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What is the *Mathematics* in Mathematics Education?

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A B S T R A C T

In this paper I tackle the question What is the mathematics in mathematics education? By providing three different frames for the word mathematics.

1. Frame 1: Mathematics as an abstract body of knowledge/ideas, the organization of that into systems and structures, and a set of methods for reaching conclusions.

2. Frame 2: Mathematics as contextual, ever present, as a lens or language to make sense of the world.

3. Frame 3: Mathematics as a verb (not a noun), a human activity, part of one's identity.

After introducing the frames and examining their distinction and their overlap, I discuss their implication with respect to student-centered classroom, context, and culture.

1. Introduction

In this paper I tackle the question of *What is the mathematics in mathematics education?* This question is motivated by discussions I found myself having with other mathematics educators. Typically, in these discussions we struggled to clearly communicate how we defined mathematics to each other. Yet, how we define mathematics determines what we might include/exclude in various contexts. One such example is this special issue in the *Journal of Mathematical Behavior: Mathematics In Society: Exploring The Mathematics That Underpins Social Issue.* We co-editors of *this* special issue struggled sometimes to define what we meant by mathematics to each other. However, what we meant by mathematics was one of the key factors underlying the decisions for inclusion/exclusion in this special issue.

I contextualize the question in an ongoing debate in the field of mathematics education. I follow this with first positioning myself and then introducing the construct of frames to distinguish different meanings for the same word. Then, I introduce three frames for the word mathematics, examine their distinction and their overlap, illustrate the frames with an example, and discuss the implications of each frame on student-centered instruction. I close by examining communication across frames.

To answer the question *What is the mathematics in mathematics education?* I first examined how the people in the field of mathematics define of mathematics. What I found was that a concise and meaningful definition of mathematics is difficult, and the field of mathematics is in agreement that there is **no joint definition of what mathematics is** (Ascher & D'Ambrosio, 1994; Joseph, 2010;

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Stinson & Walshaw, 2017). Ascher, a mathematician, and ethno-mathematician stated that "mathematicians, as such, rarely define mathematics" (p.37) and if they do it is either very broad or very narrow. Ascher explained further,

> One of the hardest problems I encountered [as a mathematician], particularly because I began by working with a person who was not a mathematician, was that he felt I should or could state for him a definition of mathematics. *But there is no clear definition of what mathematics is* [emphasis added]. The problem then becomes how to say that something is mathematical or a good illustrative case. It meant resorting to a belief that, from my training as a mathematician, I would know it when I saw it even if I couldn't say what it was. (Ascher & D'Ambrosio, 1994)

The field of mathematics education engaged in a discussion about mathematics over a decade ago, and found itself in a "significant divide" (Wagner, 2017, p. 292) over the question: *Where's the Math (in Mathematics Education Research)?* (Heid, 2010). Wagner explains the split as generally between scholars who attend to sociocultural aspects and they who do not" (Wagner, 2017, p. 296). Kathy Heid, the then editor of JRME, one of the main journals for mathematics education research in the United States (Nivens & Otten, 2017; Williams & Leatham, 2017) stated.

> JRME publishes research in which mathematics is an essential component rather than being a backdrop for another area of inquiry. I encourage readers to continue to examine articles in JRME with the "Where's the math?" question in mind." (Heid, 2010, p. 103).

