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Teaching routines and student-centered mathematics instruction: The essential role of conferring to understand student thinking and reasoning☆

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A R T I C L E I N F O

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A B S T R A C T

We compare two lessons with respect to how a teacher centers student mathematical thinking to move instruction forward through enactment of five mathematically productive teaching routines: Conferring To Understand Student Thinking and Reasoning, Structuring Mathematical Student Talk, Working With Selected and Sequenced Student Math Ideas, Working with Public Records of Students' Mathematical Thinking, and Orchestrating Mathematical Discussion. Findings show that the lessons differ in the enactment of teaching routines, especially Conferring to Understand Student Thinking and Reasoning which resulted in a difference in student-centeredness of the instruction. This difference highlights whose mathematics was being centralized in the classroom and whether the focus was on correct answers and procedures or on students' mathematical thinking and justifying.

1. Introduction

Mathematics education scholars and professional organizations advocate for classrooms centered on student mathematical thinking. That is, student mathematical thinking is the means to move instruction forward (Jacobs & Spangler, 2017; Nasir & Cobb, 2006; Schoenfeld, 2010; Turner et al., 2013). Two core practices have been identified for student-centered mathematics instruction: "*teacher noticing* [of student mathematical thinking] and *leading discussions* [centered around student mathematical thinking]." (Jacobs & Spangler, 2017, p. 768). These practices are also reflected in NCTM's *Principles to Actions (2014)*: "elicit and use evidence of student thinking" and "facilitate meaningful mathematical discourse" (p. 10).

In this paper, we use the term "student-centered mathematics instruction" to describe instruction where (a) student mathematical thinking is made public, (b) students engage with each other's mathematical thinking, and (c) student mathematical sense-making, conjecturing, and justifying drive the instruction. We examine a set of five *mathematically productive teaching routines* designed for enacting student-centered mathematics instruction: *Conferring To Understand Student Thinking and Reasoning, Structuring Mathematical Student Talk*, *Working With Selected and Sequenced Student Math Ideas*, *Working with Public Records of Students' Mathematical Thinking,* and *Orchestrating Mathematical Discussion*. We have observed that the existence of such routines in a teachers' practice was not always

☆ "The questions are [asked] to give you [the teacher] ideas where they [the students] are at and not to teach them. That is something I never thought of." [Hannah, fourth grade teacher]

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reflective of student-centered mathematics instruction. Thus, we focus this exploration on: How might differing enactments of teaching routines support or constrain student-centered instruction?

2. Literature

Standards set by national organizations call for collaborative and student-centered classrooms that provide opportunities for students to reason about mathematics and construct an understanding of mathematics as a part of a learning community (Boaler & Staples, 2008, 2007; National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010). To this end, researchers have documented many classrooms that evidence the types of rich mathematical engagement that is valued. In these classrooms, students are positioned as capable mathematics learners through the teachers' support of equitable and meaningful participation in classroom mathematical discourse (Turner et al., 2013). Teachers have clear mathematical goals (Bandura, 1986; Schunk, 1990, 1991), implement high cognitive demand tasks (Stein & Smith, 1998), encourage the explorations of multidimensional solutions (Carpenter et al., 1996; Fennema et al., 1996), support argumentation (Conner et al., 2014), and create a supportive and equitable community (Cohen et al., 1999).

We note that the term mathematics is used in this paper as the apprenticeship into the discipline of mathematics (Weber & Melhuish, 2022) as typically taught in K-12 mathematics in the United States. This means that justification is seen as the goal to achieve in the K-12 mathematics classroom (National Council of Teachers of Mathematics, 2000; National Governors Association, 2010). The National Research Council (NRC) describes "mathematical proficiency" as five interweaving strands that make up the complex whole of mathematical proficiency:

> *conceptual understanding*—comprehension of mathematical concepts, operations, and relations*, procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, *strategic competence*—ability to formulate, represent, and solve mathematical problems, *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification, and *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (Kilpatrick et al., 2001, p. 16).

We differentiate this view of mathematics from other definitions (Thanheiser, 2023) such as *mathematics as contextual, ever present, as a language to make sense of the world (*Ani, 2021*)* or *mathematics as a verb (not a noun), a human activity, part of one's identity (*Aguirre et al., 2013; Gutierrez, 2018) to acknowledge the limitations in our focus.[1] With the *apprenticeship into the discipline* notion of mathematics in mind, enacting student-centered instruction focused on justification is complex, requires intentional and strategic practices, and persistence in enacting these practices over time (Boaler & Staples, 2008; Franke et al., 2007; Hufferd-Ackles et al., 2004; Staples, 2007; Stein et al., 2008). These practices require extensive knowledge of students' progression of learning, and how to make productive problem-solving visible. Furthermore, these practices incorporate not just teaching mathematical reasoning but also engendering classroom mathematical norms of sharing thinking, questioning, error analysis, and making sense of other students' ideas (Boaler & Staples, 2008; Hufferd-Ackles et al., 2004; Lampert et al., 2013; Staples, 2007).

To focus on student-centered mathematics instruction with an explicit mathematical goal, teachers need to identify the types of student thinking that can be leveraged in whole class discussion (e.g., van Zoest et al., 2017) to move instruction forward. Instructors need to identify this mathematical thinking in the classroom, make it public if needed, and engage students in discussion around each other's ideas (e.g., Stein et al., 2008). Such classrooms become math talk communities (Hufferd-Ackles et al., 2004) where students may be increasingly responsible for their mathematics learning and engaging with each other's mathematical thinking. This interactive culture is linked to deeper mathematical thinking as students engage in inquiry and argumentation (Woods et al., 2006).

Due to the high degree of complexity involved in facilitating a student-centered mathematics classroom, many researchers have identified mechanisms which can concretize this work and provide structured ways to plan and enact more student-centered instruction ideas (e.g., Stein et al., 2008). In this section, we focus on our five focal teaching routines that help facilitate such classrooms and conclude with discussion of student-centered mathematics instruction.

2.1. Teaching routines

A teaching routine can be operationalized as "recurring, patterned sequences of interaction teachers and students jointly enact to organize opportunities for student learning in classrooms" (DeBarger et al., 2011, p. 244), a "predictable set of actions" (Kelemanik et al., 2016, p. 18). Teaching routines allow predictability for the teacher and students in how to engage with the content in a way that is student centered. Kelemanik et al. describe the core of such routines as allowing students to engage individually, in small groups, and in the whole class with ideas generated by the students themselves. This purposeful combination of individual, partner, and whole-group interactions supports a range of learners by creating opportunities and varied structures for them to process their mathematical ideas and their learning as well as the mathematical ideas of others. (p. 25)

In this paper we are focusing on five teaching routines developed for accessing and working with student mathematical thinking to

[1] We also note that research has shown that the apprenticeship into the discipline type of focus may not serve all students equally and thus is not sufficient to teach all students (Melhuish et al., 2022). We include this comment here to highlight that while the results of this paper show what an increased focus on student-centered mathematics instruction looks like, we acknowledge that it might not serve all students the same and further work needs to be done to examine ways to serve all students.

create student-centered instruction: *Conferring To Understand Student Thinking and Reasoning, Structuring Mathematical Student Talk, Working with Public Records of Students' Mathematical Thinking, Working With Selected and Sequenced Student Math Ideas, and Orchestrating Mathematical Discussion.* (Teachers Development Group, 2013) (See Table 1 for shortened research definitions of the teaching routines).

In this section, we further unpack these five teaching routines and connect them to the larger literature base on student-centered mathematics instruction.

The goal of *Conferring to Understand Student Thinking and Reasoning* is to gain formative assessment information about the depth, boundaries, firmness, and/or tentativeness of students' thinking as well as to scaffold one or more students' mathematical thinking to expand their repertoire of strategies and/or the boundaries of their thinking. *Conferring to Understand Student Thinking and Reasoning* can take on a variety of forms such as talking to a student one-on-one or listening in on or talking to small groups of students. This routine emphasizes the role of teachers as researching student matheamtical thinking to productively notice (attend to, interpret, and respond to) student thinking (Jacobs & Spangler, 2017; Jacobs et al., 2010; Munson, 2018; Sherin & van Es, 2005). By engaging with students and questioning their thinking, teachers can be put into a position to incorporate and respond to students' mathematical thinking. Teachers taking up student maatheamtical ideas and building on them allows students to build conceptual understanding (Carpenter et al., 1989).

The goal of *Structuring Mathematical Student Talk* is to support the development of student-to-student interaction about mathematics as well as to provide formative assessment information for the teacher that drives instructional decisions. *Structuring Mathematical Student Talk* involves two key components: providing a structure or protocol for students talking to partners or small groups (such as assigning roles) and a mathematically meaningful prompt to drive the discussion resulting in mathematically worthwhile talk. During *Structuring Mathematical Student Talk* Students work with partners or in small groups to explain their reasoning (Anthony & Walshaw, 2009), listen to understand each other (Hintz & Tyson, 2015; Hoyles, 1985), and ask genuine questions. This allows students to generate ideas and make sense of each other's explanations. Listening to interpret and collaborate (rather than just evaluate) (e.g., Hintz & Tyson, 2015) leads students to ask genuine questions and make sense of each other's ideas. This positions the students to not just decide if their peer has a "correct" answer, but to engage genuinely with their thinking. *Structuring Mathematical Student Talk* allows all students access to mathematical ideas (Kelemanik et al., 2016; White, 2003) and has been shown to benefit students from traditionally marginalized groups (Malloy, 1997).

