that

$$\|v_h(\tau)\|^2_{M(\tau)} = \|v_h(0)\|^2_{M(0)} - \int_0^\tau d(v_h(\hat{t}), v_h(\hat{t}))\, d\hat{t}.$$

Since $d(v_h(\hat{t}), v_h(\hat{t})) \geqslant 0$ by (3.8) of Lemma 3.2, the proof is complete. □

3.3. **Local error in a tent.** To estimate the error in the semidiscrete solution, we use, like previous authors [5], the spatial L^2 projection into the DG space V_h^{v}. Let $P_h : L^{\mathrm{v}} \to V_h^{\mathrm{v}}$ be defined by $(P_h v, w)_{\mathrm{v}} = (v, w)_{\mathrm{v}}$ for all $v \in L^{\mathrm{v}}$ and $w \in V_h^{\mathrm{v}}$. Define

$$|v|_d = d(v, v)^{1/2}, \qquad v \in H_h^{\mathrm{v}}.$$

This is a seminorm by (3.8) of Lemma 3.2 and our assumptions (2.11b) and (2.11d). Let $h_K = \operatorname{diam} K$ for any spatial element K. The next lemma also uses the broken Sobolev space $H^s(\Omega_h^{\mathrm{v}}) = \Pi_{K \in \Omega_h^{\mathrm{v}}} H^s(K)$, and

$$h_{\mathrm{v}} = \max_{K \in \Omega_h^{\mathrm{v}}} h_K, \qquad |w|^2_{H^s(\Omega_h^{\mathrm{v}})^L} = \sum_{K \in \Omega_h^{\mathrm{v}}} |w|^2_{H^s(K)^L}.$$

Lemma 3.7. *If $w \in H^l(\Omega_h^{\mathrm{v}})^L$ for some $1 \leqslant l \leqslant p+1$, then for any $v_h \in V_h^{\mathrm{v}}$,*

$$a(w - P_h w, v_h) \lesssim h_{\mathrm{v}}^l\, |w|_{H^l(\Omega_h^{\mathrm{v}})^L} |v_h|_d.$$

Proof. Let $e = w - P_h w$. Then the first term on the right hand side of

$$a(e, v_h) = \sum_{K \in \Omega_h^{\mathtt{v}}} \sum_{j=1}^{N} (\delta \mathcal{L}^{(j)} e, \partial_j v_h)_K - \sum_{K \in \Omega_h^{\mathtt{v}}} (\delta \hat{F}_e^n, v_h)_{\partial K}$$

must vanish, because $\delta \partial_j v_h|_K$ is a polynomial of degree at most p and $\mathcal{L}^{(j)}$ is constant due to assumption (2.6). Hence

$$\begin{aligned} a(e, v_h) &= - \sum_{K \in \Omega_h^{\mathtt{v}}} (\delta(\mathcal{D}\{e\} + S[\![e]\!]), v_h)_{\partial K \setminus \partial \Omega} + \frac{1}{2}(\delta(\mathcal{D} + B)e, v_h)_{\partial K \cap \partial \Omega} \\ &= \sum_{F \in \mathcal{F}_i^{\mathtt{v}}} (\delta \mathcal{D}^{(n_F)}\{e\}, [\![v_h]\!]_F)_F - (\delta S [\![e]\!]_F, [\![v_h]\!]_F)_F - \sum_{F \in \mathcal{F}_b^{\mathtt{v}}} \frac{1}{2}(\delta(\mathcal{D} + B)e, v_h)_F \\ &\lesssim \sum_{F \in \mathcal{F}_i^{\mathtt{v}}} \int_F \delta \|e\|_2 \left|[\![v_h]\!]\right|_S + \sum_{F \in \mathcal{F}_b^{\mathtt{v}}} \int_F \delta \|e\|_2 \, |v_h|_B \end{aligned}$$

due to assumptions (2.11e), (2.11f), and (2.11c). On any facet F adjacent to an element K, by shape regularity and the well-known properties of L^2 projectors, $h_{\mathtt{v}}^{1/2} \|e\|_F \lesssim h_{\mathtt{v}}^l |w|_{H^l(K)^L}$. Since $\delta \lesssim h_{\mathtt{v}}$, the result now follows after applying Cauchy-Schwarz inequality and (3.8) of Lemma 3.2. □

The next lemma provides control of the local error at any pseudotime τ in terms of the initial error. To measure the regularity of functions w on a tent $T^{\mathtt{v}}$, we find it convenient to use (semi)norms computed using the pull back $w \circ \Phi$ on $\hat{T}^{\mathtt{v}}$, defined by

$$|w|_{\mathtt{v},l} = \sup_{0 \leqslant \tau \leqslant 1} |(w \circ \Phi)(\tau)|_{H^l(\Omega_h^{\mathtt{v}})^L}, \qquad \|w\|_{\mathtt{v},l} = \sup_{0 \leqslant \tau \leqslant 1} \|(w \circ \Phi)(\tau)\|_{H^l(\Omega_h^{\mathtt{v}})^L}. \tag{3.15}$$

Clearly, these are bounded when $\hat{w} = w \circ \Phi$ is in $C^0(0, 1, H^l(\Omega_h^{\mathtt{v}})^L)$.

Lemma 3.8 (Local error bound)**.** *Let u be the exact solution of* (3.12) *on a causal tent $T^{\mathtt{v}}$, $\hat{u} = u \circ \Phi \in C^1(0, 1, H^{\mathtt{v}} \cap H^{p+1}(\Omega_h^{\mathtt{v}})^L)$, and let $\hat{u}_h(\tau) = R_h^{\text{sem}}(\tau)\hat{u}_h^0$ for any $\hat{u}_h^0 \in V_h^{\mathtt{v}}$. Then*

$$\|\hat{u}(\tau) - \hat{u}_h(\tau)\|_{M(\tau)} \lesssim \|\hat{u}(0) - \hat{u}_h^0\|_{M(0)} + h_{\mathtt{v}}^{p+1} |u|_{\mathtt{v},p+1}.$$

Proof. Integrating (3.14) of Definition 3.5, we see that the semidiscrete solution $\hat{u}_h$ satisfies $b_\tau(\hat{u}_h, w_h) = 0$ for all $w_h \in C^1(0, 1, V_h^{\mathtt{v}})$. We have also shown that the exact solution $\hat{u}$ satisfies a similar identity, namely (3.13). Subtracting these identities, we have

$$b_\tau(\hat{u} - \hat{u}_h, w_h) = 0 \qquad \text{for all } w_h \in C^1(0, 1, V_h^{\mathtt{v}}). \tag{3.16}$$

Let $e_h(x, t)$ in $C^1(0, 1, V_h^{\mathtt{v}})$ denote the function whose time slices are defined by $e_h(\tau) = \hat{u}_h(\tau) - P_h \hat{u}(\tau)$ for each $0 \leqslant \tau \leqslant 1$. Equation (3.16) implies that $b_\tau(e_h, e_h) = b_\tau(\hat{u} - P_h \hat{u}, e_h) = b_\tau(e, e_h)$, where we have set $e = \hat{u} - P_h \hat{u}$. Therefore, together with Lemma 3.3, we obtain

$$\begin{aligned} &\frac{1}{2}\left(\|e_h(\tau)\|_{M(\tau)}^2 - \|e_h(0)\|_{M(0)}^2 + \int_0^\tau \left|e_h(\hat{t})\right|_d^2 d\hat{t} \right) = b_\tau(e_h, e_h) = b_\tau(e, e_h) \\ &\quad = \int_0^\tau \left[(\partial_{\hat{t}}[M(\hat{t})e], e_h)_{\mathtt{v}} - a(e(\hat{t}), e_h(\hat{t})) \right] d\hat{t} \\ &\quad = (M(\tau)e(\tau), e_h(\tau))_{\mathtt{v}} - (M_0 e(0), e_h(0))_{\mathtt{v}} - \int_0^\tau \left[(M(\hat{t})e, \partial_{\hat{t}}\, e_h)_{\mathtt{v}} + a(e(\hat{t}), e_h(\hat{t})) \right] d\hat{t}. \end{aligned}$$

Since $\partial_{\hat{t}}\, e_h$ is of degree at most p on each element and $(M(\hat{t})e, \partial_{\hat{t}}\, e_h)_{\mathtt{v}} = (\mathcal{G}e, \partial_{\hat{t}}\, e_h)_{\mathtt{v}} - \sum_{j=1}^N ((\partial_j \varphi)\, \mathcal{L}^{(j)} e, \partial_{\hat{t}}\, e_h)_{\mathtt{v}} = 0$ by (2.6) and the orthogonality property of the projection error. Applying Lemma 3.7 to the last term,

$$\|e_h(\tau)\|^2_{M(\tau)} - \|e_h(0)\|^2_{M(0)} + \int_0^\tau \left|e_h(\hat{t})\right|_d^2 \, d\hat{t} \;\lesssim$$
$$(M(\tau)e(\tau), e_h(\tau))_{\mathtt{v}} - (M(0)e(0), e_h(0))_{\mathtt{v}} + h_{\mathtt{v}}^{p+1} \int_0^\tau |\hat{u}(\hat{t})|_{H^{p+1}(\varOmega_h^{\mathtt{v}})^L} |e_h(\hat{t})|_d \, d\hat{t}.$$

By Cauchy-Schwarz inequality in the inner product (see Lemma 3.1) generated by $M(\tau)$,

$$(M(\tau)e(\tau), e_h(\tau))_{\mathtt{v}} \;\lesssim\; h_{\mathtt{v}}^{p+1} |\hat{u}(\tau)|_{H^{p+1}(\varOmega_h^{\mathtt{v}})^L} \|e_h(\tau)\|_{M(\tau)},$$

which holds also when $\tau = 0$. Further applications of Cauchy-Schwarz and Young's inequalities yield

$$\begin{aligned}
&\|e_h(\tau)\|^2_{M(\tau)} + \int_0^\tau \left|e_h(\hat{t})\right|_d^2 \, d\hat{t} \\
&\quad\lesssim \|e_h(0)\|^2_{M(0)} \\
&\quad + {h_{\mathtt{v}}}^{2(p+1)} \left(|\hat{u}(0)|^2_{H^{p+1}(\varOmega_h^{\mathtt{v}})^L} + |\hat{u}(\tau)|^2_{H^{p+1}(\varOmega_h^{\mathtt{v}})^L} + \int_0^\tau \left|\hat{u}(\hat{t})\right|^2_{H^{p+1}(\varOmega_h^{\mathtt{v}})^L} d\hat{t} \right) \\
&\quad\lesssim \|e_h(0)\|^2_{M(0)} + {h_{\mathtt{v}}}^{2(p+1)} |u|^2_{\mathtt{v},p+1}.
\end{aligned}$$

Finally, using the well known error bounds for the L^2 projection and the triangle inequality, we obtain the result of the lemma. □

3.4. **Global error bound.** Recall the advancing front C_i defined by (1.1) and the layer L_i defined by (1.2). We will use the following "`T2G`" procedure several times in the sequel.