Around the same time Harel (2010) worried that 'current' papers in mathematics education were "adscititious" (p. 4) to mathematics and the "special nature of the learning and teaching of mathematics" (p. 4). On the other side of the split Martin, Gholson, and Leonard (2010) and Foote and Bartell (2011) raised questions such as: "Whose mathematics" were Heid & Harel referring to? And, who gets to decide what is and is not mathematics in mathematics education research? Martin et al. (2010) explicated that "issues of identity, language, power, racialization, and socialization" (p. 15) were often left out of mathematics education research and wondered whether contributing to "equitable mathematical experiences and outcomes" should be a criterion for mathematics education research. This point, in particular, stands out to me, the author of this paper, who recently published a paper in JRME which showed inequitable student achievement results for some students only *after* data was disaggregated (Melhuish et al., 2022). Without purposeful disaggregation the research would have come to a different conclusion and may have been published as such, which would have kept the inequity invisible. This raises the question, how often have we not done due diligence in disaggregating data and examining intersectionality in our classrooms? Other recent work has also raised the issue of mathematics teacher educators (MTEs) looking beyond aggregate results with well-known professional development programs such as CGI (Rodríguez, Jessup, Myers, Louie, & Chao, 2022) and Active Learning (Reinholz et al., 2022) among others.

The field of mathematics education continues the "conversation about equity-focused research in mathematics education" (Matthews, P.; Herbst, P.; Crespo, S.; & Lichtenstein, E., 2022, p. 342) in the most recent editorial of JRME where the editors ask, "Does the equity limb [of the mathematics education research tree], in fact, grow at a distinct angle?" (p. 346) countering a commentary written by Louie and Zhan (2022) in the same issue where they "emphasize that the equity limb does not grow at a distinct angle from other limbs of the mathematics education research tree; rather, these limbs can, should, and sometimes do intertwine and mutually support one another" (p. 365).

Building on the responses to the question *Where is the math in mathematics education research?* described above, I shift the question from *Where* to *What* and ask *What is the mathematics in mathematics education?* I want to acknowledge that people might hold different answers to this question either at the same time or over their careers. However, explicating a current notion of (a) how we define mathematics and (b) what the implications of those definitions are will help us to communicate with each other, even, or perhaps especially as we define it differently.

2. Positioning myself

I am a mathematics educator who had always been successful in mathematics even though I struggled with almost every other subject in school. Successful to me meant I understood why things worked (at least why procedures worked and when to implement them). I assumed that this procedural fluency or conceptual understanding of mathematics is what everyone was aiming for but realized later that was not the case.

I grew up in Germany as a Jewish daughter to a Hungarian immigrant single mother. I am an immigrant to the United States and English is my third language. I moved to the United States during my university education intending to become an English and Math teacher for ages 11–18 in Germany. I have a master's degree in mathematics (with a focus on abstract algebra to be precise) and a PhD in mathematics education (with a dissertation focused on place value).

I have a job as a mathematics educator in a mathematics department. In my situation, this means I teach mathematics content courses to preservice elementary and middle school teachers. This also means that most of my colleagues are mathematicians. Currently I am the only tenured woman faculty in my department of over 20 tenured faculty. At the beginning of my career as a mathematics educator I focused on preservice teachers' conceptions of multidigit whole numbers (Thanheiser, 2009, 2010, 2012, 2018). I would characterize this as a focus on abstract mathematical knowledge. This research focus was followed by a focus on learning and motivation (Thanheiser, 2014, 2018). I would characterize this as a focus on how abstract mathematical knowledge *develops*. Currently my focus is moving towards critical mathematics education (Skovsmose, 1985) such as disaggregating data to examine whether outcomes are similar for various groups (Melhuish et al., 2022) and critically examining what it means to learn and to teach mathematics (Thanheiser & Koestler, 2021; Thanheiser & Sugimoto, 2020). I now focus on connecting mathematics to the real world (Conway et al., 2022; Han et al., 2022; Robinson et al., 2022) and antibias mathematics teaching (Yeh et al., 2022). I would characterize this as a focus on contextual mathematics and a focus on student identity, community, etc. In this paper I draw on this shift from an abstract view of mathematics to a more contextual view that centers on student identity. I focus on how I would have answered the question *What is the math in mathematics education?* at various points in my career, and on the implications of each definition teaching and learning. I use framing as a way to differentiate responses.