The goal of *Working with Public Records of Students' Mathematical Thinking* is to make student mathematical thinking available to all students to work with it (Stein & Smith, 2011). This may look like recording student ideas (Cengiz et al., 2011; Staples, 2007) or having students share their own work on the board, the overhead, or posters and engaging the class to work with these public ideas. Making students' thinking public can serve several goals including making mathematical reasoning visible for students (Ghousseini, 2009), and maintaining common ground (Staples, 2007). Publicizing student work has the potential to position students as contributors to

Table 1
Five Teaching Routines for accessing and working with student mathematical ideas to create student-centered instruction.

Teaching routine for accessing and working with student mathematical ideas with description.	Implementation, and goal
Teaching routine: CONFERRING TO UNDERSTAND STUDENTS' MATHEMATICAL THINKING *To implement this routine, the teacher:* *confers with students one-on-one or within small groups using genuine questions, statements, and actions to learn about (not funnel) student thinking.*	**Small group**: Accessing student mathematical thinking and engaging students with each other's ideas.
Teaching routine: STRUCTURE MATHEMATICALLY WORTHWHILE STUDENT TALK *To implement this routine, the teacher:* 1. purposefully structures student-to-student (pairs or small group) interaction, and 2. provides a targeted mathematical task or question in relation to the current task	
Teaching routine: WORKING WITH PUBLIC RECORDS OF STUDENTS' MATHEMATICAL THINKING *To implement this routine, the teacher makes students' mathematical ideas available to the class by moving visual records of their ideas into the public space and prompting student engagement with the ideas while the public record still available.*	**Whole class:** Focusing the whole class on students' thinking and engage students in making sense of each other's ideas.
Teaching routine: WORKING WITH SELECTED AND SEQUENCED STUDENT MATH IDEAS *To implement this routine,* 1. the teacher selects student ideas to be considered, and 2. sequences the presentation of the ideas.	
Teaching routine: ORCHESTRATING MATHEMATICAL DISCUSSIONS *To implement this routine, the teacher has to have* 1. students contribute mathematical ideas 2. respond to each other's' ideas, with 3. at least three students contributing.	

mathematics (Cohen, 1994).

The goal of *Working With Selected and Sequenced Student Math Ideas* is to provide a structure for sharing public records in order to advance student understanding by fostering connections related to the core mathematical ideas on which the lesson/task focuses. Once teachers have conferred with their students, they choose how to build on student ideas with the whole class by selecting and sequencing how student ideas are shared (Stein & Smith, 2011; Stein et al., 2008). These selections can be in service of mathematical goals, social goals, and other goals the teacher has. *Working With Selected and Sequenced Student Math Ideas* can take on a variety of forms such as the teacher purposely selecting a more visual or a more symbolic representation to go first or last, sharing increasingly more or less complex strategies, or ordering the presentation of student solutions based on some criteria.

The goal of *Orchestrating Mathematical Discussion* is to allow collective engagement (or for our purpose at least three students) in creating mathematical meaning and building and evaluating mathematical ideas. The goal of this teaching routine (with the support of others) is to allow student thinking to be made public and discussed to collectively build understanding. “In instructionally productive discussions, the teacher and a wide range of students contribute orally, listen actively, and respond to and learn from others’ contributions” (Reisman et al., 2018, p. 279). Through such discussions, students can be supported in mathematical practices such as argumentation including conjecturing and justifying (Conner et al., 2014; Forman et al., 1998; Staples, 2007).

The five teaching routines build on Smith & Stein’s (2011) five practices for orchestrating productive mathematics discussion. *Conferring To Understand Student Thinking and Reasoning* connects to Practice 1 “anticipating likely student responses to challenging math tasks” (p. 8) and Practice 2 “monitoring students’ actual responses to the tasks (while students work on the tasks in pairs or small groups.” (p. 8). *Structuring Mathematical Student Talk* connects to Practice 2 in that it allows for monitoring to take place. *Working With Selected and Sequenced Student Math Ideas* connects to Practice 3 “selecting particular students to present their mathematical work during the whole class discussion” and Practice 4 “sequencing the students’ responses that will be displayed in a specific order. *Working with Public Records of Students’ Mathematical Thinking* connects to Practices 3 and 4 as well as it allows student thinking to enter the whole class discussion. *Orchestrating Mathematical Discussion* connects to Practice 5 ”connecting different students’ responses and connecting the reposes to key mathematical ideas” (p. 8).

These five teaching routines also address both core practices named above: teacher noticing and the leading discussion (see Table 2).

2.2. Student-centered mathematics instruction

Our operationalization of student-centered mathematics instruction has three components: (a) student mathematical thinking is made public, (b) students engage with each other’s mathematical thinking, and (c) student mathematical sense-making, conjecturing, and justifying drive the instruction. The first two components serve the third. Sensemaking, conjecturing, and justifying allows students to take on the responsibility of meaning making (e.g., Stephens et al., 2017). Through sense-making, conjecturing, and justifying, students are able to create and develop understanding of their mathematical knowledge (e.g., Stephens et al., 2017).

Justifying is focal in policy documents (National Council of Teachers of Mathematics, 2000; National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010) as well as in the mathematics education research community (e.g., Blanton & Kaput, 2005; Ellis, 2011; Jacobs et al., 2007; Stephens et al., 2017). We build on Stylianides (2007) definition of proof as using established statements, using known and valid reasoning forms, and using appropriate communication forms to define justifying as follows:

- ***Justifying.*** Students create a logical argument that uses math structures (definitions, properties, meanings) and/or provides a counterexample to invalidate a claim to justify why a solution or mathematical statement is correct/true or incorrect/untrue. [Note that justifying does not need to be correct or complete to be counted as such.]

To enable teachers to engage students in justifying, they need to know what counts as a justification, what students struggle with, how topics connect with one another, and how to teach in a way that supports reasoning (Lo & McCrory, 2009). Most importantly, teachers need to assess student thinking so they can build on initial attempts to make sense, conjecture, and justify. Teacher noticing of sense-making, conjecturing, and justifying is complex (Melhuish et al., 2020). The complexity lies in the fact that sense-making, conjecturing, and justifying serve both as “means by which students enhance their understanding of mathematics” (Staples et al., 2012, p. 447) and a way of doing mathematics.

Table 2
Linking *Mathematically* productive teaching routines to core pr*actices*.

Core Practices	Mathematically Productive Teaching Routines
Teacher noticing	• Conferring To Understand Student Thinking and Reasoning, allows teachers to notice student thinking. • Structuring Mathematical Student Talk, allows teachers to listen in and as such notice student thinking.
Leading discussion	• Working With Selected and Sequenced Student Math Ideas, allows teachers to orchestrate the discussion around certain ideas. • Working with Public Records of Students’ Mathematical Thinking, allows teachers to orchestrate the discussion around student mathematical ideas. • Orchestrating Mathematical Discussion.

This study adds to the existing literature about the complexities of high-quality student-centered mathematics instruction by:

1. Operationalizing key teaching routines that can serve to support student-centered mathematics instruction.
2. Analyzing the different enactment of these teaching routines in the mathematics classroom to better understand when and how such routines promote student-centered instruction.

3. Framing: student-centered mathematics instruction, teaching routines, cognitive demand of tasks, and lesson cohesion

In this study, we assume that student-centered mathematics instruction provides the means to support students in learning mathematics deeply and meaningfully. In this framing, we elaborate the theoretical mechanisms for how the teaching routines may support student-centered mathematics instruction. We then consider two other essential dimensions of student-centered classrooms: Cognitive Demand and Lesson Cohesion.

3.1. Connecting teaching routines and student-centered mathematics instruction

Conferring to Understand Student Thinking and Reasoning is at the core of student-centered instruction as teachers need to access student mathematical thinking to allow them to work with it. The other routines support this one. *Structuring Mathematical Student Talk* allows the teacher to access student mathematical thinking while walking around and listening in on conversations. As such it makes student mathematical thinking public to the teacher. It also allows students to engage with each other's mathematical thinking in small groups and as such makes student thinking public to each other in the small groups. *Working with Public Records of Students' Mathematical Thinking* makes student thinking public to the whole class. *Orchestrating Mathematical Discussion* while *Working With Selected and Sequenced Student Math Ideas* and *Working with Public Records of Students' Mathematical Thinking* allows students to engage with each other's ideas in the whole class setting. All three routines combined can lead to establishing common ground (Staples, 2007) in the mathematics classroom. Common ground is defined as those mathematical ideas that are held in common by the classroom community. Comparing across strategies (e.g., Durkin et al., 2017) and engaging in critiquing and argumentation (Yackel & Cobb, 1996) can be particularly productive. This allows for more opportunities to make meaning, make connections (Jacobs & Spangler, 2017) and thus develop understanding.

Finally, the goal is for students to determine the correctness or sensibility of an idea or solution as they discuss which solutions make sense and provide evidence to support their conclusions. This contrasts with appealing to the teacher or textbook as the standard by which to determine correct solutions. By appealing to mathematical reasoning, students are positioned as sense-makers and have the opportunity to compare and critique in ways that support greater mathematical understanding.