Definition 3.9 (Tent propagators to global propagators: `T2G`)**.** Suppose we are given a collection of operators $\mathcal{R}$, one for each tent. The element of $\mathcal{R}$ corresponding to a tent $T^{\mathtt{v}}$ is an operator $R^{T^{\mathtt{v}}} : L^{\mathtt{v}} \to L^{\mathtt{v}}$, which we refer to as the given *tent propagator* on $T^{\mathtt{v}}$, or more precisely, on its preimage $\hat{T}^{\mathtt{v}}$. We think of $R^{T^{\mathtt{v}}}$ as transforming functions given at the bottom of $\hat{T}^{\mathtt{v}}$ to functions at the top of $\hat{T}^{\mathtt{v}}$ by some specific discrete process or by the exact solution operator. To produce global propagation operators from the collection $\mathcal{R}$, we start by mapping functions on C_{i-1} to functions on C_i, or equivalently per the advancing front definition (1.1), by mapping functions of $(x, \varphi_{i-1}(x))$ to functions of $(x, \varphi_i(x))$. The *layer propagator* of the layer L_i generated by $\mathcal{R}$, denoted by $G^{i,i-1} : L^2(C_{i-1})^L \to L^2(C_i)^L$, is defined by first considering points on the front C_i which have not advanced in time, where $G^{i,i-1}w$ simply coincides with w, and then considering the remaining points $(x, \varphi_i(x))$ on C_i which are separated from $(x, \varphi_{i-1}(x))$ on C_{i-1} by a tent, say $T^{\mathtt{v}}$, where we use the tent propagator of $T^{\mathtt{v}}$ (see Figure 2). The next formula states this precisely. For any $w \in L^2(C_{i-1})^L$,

$$(G^{i,i-1}w)(x, \varphi_i(x)) = \begin{cases} w(x, \varphi_{i-1}(x)) & \text{at } x \in \varOmega \text{ where } \varphi_i(x) = \varphi_{i-1}(x), \\ (R^{T^{\mathtt{v}}} \hat{w}_{\mathtt{v}})(x) & \text{if } x \in \varOmega^{\mathtt{v}} \text{ for some } \mathtt{v} \in V_i, \end{cases}$$

where $\hat{w}_{\mathtt{v}}(x) = w(x, \varphi_{i-1}(x))|_{\varOmega^{\mathtt{v}}}$. Finally, for a pair i, j with $i > j \geqslant 0$, the *global propagator* generated by $\mathcal{R}$ is the operator $G^{i,j} : L^2(C_j)^L \to L^2(C_i)^L$, defined by

$$G^{i,j} = G^{i,i-1} \circ G^{i-1,i-2} \circ \cdots \circ G^{j+1,j}.$$

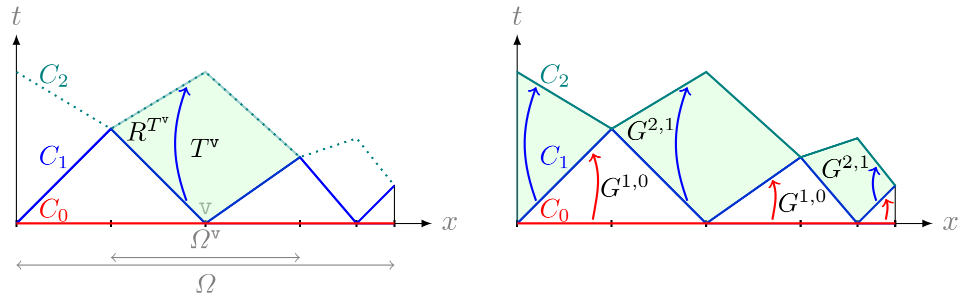


Figure 2. Schematic of a tent propagator (left) and two layer propagators (right).

Let `T2G` denote this process of producing global propagators from a collection of tent propagators, i.e., we define $\mathtt{T2G}(i, j, \mathcal{R})$ to be the $G^{i,j}$ above.

For the semidiscretization, the tent propagator on T^{v} is the operator $R_h^{\mathrm{sem}}(1) \circ P_h : L^{\mathrm{v}} \to V_h^{\mathrm{v}} \subset L^{\mathrm{v}}$, set using the operator $R_h^{\mathrm{sem}}(\tau)$ of Definition 3.5, evaluated at pseudotime $\tau = 1$ (corresponding to the tent top). Collecting these semidiscrete tent propagators into $\mathcal{R}_h$ we use Definition 3.9 to set the corresponding semidiscrete global propagators $R_h^{i,j} = \mathtt{T2G}(i, j, \mathcal{R}_h)$. The exact propagator $R^{i,j}$ is defined similarly, replacing R_h^{sem} by the exact propagator of the hyperbolic system on tents (without projecting tent bottom data), so that if $u(x, t)$ is the global exact solution of the hyperbolic system on $\Omega \times [0, T]$, then

$$R^{i,j}(u|_{C_j}) = u|_{C_i}. \tag{3.17}$$

The semidiscrete error propagation operators across layers can now be defined by

$$E_h^{i,j} = R^{i,j} - R_h^{i,j}.$$

Letting C_m denote the final front and C_0 the first, we are interested in bounding the error at the final front, which is simply $E_h^{m,0} u^0$. Setting $R^{0,0}$ and $R_h^{m,m}$ to the trivial identity operators, we have the following lemma.

Lemma 3.10. $$E_h^{m,0} = \sum_{j=1}^{m} R_h^{m,j} E_h^{j,j-1} R^{j-1,0}.$$

Proof. Adding and subtracting $R_h^{m,m-1} \circ R^{m-1,0}$,

$$\begin{aligned} E_h^{m,0} &= R^{m,0} - R_h^{m,0} = R^{m,m-1} \circ R^{m-1,0} - R_h^{m,m-1} \circ R_h^{m-1,0} \\ &= (R^{m,m-1} - R_h^{m,m-1}) \circ R^{m-1,0} + R_h^{m,m-1} \circ (R^{m-1,0} - R_h^{m-1,0}), \end{aligned}$$

i.e.,

$$E_h^{m,0} = E_h^{m,m-1} R^{m-1,0} + R_h^{m,m-1} E_h^{m-1,0}.$$

The last term admits a recursive application of the same identity. Doing so $m - 1$ times, the lemma is proved. □

Our global error analysis proceeds in a norm on advancing fronts defined by

$$\|w\|_{C_i}^2 = \int_\Omega \big[g(w(x, \varphi_i(x))) - f(w(x, \varphi_i(x)))\,(\operatorname{grad}_x \varphi_i)(x)\big] \cdot w(x, \varphi_i(x)).$$

Let $\|w\|_{C_i,\mathtt{v}}$ be defined by the same equality after replacing the integral over Ω by integral over $\Omega^{\mathtt{v}}$. Since the first and last fronts, C_0 and C_m, respectively, are flat

$$\|w\|_{C_0}^2 = (\mathcal{G}w(0), w(0))_\Omega \quad \text{and} \quad \|w\|_{C_m}^2 = (\mathcal{G}w(T), w(T))_\Omega, \tag{3.18}$$

where, as before, T is the final time.

Lemma 3.11. *For all $w \in L^2(C_j)^L$ and $i > j$, we have*

$$\|R_h^{i,j} w\|_{C_i} \leqslant \|w\|_{C_j}.$$

Proof. First consider the case $j = i-1$ and a $\mathtt{v} \in V_i$. Applying Lemma 3.6 on tent $T^{\mathtt{v}}$ in L_i, we obtain that $\hat{r}_h = (R_h^{i,i-1} w) \circ \Phi$ and $\hat{w} = w \circ \Phi$ satisfies

$$\|\hat{r}_h\|_{M(1)}^2 \leqslant \|P_h \hat{w}\|_{M(0)}^2. \tag{3.19}$$

By (2.6),

$$\begin{aligned} \|P_h \hat{w}\|_{M(0)}^2 &= (\mathcal{G} P_h \hat{w}, P_h \hat{w})_{\mathtt{v}} - \sum_{j=1}^{N} (\partial_j \varphi_{\mathrm{bot}} \mathcal{L}^{(j)} P_h \hat{w}, P_h \hat{w})_{\mathtt{v}} \\ &= (M(0) P_h \hat{w}, \hat{w})_{\mathtt{v}} \leqslant \|P_h \hat{w}\|_{M(0)} \|\hat{w}\|_{M(0)}, \end{aligned}$$

so (3.19) implies that $\|\hat{r}_h\|_{M(1)}^2 \leqslant \|\hat{w}\|_{M(0)}^2$, which is the same as $\|R_h^{i,i-1} w\|_{C_i,\mathtt{v}}^2 \leqslant \|w\|_{C_{i-1},\mathtt{v}}^2$. Summing over $\mathtt{v} \in V_i$, we prove that

$$\|R_h^{i,i-1} w\|_{C_i} \leqslant \|w\|_{C_{i-1}}. \tag{3.20}$$

Repeatedly applying this inequality on any further layers in between $i-1$ and j proves the lemma. ☐

In the subsequent statements of error estimates like in the next theorem, we will tacitly assume that the exact solution is smooth enough for the seminorms on the right hand side to be finite.

Theorem 3.12 (Error estimate for the semidiscretization)**.** *Suppose $\Omega \times (0,T)$ is meshed by m layers of tents satisfying the causality condition* (1.4)*. At the final time T, the difference between the exact solution $u(T)$ and the semidiscrete MTP solution $u_h(T) \in V_h$ satisfies*

$$\|u(T) - u_h(T)\|_\Omega \lesssim \left(\sum_{j=1}^{m} h_j \right)^{1/2} \left(\sum_{j=1}^{m} \sum_{\mathtt{v} \in V_j} h_{\mathtt{v}}^{2p+1} |u|_{\mathtt{v},p+1}^2 \right)^{1/2},$$

where $h_j = \max_{\mathtt{v} \in V_j} h_{\mathtt{v}}$.

Proof. Let $u_j = u|_{C_j}$. Then, per (3.17), $R^{j-1,0} u^0 = u_{j-1}$. Therefore,

$$\begin{aligned} \|u(T) - u_h(T)\|_\Omega \lesssim \|u - u_h\|_{C_m} &= \|E_h^{m,0} u^0\|_{C_m} && \text{by (3.18)} \\ &\leqslant \sum_{j=1}^{m} \|R_h^{m,j} E_h^{j,j-1} u_{j-1}\|_{C_m} && \text{by Lemma 3.10} \\ &\leqslant \sum_{j=1}^{m} \|E_h^{j,j-1} u_{j-1}\|_{C_j} && \text{by Lemma 3.11.} \end{aligned} \tag{3.21}$$

Since the spatial projection of the support of $E_h^{j,j-1}u_j$ can be subdivided into the union of non-overlapping vertex patches $\Omega^{\mathtt{v}}$ for all pitch vertices $\mathtt{v} \in V_j$,

$$\|E_h^{j,j-1}u_{j-1}\|_{C_j}^2 = \sum_{\mathtt{v}\in V_j} \|E_h^{j,j-1}u_{j-1}\|_{C_j,\mathtt{v}}^2.$$

On a tent $T^{\mathtt{v}}$ with $\mathtt{v} \in V_j$, note that $E_h^{j,j-1}u_{j-1}|_{\partial_{\mathrm{top}}T^{\mathtt{v}}} = (u_j - R_h^{j,j-1}u_{j-1})|_{\partial_{\mathrm{top}}T^{\mathtt{v}}} = u|_{\partial_{\mathrm{top}}T^{\mathtt{v}}} - R_h^{T^{\mathtt{v}}}(u|_{\partial_{\mathrm{bot}}T^{\mathtt{v}}} \circ \Phi^{-1})$. Putting $\hat{u} = u|_{T^{\mathtt{v}}} \circ \Phi$ and $\hat{u}_h = R_h^{T^{\mathtt{v}}}\hat{u}(0) = R_h^{\mathrm{sem}}(1) \circ P_h\hat{u}(0)$, applying Lemma 3.8 with $\tau = 1$ and $\hat{u}_h^0 = P_h\hat{u}(0)$ yields

$$\begin{aligned}\|E_h^{j,j-1}u_{j-1}\|_{C_j,\mathtt{v}} &= \|\hat{u}(1) - \hat{u}_h(1)\|_{M(1)} \\ &\lesssim \|\hat{u}(0) - P_h\hat{u}(0)\|_{M(0)} + h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1} \lesssim h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1}.\end{aligned}$$

Using this in (3.21),

$$\|u(T) - u_h(T)\|_{\Omega} \lesssim \sum_{j=1}^m \left(\sum_{\mathtt{v}\in V_j} h_{\mathtt{v}}^{2p+2}|u|_{\mathtt{v},p+1}^2\right)^{1/2} \lesssim \sum_{j=1}^m h_j^{1/2}\left(\sum_{\mathtt{v}\in V_j} h_{\mathtt{v}}^{2p+1}|u|_{\mathtt{v},p+1}^2\right)^{1/2},$$

so the proof is finished by applying the Cauchy-Schwarz inequality. □

Remark 3.13. Note that h_j may be interpreted as the "layer height" of L_j due to the causality condition. Suppose

$$\sum_{i=1}^m h_i \lesssim T. \tag{3.22}$$

Theorem 3.12 then yields $O(h^{p+1/2})$-rate of convergence with $h = \max_{\mathtt{v}} h_{\mathtt{v}}$. Of course, (3.22) can be violated by choosing very sparse layers (e.g., with one tent per layer), but this is not useful to get the best estimate from Theorem 3.12, nor is it useful in practice: indeed, a large number of non-interacting tents (such as the tents of the same color in Figures 1b–1e) in each layer allows for better parallelism.