3. Framing

In this paper I draw on *framing* (Lakoff, 2006; Louie, 2017) and explore the frames of mathematics and the implications of different frames of mathematics. Lakoff explained "frames are mental structures that shape the way we see the world" (Lakoff, 2014, p. ix). He explained that every word evokes at least one frame, but some words can evoke multiple frames. "Most frames are unconscious and have just developed naturally and haphazardly and have come into the public's mind through common use." (Lakoff, 2006, p. 2). Frames, like other ideas people hold in their heads, cannot be directly examined but are known by their consequences. In the case of *mathematics*, we cannot directly examine how people define mathematics but need to understand what they do and do not see as part of mathematics. Reframing requires accessing what we already believe and making it conscious (Lakoff, 2014; Louie, 2017). The questions I pondered for this article are:

- What are some of the different frames for the word mathematics in the field of mathematics education?
- What are some of the implications of those frames?

I note that the selection of the frames examined in this paper chronicles my own development as a mathematics educator from framing mathematics as *an abstract body of knowledge/ideas, the organization of that body of knowledge into systems and structures, and a set of methods for reaching conclusions*. To mathematics as *contextual, ever present, as a lens or language to make sense of the world.* And finally to mathematics *as a verb (not a noun), a human activity, part of one's identity.* It also chronicles the struggles I experienced when discussing mathematics with my mathematics and mathematics education colleagues.

4. Frame 1

4.1. Math as an abstract body of knowledge/ideas, the organization of that into systems and structures, and a set of methods for reaching conclusions

Mathematics is often described as the "science of patterns" (Steen, 1988), an "intellectual activity which calls for both intuition and imagination in deriving "proofs" and reaching conclusions" (Joseph, 2010, p. 3). Authentic mathematics is seen as an apprenticeship into the discipline of mathematics (Weber & Melhuish, 2022). Justification and generalization are often seen as the ultimate goal to achieve in the K-12 mathematics classroom (National Council of Teachers of Mathematics, 2000; National Governors Association, 2010). These goals are achieved by exploring different topics including "a body of knowledge relating to number and space, and … [prescribing] a set of methods for reaching conclusions about the physical world" (p. 3) or "those ideas have to do with number, logic, and spatial configuration and, very important, the combination or organization of those into systems and structures." (Ascher & D'Ambrosio, 1994).

Mathematics as a list of topics and practices that culminate in proofs or justification and generalization is taught in every year through the K-12 curriculum and is also an academic field of study. The National Research Council (NRC) describes mathematics as "one of humanity's great achievements," (Kilpatrick, Swafford, Findell, & research, 2001, p. 1), "so much a part of modern life that anyone who wishes to be a fully participating member of society must know basic mathematics," and "efforts made over thousands of years by every civilization to comprehend nature and bring order to human affairs." (Kilpatrick et al., 2001, p. 15). These descriptions tightly connect mathematics to context, culture, and society. However, when these authors defined "mathematical proficiency" they focused mostly on the abstract (vs contextual or culturally based). They discussed five interweaving strands that make up the complex whole of mathematical proficiency:

> *conceptual understanding*—comprehension of mathematical concepts, operations, and relations, *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately, *strategic competence*—ability to formulate, represent, and solve mathematical problems, *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification, and *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy. (Kilpatrick et al., 2001, p. 16).

Much of school mathematics is seen as abstract rules and procedures (Adiredja, 2019) as laid out in national standards, for example the Common Core State Standards in the United States (National Governors Association, 2010). The standards are full of procedures on how to calculate things and how to understand those calculations conceptually. Whether with conceptual understanding (Hiebert & Lefevre, 1986), a thorough understand of why the procedures work, or procedural fluency (Star, 2005, 2007), knowing when to use which rule, the standards focus on abstract rules and then some applications.