Below we relate the components of student-centered instruction to the teaching routines (see Table 3).

3.2. Mathematical goal and cohesion of the lesson

While the teaching routines serve to connect instructional practice and students' mathematical thinking and discussion within a lesson, there are also key elements of planning lessons and tasks that contextualize and provide insight into the mathematical richness of a lesson. In particular, mathematically rich lessons center around a specific mathematics goal. For example, identifying a core mathematical goal from curriculum materials can be supportive in developing a mathematical productive classroom (Remillard, 1996). By identifying a mathematical goal, a lesson can shift away from implementation of procedures to find answers to the underlying mathematics. Ideally, goals are specific, within reach, and offer some degree of challenge (Bandura, 1986; Schunk, 1990, 1991). To orchestrate a productive discussion "teachers need to have clear learning goals for what they are trying to accomplish in the lesson," (Stein & Smith, 2011, p. 13).

With an established goal the lesson can be examined for cohesion around that goal. Instructional cohesion is an important feature of implementing mathematics lessons (Cai et al., 2009; Stigler & Hiebert, 1999). The cohesion of the lesson captures the degree to which

Table 3
Relating the five Teaching Routines to student-centered instruction.

Student-Centered Instruction	Teaching Routines
(a) student thinking is made public	• TR: Structuring Mathematical Student Talk (public to the small group) • TR: Conferring To Understand Student Thinking and Reasoning, (public to the teacher) • TR: Working with Public Records of Students' Mathematical Thinking (public to the whole class)
(b) students engage with each other's thinking,	• TR: *Structuring Mathematical Student Talk* (students engage with each other in small groups) • TR: *Working with Public Records of Students' Mathematical Thinking* (students engage with shared student work in whole class setting) • TR: *Working With Selected and Sequenced Student Math Ideas* (students engage with each other in the whole class groups)
(c) student sense-making, conjecturing, and justifying drive the instruction	• TR: Orchestrating Mathematical Discussion

different lesson activities are interconnected (Hill et al., 2008; Hill, 2014) which can be appreciated from the point of view of a focus on a single mathematical topic (Fernandez et al., 1992; Stigler & Perry, 1988) or interrelatedness of topics within a lesson (Herbel-Eisenmann & Otten, 2011).

3.3. Cognitive demand of the task

In student-centered instruction, we anticipate that students engage in high cognitive demand tasks (which should be aligned with the lesson goal.) Cognitive demand describes the kind of thinking that is asked of students (Stein & Smith, 1998). Research indicates that classrooms that support student engagement in higher-demand tasks promote greater success in measures of students' reasoning and problem solving (Stein et al., 2016). Furthermore, cognitively demanding tasks provide rich opportunities for students to engage in justifying and generalizing (Stein et al., 2016). We follow Smith and Stein's categorization of lower-level demand tasks focusing on memorization and procedures without connections, and higher-demand tasks focusing on procedures with connections and doing mathematics. Doing mathematics might include "explor[ing] and understand[ing] the nature of mathematical concepts" (Stein & Smith, 1998, p. 348).

Ultimately, we would anticipate that the literature-based teaching routines would be in service promoting student reasoning and discussion about high cognitive demand tasks and in relation to an overarching lesson goal.

4. Methods

This paper is situated in a large-scale professional development efficacy study (Melhuish et al., 2022). In this paper, we describe differences in two lessons implemented by the same teacher on a similar topic before and after engaging with the professional development (PD) focused on the *Mathematically Productive Teaching Routines* (Teachers Development Group, 2013) with the goal to support students in justifying and generalizing.

The PD included protocols and explicit planning, observing, and debriefing the teaching routines. Teaching routines were introduced via handouts with detailed descriptions and discussed and collaboratively implemented and reflected on throughout the PD (Teachers Development Group, 2013).

4.1. Context

The setting for this study was an elementary school in a mid-sized school district in the Pacific Northwest of the United States. This school had a majority of students with traditionally marginalized backgrounds and a high rate of free and reduced lunch. At the beginning of the PD the school had an enrollment of approximately 580 students with a 73 % traditionally marginalized enrollment and 79 % of children enrolled in free and reduced lunch. At this school 53 % of 5th graders were meeting the math standards as determined by the school district and published on their website.

A fourth-grade teacher, Hannah (all names are pseudonyms) served as the focal teacher for the last two years of the three-year study. This teacher was supported in developing the key principles of the PD through the PD facilitator's targeted guidance in planning regular lessons, coaching during each lesson implementation in their classroom, and ongoing support throughout the year by each school's on-site mathematics coach.

4.2. Data sources

We focus on two of Hannah's lessons. These lessons were videotaped at the end of the school year, one lesson before and one lesson after Hannah engaged with the PD. These lessons had comparable length (about one hour) and comparable content (fraction/decimal comparison strategies). However, the lessons represent significant differences across our focal constructs.

We selected these two lessons from Hannah because they maximized the opportunity to reach our explanatory aim. In Lesson 1, prior to the start of the PD, Hannah had students participating in the classroom and evidenced implementation of several teaching routines. However, the initial lesson did not appear to exhibit student-centered instruction. In contrast, Lesson 2, after the PD, included teaching routines and appeared to exhibit student-centered instruction. Thus, her classroom offered opportunities for the comparison of the quality of implementation of teaching routines.

The two lessons were our primary data source. However, we provide additional insight into our findings with Hannah's written reflections to explore alignment with how Hannah reported her own instructional growth and differences between the focal lessons.

4.3. Data analysis

Our analysis goals were to qualitatively understand the implementation of teaching routines in relation to how student-centered the instruction was (see Table 3). In addition, to allow for a more in-depth comparison of the two lessons, we examined the lesson goals and cohesion of the lesson as well as the cognitive demand.

4.3.1. Teaching routines: first phase analysis of the lesson's teaching routines

Members of the larger project research team coded for implementation of teaching routines along with modality (Whole Class, Small Group, Individual Work). To code for the teaching routines and modality, two coders independently identified the beginning and the end of each teaching routine as well as the teaching modality. The coding rubric for the teaching routines is provided in Appendix A. Once each coder coded the lessons, they met and resolved any discrepancies through discussion.

4.3.2. Teaching routines: second phase analysis of teaching routines and relation to student-centeredness

After identifying the modality of teaching and the instances of the teaching routines, we compared across the two lessons in two ways. First, we explored the difference in coded teaching routines to explore how their frequency, durations, and sequence was different across the lessons. Second, we looked within each of the routines to examine the difference in the quality of their enactment with a particular focus on how student-centered the instruction was. We examined whether and how student mathematical thinking informed the enactment of each teaching routine (see Table 4).

We examined how differences on the surface (what and when of the routines) and in the details (how student thinking was supported within them) may account for the differences we observed in the lessons. This analysis occurred over several cycles involving watching the videoed lessons and reading transcripts to create a narrative reflecting the teaching routine implementations (first author) that addressed the mathematical and instructional details. These narratives and original transcripts were then analyzed to explicitly consider the guiding questions in Table 4 by both the first and second author to elaborate, challenge, or endorse interpretations.

4.3.3. Lesson goal, lesson cohesion, and cognitive demand

To code for lesson goal and cohesion, as well as cognitive demand, each lesson was separated into segments represented by a shift in the focus of the lesson (e.g., discussing the learning target, reviewing strategies) or a shift in modality (e.g., from group discussion to private reasoning time) (see Appendix B and C for the timeline and segments for each of the two lessons). Each segment was then coded for cognitive demand (see Appendix D for the coding rubric) and the segments were examined for lesson cohesion across three levels (1) lesson incohesive, there are at least two distinct topics; (2) lesson cohesive (focused on a single topic) but without a clear goal; (3) lesson cohesive (focused on a single topic) with a clear goal. Each aspect was coded by at least two coders. Once each coder coded the lesson the coders met and resolved any discrepancies through discussion. For a more detailed description see Thanheiser et al. (2021).

5. Results & discussion

In this section, we begin by describing the focal lessons globally. Both lessons were recorded at the end of the school year and as such norms had been established in the classroom. The lessons are comparable with respect to length (60 min for Lesson 1 and 63 min for Lesson 2), grade level (4th grade) and the content taught. Both lessons focuse on comparing/ordering rational numbers, decimals in Lesson 1 and fractions in Lesson 2.

Both the classrooms for Lesson 1 and Lesson 2 seemed to be designed in a way that allowed students to interact with each other (Yeh et al., 2017). Students sat at desks in groups of four – six with two or three tables that seat two children each pushed together to form rectangles. All students were facing each other and had a clear view of the overhead projector. The walls were full of materials for mathematics and other subjects such as posters of mathematically productive habits of mind and habits of interaction (Teachers Development Group, 2013). For an image of each classroom see Fig. 1a and b. Both classrooms seemed to encourage student group work.

However, Lesson 2 differed from Lesson 1 in terms of publicly available student mathematical thinking. Lesson 2 included a poster board stand with individual posters that have the tasks that would be discussed printed on them (red box in Fig. 1b). The teacher was preparing to engage in *Working with Public Records of Student Mathematical Thinking* and to make a permanent record of the student's thinking that could be preserved (as opposed to sharing on the overhead projector only and taking it away after it has been shared).