Remark 3.14. Suppose that instead of the operators $R_h^{i,i-1}$ satisfying (3.20), we are given operators $\tilde{R}_h^{i,i-1} : L^2(C_{i-1})^L \to L^2(C_i)^L$ admitting the weaker stability bound

$$\|\tilde{R}_h^{i,i-1}w\|_{C_i} \leqslant (1 + C_{\mathrm{sta}}h_{i-1})\|w\|_{C_{i-1}} \tag{3.23}$$

with some mesh and layer independent constant $C_{\mathrm{sta}} > 0$ for all $w \in L^2(C_{i-1})^L$. For any $i > j$, consider $\tilde{R}_h^{i,j} = \tilde{R}_h^{i,i-1} \circ \tilde{R}_h^{i-1,i-2} \circ \cdots \circ \tilde{R}_h^{j+1,j}$. Note that for any $i > j$, using the arithmetic-geometric mean inequality and the inequality $(1+\alpha)^m \leqslant e^{\alpha m}$,

$$\begin{aligned}(1 + C_{\mathrm{sta}}h_j)(1 + C_{\mathrm{sta}}h_{j+1}) \cdots (1 + C_{\mathrm{sta}}h_i) &\leqslant \prod_{i=1}^m (1 + C_{\mathrm{sta}}h_i) \\ \leqslant \left[\frac{1}{m}\sum_{i=1}^m (1 + C_{\mathrm{sta}}h_i)\right]^m &\leqslant \left[1 + \frac{C_{\mathrm{sta}}}{m}\sum_{i=1}^m h_i\right]^m \leqslant \exp\Big(C_{\mathrm{sta}}\sum_{i=1}^m h_i\Big).\end{aligned}$$

Therefore, whenever (3.22) holds, iterative application of (3.23) gives the following layer-uniform bound for any $i > j$:

$$\|\tilde{R}_h^{i,j}w\|_{C_i} \leqslant e^{C_{\mathrm{sta}}T}\,\|w\|_{C_j}.$$

Using this in place of Lemma 3.11, the proof of Theorem 3.12 can be extended, replacing $R_h^{i,j}$ by $\tilde{R}_h^{i,j}$, $E^{i,j}$ by $\tilde{E}^{i,j} = R^{i,j} - \tilde{R}_h^{i,j}$, and "$\leqslant$" in (3.21) by "$\lesssim$" subsuming the T-dependent constant into the error estimates.

4. Analysis of fully discrete schemes

In this section we use time stepping schemes to arrive at practical fully discrete schemes from the semidiscretization studied in the previous section. Before studying these fully discrete schemes on a mapped tent, it is useful to quickly make a few observations on the time derivatives and Taylor expansion of the exact solution.

4.1. **Preparatory observations.** The bilinear form $a(\cdot,\cdot)$ defines an operator A from $H_h^{\mathtt{v}}$ to its dual space $(H_h^{\mathtt{v}})'$ in the usual way: $(Aw)(v) = a(w,v)$ for $w, v \in H_h^{\mathtt{v}}$. Recall the previously defined L^2 projector $P_h : L^{\mathtt{v}} \to V_h^{\mathtt{v}}$. Since $V_h^{\mathtt{v}} \subset H_h^{\mathtt{v}}$, the projector P_h extends naturally from $L^{\mathtt{v}}$ to $(H_h^{\mathtt{v}})'$, so, e.g., $P_hA : H_h^{\mathtt{v}} \to V_h^{\mathtt{v}}$ satisfies $(P_hAw, v_h)_{\mathtt{v}} = (Aw)(v_h) = a(w, v_h)$ for all $w \in H_h^{\mathtt{v}}$ and $v_h \in V_h^{\mathtt{v}}$. While describing fully discrete schemes, $A_h : V_h^{\mathtt{v}} \to V_h^{\mathtt{v}}$, defined by $(A_hw, v)_{\mathtt{v}} = a(w, v)$, for all $w, v \in V_h^{\mathtt{v}}$ will be useful. One may also consider $\tilde{A} : H^{\mathtt{v}} \to L^{\mathtt{v}}$ defined by $(\tilde{A}w, v)_{\mathtt{v}} = (\mathrm{div}_x\left[\delta f(w)\right], v)_{\mathtt{v}}$, for all $w \in H^{\mathtt{v}}$ and $v \in L^{\mathtt{v}}$. It is easy to see from (3.9) of Lemma 3.2 that A coincides with $\tilde{A}$ on functions $w \in H^{\mathtt{v}}$ with $(\mathcal{D} - B)w = 0$ on $\partial\Omega$. In particular, on such functions w, we may view Aw as a function in $L^{\mathtt{v}}$. The pull back $\hat{u}$ of the exact hyperbolic solution u from a tent $T^{\mathtt{v}}$ to the cylinder $\hat{T}^{\mathtt{v}}$ is one such function. Therefore the following equation holds in $L^{\mathtt{v}}$:

$$\partial_{\hat{t}}(M\hat{u}) = A\hat{u}, \qquad 0 \leqslant \hat{t} \leqslant 1. \tag{4.1}$$

We will proceed assuming that the exact solution $\hat{u}$ is regular enough to admit the Taylor expansion

$$\hat{u}(\tau) = \sum_{k=0}^{s} \frac{\tau^k}{k!}\hat{u}^{(k)}(0) + \rho_{s+1}(\tau), \tag{4.2}$$

for some $s \geqslant 1$. Here, $\hat{u}^{(k)}(\hat{t})$ denotes the kth order time derivative $d^k\hat{u}/d\hat{t}^k$ (which is a function in $H^{\mathtt{v}}$ when the solution is smooth—see Lemma 4.1 below), and the remainder term $\rho_{s+1}(\tau)$ can be expressed as the $H^{\mathtt{v}}$-valued Riemann integral

$$\rho_{s+1}(\tau) = \frac{\tau^{s+1}}{s!}\int_0^1 (1-\hat{t})^s \hat{u}^{(s+1)}(\hat{t}\tau)\, d\hat{t}. \tag{4.3}$$

It is well known that the expansion (4.2) holds for τ in an interval containing 0 whenever $\hat{u}$ is $s+1$ times continuously differentiable (as an $H^{\mathtt{v}}$-valued function) in that interval. When applied to a spacetime hyperbolic solution u in the physical domain, the smallness of the higher order terms in (4.2) (written there in terms of the mapped function $\hat{u}$), is evident from the following lemma, since $\delta(x) \lesssim h_{\mathtt{v}}$.

Lemma 4.1. *The function $\hat{u} = u \circ \Phi$ satisfies*

$$\hat{u}^{(k)} = (\partial_t^k u \circ \Phi)\,\delta^k.$$

Consequently, at each pseudotime $\hat{t}$, within each spatial element $K \in \Omega_h^{\mathtt{v}}$, as a function of the spatial variable x, $\hat{u}^{(k)}(\hat{t})$ is as smooth as $(\partial_t^k u)(x, \varphi(x,\hat{t}))$. Moreover, $\hat{u}^{(k)}(\hat{t})$ is in $H^{\mathtt{v}}$ if $\partial_t^k u$ is continuously differentiable in $T^{\mathtt{v}}$.

Proof. Let e denote the spacetime unit vector in the time direction i.e., all its components are zero except for the last (time) component which is 1. Then, at some fixed spacetime point $\hat{P}$ in $\hat{T}^{\mathtt{v}}$, we may write $\hat{u}^{(k)}(\hat{P}) = D^k\hat{u}(\hat{P})(e, e, \ldots, e)$, where $D^k\hat{u}$ is the multilinear form representing the kth order Fréchet derivative of

$\hat{u}$, and e is repeated k times in its argument list. Then, letting $P = \Phi(\hat{P})$ denote the mapped point in $T^{\mathtt{v}}$, by standard arguments [4] for affine maps,

$$\begin{aligned} \hat{u}^{(k)}(\hat{P}) &= D^k(u \circ \Phi)(\hat{P})(e, e, \ldots, e) \\ &= D^k u(P)([\operatorname{grad}_{x\hat{t}} \Phi]e, [\operatorname{grad}_{x\hat{t}} \Phi]e, \ldots, [\operatorname{grad}_{x\hat{t}} \Phi]e) \\ &= D^k u(P)(\delta e, \delta e, \ldots, \delta e), \end{aligned}$$

where we have used (2.7) in the last step. Since the last term above equals the product of δ^k and the derivative $\partial^k u/\partial t^k$ at P, the proof is complete. □

In view of Lemma 4.1, when the exact solution is smooth in the physical spacetime, we expect it to have the following (semi)norms finite, in addition to the ones in (3.15):

$$\begin{aligned} |w|_{\mathtt{v},l,m} &= \sup_{0\leqslant\tau\leqslant1} \sum_{k=0}^{m} \left| \hat{w}^{(k)}(\tau) \right|_{H^l(\Omega_h^{\mathtt{v}})^L}, \\ \|w\|_{\infty,\mathtt{v}} &= \sup_{0\leqslant\tau\leqslant1} \|\hat{w}(\tau)\|_{\mathtt{v}}, \qquad \|w\|_{s,\infty,\mathtt{v}} = \sum_{\ell=0}^{s} \|\partial_t^\ell w\|_{\infty,\mathtt{v}}. \end{aligned} \tag{4.4}$$

When $m = 0$, the first seminorm coincides with the seminorm in (3.15). The next result bounds the Taylor remainder term in terms of the mapped time derivative $\partial_t^s u \circ \Phi$.

Lemma 4.2. *The Taylor remainder term satisfies* $\|\rho_s\|_{\mathtt{v}} \lesssim \tau^s h_{\mathtt{v}}^s \|\partial_t^s u\|_{\infty,\mathtt{v}}$.