> School mathematics is still typically defined as a discrete set of topics taught in a linear progression that ultimately leads to calculus. This is the only version of mathematics that most people know, and thus, calculus is seen as the pinnacle of mathematical understanding." (Yeh, Tan, & Reinholz, 2021, p. 5)

Students (in this case prospective elementary teachers) often describe this type of mathematics as: boring, terrifying, daunting, about finding the right answer, a totally different language that I couldn't understand, I didn't think I really needed it in my everyday life, and the most politically neutral subject taught in school (Thanheiser & Koestler, 2021). The perceived goal of 'school mathematics' is often to open access to the next topic (in the math class), the next mathematics class, the next level of schooling, etc. Good at mathematics with respect to this definition of mathematics means to be fast and accurate at calculating (Leyva, 2022), such as having the times tables memorized. A typical high school math sequence in the US consists of Algebra 1, Geometry, Algebra 2, Precalculus or if Algebra 1 was taken in middle school the sequence starts with Geometry and ends in Calculus.

Moving beyond K-12 schooling and looking at the academy I turned to the American Mathematics Society (AMS) to look for a definition. The AMS does **not** define mathematics on their website https://www.ams.org/about-us/about. Rather they focus on their function to further the interests of mathematical research and scholarship. They state their goal is to "deliver a true sense of encouragement and connectedness, creating a community with mathematics at its center. We serve all who are interested in mathematics." However, what *that* mathematics is, is left up to the reader. Next, I turned to examine university requirements for graduation with a bachelor's degree. University requirements for a Bachelor in Mathematics vary across the United States but have a common elements which includes some or all of the following: a calculus sequence, differential equations, an introduction to mathematical analysis, group theory, and algebra. The Mathematical Association of America's (MAA) Committee on the Undergraduate Program in Mathematics also focusses on some of the topics listed above (https://www.maa.org/member-communities/committee-on-the-undergraduate-program-in-mathematics). For a specific example let's consider Portland State University's website, which states "The program is designed to provide a foundation for more advanced work and/or a basis for employment in government, industry, or secondary education" https://www.pdx.edu/math/bachelor-artsbachelor-science-mathematics and as such the goal of such a degree is access to further study or to employment rather than a goal in itself.

Many academic mathematics departments are split into abstract/pure mathematics and applied mathematics. The difference between these has been described as.

> Simply put, mathematics is the abstract study of quantity, structure, space, change, and other properties. **It has no strict universal definition**. Mathematics originated as a means of calculating, though it has developed into a field of study with a wide variety of interests. … Pure mathematics is the study of **entirely abstract mathematical concepts**. Pure mathematics has sub fields concerning the quantity, structure, space, and change. … Applied mathematics focus on the **mathematical methods used in real life applications** in engineering, sciences, economics, finance, and many more subjects. https://www.differencebetween.com/difference-between-mathematics-and-vs-applied-mathematics/

If we follow the premise that there are two types of mathematics, abstract and applied, then we accept that context can be separated from mathematics. This then begs the question of whether it makes sense to focus on abstract systems in K-12 education with the goal that students can learn a concept (without context) and then learn to apply it in various contexts? This kind of transfer of knowledge from school to real world, from abstract to context, or from one context to another has been shown to be difficult for students (Boaler, 1993; Carraher, Carraher, & Schliemann, 1985; Herbert & Pierce, 2011) and adults (Hoyles, Noss, & Pozzi, 2001; Jurdak & Shahin, 2001; Lave, 1988).

5. Frame 2

5.1. Math as contextual, ever present, as a lens or language to make sense of the world

In his book *Mathematical Enculturation: A Cultural Perspective on Mathematics Education*, Bishop identifies six general forms that mathematics can take that are all steeped in context:

- Counting, "the association of numbers with objects" (Bishop, 1988, p. 23). Answering the question "How many?"
- Locating, "code and symbolize the spatial environment" (Bishop, 1988, p. 28) Answering the question "Where?"
- Measuring, "concerned with comparing, with ordering, and with quantifying qualities which are of value and importance" (Bishop, 1988, p. 34) Answering the question "How much?"
- Designing, "concern the 'manufactured' objects, artifacts and technology which all cultures create for their home life" (Bishop, 1988, p. 38) Answering the question "What?"
- Playing, "games" (Bishop, 1988, p. 42) Answering the question "How to?"
- Explaining, "activity which lifts human cognition above the level of that associated with merely experiencing the environment." (Bishop, 1988, p. 48) Answering the question "Why?"