In addition, the image of Lesson 2 shows a poster of the strategies the students had created and used the prior day (this was recorded on poster paper and put on the wall, see red oval in Fig. 1b and a recreation of this poster in Fig. 2). Hannah regularly referred to this poster throughout the lesson. Each strategy was labeled with one of the students' names and referred to as such during the lesson. As such in Lesson 2 student mathematical thinking was centered and drove the instruction.

The two lessons also differed in the opening of the lessons. While in Lesson 1 Hannah immediately started on the first activity, in Lesson 2 Hannah took time to introduce the learning target and connect to prior learning.

Table 4
Implementation of Teaching Routines as related to student-centered instruction.

Student-centered Instruction	Questions about the implementation of teaching routines
(a) Student thinking is made public	• (How) Was student thinking made public?
(b) Students engage with each other's thinking,	• (How) Were students prompted to engage with each other's thinking, and (how) did they actually engage with each other's thinking?
(c) Student sense-making, conjecturing, and justifying drive the instruction	• What type of thinking was asked of students? (Conceptual or procedural) • (How) Were students positioned in relation to the generation of mathematical ideas in the classrooms? (Did they generate the ideas presented?) • Were students engaged in sense-making, conjecturing, justifying or did they focus on procedures?

Fig. 1. a Image of Lesson 1, b Image of Lesson 2.

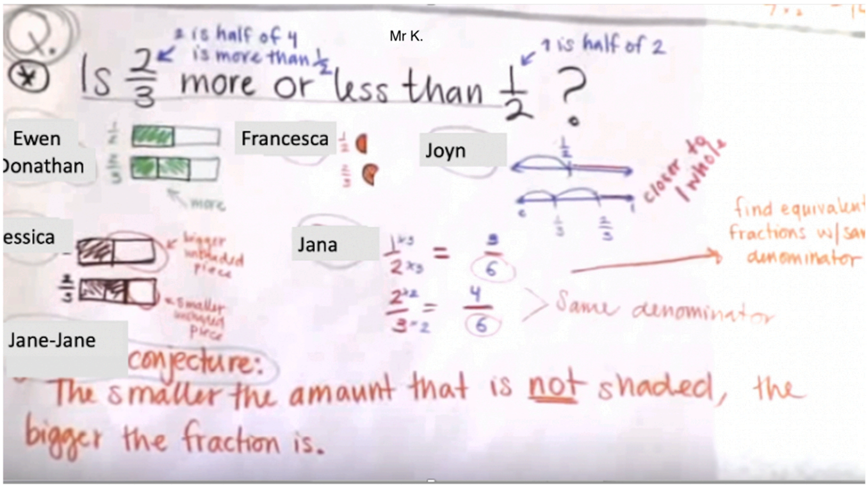

Fig. 2. Student-created strategies for comparing fractions (all names are pseudonyms).

5.1. *Implementation of teaching routines, lesson goal and cohesion of lesson, as well as cognitive demand*

In this section, we examine the implementation of the teaching routines and modalities across the two lessons. Fig. 3 shows each of the two lessons divided into one-minute segments with the teaching routines and modality of the classroom instruction.

5.1.1. *Teaching routines in Lesson 1*

Lesson 1 had two implementations of teaching routines. The first was at the beginning of the whole class discussion segment and

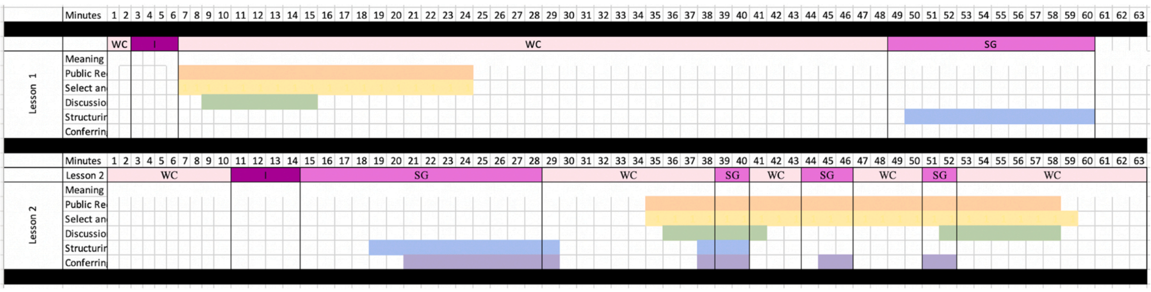

Fig. 3. Proportional Figure of Class Time spent in the various modalities and teaching routines: Modality: Whole Class (Light pink WC), Small Group (medium pink SG), Individual Work (dark pink, IW). Teaching Routines: Working with Public Records of Students' Mathematical Thinking (orange), Working With Selected and Sequenced Student Math Ideas (yellow), Orchestrating Mathematical Discussion (green), Structuring Mathematical Student Talk (blue), and Conferring To Understand Student Thinking and Reasoning (purple).

was a combination of *Working With Selected and Sequenced Student Math Ideas, Working with Public Records of Students' Mathematical Thinking,* and *Orchestrating Mathematical Discussion.* The students had a few minutes to work on the decimal ordering task (*Order the following decimal numbers from smallest to largest on the number line: 1.5, 1.1, 1.96, 1.65, 1.37*) on their own to engage with the task. Hannah used this time to walk around the room and observe the students' responses but did not confer with students to understand their thinking. She did, however, pay attention to students who ordered the decimals correctly versus incorrectly. When she walked by Joy, she stated to the class, "Remember the place value of each number." Based on what she saw while walking around, Hannah selected three students Joy, Anna, and Damon to share their work with the class. "Alright, I want - umm - let's see - I want Joy to come up here and show her work. Anna, you want to come up as well? [Anna interjects: Yeah.] Okay, Anna you can go right after Joy. And then I also want Damon." Thus, Hannah enacted the teaching routines: *Working with selected & sequenced student math ideas*. Hannah selected a student with an incorrect response to share first. This is often done to allow the class to engage with the mistake. For each of the three students who shared their responses, Hannah also enacted *Working with public records of students' mathematical thinking,* by having the students share their work publicly (on the overhead projector) and discussing this work with the class. She asked Joy to share first.

Joy began by sharing her solution (see Fig. 4), she explained.

> My smallest number would be one and one tenth. And then I went one and one fifth [corrects herself], one and five tenths, and then - one and thirty-seven hundredths - and one and sixty-five hundredths - one and ninety-six hundredths.

Even though we did see Joy's ordering, we did not hear an explanation for why she ordered the numbers this way (even with repeated prompting). Joy seemed to order decimals she saw in tenths first (1.1 and 1.5) and decimals she saw in hundredths second (1.37, 1.65, 1.962) using whole number ordering for the numbers behind the decimal point (DeWolf & Vosniadou, 2015; Durkin & Rittle-Johnson, 2015; Hiebert & Wearne, 1983).

Hannah implemented *Orchestrating Mathematical Discussion* when discussing Joy's solution, by engaging at least three students in a discussion around Joy's ordering of the decimals. She began by asking "Okay, thank you. Alright, questions or comments for Joy?" One student asked, "How do you know that's the right answer?" Joy thought about it but did not respond even with additional prompting by Hannah. When Hannah asked her, "Are you confident that it's the right answer?" Joy responded, "yeah." Joy seemed convinced that she had the correct answer. Hannah inquired again, "How do you know that it's right? What about it makes sense to you?" followed up by "Makes you really think, right? Just do your best. How do you know that it's correct? [5 sec pause]." When Joy did not come up with justification and did not seem to recognize that her response was incorrect, Hannah turned to the class, "You can give her a hint" followed by, "Alright, someone wants to help Joy out? You can pick a helpful hand." Joy called on a student who agreed with Joy's solution.

Then Hannah steered the discussion towards the procedure of adding a zero to decimals that do not have a hundredths' digit to make the comparison 'easier,' "[Joy] also wrote an equivalent for that in the hundredths - right? Cause she knows that one and tenth is equivalent to what - Damon?" Over the next three minutes the discussion focused on 1.1 being equivalent to 1.10 and 1.5 being equivalent to 1.50. At that point Conrad asked Joy, "why did you write one and thirty-seven hundredths is greater than one and fifty-hundredths?" When Hannah checked in with Joy, Joy did not seem to understand Conrad's question, so Hannah rephrased: "He's saying that since you said that one and five tenths is equivalent to one and fifty hundredths - why is one fifty - one and fifty-hundredths in front of one and thirty-seven hundredths?" After some more back-and-forth, Hannah asked, "Joy, in your opinion - what do you think you should have done? What would be the correct way of lining up your numbers?" At this point, Joy stated "I would have put a one and thirty-seven hundredths in front of the one and five tenths" correcting her procedure without offering a rationale for the correction. After Joy, two more students shared their solutions; however, Hannah did not enact *Orchestrating Mathematical Discussion* with these students' solutions.