Proof. Starting from (4.3), by Fubini's theorem and Cauchy-Schwarz inequality,

$$\begin{aligned} \tau^{-2s}\|\rho_s\|_{\mathtt{v}}^2 &\lesssim \int_0^1 (1-\hat{t})^{2s} \|\hat{u}^{(s)}(\hat{t}\tau)\|_{\mathtt{v}}^2 \, d\hat{t} \leqslant \left(\sup_{0\leqslant\tau\leqslant1} \|\hat{u}^{(s)}(\tau)\|_{\mathtt{v}}^2 \right) \int_0^1 (1-\hat{t})^{2s} \, d\hat{t} \\ &\lesssim \sup_{0\leqslant\tau\leqslant1} \|\delta^s (\partial_t^s u \circ \Phi)(\tau)\|_{\mathtt{v}}^2, \end{aligned}$$

due to Lemma 4.1. Since $\delta \lesssim h_{\mathtt{v}}$, the result follows. □

Lemma 4.3. *For any $k \geqslant 1$, whenever the exact time derivative $\hat{u}^{(k-1)}(0)$ exists in $H^{\mathtt{v}}$, we have $\hat{u}^{(k)}(0) = M_0^{-1}(A + kM_1)\,\hat{u}^{(k-1)}(0)$.*

Proof. Differentiating both sides of (4.1) $k-1$ times, $(M\hat{u})^{(k)}(\hat{t}) = A\hat{u}^{(k-1)}(\hat{t})$. Simplifying the left hand side by Leibniz rule and the linearity of $M(\hat{t})$, we have

$$M(\hat{t})\hat{u}^{(k)}(\hat{t}) - kM_1\hat{u}^{(k-1)}(\hat{t}) = A\hat{u}^{(k-1)}(\hat{t}).$$

Evaluating at $\hat{t} = 0$ and rearranging, the proof is complete. □

Note that $V_h^{\mathtt{v}}$ is an invariant subspace of the previously defined operators M_0 and M_1, due to (2.6). It will be understood from context whether we consider M_0, M_1 as operators on $L^{\mathtt{v}}$ or as operators on $V_h^{\mathtt{v}}$. For operators on $V_h^{\mathtt{v}}$, we define a discrete operator norm, analogous to (3.4), for operators $\mathcal{O}_h$ on $V_h^{\mathtt{v}}$, by

$$\|\mathcal{O}_h\|_{\mathtt{v},h} := \sup_{v_h, w_h \in V_h^{\mathtt{v}}} \frac{(\mathcal{O}_h v_h, w_h)_{\mathtt{v}}}{\|v_h\|_{\mathtt{v}} \|w_h\|_{\mathtt{v}}}$$

for all $v_h, w_h \in V_h^{\mathtt{v}}$.

Lemma 4.4. *We have* $\|A_h\|_{\mathtt{v},h} \lesssim 1,\ \|M_1\|_{\mathtt{v},h} \lesssim 1,\ \|M\|_{\mathtt{v},h} \lesssim 1,\ \|M^{-1}\|_{\mathtt{v},h} \lesssim 1.$

Proof. To prove the first inequality, consider the terms that make up $a(v_h, w_h) = (A_h v_h, w_h)_{\mathtt{v}}$ for any $v_h, w_h \in V_h^{\mathtt{v}}$. On any $K \in \Omega_h^{\mathtt{v}}$, since $\mathcal{L}^{(j)}$ is uniformly bounded and $\delta \lesssim h_K$,

$$(\delta \mathcal{L}^{(j)} v_h, \partial_j w_h)_K \lesssim \|v_h\|_K h_K \|\partial_j w_h\|_K \lesssim \|v_h\|_K \|w_h\|_K$$

where we have applied an inverse inequality in the last step. Next, consider an element boundary term in $(A_h v_h, w_h)$, restricted to say a facet $F \subset \partial K$, shared with the boundary of another element K_o in $\Omega_h^{\mathtt{v}}$:

$$(\delta\, \mathcal{D}\{v_h\}, w_h|_{\partial K})_F \lesssim \left(h_K \|v_h\|_{\partial K}^2 + h_{K_o} \|v_h\|_{\partial K_o}^2\right)^{1/2} \left(h_K^{1/2} \|w_h\|_{\partial K}\right) \lesssim \|v_h\|_{\Omega^{\mathtt{v}}} \|w_h\|_K,$$

where we have again used $\delta \lesssim h_K$ and local scaling arguments. Continuing to use similar arguments on all the remaining terms that make up $(A_h v_h, w_h)_{\mathtt{v}}$, we obtain $\|A_h\|_{\mathtt{v},h} \lesssim 1$. Finally, Lemma 3.1 shows that $\|M_1\|_{\mathtt{v},h}$, $\|M(\tau)\|_{\mathtt{v},h}$, and $\|M(\tau)^{-1}\|_{\mathtt{v},h}$ also admit mesh-independent bounds. □

The projector P_h enjoys the commutativity properties

$$M_1 P_h = P_h M_1, \quad M_0 P_h = P_h M_0, \tag{4.5}$$

because of (2.6). Although a similar commutativity identity cannot be expected of A_h, we have the following lemma.

Lemma 4.5. *For any $w \in H^l(\Omega_h^{\mathtt{v}})^L$, $1 \leqslant l \leqslant p+1$, the function $\eta_h = (A_h P_h - P_h A)w$ satisfies $\|\eta_h\|_{\mathtt{v}} \lesssim h_{\mathtt{v}}^l |w|_{H^l(\Omega_h^{\mathtt{v}})^L}$.*

Proof. Since $\|\eta_h\|_{\mathtt{v}}^2 = (A_h P_h w, \eta_h)_{\mathtt{v}} - (P_h A w, \eta_h)_{\mathtt{v}} = a(P_h w, \eta_h)_{\mathtt{v}} - a(w, \eta_h)_{\mathtt{v}}$,

$$\|\eta_h\|_{\mathtt{v}}^2 = a(P_h w - w, \eta_h) \lesssim h_{\mathtt{v}}^l |w|_{H^l(\Omega_h^{\mathtt{v}})^L} |\eta_h|_d$$

by Lemma 3.7. By Lemma 4.4, $|\eta_h|_d^2 = -((2A_h + M_1)\eta_h, \eta_h)_{\mathtt{v}} \lesssim \|\eta_h\|_{\mathtt{v}}^2$, so the inequality of the lemma follows. □

4.2. **Lowest order tent-implicit scheme.** While the overall MTP strategy is a tent-by-tent time-marching strategy akin to explicit methods, within a mapped tent, one may choose between explicit or implicit schemes. By a "tent-implicit" scheme, we mean a method that solves the semidiscretization (2.9) on a mapped tent using implicit time stepping. Although this requires matrix inversion, the size of the matrix is only as large as the number of spatial degrees of freedom in one tent (much smaller than the size of the global matrix that needs to be inverted in standard implicit schemes for method of lines discretizations). Numerical results using tent-implicit schemes of various orders were reported first in [13, §5.4]. In this subsection, we provide a convergence analysis of the lowest-order case.

To derive the lowest order tent-implicit method, we begin by rewriting (2.9) in a form analogous to (4.1), i.e.,

$$\partial_{\hat{t}}(M \hat{u}_h) = A_h \hat{u}_h, \quad 0 \leqslant \hat{t} \leqslant 1.$$

Then, putting $y_h = M\hat{u}_h$, we have $\partial_{\hat{t}}\, y_h = A_h M^{-1} y_h$. The implicit Euler method applied to this defines an approximation $y_{h1}(\tau)$ to $y_h(\tau)$ given by $y_{h1}(\tau) - y_h(0) = \tau A_h M^{-1} y_{h1}(\tau)$. Since $\hat{u}_h = M^{-1} y_h$, an approximation to $\hat{u}_h(\tau)$ is furnished by $M^{-1} y_{h1}(\tau)$, which after simplification becomes $M^{-1}(I - \tau A_h M^{-1})^{-1} M_0 \hat{u}(0)$. This motivates the following definition of the discrete propagator.

Definition 4.6 (Lowest order tent-implicit flow: $R_{h1}^{\text{imp}}(\tau)$)**.** Define $R_{h1}^{\text{imp}}(\tau) : V_h^{\mathtt{v}} \to V_h^{\mathtt{v}}$ by

$$R_{h1}^{\text{imp}}(\tau) = M(\tau)^{-1}(I - \tau A_h M(\tau)^{-1})^{-1} M_0.$$

The two inverses required for this definition are both well defined: first, M is invertible by Lemma 3.1; second, $I - \tau A_h M^{-1}$ is invertible because

$$(I - \tau A_h M^{-1})M = M - \tau A_h = (M_0 - \frac{\tau}{2} M_1) - \frac{\tau}{2}(2A_h + M_1),$$

together with Lemmas 3.2 and 3.1 imply that for any $0 \neq v \in V_h^{\mathtt{v}}$,

$$((I - \tau A_h M^{-1})Mv, v)_{\mathtt{v}} \geqslant \|v\|^2_{M(\tau/2)} + |v|_d^2 > 0.$$

We proceed to prove convergence of the scheme, beginning with the next stability result that closely resembles the inequality of Lemma 3.6.

Lemma 4.7 (Unconditional strong stability)**.** *For any $v \in V_h^{\mathtt{v}}$ and any $0 \leqslant \tau \leqslant 1$,*

$$\|R_{h1}^{\text{imp}}(\tau) v\|^2_{M(\tau)} \leqslant \|v\|^2_{M_0}.$$

Proof. Let $v_\tau = R_{h1}^{\text{imp}}(\tau) v$. Then $(M - \tau A_h) v_\tau = M_0 v$. Taking the inner product with v_τ on both sides,

$$\begin{aligned}
\|v_\tau\|_M^2 &= (M_0 v, v_\tau)_{\mathtt{v}} + \tau (A_h v_\tau, v_\tau)_{\mathtt{v}} \\
&\leqslant \frac{1}{2}\|v\|^2_{M_0} + \frac{1}{2}\|v_\tau\|^2_{M_0} + \tau(A_h v_\tau, v_\tau)_{\mathtt{v}} \\
&= \frac{1}{2}\|v\|^2_{M_0} + \frac{1}{2}\|v_\tau\|^2_{M} + \frac{\tau}{2}((2A_h + M_1)v_\tau, v_\tau)_{\mathtt{v}}.
\end{aligned}$$

Now, since $((2A_h + M_1)v_\tau, v_\tau)_{\mathtt{v}} = -|v_\tau|_d^2$ (see Lemma 3.2), the proof is complete. ☐

When using any (spatial) polynomial degree $p \geqslant 0$, we obtain the following bound for the lowest order method (showing that the rate is limited by the time discretization error), which uses the (semi)norms defined in (3.15) and (4.4).

Lemma 4.8 (Local error bound)**.** *Let $\hat{u}$ denote the exact solution on $\hat{T}^{\mathtt{v}}$ and let $\hat{u}_{h1}^{\text{imp}}(\tau) = R_{h1}^{\text{imp}}(\tau)\hat{u}_h^0$ for some $\hat{u}_h^0 \in V_h^{\mathtt{v}}$. Then,*

$$\|\hat{u}(\tau) - \hat{u}_{h1}^{\text{imp}}(\tau)\|_{M(\tau)} \lesssim \|\hat{u}(0) - \hat{u}_h^0\|_{M(0)} + h_{\mathtt{v}}\big(\|u\|_{2,\infty,\mathtt{v}} + |u|_{\mathtt{v},1}\big).$$

Proof. Let $X_h = M_0^{-1}(A_h + M_1)$ and $X = M_0^{-1}(A + M_1)$. By (4.5),

$$P_h X - X_h P_h = M_0^{-1}(P_h A - A_h P_h). \tag{4.6}$$

An alternate expression for the discrete propagator will also be useful: $R_{h1}^{\text{imp}} = M^{-1}(I - \tau A_h M^{-1})^{-1} M_0 = (M - \tau A_h)^{-1} M_0 = (M_0 - \tau(A_h + M_1))^{-1} M_0$, i.e.,

$$R_{h1}^{\text{imp}}(\tau) = (I - \tau X_h)^{-1}. \tag{4.7}$$

With these preparations, we derive an "error equation" for $\varepsilon_h = \hat{u}_{h1}^{\text{imp}}(\tau) - P_h \hat{u}(\tau)$. Note that ε_h is a function in $V_h^{\mathtt{v}}$ for each τ. Writing

$$\varepsilon_h = R_{h1}^{\text{imp}}\big[\hat{u}_h^0 - P_h \hat{u}(0)\big] + \phi_h,$$

with $\phi_h = R_{h1}^{\text{imp}} P_h \hat{u}(0) - P_h \hat{u}(\tau)$, we analyze ϕ_h further as follows.

$$\begin{aligned}
\phi_h &= (I - \tau X_h)^{-1} P_h \hat{u}(0) - P_h \hat{u}(\tau) && \text{by (4.7)}, \\
&= \big[(I - \tau X_h)^{-1} - I\big] P_h \hat{u}(0) - \tau P_h \hat{u}^{(1)}(0) - P_h \rho_2 && \text{by (4.2)},
\end{aligned}$$

$$
\begin{aligned}
&= \tau(I-\tau X_h)^{-1} X_h P_h \hat{u}(0) - \tau P_h \hat{u}^{(1)}(0) - P_h \rho_2 \\
&= \tau(I-\tau X_h)^{-1}\big[P_h X - M_0^{-1}(P_h A - A_h P_h)\big]\hat{u}(0) \\
&\quad - \tau P_h \hat{u}^{(1)}(0) - P_h\rho_2 \qquad\qquad \text{by (4.6)}, \\
&= \tau^2 X_h (I-\tau X_h)^{-1} P_h \hat{u}^{(1)}(0) - \tau (I-\tau X_h)^{-1} M_0^{-1}\eta_h - P_h\rho_2
\end{aligned}
$$

with $\eta_h = (P_h A - A_h P_h)\hat{u}(0)$. We have used Lemma 4.3 in the last step.