While this is also a list of topics culminating in justification this list is deeply steeped in context and in where mathematics came from. Historically abstract mathematics grew out of context via abstraction (Ascher & D'Ambrosio, 1994). Thus, mathematics resides in the contexts before the abstraction. The field of ethnomathematics examines the connection between mathematics and culture. One of the founders of this field, Marcia Ascher stated.

> Most mathematicians believe that what gives mathematics its power is the manipulation of symbols standing for anything and having no context. In a problem, x is x and nothing more. However, I believe mathematics could be even more powerful by retaining some recognition of what the symbols stand for and gearing the approaches used to that. If, for example, you are dealing with X's referring to numbers of human beings, you should only be seeking integer solutions and selecting solution methods accordingly … I also think that this is one of the causes of the dislike of mathematics among young people. There is a feeling, often articulated by those alienated by mathematics, that it is emotionless and lacks feeling. Even students who like mathematics seem to associate it with a certain inhumanity" (Ascher & D'Ambrosio, 1994)

Some define mathematics as a language which begs the question whether a language can be learned by exclusively focusing on the abstract (grammar rules and vocabulary) or whether languages need to be learned by a combination of focusing on grammar and vocabulary (abstract) as well as immersion into the language and culture (context). A balanced approach of learning bits of words, and then using them in context, will allow success (Ani, 2021).

In his book *Dear Citizen Math*, Ani (2021) considers mathematics as a tool to think about questions we confront as a society, a tool for exploring questions about real life, a lens for viewing reality. He states "math isn't some arbitrary collection of skills to memorize. It's a prism for looking at the world, for analyzing how it works and reimagining how it could" (p. 11). He continues "mathematics is a tool that we humans have invented to explore the world around us and to think more creatively about it. … Math is a way of thinking" (p. 14). Ani states we need to consider both when talking about math, math as an object we learn (abstract) and math as a lens we perceive our surroundings with (contextual or as a language).

Critical mathematics educators (Skovsmose, 1994) often define mathematics as contextual and inseparable from the context. Building on Freire (1970, 1994) distinction between reading the word and reading the world, Gutstein (2006) defined reading the world with mathematics as.

> to use mathematics to understand relations of power, resource inequities, and disparate opportunities between different social groups and to understand explicit discrimination based on race, class, gender, language, and other differences. Further it means to dissect and deconstruct media and other forms of representation. It means to use mathematics to examine these various phenomena both in one's immediate life and in the broader social world and to identify relationships and make connections between them. (Gutstein, 2012, p. 26).

Both Freire and Gutstein see reading the word and the world as inextricably linked.

Ethnomathematicians look at cultures and try to understand how various cultures use mathematics. The use of mathematics tells us "Simultaneously of the people's ideas regarding number [or other math concept under consideration], something of their language, and something about how they categorize the world around them" (Ascher, 2017, p. 186). As such culture, context, and mathematics are intertwined. Increasing ones understanding of a culture including mathematics different from one's own increases ones understanding of one's own culture including mathematics "by shedding more light on assumptions we make which could, in fact, be otherwise" (Ascher, 2017, p. 186). For example, the Incas used data, number, space, and logic to encode their quipus (a system of knotted strings).

- In this system *each string* could represent a different referent (i.e. population of different villages). In our system each string would be equivalent to a different number.
- Within a string *the placement* of a knot indicated the size of the referent unit, much like in our base-ten place value system the location of a digit refers to the size of its referent unit. In 234, for example, *the location* of the 3 shows us that the referent unit is tens.
- *The number of knots* indicated the number of referent units, much like in our base ten place value system the *value of the digits* refers to the *number of referent units*. In 234 for example the value of the digit 3 shows us we use 3 referents of size ten, thus the value is 3×10 or 30.