While Joy's answer was made public, her thinking was not. And while students were asked to engage with Joy's answer, the focus was on procedural thinking rather than sense-making, conjecturing, and justifying. Hannah's move to center Joy's response first seemed targeted to address her incorrect response which appeared to be held by at least some other students in the class. Anna, who shared next, explained that her original solution was like Joy's, but by listening to Joy she switched the 1.5 and the 1.37. While Hannah enacted three teaching routines simultaneously, the focus of the exchange seemed to be to get to the correct answer via filling in zeros to show the equivalence of 1.1 and 1.10 and 1.5 and 1.50 and then use whole number reasoning to order the decimals, rather than comparing the values of the decimal using sense making strategies. Other than procedural reasoning, we did not get an insight into the

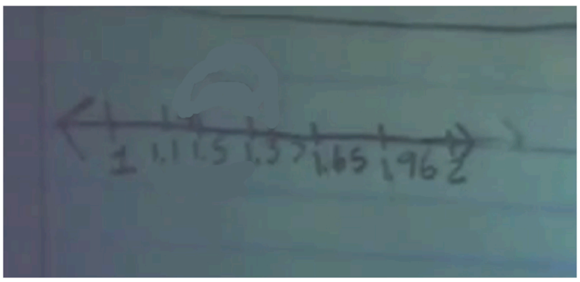

Fig. 4. Joy's initial solution.

children's mathematical thinking in this classroom.

This was also true for the third solution that was shared. Damon rewrote each decimal as a fraction, 1.1 as 1 and 1/10 and 1.37 as 1 and 37/100 (see Fig. 5).

The class then discussed the need for common denominators to make the comparisons. While Damon had the decimals ordered correctly on paper, he did not explain how he knew that 1.37 was less than 1.5 and reversed the order when explaining.

> So what I did is that I saw the place values of all these numbers - and you told us to put it from smallest to biggest. And I saw one and one tenths - so I know that's ten - so I put that in the - in the first number line - **and then I saw one and thirty-seven hundredths is greater than one and five tenths - so I put one and thirty-seven hundredths behind one and five tenths** - and one and sixty-five hundredths - is less than one and ninety-six hundredths - so that's why I put one and sixty-five hundredths behind one and ninety-six hundredths."

This was not taken up in the class during the discussion. Rather the discussion focused on the need for the common denominator to explain the order. The quality of the enactment of these three teaching routines was not focused on student sense making, conjecturing, and justifying but was focused on procedures and correct answers.

The next part of her class (24 min) was situated in whole class form without any of the teaching routines being implemented. The focus was on a game in which students had to either round a number or write a decimal number with the use of a place value chart. Hannah set up the game by splitting the classroom in half and the two halves competed against each other. In each turn, one person from each half of the class would come to the board and "write down whatever Ms. Hannah says" which included writing decimals or rounding decimals as fast as possible, with the first person to have the correct answer winning a point for their group. The rest of the class was expected to do the same in their journals. They worked on: 1) Write 1 1/100 as a decimal, 2) round 32 5/100 to the nearest tenth, 3) write 4 7/10 as a decimal, 4) round 5.55 to the nearest whole, 5) Write 1001 1/4th as a decimal.

Towards the end of Lesson 1, Hannah put the class into their small table groups and enacted *Structuring Mathematical Student Talk*. Each group was given a poster with a different word problem already glued onto the poster (in the bottom right corner) (see Fig. 6).

Hannah instructed the class.

> I'm going to have you guys spend the next about seven minutes - the next about seven minutes - to solve one story problem with your group. I have a poster paper, that is right here, so turn your eyes right here please [Hannah is holding. A poster paper up] … You have a story problem. … - you have a story problem [motions to the bottom right of the poster]. We are going to split this [poster paper] into four. You have your explanation with words [bottom left], with …, with numbers [top right] and with pictures [top left]. … You guys - the first thing you should do is read the problem. The second thing is talk about it - and then decide who is going to write the answer. …

Hannah enacted the *Structuring Mathematical Student Talk* teaching routine by asking students to work in together groups on word problems (targeted math prompt) with directions to first "read the problem," "talk about it" and decide who was to "write the answer" (structured student-to-student interaction). The group had a targeted math prompt and a structure for their interaction. However, we note that the structure for interaction is about producing the answer with engagement directions left vague for the problem-solving portion. Thus, the quality of the enactment of this teaching routine was not focused on student-centered instruction. Hannah checked in with groups and primarily redirected students to continue to work on the poster. While the students were working in groups, we did not gain insight into their thinking. Hannah did not confer to understand their thinking but focused on keeping the students on task. The lesson wrapped up with Hannah checking in with groups on whether they were done.

While Hannah implemented *Structuring Mathematical Student Talk*, her goal seemed to be to have groups produce a poster, it was unclear how the students were thinking about the problems. Thus, students were sharing their answers and collaborating in this class, but the focus was on correct answers rather than student mathematical thinking.

In summary, Hannah implemented several teaching routines. Instead of focusing on student mathematical ideas and having students build on each other's thinking or strategies (reflecting student-centered instruction), she focused on procedures to get the correct answer. When the students shared, we did not see much of their thinking. And the teaching routine *Conferring to Understand Students' Mathematical Thinking & Reasoning* was not implemented at all.

We claim Hannah did not implement student-centered mathematics instruction. While some students' answers were made public, the focus was not on student mathematical thinking but rather on answers and procedures. When students engaged with each other's

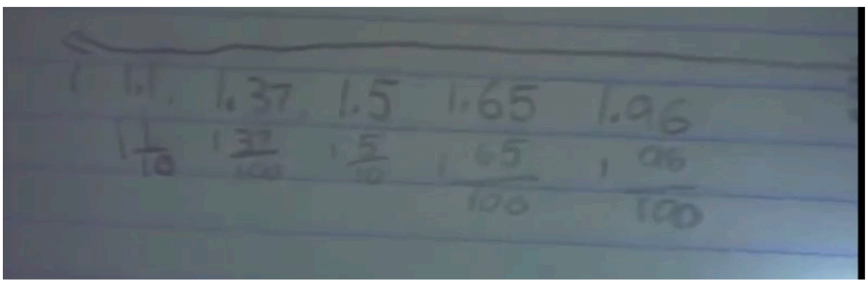

Fig. 5. Damon's solution to the decimal ordering task.

Word Problems on Posters
Today the grade 4 runners ran seventy-one hundredths km. The grade 6 runners ran six hundred seventy-five thousandths km. Which grade ran more?
Dear math students, I have some squash seeds and tomato seeds to plant, but I don't know which is which. My garden book says that squash seeds are about 0.7 of a centimeter long, and tomato seeds are about 0.25 of a centimeter long. Since 25 is bigger than 7, I would say that the tomato seed is the bigger seed. My friend says that's wrong. Who is right? Can you explain why? Thank you, Puzzled Penguin.
Samantha had a quarter, a dime, and two pennies in her pocket. What decimal part of a dollar does she have?
Chad is 1.78 meters tall. His friend Rob is 1.8 meters tall, and his friend Brent is 1.735 meters tall. Make a chart showing their heights rounded to the nearest hundredth. List the rounded heights in order from tallest to shortest.

Fig. 6. Word problems handed out on the posters.

answers the focus was on correcting rather than sense making. Thus, none of the three aspects of student-centered mathematics instruction (see Section 1) were observed in Lesson 1.

5.1.2. Lesson goal and cohesion of lesson, as well as cognitive demand in Lesson 1

Most of the lesson focused on procedures without a focus on the concepts that underlie those procedures and as such was coded as low cognitive demand. In this lesson the learning target was not explicated to the class. From observing the lesson various potential learning targets can be identified, namely ordering decimals, rounding decimals, writing mixed numbers as decimals, making sense of word problems, solving word problems, connecting words, pictures, and numbers, and working with money. The connection between the different potential learning targets was not explicated and not obviously observable. Thus, the lesson cohesion was coded as mid-level (2).

5.1.3. Teaching routines in Lesson 2

Lesson 2 had prolonged and denser (more teaching routines were implemented simultaneously) implementation of teaching routines. Hannah began the lesson by introducing the learning target and by reviewing six student generated strategies for comparing fractions from the previous lesson (see Fig. 2). This was followed by Hannah sharing a formative assessment she gave at the end of the last class where only seven of the 23 students produced a correct answer to three fraction comparison problems (are $^{1}/_{8}$, 24/42, and 6/11 respectively less than, equal to, or greater than ½). Thus, the class was not ready to move on, but would spend another day working on the fraction comparison problems as Hannah explained, “As a group we want to move on when most of us get it.” Then Hannah gave the students some time to review their work from the prior day. After this time, Hannah implemented *Structuring Mathematical Student Talk* combined with *Conferring to Understand Student Thinking and Reasoning* during small group work. Hannah paired the students up and explained:

> Number One, you are going to find a partner. Number Two, you are going to look, compare your work, and you are going to find either a problem that you don’t agree on- you got different answers or a problem where you used different strategies to solve. So-compare your work- compare, and then find one problem that is different either with your answer or different with a strategy

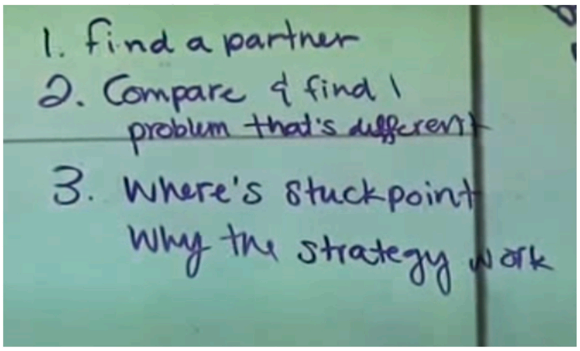

Fig. 7. Instructions for group work.