To bound ϕ_h, first note that by Lemma 4.4, $\|X_h\|_{\mathtt{v},h} \lesssim 1$. Also, by (4.7) and Lemma 4.7, $\|R^{\mathrm{imp}}_{h1}(\tau)\|_{\mathtt{v},h} = \|(I-\tau X_h)^{-1}\|_{\mathtt{v},h} \lesssim 1$, so $\|\phi_h\|_{\mathtt{v}} \lesssim \tau^2\|\hat{u}^{(1)}(0)\|_{\mathtt{v}} + \tau\|\eta_h\|_{\mathtt{v}} + \|\rho_2\|_{\mathtt{v}}$. Now, applying Lemmas 4.1, 4.5 and 4.2,

$$\|\phi_h\|_{\mathtt{v}} \lesssim \tau^2 h_{\mathtt{v}} \|\partial_t u\|_{\infty,\mathtt{v}} + \tau h_{\mathtt{v}} |u|_{\mathtt{v},1} + \tau^2 h_{\mathtt{v}}^2 \|\partial_t^2 u\|_{\infty,\mathtt{v}}.$$

Together with the stability result of Lemma 4.7, this proves

$$\|\varepsilon_h\|_M \lesssim \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M_0} + h_{\mathtt{v}}\big(\|\partial_t u\|_{\infty,\mathtt{v}} + \|\partial_t^2 u\|_{\infty,\mathtt{v}} + |u|_{\mathtt{v},1}\big).$$

Using the triangle inequality, $\|\hat{u}(\tau) - \hat{u}^{\mathrm{imp}}_{h1}(\tau)\|_M \leqslant \|\hat{u}(\tau) - P_h\hat{u}(\tau)\|_M + \|\varepsilon_h\|_M$, and the standard estimate for L^2 projection, $\|\hat{u}(\tau) - P_h\hat{u}(\tau)\|_{\mathtt{v}} \lesssim h_{\mathtt{v}}|u|_{\mathtt{v},1}$, the proof can now be completed. □

The previous two lemmas lead to a global convergence theorem, as we shall now see. The implicit scheme's tent propagator on $T^{\mathtt{v}}$ is the operator $R^{\mathrm{imp}}_{h1}(1) \circ P_h : L^{\mathtt{v}} \to V_h^{\mathtt{v}}$, set using $R^{\mathrm{imp}}_{h1}(\tau)$ evaluated at pseudotime $\tau = 1$ corresponding to the tent top. Letting $\mathcal{R}^{\mathrm{imp}}_h$ denote the collection of such tent propagators over all tents, we use Definition 3.9 to set the global propagator $R^{i,j}_{h,\mathrm{imp}} = \mathtt{T2G}(i, j, \mathcal{R}^{\mathrm{imp}}_h)$, and consider the discrete solution $u^{\mathrm{imp}}_{h1} = R^{m,0}_{h,\mathrm{imp}} u^0$ at the final time T.

Theorem 4.9 (Error estimate for the lowest order tent-implicit scheme)**.** *Under the same conditions as Theorem 3.12, for any spatial degree $p \geqslant 0$, the fully discrete solution u^{imp}_{h1} satisfies*

$$\|u(T) - u^{\mathrm{imp}}_{h1}\|_{\Omega} \;\lesssim\; \left(\sum_{j=1}^m h_j\right)^{1/2} \left[\sum_{j=1}^m \sum_{\mathtt{v}\in V_j} h_{\mathtt{v}} \big(\|u\|_{2,\infty,\mathtt{v}} + |u|_{\mathtt{v},1}\big)^2\right]^{1/2}.$$

Proof. First, due to Lemma 4.7, we observe that in complete analogy with Lemma 3.11, one can prove that for $i > j$,

$$\|R^{i,j}_{h,\mathrm{imp}} w\|_{C_i} \leqslant \|w\|_{C_j}.$$

Defining $E^{i,j}_{h,\mathrm{imp}} = R^{i,j} - R^{i,j}_{h,\mathrm{imp}}$, in analogy with Lemma 3.10, we can show that

$$E^{m,0}_{h,\mathrm{imp}} = \sum_{j=1}^m R^{m,j}_{h,\mathrm{imp}} E^{j,j-1}_{h,\mathrm{imp}} R^{j-1,0}.$$

Hence the theorem can be proved along the same lines as the proof of Theorem 3.12, using Lemma 4.8 in place of Lemma 3.8. □

As before, under the further assumption that (3.22) holds, Theorem 4.9 gives an $O(h^{1/2})$ rate of convergence for the solution at the final time.

4.3. **Lowest order explicit scheme.** A perhaps nonstandard route to derive an explicit scheme is to view it as an iterative method for solving the equations of an implicit scheme. Pursuing this approach using the tent-implicit scheme of §4.2, we write $v_\infty = R^{\mathrm{imp}}_{h1}(\tau)v_0$, or equivalently, using the operator $X_h = M_0^{-1}(A_h + M_1)$ in (4.7),

$$(I - \tau X_h)v_\infty = v_0.$$

Hence the Richardson iteration for solving this linear system for v_∞ takes the form

$$v_{\ell+1} = v_\ell + (v_0 - (I - \tau X_h)v_\ell), \qquad \ell = 0, 1, \ldots. \tag{4.8}$$

Definition 4.10 (Lowest order explicit discrete flows: $R^{\exp}_{h1}(\tau)$ and $R^{\exp}_{h1q}(\tau)$)**.** Let $v_0 \in V_h^{\mathrm{v}}$. The result v_1 after one iteration of (4.8) defines the operator $R^{\exp}_{h1}(\tau) : V_h^{\mathrm{v}} \to V_h^{\mathrm{v}}$:

$$R^{\exp}_{h1}(\tau)v_0 = v_1 = (I + \tau X_h)v_0.$$

The result v_q obtained after performing $q \geqslant 1$ iterations defines $R^{\exp}_{h1q}(\tau) : V_h^{\mathrm{v}} \to V_h^{\mathrm{v}}$ by

$$R^{\exp}_{h1q}(\tau)v_0 = v_q = v_0 + \tau X_h v_{q-1}.$$

Note that no matrix inversions are required for conducting these q iterations, except for one local mass matrix inversion (M_0^{-1}) per tent.

Unlike the tent-implicit scheme, we are now able to obtain stability for the explicit scheme only under further conditions. From Lemma 4.4, we know that $\|X_h\|_{M_0} \lesssim 1$. Hence the condition (4.9) in the next result can be met by performing sufficiently many iterations.

Lemma 4.11 (Conditional stability)**.** *If q is large enough to admit*

$$\|X_h\|_{M_0} \lesssim h_{\mathrm{v}}^{1/(q+1)}, \tag{4.9}$$

then there is a $c_q > 0$ independent of h_{v} such that for all $v_0 \in V_h^{\mathrm{v}}$,

$$\|R^{\exp}_{h1q}(\tau)v_0\|_{M(\tau)} \leqslant (1 + c_q h_{\mathrm{v}})\|v_0\|_{M_0}.$$

Proof. Recursively expanding $v_q = v_0 + \tau X_h v_{q-1}$, we obtain

$$v_q = \sum_{j=0}^{q} (\tau X_h)^j v_0.$$

Rewriting this, using (4.7), as

$$v_q = (I - \tau X_h)^{-1}\big[I - (\tau X_h)^{q+1}\big]v_0 = R^{\mathrm{imp}}_{h1}\big[1 - (\tau X_h)^{q+1}\big]v_0, \tag{4.10}$$

we apply Lemma 4.7. Hence

$$\|v_q\|_M \leqslant \big\|v_0 - (\tau X_h)^{q+1}v_0\big\|_{M_0} \leqslant \|v_0\|_{M_0} + \|X_h\|^{q+1}_{M_0}\|v_0\|_{M_0}$$

and the result follows using (4.9). □

Lemma 4.12 (Local error bound)**.** *Let $\hat{u}$ denote the exact solution on $\hat{T}^{\mathrm{v}}$, let $\hat{u}^{\exp}_{h1q}(\tau) = R^{\exp}_{h1q}(\tau)\hat{u}^0_h$ for some $\hat{u}^0_h \in V_h^{\mathrm{v}}$, and suppose* (4.9) *holds. Then,*

$$\|\hat{u}(\tau) - \hat{u}^{\exp}_{h1q}(\tau)\|_{M(\tau)} \;\lesssim\; \|\hat{u}(0) - \hat{u}^0_h\|_{M(0)} + h_{\mathrm{v}}\big(\|u\|_{2,\infty,\mathrm{v}} + \|u\|_{\mathrm{v},1}\big).$$

Proof. Let $e_h = \hat u_{h1}^{\mathrm{imp}}(\tau) - \hat u_{h1q}^{\mathrm{exp}}(\tau)$. Since $\hat u(\tau) - \hat u_{h1q}^{\mathrm{exp}}(\tau) - e_h = \hat u(\tau) - \hat u_{h1}^{\mathrm{imp}}(\tau)$ can be bounded by Lemma 4.8, it suffices to bound e_h. By (4.10),

$$\begin{aligned} e_h &= (I - \tau X_h)^{-1}\hat u_h^0 - (I-\tau X_h)^{-1}\big[I - (\tau X_h)^{q+1}\big]\hat u_h^0 \\ &= (I-\tau X_h)^{-1}(\tau X_h)^{q+1}\hat u_h^0. \end{aligned}$$

Thus, by Lemma 4.7 and (4.9), $\|e_h\|_M \lesssim h_{\mathtt v}\|\hat u_h^0\|_{M_0}$. We may further write $\hat u_h^0$ as the sum of $\hat u_h^0 - \hat u(0)$ and $\hat u(0)$ and apply triangle inequality to obtain the right hand side of the stated bound. □

Letting $\mathcal R_{h1q}^{\mathrm{exp}}$ denote the collection of explicit tent propagator operators $R_{h1q}^{\mathrm{exp}}(1)\circ P_h : L^{\mathtt v} \to V_h^{\mathtt v}$ on all tents, we use Definition 3.9 to set the global propagators $\mathtt{T2G}(i, j, \mathcal R_{h1q}^{\mathrm{exp}})$, and consider the discrete solution $u_{h1q}^{\mathrm{exp}} = \mathtt{T2G}(m, 0, \mathcal R_{h1q}^{\mathrm{exp}})u^0$ at the final time T.