As such different information could be encoded into quipus and they could be sent via runners from place to place. While the quipus represented numbers, without knowing what each string represented the numbers were meaningless. Quipus can be used to elucidate our base ten number system by explicating the referents (see also, Thanheiser & Melhuish, 2019). This frame of viewing mathematics allows us to connect with context but does not necessarily yet factor in the students who are engaging with the mathematics.

6. Frame 3

6.1. *Mathematics as a verb (not a noun), a human activity, part of one's identity*

Rochelle Gutierrez asked us to question the "very nature of mathematics" (Gutiérrez, 2017, p. 2). She described mathematics as something that is not "removed from humans, out there to be discovered" but as "a human practice, it's a practice of living beings, and so first of all, we need to redefine it from being a noun to being a verb." (Gutierrez, 2018). Gutierrez builds on "indigenous perspective, [where] there's not a separation between humans and nature … Indigenous knowings also never separate ethics from moving through the world."

This shift from seeing mathematics as a noun (a stable thing that one needs to learn about and understand, and as something that exists without human activity) to seeing mathematics as a verb (a thing that is created and enacted by humans), allows teachers to work with students as the authors of their own mathematical understanding. Berry (2022) defines “mathematical identity is how one sees themselves as a doer of mathematics.” Thus, mathematics shifts from something outside oneself to something that is a human activity, part of oneself. Berry continues “I can support the construction of a student’s identity through the kind of work I ask them [to do]” As such mathematics becomes inextricably linked to one’s identity and how one sees oneself and the world when making sense of it. Aguirre, Mayfield-Ingram, and Martin (2013) “argue that students need to learn mathematics in light of who they are and the diverse gifts that they bring to their experiences every day.” (pp. 9–10).

For many students at the K-12 and the undergraduate level, even math majors, mathematics is a noun, often involving words like axioms, logic, and numbers, as a steppingstone to the next topics or other STEM fields or college. However, considering history and culture and how mathematics is and has been performed differently across cultures (Bishop, 1988) shows that it is something people *do*, rather than something people *discover*. This links mathematics to the way people see themselves with respect to mathematical sense making. Returning to the strands of mathematical proficiency one of the strands, *productive disposition*, focused on sense making. However, the National Research Council (NRC) confined *productive disposition* to their narrower definition of mathematics described above rather than broadly as a sense making stance with respect to the world. I would argue that a sense making stance with respect to the world is part of one’s mathematics identity.

Identity, like math, has been considered as something we do rather than something that is or something we are (Butler, 2005). If how we look at the world is (and must be) informed by mathematics, then mathematics is part of our identity. Mathematical identity is part of who we *a re* . Martin (2006) defines mathematical identity as:

> the dispositions and deeply held beliefs that individuals develop, within their overall self-concept, about their ability to participate and perform effectively in mathematical contexts and to use mathematics to change the conditions of their lives. A mathematics identity encompasses a person’s self-understanding of himself or herself in the context of doing mathematics (i.e., usually a choice between a competent performer who is able to do mathematics or an incompetent performer unable to do mathematics, but often flowing back and forth). It also encompasses how others “construct” us in relation to mathematics. As a result, a mathematics identity is expressed in its narrative form as a negotiated self, the results of our own assertions and the sometimes-contested external ascriptions of others. The development of particular kinds of mathematics identities reflects how mathematics socialization experiences are interpreted and internalized to shape people’s beliefs about mathematics and themselves as doers of mathematics. (pp. 206–207).