> that you picked, or that you used. And Three: I want you to, if it is a different answer - I want you to figure out who is right and who is wrong, or- why that strategy worked.

Hannah also wrote these instructors down (see Fig. 7).

Hannah continued to explain how students should interact:

> Instead of saying who is right and who is wrong, let's say this- where is the stuck-point? Figure out the stuck point. Or- why the strategy works. Why does it work mathematically? Kind of going deeper and thinking about justifying. Why this strategy works.

The students had both a focal mathematical goal and structure around how to interact in their group. Thus, the Teaching Routine: *Structuring Mathematical Student Talk* was implemented with an explicit focus on student-centered mathematics instruction by prompting students to engage with each other's thinking focused on sense making.

While the students worked, Hannah walked around and conferred with the students to understand their thinking. Hannah's focus had moved beyond correctness (as observed in Lesson 1) as can be seen in the following exchange and focused on sense making:

> **Hannah:** Ok, so how do you know that is right?
>
> **S:** Because I am smart
>
> **Hannah:** That is not math reasoning. Math reasoning is the authority in this classroom. 'I am smart' does not tell me anything. 'I am smart' tells me that you think too much of yourself. So- mathematically. Why does this make sense? And what strategy did you use to solve it?

When Hannah brought the groups back together, she spent about four minutes reflecting with them on their interactions with each other's thinking. This was followed by her enacting the teaching routines, *Working With Selected and Sequenced Student Math Ideas, Working with Public Records of Students' Mathematical Thinking,* and *Orchestrating Mathematical Discussion.* Note, these were the same routines to occur simultaneously in Lesson 1. Hannah explained that she selected specific students to share "a specific strategy for a specific problem." This indicated that she did not simply attend to correct/incorrect but was looking for specific strategies. She instructed the students:

> So, our goal for this time when we have students come up and share their strategies and their thinking is for us to … couple of things, number 1: make sense of their thinking and number 2: instead of Miss Hannah doing all the questioning I want you guys to question the students. And really question about why their strategy works, and why they use that strategy. So that, especially if it's a strategy that you guys are struggling with, …, ask a lot of questions that would help you understand. Because hey, maybe that is a strategy that would be useful to you, ok?

Hannah engaged the students with each other's thinking and shifted the responsibility of sense making, conjecturing, and justifying to the students. The individual students (selected by Hannah) shared how they compared fractions by putting their work on the overhead projector. In addition, Hannah recorded the student's thinking on a prepared poster board so it would still be available after the student sat back down. Hannah also enacted *Orchestrating Mathematical Discussion* including more than three students in the discussion twice while discussing strategies. For each strategy shared, the class was asked to identify which of the student-name-strategies (reviewed at the beginning of the lesson) underlaid the approach (see Fig. 2). Several strategies were regularly referred to a) Donathan: focused on the size of shaded piece, b) Jessica: focused on the size of unshaded piece - "the smaller the amount that is not shaded, the bigger the fraction is," and c) Joyn: focused on the distance from one.

Aina shared her strategy for comparing $^1/_8$ to ½ (see Fig. 8), "What I did is … I draw a rectangle and I draw 1/8, I shaded 1, and I draw 1/2 and shaded 1."

Some discussion ensued on whether Aina used Jessica's (focus on unshaded pieces) or Donathan's (focus on shaded pieces) strategy (see Fig. 2). Hannah asked Jessica to re-explain her strategy. Jessica responded, "So what it is, that, the part that is not shaded is different and the other part that is not shaded is smaller so that [inaudible] that is the whole that is not shaded." To which Hannah inquired "So you're looking at this part that's not shaded." Jessica agreed. Hannah then asked Aina to explain her thinking again. Aina struggled, so Hannah called on another student who also agreed it was Jessica's strategy.

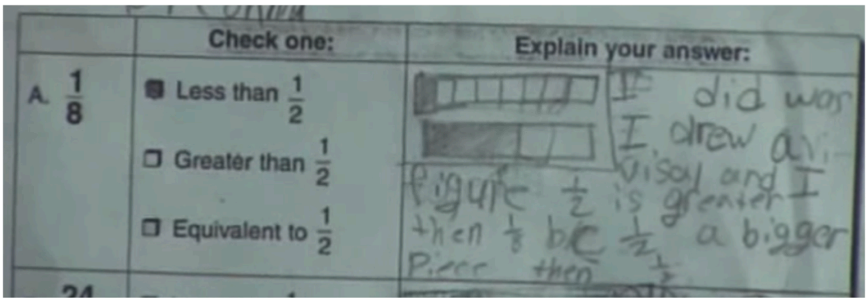

Fig. 8. Aina's solution to comparing $^1/_8$ to ½.

Hannah responded,

> When I looked at this [Aina's work], I thought that she wasn't using Jessica's strategy, I thought she was using … [inaudible class comments] … Yeah, Donathan's strategy. How many of you guys make a connection with Donathan's strategy there?

Hannah then asked Jana to explain. Jana stated "Donathan's line of strategy is when you draw both models out, and the piece that is shaded more is the bigger piece." Jessica had compared both fractions to 1, and thus looked at which unshaded part was larger while Donathan had looked at the shaded part and looked at which one was larger. Of note, there is no discussion of whether a strategy is correct/incorrect, but rather the focus is on student reasoning and making sense of how two students' strategies compare.

After the next student (Carlotta) put up her strategy (see Fig. 9). Hannah enacted *Structuring Mathematical Student Talk* and *Conferring to Understand Student Thinking and Reasoning.*

Hannah instructed the class.

> And think about the strategy she's using out of our list of strategies there [the one she reviewed at the beginning of the lesson]. So, whisper to someone in your group what strategy you think she's using out of all of these, who's strategy? Go.

Hannah walked through the room and conferred with individual students and groups, "So which one do you guys think?" Most stated that Carlotta used Joyn's strategy to which Hannah asked questions like, "You think Joyn? Why?" Joyn had compared to the benchmark 1. Hannah called the class back together and continued to enact *Orchestrating Mathematical Discussion.* She disagreed with the class, "I don't know, I disagree with you. I disagree with you," and called on a student who said it was Mr. K.'s strategy. Mr. K. had compared both fractions to the benchmark ½ determining that in $^{2}/_{3}$, 2 is half of four, and thus $^{2}/_{3}$ has to be larger than ½. Hannah had introduced Mr. K.'s strategy during the prior lesson since it had not come up as one of the strategies students created.

> A lot of you said Joyn's. I have to disagree, I don't think she is using Joyn's strategy, cause Joyn was not- never said that- Oh! It's not 1/2- Elijah? You had a different idea, what was it? Just say it.

Elijah confirmed he thought of Mr. K.'s strategy. Based on her conferring Hannah knew who had used the strategy she wanted shared and thus was able to allow a student to share instead of her sharing. Hannah then called on Carlotta "you're the queen of that strategy" when Carlotta stated, "I was thinking about the half of the whole." Hannah expanded, "Is the numerator half of the denominator? Or is it less, or more? So Carlotta's way of thinking is- one- is not- half of eight."

Next Hannah asked Polly to share her strategy comparing a different set of fractions: 24/42 and ½ (see Fig. 10). Polly put her work on the overhead projector.

Before discussing Polly's strategy, Hannah asked the students to work in groups but did not implement *Structuring Mathematical Student Talk* because she did not specifically tell students how to interact: "So, talk with your groups- whose strategy did she use? Go." Students were expected to work with each other and engage in making sense of Polly's strategy. During the group work Hannah implemented *Conferring To Understand Student Thinking and Reasoning* asking students why they thought it was a particular strategy. Both Mr. Kemp and Joyn's strategies were mentioned.

When Hannah called the class back together, she turned it over to Polly to explain her strategy. Polly justified her solution:

> So when I drew the number line and I drew one-half I looked at the 2 on the bottom of the one-half. And I knew that since there was two different pieces, I divided each of them by two, and I got 21 [half of 42 = 21]. And since the numerator was 24, I knew that- since the numerator was 24 I knew that - it [24/42] is greater than [½] - cause it [24 is] above the half of 42.

Hannah asked several students to revoice Polly's explanation –engaging them with her thinking. Then Hannah directed the students to talk with each other again to make sense of Polly's strategy enacting *Conferring To Understand Student Thinking and Reasoning.* She asked meaningful questions such as, "What is the meaning of 2?" and "Why would she divide by 2?" Hannah then implemented *Orchestrating Mathematical Discussion* in the whole class by first engaging the class in making sense of Polly's strategy, and then in reflecting on what made sense to them about the strategy.

With respect to student-centered mathematics instruction, in Lesson 2 (a) student thinking was made public by referring to student invented strategies from the prior day, by sharing student strategies on the overhead projector, and by creating a permanent record of students' strategies, (b) students engaged with each other's thinking both in small groups and in whole class and (c) student sense-making, conjecturing, and justifying drove the instruction. Hannah built the lesson around student strategies.