Theorem 4.13 (Error estimate for iterated lowest order explicit scheme)**.** *Suppose* (4.9)*,* (3.22)*, and the conditions of Theorem 3.12 hold. Then for any spatial degree $p \geqslant 0$, the fully discrete explicit solution u_{h1q}^{exp} satisfies*

$$\|u(T) - u_{h1q}^{\mathrm{exp}}\|_\Omega^2 \;\lesssim\; \sum_{j=1}^m \sum_{\mathtt v\in V_j} h_{\mathtt v}\big(\|u\|_{2,\infty,\mathtt v} + \|u\|_{\mathtt v,1}\big)^2.$$

Proof. The proof proceeds along the lines of the proof of Theorem 3.12, replacing the applications of Lemmas 3.6 and 3.8, respectively, by those of Lemmas 4.11 and 4.12 instead. The main difference is that we must now invoke the argument of Remark 3.14 due to the weaker stability estimate of Lemma 4.11. □

The $O(h^{1/2})$ rate of convergence given by Theorem 4.13 is the same as the rate given by Theorem 4.9 for the lowest order tent-implicit scheme. Increasing the iteration number q can improve stability but does not generally improve the order of convergence.

4.4. **Arbitrary order SAT schemes.** Letting $X_h^{(0)}$ denote the identity operator on $V_h^{\mathtt v}$, recursively define further operators on $V_h^{\mathtt v}$ by

$$X_h^{(k)} = M_0^{-1}(A_h + kM_1)X_h^{(k-1)}, \quad k \geqslant 1. \tag{4.11}$$

Similarly, let $X^{(0)} = 1$ and $X^{(k)} = M_0^{-1}(A + kM_1)X^{(k-1)}$ for $k \geqslant 1$. By Lemma 4.3, the time derivative of the exact solution satisfies $\hat u^{(k)}(0) = X^{(k)}\hat u(0)$. Hence the expansion (4.2) may be written as

$$\hat u(\tau) = \sum_{k=0}^s \frac{\tau^k}{k!}X^{(k)}\hat u(0) + \rho_{s+1}(\tau), \tag{4.12}$$

This motivates us to define the SAT flow by replacing $X^{(k)}$ with the discrete operator $X_h^{(k)}$ as follows. (A gentler derivation can be found in [12] and it can be seen easily that the discrete flow defined there coincides with the one in the next definition.)

Definition 4.14 (Discrete s-stage SAT flow: $R_{hs}^{\mathrm{sat}}(\tau)$ for $s \geqslant 1$)**.** Define $R_{hs}^{\mathrm{sat}}(\tau) : V_h^{\mathtt v} \to V_h^{\mathtt v}$ by

$$R_{hs}^{\mathrm{sat}}(\tau)v = \sum_{k=0}^{s-1}\frac{\tau^k}{k!}X_h^{(k)}v + \frac{\tau^s}{s!}M(\tau)^{-1}M_0X_h^{(s)}v, \qquad v \in V_h^{\mathtt v}.$$

Lemma 4.15. *Let u be the exact solution of (3.12) on a causal tent T^{v} and $\hat{u} = u \circ \Phi \in C^{s+1}(0,1,H^{\mathrm{v}} \cap H^{p+1}(\Omega_h^{\mathrm{v}})^L)$. Then for any $s \geqslant 1$,*

$$\|R_{hs}^{\mathrm{sat}}(\tau)\hat{u}_h^0 - P_h\hat{u}(\tau)\|_{M(\tau)} \lesssim \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M(0)} + \tau^s h_{\mathrm{v}}^s \|u\|_{s+1,\infty,\mathrm{v}} + \tau h_{\mathrm{v}}^{p+1}|u|_{\mathrm{v},p+1,s-1}.$$

Proof. Let $\hat{u}_{hs}(\tau) = R_{hs}^{\mathrm{sat}}(\tau)\hat{u}_h^0$ and let $\varepsilon_k = (X_h^{(k)}P_h - P_hX^{(k)})\hat{u}(0)$. Then projecting and subtracting (4.12) from the expansion defining $R_{hs}^{\mathrm{sat}}(\tau)\hat{u}_h^0$, we obtain

$$\begin{aligned}\hat{u}_{hs}(\tau) - P_h\hat{u}(\tau) &= \sum_{k=0}^{s-1}\frac{\tau^k}{k!}\big[X_h^{(k)}(\hat{u}_h^0 - P_h\hat{u}(0)) + \varepsilon_k\big]\\ &\quad + \frac{\tau^s}{s!}M^{-1}M_0\big[X_h^{(s)}(\hat{u}_h^0 - P_h\hat{u}(0))\big]\\ &\quad + \frac{\tau^s}{s!}\big[(M^{-1}M_0 - I)P_h\hat{u}^{(s)}(0) + M^{-1}M_0\varepsilon_s\big] - P_h\rho_{s+1}.\end{aligned}$$

Letting $\mu_s = (M^{-1}M_0 - I)P_h\hat{u}^{(s)}(0)$, and noting that $\varepsilon_0 = 0$,

$$\hat{u}_{hs}(\tau) - P_h\hat{u}(\tau) = R_{hs}^{\mathrm{sat}}(\tau)\big[\hat{u}_h^0 - P_h\hat{u}(0)\big] + \sum_{k=1}^{s}\frac{\tau^k}{k!}\varepsilon_k + \frac{\tau^s}{s!}\big[\mu_s + M^{-1}M_0\varepsilon_s\big] - P_h\rho_{s+1}. \tag{4.13}$$

To estimate the terms on the right hand side, we first use Lemma 4.4 to conclude that $\|R_{hs}^{\mathrm{sat}}(\tau)[\hat{u}_h^0 - P_h\hat{u}(0)]\|_M \lesssim \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M_0}$. By Lemma 4.1, $\|\mu_s\|_{\mathrm{v}} \lesssim h_{\mathrm{v}}^s\|\partial_t^s u\|_{\infty,\mathrm{v}}$. To bound ε_k, note that

$$\varepsilon_k = M_0^{-1}(A_h + kM_1)\varepsilon_{k-1} + \eta_{k-1} \tag{4.14}$$

where $\eta_j = M_0^{-1}(A_hP_h - P_hA)\hat{u}^{(j)}(0)$. By Lemma 4.5, $\|\eta_j\|_{\mathrm{v}} \lesssim h_{\mathrm{v}}^{p+1}|u|_{\mathrm{v},p+1,j}$. Hence recursively bounding $\|\varepsilon_k\|_{\mathrm{v}}$ by $\|\varepsilon_{k-1}\|_{\mathrm{v}}$ using (4.14), and noting that $\varepsilon_0 = 0$, we have $\|\varepsilon_k\|_M \lesssim h_{\mathrm{v}}^{p+1}|u|_{\mathrm{v},p+1,k-1}$. The final term in (4.13) can be treated using Lemma 4.2, which yields $\|P_h\rho_{s+1}(\tau)\|_{\mathrm{v}} \lesssim \tau^{s+1}h_{\mathrm{v}}^{s+1}\|\partial_t^{s+1}u\|_{\infty,\mathrm{v}}$. When these estimates are used to bound the terms in the right hand side of (4.13) (and noting that τ is a common factor in all terms except the first), we obtain the stated inequality. □

In order to improve the stability of these explicit SAT schemes, we shall now divide each tent into r subtents and apply the SAT scheme in each subtent.

Definition 4.16 (Subtents)**.** Subdivide a tent T^{v} into r subtents as follows. For $\ell = 1, \ldots, r$, define the ℓth subtent by

$$T_{[\ell]}^{\mathrm{v}} = \{(x,t) : x \in \Omega^{\mathrm{v}},\ \varphi^{[\ell]}(x) \leqslant t \leqslant \varphi^{[\ell+1]}(x)\},$$

where $\hat{t}^{[\ell]} = (\ell-1)/r$, $\varphi^{[\ell]} = \varphi(x, \hat{t}^{[\ell]})$. Let $\delta^{[\ell]} = \varphi^{[\ell+1]} - \varphi^{[\ell]}$. Using $\delta^{[\ell]}$ in place of δ in (3.1) and (3.3), we define $a^{[\ell]}(w,v)$ and $M_1^{[\ell]}$, respectively, and let $A_h^{[\ell]} : V_h^{\mathrm{v}} \to V_h^{\mathrm{v}}$ be defined by $(A_h^{[\ell]}w, v)_{\mathrm{v}} = a^{[\ell]}(w,v)$ for $w, v \in V_h^{\mathrm{v}}$. Finally let $M_0^{[\ell]}$ be defined by (3.2) after replacing φ_{bot} there by $\varphi^{[\ell]}$ and let $M^{[\ell]}(\tau) = M_0^{[\ell]} - \tau M_1^{[\ell]}$. It is easy to see that $\delta^{[\ell]} = \delta/r$ and

$$A_h^{[\ell]} = \frac{1}{r}A_h, \quad M_1^{[\ell]} = \frac{1}{r}M_1, \quad M^{[\ell]}(0) = M(\hat{t}^{[\ell]}), \quad M^{[\ell]}(1) = M(\hat{t}^{[\ell+1]}). \tag{4.15}$$

Definition 4.17 (Discrete s-stage SAT propagator using r subtents: R_{rhs}^{sat})**.** Define $X_{h,[\ell]}^{(k)}$ by replacing A_h, M_1, and M_0, by $A_h^{[\ell]}, M_1^{[\ell]}$, and $M_0^{[\ell]}$, respectively, in (4.11).

Define $R^{\text{sat}}_{[\ell],hs}(\tau)$ on a subtent $T^{\mathtt{v}}_{[\ell]}$ by replacing $M_0, X_h^{(k)}$ and M by $M_0^{[\ell]}, X_{h,[\ell]}^{(k)}$ and $M^{[\ell]}$, respectively, in Definition 4.14. Applying on the r subtents successively, we define $R^{\text{sat}}_{rhs} = R^{\text{sat}}_{[r],hs}(1) \circ R^{\text{sat}}_{[r-1],hs}(1) \cdots \circ R^{\text{sat}}_{[1],hs}(1)$.

Note that the constant in "$\lesssim$" will not be allowed to depend on r (so that we may admit examples with $h_{\mathtt{v}}$-dependent r), as emphasized in the next lemma.

Lemma 4.18 (Local error in a tent)**.** *Let u be the exact solution of* (3.12) *on a causal tent $T^{\mathtt{v}}$, $\hat{u} = u \circ \Phi \in C^{s+1}(0,1,H^{\mathtt{v}} \cap H^{p+1}(\Omega_h^{\mathtt{v}})^L)$, and let $\hat{u}_{rhs} = R^{\text{sat}}_{rhs}\hat{u}_h^0$. Then there is a mesh-independent constant $c_{s,p}$ that is also independent of r such that*

$$c_{s,p}\|\hat{u}_{rhs} - \hat{u}(1)\|_{M(1)} \leqslant \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M(0)} + h_{\mathtt{v}}^s\|u\|_{s+1,\infty,\mathtt{v}} + h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1,s-1}.$$

Proof. Denoting the discrete solutions by $\hat{u}_{1,hs} = R^{\text{sat}}_{[1],hs}\hat{u}_h^0$ and $\hat{u}_{\ell,hs} = R^{\text{sat}}_{[\ell],hs}\hat{u}_{\ell-1,hs}$ for $1 \leqslant \ell \leqslant r$, we compare them with the subtent exact solutions, denoted by $\hat{u}_\ell = \hat{u}(\hat{t}^{[\ell+1]})$. In the pseudotime coordinate of the un-split tent $T^{\mathtt{v}}$, the value $\tau = 1/r$ corresponds to the top of the first subtent, where the exact solution is $\hat{u}_1 = \hat{u}(1/r)$. Thus Lemma 4.15 with $\tau = 1/r$ gives

$$\|\hat{u}_{1,hs} - P_h\hat{u}_1\|_{M(1/r)} \lesssim \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M(0)} + \frac{1}{r}\left(h_{\mathtt{v}}^s\|u\|_{s+1,\infty,\mathtt{v}} + h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1,s-1}\right).$$