If we take on the assumption that mathematics is part of who we are and how we look at the world, then how we are perceived and whether we are given access to sense making is inextricably linked to mathematics teaching and learning. The question “*Where is the math*?” as such is deeply frustrating within this frame of mathematics because “we understood that the teaching and learning of mathematics has always depended on issues that, although not directly focused on content, deeply affected how the content was taught and who received access to it.” (Frank, 2020, pp. 428–429). Frank continues to explore that one of the foci of mathematics education therefore must be “dismantling structures that impede student success and participation rather than setting achievement, or lack thereof, at the feet of students” (Frank, 2020, p. 429). Gutierrez agrees with this notion and in the sense of reframing being difficult if we don’t change words (Lakoff, 2014) introduces the word *Mathematx* which she defines it as.

> a vision of practicing mathematics that might move past previous notions of Western versus other mathematics, past an idea of mathematics as either oppressing or liberating, beyond a mathematics that is either discovered or invented, towards an idea that allows us to deal with today’s complexity and uncertainties. (Gutiérrez, 2017, p. 12)

Gutierrez highlights three aspects of Mathematx:

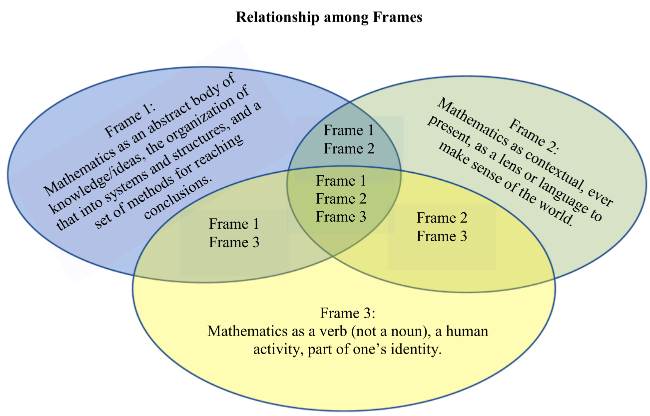

Fig. 1. Relationship among Frames of Mathematics.

1. First Mathematx is a way of seeking, acknowledging, and creating patterns for the purpose of solving problems (e.g., survival) and experiencing joy.
2. Second, whereas mathematics tends to be thought of as a noun (e.g., a body of knowledge, a science of patterns, a universal language), mathematx is performance and, therefore, a verb. … mathematx emphasizes the guiding principles and the process as opposed to the product.
3. Drawing upon the concept of reciprocity, mathematx is an intervention-in-reality (action) as opposed to a representation-in-reality (explanation) (de Sousa Santos, 2007). … Living mathematx means both that we live a version of mathematx as well as we are a living version of mathematx." (Gutiérrez, 2017, p. 14)

Mathematics within this frame then is not something that lives outside the person who engages with it but is something that is produced by the person who engages with it. As such a sense-making stance to the world needs to be part of who students are for them to engage with this type of mathematics and as such mathematics is inextricably linked to identity. In summary it is impossible to look at mathematics without including the students/people who are engaging with the mathematics examining that engagement.

Each of the frames described above can be viewed on its own or in conjunction with either or both other frames. The center of the Venn Diagram in Fig. 1 represents a view of mathematics that includes all three frames described in this paper. Considering the three frames, we could view Frame 1 as "I hear and see math," Frame 2 as "I hear and see the world with math" and Frame 3 as "I do math, I can use math to change the world, I am math" joining all three in the center of the Venn Diagram. It is important to note that frames can overlap. Frame 2 and Frame 3 can be developed in conjunction with Frame 1.

7. Example

To examine the various frames let's start with an example. Read the sentence "Today, the United States remains nowhere close to racial equality. African Americans make up 40 % of the incarcerated population." (Reynolds & Kendi, 2020, p. xii). Would you consider reading this sentence as doing mathematics? How about engaging with the sentence? What type of engagement with the sentence (if any) do you see as doing mathematics? What about either representation in Fig. 2 based on *The Sentencing Project* (Nellis (2016)? Would you consider that mathematics?