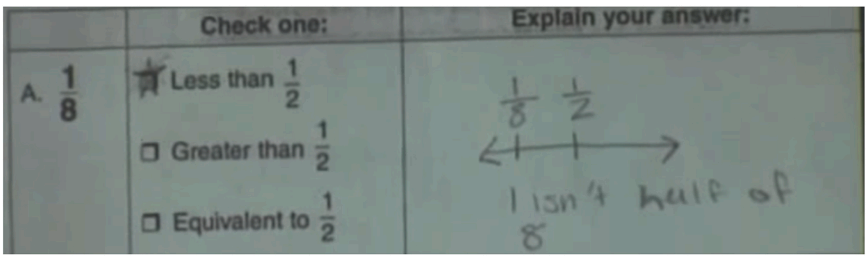

Fig. 9. Carlotta's solution to comparing $^{1}/_{8}$ to ½.

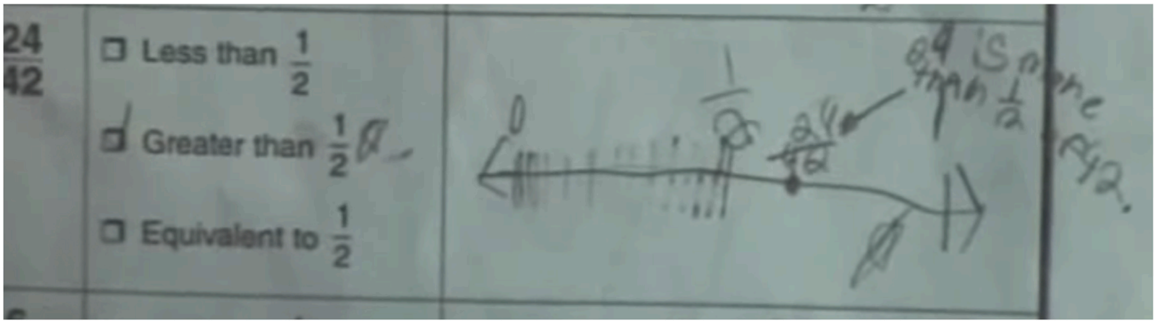

Fig. 10. Polly's solution to comparing 24/42 to ½.

5.1.4. Cognitive demand and lesson cohesion in Lesson 2

Most of the lesson focused on concepts and students making sense of each other's current and previously established comparison strategies. Students were also asked to compare and contrast between strategies. Additionally, students engaged in justifying their solutions. As such, this lesson was coded as high cognitive demand. In this lesson, the learning target was publicly stated to the class and referred to several times throughout the lesson. The whole lesson was focused on reviewing and working on fraction comparison strategies using benchmarks and number lines. Thus, the lesson cohesion was coded cohesive (Level 3).

5.2. Contrasting across Lesson 1 and Lesson 2

Both lessons focused on comparing fractions or decimals, were of similar length (60 min Lesson 1, 63 min Lesson 2) and similar structure (mostly whole class interactions, some small group interactions, and a little bit of individual think time). However, although students' work was shared in Lesson 1, none of the components of student-centered instruction were fully implemented and the focus was on applying procedures and correct answers. In Lesson 2, Hannah's instruction reflected all three components of student-centered mathematics instruction which resulted in students engaging in sense-making, conjecturing, and justifying. While the topics of the lessons were comparable, the tasks in Lesson 1 had a lower cognitive demand than the task in Lesson 2 and Lesson 1 had less clear cohesion than Lesson 2.

With respect to the implementation of teaching routines, Hannah enacted *Working With Selected and Sequenced Student Math Ideas* and *Working with Public Records of Students' Mathematical Thinking* together in both lessons. Hannah selected students who then shared their work on the overhead project. In addition, Hannah also implemented the teaching routine *Orchestrating Mathematical Discussion* in coordination with the two prior routines in both lessons. On the surface, the combination of routines seems quite similar in Lesson 1 and Lesson 2; however, there are some important differences to note with respect to student-centered implementation.

First, the length of the implementation of the teaching routines (shown in Table 5) shows that in Lesson 2 student thinking was driving more of the instruction as more time was spent in all teaching routines but especially in *Conferring To Understand Student Thinking and Reasoning* and *Orchestrating Mathematical Discussion*.

The timing of the teaching routines also differed. While *Structuring Mathematical Student Talk* was used for about the same amount of time in both lessons in Lesson 2, it was strategically used before and during whole class discussion to allow students to engage with each other in small groups before they did so in the larger class. Students actively collaborated several times later in the lesson, and this early structuring likely informed the productivity of small groups in general. Further, structuring talk early meant the ideas generated during small group work can directly inform the later discussion. In Lesson 1 the talk was structured at the end for students to have some practice in small group work and did not lead into a whole class discussion.

While both lessons used individual think time, the structure of the whole class and small groups usage was quite different. In the first lesson, small groups serve the role of wrap up and practice during the implementation of *Structuring Mathematical Student Talk*. In the second lesson, the small group and whole class time served complementary roles and were implemented during *Structuring*

Table 5
Time spent in each of the Five Teaching Routines in Lessons 1 and 2.

Teaching routine for accessing and working with student mathematical ideas with description	Lesson 1	Lesson 2
Teaching routine: CONFERRING TO UNDERSTAND STUDENTS' MATHEMATICAL THINKING	0	9 min 3 min 2 min 2 min
Teaching routine: STRUCTURE MATHEMATICALLY WORTHWHILE STUDENT TALK	11 min	11 min 3 min
Teaching routine: WORKING WITH PUBLIC RECORDS OF STUDENTS' MATHEMATICAL THINKING Teaching routine: WORKING WITH SELECTED AND SEQUENCED STUDENT MATH IDEAS	18 min	24 min
Teaching routine: Orchestrating Mathematical Discussions	7 min	7 min 6 min

Mathematical Student Talk as well as in between different strategies to allow the teacher to implement *Conferring To Understand Student Thinking and Reasoning.* This back-and-forth between small-group and whole-class time allowed the whole class time to be student centered. Thus, the goal for and the implementation of the modalities were markedly different.

Small group interactions allow students to engage with a task and talk to each other. In the second lesson, this talk was structured so that students engaged in a focus on sensemaking rather than arriving at a correct answer. It also allowed the teacher to listen in and confer with the students. This made it possible for the teacher to draw on student thinking during the whole class discussion. The back-and-forth between small group and whole class discussion allowed more student voices to be heard within the small groups and the teacher to engage with student thinking during small groups. Students could then engage with ideas generated during group work during the whole class discussion.

The most striking difference between Lesson 1 and Lesson 2 is that the teaching routine *Conferring To Understand Student Thinking and Reasoning* was not implemented at all in Lesson 1, but was implemented four times (for a total of 16 min) in Lesson 2. It is this routine that allows the teacher access to student mathematical thinking, and thus this routine is essential to center the class around student mathematical thinking. In Lesson 1, students participated in class and worked in groups, but Hannah was not in a position to center their thinking because she did not confer with students during their mathematical activity.

While the teaching routines were observed in both lesson there was a qualitative difference between their enactment, focusing on procedures and correct answers in Lesson 1 and focusing on student thinking and justifying in Lesson 2. We provide an overview of the qualitative differences of the routines Table 6.

In summary, Hannah enacted teaching routines for roughly the same amount of time in both Lesson 2 (49 % of the time) and Lesson 1 (45 % of the time) (see Fig. 3 and Table 5); however, the density of the teaching routines (number of teaching routines enacted simultaneously) was much higher in Lesson 2 than Lesson 1 and the nature of the implementation of the teaching routines much more student centered (see Table 4).

Hannah not only engaged more actively with the teaching routines in Lesson 2, but she also focused more on student thinking and student engagement with each other's thinking. The teaching routine *Conferring to Understand Student Thinking and Reasoning* was only present in Lesson 2 and appeared to play an essential role in how the later routines played out.

In Lesson 1 the focus seemed to be to 'lead' the students to a series of correct answers without a clearly explicated learning goal/target. Whereas in Lesson 2 the focus seemed to be to make sense of each other's thinking and ask questions with a clear learning goal/target of making sense of fraction comparison strategies. This difference in focus is reflected in both the observable goals in the selected routines and the isolated way that teaching routines unfolded in Lesson 1 versus the interconnected nature in Lesson 2. The lynchpin routine *Conferring To Understand Student Thinking and Reasoning* served to link the structured talk and small group work with the ideas selected, sequenced, and discussed in the whole class time.

At the beginning of this paper, we defined student-centered mathematics instruction as classrooms in which (a) student mathematical thinking is made public, (b) students engage with each other's thinking, and (c) student sense-making, conjecturing, and justifying drive the instruction. Returning to this definition, we note that Lesson 1 did not incorporate any of the three aspects, while Lesson 2 incorporated all three aspects (see Table 7).