Similarly, on the ℓth subtent, for $\ell = 1,2,\ldots,r$,

$$\begin{aligned}\|\hat{u}_{\ell,hs} - P_h\hat{u}_\ell\|_{M(\hat{t}^{[\ell+1]})} &\lesssim \|\hat{u}_{\ell-1,hs} - P_h\hat{u}_{\ell-1}\|_{M(\hat{t}^{[\ell]})} \\ &\quad + \frac{1}{r}\left(h_{\mathtt{v}}^s\|u\|_{s+1,\infty,\mathtt{v}} + h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1,s-1}\right).\end{aligned}$$

Applying this estimate for $\ell = r, r-1, \ldots, 1$, successively in that order, where at each step the first term on the right hand side is bounded using the next estimate,

$$\|\hat{u}_{rhs} - P_h\hat{u}(1)\|_{M(1)} \lesssim \|\hat{u}_h^0 - P_h\hat{u}(0)\|_{M(0)} + \left(h_{\mathtt{v}}^s\|u\|_{s+1,\infty,\mathtt{v}} + h_{\mathtt{v}}^{p+1}|u|_{\mathtt{v},p+1,s-1}\right)\sum_{\ell=1}^r \frac{1}{r}$$

which completes the proof. □

Letting $\mathcal{R}^{\text{sat}}_{rhs}$ denote the collection of explicit tent propagators $R^{\text{sat}}_{rhs} \circ P_h : L^{\mathtt{v}} \to V_h^{\mathtt{v}}$ on all tents, we use Definition 3.9 to set the global propagators $\mathtt{T2G}(i,j,\mathcal{R}^{\text{sat}}_{rhs})$, and consider the discrete solution $u^{\text{sat}}_{rhs} = \mathtt{T2G}(m,0,\mathcal{R}^{\text{sat}}_{rhs})u^0$ at the final time T.

Theorem 4.19 (Error estimate for the SAT scheme)**.** *Assume that there is a mesh-independent $C_{\text{sta}} \geqslant 0$ such that*

$$\|R^{\text{sat}}_{rhs}v\|_{M(1)} \leqslant (1 + C_{\text{sta}}h_{\mathtt{v}})\|v\|_{M(0)} \tag{4.16}$$

for all $v \in V_h^{\mathtt{v}}$ on all tents $T^{\mathtt{v}}$. Suppose also that (3.22) *and the conditions of Theorem 3.12 hold. Then the fully discrete explicit s-stage SAT solution u^{sat}_{rhs}, obtained using spatial polynomial degree p, satisfies*

$$\|u(T) - u^{\text{sat}}_{rhs}\|^2_{\Omega} \lesssim \sum_{j=1}^m \sum_{\mathtt{v}\in V_j} h_{\mathtt{v}}^{2s-1}\|u\|^2_{s+1,\infty,\mathtt{v}} + h_{\mathtt{v}}^{2p+1}\|u\|^2_{\mathtt{v},p+1,s-1}.$$

Proof. The proof proceeds along the lines of the extension of the proof of Theorem 3.12 mentioned in Remark 3.14, replacing the application of Lemma 3.6 by (4.16), and replacing the application of Lemma 3.8 by Lemma 4.18. □

Theorem 4.19 bounds the error by terms that converge to zero at the same rate, provided the number of stages in the SAT scheme is tied to the spatial degree by $s = p + 1$. Then, the convergence rate given by Theorem 4.19 is $O(h^{p+1/2})$, the same rate we obtained for the semidiscretization (in Theorem 3.12). In §4.4.1 we show, through a numerical example, that this rate is generally un-improvable.

One can solve a local eigenproblem on a tent to computationally check if the stability assumption (4.16) is satisfied. Since this eigenvalue computation is described in detail in [14, §6.1], we shall not comment further on this computational avenue for stability verification. In §4.4.2 and §4.4.3, we describe two cases where stability can be proved staying within the framework of the general symmetric linear hyperbolic systems we have been considering.

4.4.1. *Numerical observations on the convergence rate.* It is natural to wonder if the convergence rate of $O(h^{p+1/2})$, given by Theorem 4.19 (and Theorem 3.12), is improvable. Our numerical experience from computations with various hyperbolic systems suggests that one is likely to observe a higher convergence rate of $O(h^{p+1})$ on generic examples and meshes. Yet, as we show now, there is at least one family of tent meshes in the $N = 2$ case where $O(h^{p+1/2})$ rate of convergence is observed. Such tent meshes are created by selecting the spatial mesh Ω_h from the mesh families described in [19], where it is shown that the standard $O(h^{p+1/2})$ error estimate for the DG method for *stationary* advection equation cannot be improved. Building causal tents atop such a mesh, we show that our $O(h^{p+1/2})$ estimate for the time-dependent advection problem also cannot be improved.

The structured spatial meshes we borrow from [19] consist of horizontal layers of right triangles grouped in vertical bands. As the mesh is refined, the number of vertical bands is controlled by a parameter $\sigma \in [0, 1]$. We used the MTP discretization with polynomial orders p varying from 0 to 3, together with SAT time stepping with $r = \max\{1, 2p\}$ and $s = p + 1$ to solve the advection problem of Example 2.3 (modified to take a nonhomogeneous inflow boundary condition). The domain Ω is set to the unit square, the advective vector field b is set to the constant vector $b = [0, 1]^t$, so that $\partial_{\text{in}}\Omega = \{(x_1, x_2) : 0 \leqslant x_2 \leqslant 1\}$, and the inflow boundary condition is set by $u = x_1^{p+1}$ on $\partial_{\text{in}}\Omega$. The initial condition is $u^0(x_1, x_2) \equiv x_1^{p+1}$. At $t = T = 1$, the MTP solution approximated the exact solution $u(x_1, x_2, t) = x_1^{p+1}$ at all spatial points $(x_1, x_2) \in \Omega$.

We obtained different convergence rates for different choices of σ, but in all cases, the rates are bounded between $O(h^{p+1/2})$ and $O(h^{p+1})$. We obtained the minimal convergence rate (largest errors) when $\sigma = 3/4$ and $\sigma = 1/2$ for $p \geqslant 1$ and $p = 0$, respectively. The errors and rates observed for these values of σ are plotted in Figure 3, which clearly show $O(h^{p+1/2})$ rate of convergence. We note that our rate-minimizing σ-value of 3/4 is the same value of σ used in [19] for the $p = 1$ case (the only case where numerical results are given there).

4.4.2. *Stability verification in the $p = 0$, $s = 1$ case.* This case is motivated by the many studies of the $p = 0$ case in the DG literature (see e.g. [3, 7, 21]), often called the finite volume case, and is illustrative of why special cases are worth pursuing. We focus on the operator of the SAT scheme, obtained by setting $s = 1$ in Definition 4.14, which can be simplified to

$$R_{h1}^{\text{sat}}(\tau) = I + \tau M(\tau)^{-1}(A_h + M_1),$$

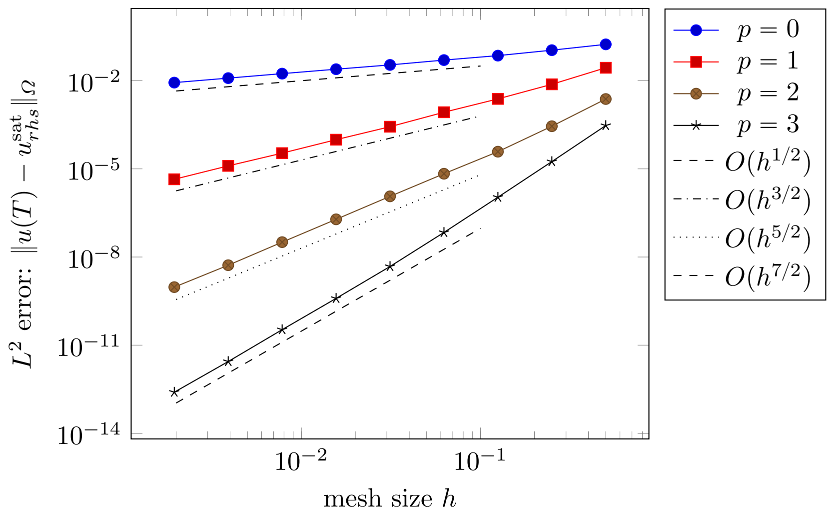


Figure 3. Convergence rates observed when solving the advection problem

and the corresponding operator R_{rh1}^{sat} obtained using r subtents, per Defintion 4.17. Note that $R_{h1}^{\mathrm{sat}}(\tau)$ differs slightly from the $q = 1$ case of Definition 4.10, namely, $R_{h1}^{\mathrm{exp}}(\tau) = I + \tau M_0^{-1}(A_h + M_1)$. While R_{h1q}^{exp} only requires one local mass matrix inversion for q iterations within a tent, the application of R_{rh1}^{sat} requires one local inversion per subtent. However, R_{rh1}^{sat} admits a stronger stability estimate that we shall prove after making the following observation.

Lemma 4.20. *When $p = 0$, we have, for all $v, w \in V_h^{\mathtt{v}}$,*

$$(A_h w, v)_{\mathtt{v}} \lesssim \|w\|_{\mathtt{v}} |v|_d, \tag{4.17}$$

$$\|(A_h + M_1)v\|_{\mathtt{v}} \lesssim |v|_d. \tag{4.18}$$

Proof. When $p = 0$, the derivative terms in (3.1) vanish, so

$$\begin{aligned}(A_h w, v)_{\mathtt{v}} &= \sum_{K \in \Omega_h^{\mathtt{v}}} -(\delta \hat{F}_w^n, v)_{\partial K} \\ &= -\sum_{F \in \mathcal{F}_b^{\mathtt{v}}} \frac{1}{2}(\delta(\mathcal{D}^{(n)} + B)w, v)_F \\ &\quad + \sum_{F \in \mathcal{F}_i^{\mathtt{v}}} (\delta \mathcal{D}^{(n_F)}\{w\}, [\![v]\!]_F)_F - (\delta S [\![w]\!]_F, [\![v]\!]_F)_F\end{aligned}$$

where we have rearranged the sum to run over the mesh facets. Now, by Cauchy-Schwarz inequality, (2.11), and Lemma 3.2, the estimate of (4.17) follows.

Of course, (4.17) can also be written as $(w, A_h^t v)_{\mathtt{v}} \lesssim \|w\|_{\mathtt{v}} |v|_d$ where A_h^t is the $L^{\mathtt{v}}$-adjoint of A_h. When this is added to the obvious inequality

$$(-(A_h + A_h^t + M_1)v, w)_{\mathtt{v}} \leqslant (-(2A_h + M_1)v, v)_{\mathtt{v}}^{1/2}(-(2A_h + M_1)w, w)_{\mathtt{v}}^{1/2} = |v|_d |w|_d,$$

we obtain $(-(A_h + M_1)v, w)_{\mathtt{v}} \lesssim |v|_d \|w\|_{\mathtt{v}}$, so

$$\|(A_h + M_1)v\|_{\mathtt{v}} = \sup_{0 \neq w \in V_h^{\mathtt{v}}} \frac{((A_h + M_1)v, w)_{\mathtt{v}}}{\|w\|_{\mathtt{v}}} \lesssim |v|_d$$

proves (4.18). □

Let K_h denote the kernel of $A_h + A_h^t + M_1 : V_h^{\mathtt{v}} \to V_h^{\mathtt{v}}$ and let $K_h^\perp$ denote its $L^{\mathtt{v}}$-orthogonal complement in $V_h^{\mathtt{v}}$. Set

$$\kappa = \sup_{0\leqslant\tau\leqslant 1} \sup_{0\neq v\in K_h^\perp} \frac{\|M(\tau)^{-1}(A_h + M_1)v\|^2_{M(\tau)}}{|v|_d^2}. \tag{4.19}$$

Proposition 4.21 (Conditional strong stability)**.** *In the case $p = 0$ and $s = 1$, the constant κ of* (4.19) *satisfies $\kappa \lesssim 1$. For all*

$$0 \leqslant \tau \leqslant 1/\kappa \tag{4.20}$$

and all $v \in V_h^{\mathtt{v}}$, we have

$$\|R_{h1}^{\mathrm{sat}}(\tau)v\|_{M(\tau)} \leqslant \|v\|_{M_0}. \tag{4.21}$$

Furthermore, if $r \geqslant \kappa$ subtents are used, then

$$\|R_{rh1}^{\mathrm{sat}}v\|_{M(\tau)} \leqslant \|v\|_{M_0}, \tag{4.22}$$

i.e., the stability assumption (4.16) *of Theorem 4.19 holds with $C_{\mathrm{sta}} = 0$.*

Proof. Since $\|M(\tau)^{-1}\|_{\mathtt{v},h} \lesssim 1$ by Lemma 4.4, the estimate (4.18) of Lemma 4.20 shows that $\kappa \lesssim 1$ whenever $p = 0$.