In response to our research question, "How might differing enactments of teaching routines support or constrain student-centered instruction?," we can say that the inclusion of *Conferring To Understand Student Thinking and Reasoning* combined with the explicit focus on students' mathematical strategy and mathematical thinking rather than procedure and answer, led to a lesson which would satisfy all three criteria of student-centered mathematics instruction. We conjecture it was both the substance (over form) and integration of the routines that account for the differences. Simply meeting the requirement to enact a teaching routine (form) was not a sufficient condition for the routines to meet their implicit intention of serving to promote a student-centered mathematics instruction. Further, without the routines building on one another (structuring talk for students to generate ideas; conferring for the teacher to observe and

Table 6
Qualitative overview of the differences in teaching routine implementation.

Teaching routine	Lesson 1	Lesson 2
Nature of *Working With Selected and Sequenced Student Math Ideas*	Explicate and correct incorrect responses and procedures.	Compare and connect different student strategies
Nature of *Working with Public Records of Students' Mathematical Thinking*	Students are prompted to provide "hints" and correct a mistake. Students are prompted to provide "compliments."	Students are prompted to identify the type of strategy in a student record. Students are prompted to revoice strategies in the record. Students are prompted to ask questions. Students are prompted to make sense of components of the strategy.
Nature of *Structuring Mathematical Student Talk*	Solve a word problem and prepare a response. Structuring included a role assignment based on producing an answer. Purpose was to wrap-up at the end of class.	Compare strategies responses. Structuring included comparing strategies, arriving and consensus, and mutual learning. Purpose was to generate ideas to be used throughout the lessons.
Nature of *Conferring To Understand Student Thinking and Reasoning*	No conferring	Research students' thinking. Ask them to explain "why" and listen in on how they are thinking.
Nature of *Orchestrating Mathematical Discussion*	Correct a mistake	Come to consensus on what type of strategy is being used Make sense of the components of a particular strategy

Table 7
Student-centeredness of instruction in Lessons 1 and 2.

Student-centered Instruction	Lesson 1	Lesson 2
(a) student thinking is made public	The classroom was setup in a way that allowed students to collaborate and student work was displayed on the overhead projector. The work made public reflected mathematical procedures and was based on correct/incorrect.	The classroom was setup in a way that allowed students to collaborate, and student work was displayed on the overhead projector and on permanent records (posters). The teacher conferred with students to understand their thinking and select who would share. The work made public reflected students' mathematical strategies.
(b) Students engage with each other's thinking,	In whole class discussion, students engaged with each other for the purpose of correcting. In small groups, talk was structured to solve a problem and complete a poster but not engage with each other's thinking.	In whole class discussion, students engage with other for the purpose of comparing strategies making sense of student work. In small groups, students engaged in discussion of each other's ideas including analyzing ideas made public and comparing student strategies.
(c) Student sense-making, conjecturing, and justifying drive the instruction	Instruction focused on a pre-set correct mathematical procedure	Instruction focused on sense making, comparing reasoning, and justifying student-generated strategies

understand student ideas; and public records, selecting and sequencing, and orchestrating discussion of these ideas), the class is unlikely to be driven by student mathematical thinking.

Finally, we note the differences in cognitive demand of tasks and lesson cohesion. Our analysis shows that teaching routines can be implemented in ways that meet their descriptive qualities, but without high cognitive demand tasks or lesson cohesion. The routines do not operate independently from these dimensions. That is, higher cognitive demand tasks in the second lessons allowed for the teaching routines to be more likely to meet our student-centered classroom definition. Furthermore, lesson cohesion was reflected in the ways that Hannah intentionally selected student work, orchestrated discussions, and most importantly conferred to identify student reasoning that was aligned with lesson aims. We suggest that teaching routines can operate more fruitfully with high cognitive demand tasks and focus on overarching lesson goals.

5.3. Insights into the differences across Lesson 1 and Lesson 2

While the focus of this paper was to document the ways Hannah's implementation of teaching routines differed across the two lessons, we also briefly return to other data from the larger study to provide insight into the results with Hannah's own words throughout the PD. Promoting and working with students' mathematical thinking was central to the focus of the PD. Versions of the teaching routines were operationalized, planned for, and practiced with the teachers (see Appendix E).

Lesson 2 provided a stark difference to Lesson 1 in terms of teaching routines, especially the implementation of *Conferring to Understand Students' Mathematical Thinking & Reasoning. Conferring to Understand Students' Mathematical Thinking & Reasoning* was the main focus of 6 of the 15 Studio PD sessions. The PD focused on researching student mathematical thinking to build the teacher's understanding of their students' mathematical sense-making. In Lesson 1 and at the beginning of her engagement with the PD, Hannah's questioning did not model this focus. Hannah came into the PD as a teacher who cared deeply about her practice and whose classroom included a great deal of active participation on the part of her students. However, as she reflected at the end of the first year of the PD, it had not occurred to her to confer with students for the purpose of researching their thinking. She explained, "The questions are [asked] to give you [the teacher] ideas where they [the students] are at and not to teach them. That is something I never thought of." This was a powerful insight for Hannah in terms of shifting to noticing students' mathematical thinking.

In the beginning of the second year, we can see Hannah continuing this focus. She responded to the prompt, *What are key elements of your professional learning from today's collaborative inquiry?* with, "Plan on asking specific questions during conferring [with the students] – research first and then advance their thinking." In the third year of the PD, she similarly explained a learning tied to focusing on student thinking responding to a prompt about what is something she did not know prior to the PD with "Pushing students to show their thinking rather than just having a correct answer." In each of these reflections from across the PD, we can see Hannah first shifting to attending to thinking, then reflecting on the role of attending to thinking prior to advancing thinking, and finally situating mathematical thinking as an important goal for students themselves.

The PD emphasized the development of student-centered instruction where student mathematical thinking drives the lessons, student argumentation is the means for establishing mathematical facts, and students richly engage with each other's ideas. The teaching routines provided a mechanism to support teachers in their professional learning. In the case of Hannah, we had a teacher who was already leading a classroom where student voices were heard and valued – and her instruction included most of the teaching routines in some form. However, prior to the PD, the mathematics often remained at a low cognitive demand/procedural level, and student thinking played a perfunctory role. After the PD, and according to Hannah's own reflections, she deepened her practice through major shifts including more productively enacting the teaching routines, noticing student thinking via conferring, and pressing students to justify and engage with each other's ideas thus also increasing the cognitive demand of the tasks.

6. Conclusions

In this paper, we defined student-centered mathematics instruction as instruction that supports classrooms where (a) student mathematical thinking is made public, (b) students engage with each other's mathematical thinking, and (c) student mathematical sense-making, conjecturing, and justifying drive the instruction.

One of the main takeaways from this paper is that simply implementing teaching routines will not result in student-centered instruction. Lesson 1 exemplifies the implementation of teaching routines with a focus on procedures and correct answers which did not result in student-centered instruction. Lesson 2 exemplifies the implementation of teaching routines with a focus on student mathematical thinking (sense making, conjecturing, and justifying), resulting in student-centered instruction.

In addition to the quality of implementation, another difference between the lessons was the implementation of the teaching routine *Conferring To Understand Student Thinking and Reasoning.* This teaching routine plays a particularly important role for student-centered instruction as it serves to allow the teacher to learn about the students' thinking. That is, *Conferring To Understand Student Thinking and Reasoning* supports the other teaching routines.

7. Implications, limitations, and future research

In this study, we compared two lessons because Hannah provided a rather unique situation: a substantially shifted instructional approach that still contained many common elements in terms of form and routines. With this combination of similarities and differences, we were able to attend to ways that teaching routines may or may not be student-centered. However, such specificity involves a tradeoff in terms of knowledge generated. Additional research may attend to differences in lessons between teachers who had not integrated any teaching routines and novice integration. Further, while this analysis was situated in a PD, we do not attempt to make any causal claims for the differences in lesson. Future research can serve to unpack the mechanisms of instructional change.

We also note that only *Conferring To Understand Student Thinking and Reasoning* included explicit attention to its purpose in its name (researching student thinking). Explicating the purpose of each teaching routine and connecting it to one or several of the three student-centered criteria may help support richer enactment as seen in Lesson 2. For example, a routine like working with public records of student thinking could be revised to: Engaging students with public records of their peer's sense-making, strategies, and arguments.

Hannah's case provided a context to explore differences in practice at a deep level. Similar to Gruver and Hawthorne (2022), we evidenced that although these routines were developed from the literature on supporting student-centered mathematics instruction, it is possible to enact them in ways that remain procedurally focused (as in Hannah's first lesson). Hannah's case provides insight into the qualitative differences needed to enact them in such a way as to support a student-centered mathematics instruction.

Finally, we note that the focus of all teaching routines was on student mathematical thinking (as opposed to, for example, including student identity). Conferring played a critical role in noticing student thinking and supporting its integration into the classroom. However, centralizing students' mathematical thinking is just one element involved in noticing and supporting all students (e.g, Louie et al., 2021; van Es et al., 2022). Furthermore, attending only to mathematical thinking may reinforce that apprenticeship into the discipline of mathematics (Weber & Melhuish, 2022) as the only means of mathematical success to the exclusion of many. We suggest there is a further need to consider whose mathematics is valued, what meanings of mathematics underlie implementation of the TRs, and how TRs could adapt to other views of mathematics and students (Thanheiser, 2023) to allow for a border impact.

Declarations of interest

None.

Appendices A–E. Supporting information

Supplementary data associated with this article can be found in the online version at doi:10.1016/j.jmathb.2023.101032.

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