Let $v_\tau = R_{h1}^{\mathrm{sat}}(\tau)v = v + \tau M^{-1}(A_h + M_1)v$. Then expanding $\|v_\tau\|_M^2$,

$$\begin{aligned}
\|v_\tau\|_M^2 &= \|v\|_M^2 + 2\tau((A_h + M_1)v, v)_{\mathtt{v}} + \tau^2\|M^{-1}(A_h + M_1)v\|_M^2 \\
&= \|v\|_{M_0}^2 + \tau((2A_h + M_1)v, v)_{\mathtt{v}} + \tau^2\|M^{-1}(A_h + M_1)v\|_M^2 \\
&\leqslant \|v\|_{M_0}^2 - \tau|v|_d^2 + \tau^2\kappa|v|_d^2
\end{aligned}$$

so (4.21) follows when $1 - \tau\kappa \geqslant 0$.

Next, consider a subtent $T_{[\ell]}^{\mathtt{v}}$. By (4.15), $R_{[\ell],h1}^{\mathrm{sat}}(1) = I + M^{[\ell]}(1)^{-1}(A_h^{[\ell]} + M_1^{[\ell]}) = I + r^{-1}M(\hat{t}^{[\ell+1]})(A_h + M_1)$, so translating (4.21) with $\tau = 1/r \leqslant 1/\kappa$ to this subtent, we obtain $\|R_{[\ell],h1}^{\mathrm{sat}}(1)v\|_{M(t^{[\ell+1]})} \leqslant \|v\|_{M(t^{[\ell]})}$. Successively applying these estimates over all subtents, (4.22) is proved. □

Note that the inverse of κ appearing in (4.20) will stay away from zero (since $\kappa \lesssim 1$) allowing for a nontrivial advance in τ. One can view (4.20) as the analogue of a traditional "CFL condition" *within* a tent. Indeed, the pseudotime restriction (4.20) may be interpreted as a restriction on time advance in the physical spacetime by a small subtent whose tent pole height is a scalar multiple of $h_{\mathtt{v}}$. Even in the event (4.20) forbids us to reach the tent-top pseudotime (i.e., when $\tau = 1$ does not satisfy (4.20)), splitting the tent into smaller subtents does allow the analogue of (4.20) to hold throughout every subtent.

4.4.3. *Stability verification in the $s = 2$ case.* We will now show how to prove stability under a stronger CFL condition in the two-stage case. Definition 4.14 with $s = 2$ yields

$$R_{h2}^{\mathrm{sat}}(\tau) = I + \tau X_h^{(1)} + \frac{\tau^2}{2}M^{-1}M_0X_h^{(2)}.$$

Lemma 4.22. *For any $v \in V_h^{\mathtt{v}}$,*

$$\|R_{h2}^{\mathrm{sat}}(\tau)v\|_{M(\tau)}^2 = \|v\|_{M_0}^2 - \tau\left|v + \frac{\tau}{2}X_h^{(1)}v\right|_d^2 + \tau^3\,Z(\tau, v),$$

where $Z(\tau, v) = [2(M_1 X_h^{(1)} v, X_h^{(1)} v)_{\mathtt{v}} - |X_h^{(1)} v|_d^2 + \tau \|M(\tau)^{-1} M_0 X_h^{(2)} v\|_M^2]/4.$

Proof. Let $w = \tau X_h^{(1)} v + (\tau^2/2) M^{-1} M_0 X_h^{(2)} v$. Then $v_\tau = R_{h2}^{\text{sat}}(\tau) v$ can be written as $v_\tau = v + w$. Expanding $\|v + w\|_M^2$,

$$\begin{aligned}\|v_\tau\|_M^2 &= \|v\|_{M_0}^2 - \tau(M_1 v, v)_{\mathtt{v}} + 2(w, v)_M + \|w\|_M^2 \\ &= \|v\|_{M_0}^2 - \tau(M_1 v, v)_{\mathtt{v}} + 2\tau((M_0 - \tau M_1) X_h^{(1)} v, v)_{\mathtt{v}} + \tau^2 (M_0 X_h^{(2)} v, v)_{\mathtt{v}} + \|w\|_M^2 \\ &= \|v\|_{M_0}^2 + \tau((2A_h + M_1) v, v)_{\mathtt{v}} + \tau^2 (A_h X_h^{(1)} v, v)_{\mathtt{v}} + \|w\|_M^2.\end{aligned}$$

Note that $(A_h X_h^{(1)} v, v)_{\mathtt{v}} = (X_h^{(1)} v, A_h^t v)_{\mathtt{v}} = (X_h^{(1)} v, (A_h^t + A_h + M_1) v)_{\mathtt{v}} - \|X_h^{(1)} v\|_{M_0}^2$. Since $d(y, z) = -((A_h^t + A_h + M_1) y, z)_{\mathtt{v}}$, we have

$$\|v_\tau\|_M^2 = \|v\|_{M_0}^2 - \tau |v|_d^2 - \tau^2 d(X_h^{(1)} v, v) - \tau^2 \|X_h^{(1)} v\|_{M_0}^2 + \|w\|_M^2.$$

Next, letting $z = (1/2) M^{-1} M_0 X_h^{(2)} v$, expanding the last term above $\|w\|_M^2 = \tau^2 \|X_h^{(1)} v\|_M^2 + \tau^3 (X_h^{(1)} v, M_0 X_h^{(2)} v)_{\mathtt{v}} + \tau^4 \|z\|_M^2$, noting that $M_0 X_h^{(2)} = (A_h + 2M_1) X_h^{(1)}$, and simplifying,

$$\begin{aligned}\|v_\tau\|_M^2 &= \|v\|_{M_0}^2 - \tau |v|_d^2 - \tau^2 d(X_h^{(1)} v, v) + \tau^3 ((A_h + M_1) X_h^{(1)} v, X_h^{(1)} v)_{\mathtt{v}} + \tau^4 \|z\|_M^2 \\ &= \|v\|_{M_0}^2 - \tau \Big| v + \frac{\tau}{2} X_h^{(1)} v \Big|_d^2 + \frac{\tau^3}{4} ((2A_h + 3M_1) X_h^{(1)} v, X_h^{(1)} v)_{\mathtt{v}} + \tau^4 \|z\|_M^2\end{aligned}$$

from which the stated identity follows. □

Proposition 4.23. *Let* $v \in V_h^{\mathtt{v}}$ *and* r *be chosen as the smallest integer not smaller than* $\kappa_2^{1/3}/h_{\mathtt{v}}^{1/2}$ *where* κ_2 *is defined using* $Z(\tau, v)$ *of Lemma 4.22 by*

$$\kappa_2 = \sup_{0 \leqslant \tau \leqslant 1} \sup_{0 \neq v \in V_h^{\mathtt{v}}} \frac{Z(\tau, v)}{\|v\|_{M_0}^2}.$$

Then

$$\|R_{h2}^{\text{sat}}(\tau) v\|_{M(\tau)} \leqslant (1 + h_{\mathtt{v}}^{3/2})^{1/2} \|v\|_{M_0} \qquad \textit{for all } \tau \leqslant 1/r,$$

and R_{rh2}^{sat} *satisfies the stability assumption* (4.16) *of Theorem 4.19.*

Proof. By Lemma 4.22 and the definition of κ_2,

$$\|R_{h2}^{\text{sat}}(\tau) v\|_{M(\tau)}^2 \leqslant \|v\|_{M_0}^2 + \tau^3 \kappa_2 \|v\|_{M_0}^2 \leqslant (1 + h_{\mathtt{v}}^{3/2}) \|v\|_{M_0}^2,$$

since $\tau^3 \kappa_2 = \kappa_2 / r^3 \leqslant h_{\mathtt{v}}^{3/2}$. Applying this successively on each subtent, we obtain

$$\|R_{rh2}^{\text{sat}} v\|_{M(1)} \leqslant (1 + h_{\mathtt{v}}^{3/2})^{r/2} \|v\|_{M_0}.$$

Next, we use the bound $(1 + h_{\mathtt{v}}^{3/2})^{r/2} \leqslant \exp(h_{\mathtt{v}}^{3/2} r/2)$. Since the argument of the exponential is bounded, $\exp(h^{3/2} r/2) - 1 \lesssim h_{\mathtt{v}}^{3/2} r/2 \lesssim h_{\mathtt{v}}$. Thus there is an $h_{\mathtt{v}}$-independent constant $C > 0$ such that $\|R_{rh2}^{\text{sat}} v\|_{M(1)} \leqslant (1 + C h_{\mathtt{v}}) \|v\|_{M_0}$. □

Note that by Lemma 4.4, the constant κ_2 satisfies $\kappa_2 \lesssim 1$. Hence Proposition 4.23 gives stability under a so-called "3/2-CFL" condition. The latter term is an adaptation of the terminology on CFL conditions in [3] for our tents, in view of the fact that our $\tau \leqslant 1/r$ condition, with r as in Proposition 4.23, implies that the amount of time advance along a tent pole $(\tau\delta)$ is limited by $O(h_{\mathtt{v}}^{3/2})$.

5. Conclusion

We have developed a convergence theory for MTP schemes for a large class of linear hyperbolic systems, covering the semidiscrete case (§3), as well as a few fully discrete schemes (§4). The convergence rate for the semidiscretization was established to be $O(h^{p+1/2})$ in Theorem 3.12 under reasonable assumptions. When the number of stages $s = p + 1$, the fully discrete SAT scheme also gave the same convergence rate (Theorem 4.19) under the stability assumption (4.16). Through a selected numerical example, we showed in §4.4.1 that this convergence rate cannot be improved in general. The stability of SAT scheme was verified in §4.4.2 for the $p = 0$, $s = 1$ case and in §4.4.3 for the (arbitrary p) $s = 2$ case. Proving the stability of SAT schemes (verifying (4.16)) for other values of s is currently an open problem. It is however possible to computationally verify stability within each tent by solving a small eigenvalue problem as shown in [14]. The numerical results there suggest that an estimate of the form $\|R^{\rm sat}_{rhs} v\|_M \leqslant (1 + Cr^{-s})\|v\|_{M_0}$ might hold for general r and s. If this is provable, then for larger s, a slight modification of the argument of Proposition 4.23 would prove stability under a less stringent $(1+1/s)$-CFL condition, which limits the amount of time advance by a scalar multiple of $h_{\rm v}^{1+1/s}$. Also, if our analysis in §4.4.2 is any indication, it might be a worthwhile future pursuit to seek further special cases where stability holds under even weaker CFL conditions within a tent. The simplest cases of the fully discrete analyses we presented are those of the lowest order tent-implicit scheme in §4.2 and the lowest order iterated explicit scheme in §4.3. The latter was obtained from a nontraditional viewpoint of explicit schemes as iterative solvers for implicit schemes.

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