I came across this sentence above when reading *Stamped: Racism, antiracism, and you: A remix of the National Book Award-winning Stamped from the beginning* with my daughter. In Table 1 I present several stages of engagement with the sentence and discuss how I view them as doing mathematics.

Below I elaborate how mathematics is viewed in each of three frames and relate those back to the example.

7.1. Let's begin with Frame 1

Mathematics as an abstract body of knowledge/ideas, the organization of that into systems and structures, and a set of methods for reaching conclusions. Applying this frame, one would consider Step 5 to be mathematics. Step 5 can be linked to at least one specific Common Core Standards and as such specifically addresses a topic in the sequence of topics to be taught in K-12 mathematics.

7.2. Considering Frame 2

Mathematics as contextual, ever present, as a lens or language to make sense of the world. Applying this frame, one would consider all 6 steps would be mathematics. In their totality all 6 steps contribute to exploring questions about real life and viewing math as a lens or language to make sense of the world. In this frame mathematics is seen as "a tool that allows us to think more clearly about the questions that we confront as a society" (Ani, 2021, p. 7). In this example this includes understanding the over-representation of minoritized communities in the US prison system. Good at mathematics with respect to this frame means interrogating the information and understanding it within the sociohistorical and political contexts of slavery in the US, white supremacy, and antiblackness (Bell, 2018; D'ignazio & Klein, 2020; DiAngelo, 2018; Kendall, 2021; Kendi, 2016; Love, 2019; Rothstein, 2017; Saad, 2020; Steele, 2011; Tatum, 2017; Wilkerson, 2020). The assumptions underlying this example are that if things were fair the prison population should

Race of US Inmates	Percent
Black	37.8
Native American	2
Asian	1.5
White	58.8

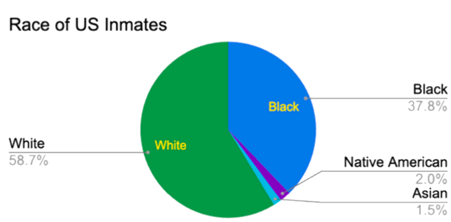

Fig. 2. Race of US inmates based on *The Sentencing Project* (Nellis, 2016).

Table 1
Engaging with "Today, the United States remains nowhere close to racial equality. African Americans make up 40 % of the incarcerated population." (Reynolds & Kendi, 2020, p. xii) in various ways.

Example/Engagement with the sentence.	"Today, the United States remains nowhere close to racial equality. African Americans make up 40 % of the incarcerated population." (Reynolds & Kendi, 2020, p. xii).		
Step 1: Asking Questions	What does it really mean that African Americans make up 40 % of the incarcerated population. Is that fair? If not, how unfair is this? What portion of the population is African American? What proportion of the prison population do other populations (for example white prisoners) make up?		
Step 2: Finding Data to answer the questions	Census data can provide us numbers for the US population at large. As of 2020, 60 % of the population in the US identifies as white (and not Hispanic) and about 13 % of the population identifies as Black[a] (and not Hispanic). https://www.census.gov/newsroom/press-releases/2021/population-changes-nations-diversity.html and https://usafacts.org/state-of-the-union/population/?utm_source=bing&utm_medium=cpc&utm_campaign=ND-StatsData&msclkid=76c07f3d03641269fa7e1ef6b3b2b01b The Sentencing Project (Nellis, 2016) provides data on incarceration. Per 100,000 residents in the United States, the rate of Black prisoners is 1240 and the rate of white prisoners is 261.		
Step 3: Articulating Assumptions	**Assumption:** If things were fair, the population at large would be proportional to the incarcerated population.		
Step 4: Identifying given information (based on data in Step 2) and organization of data.	**Given:** Population that Identifies as Black (and not Hispanic) is 13 % of US population **Given** Population that identifies as white (and not Hispanic) is about 60 % **Given** Incarcerated population rate of Black (and not Hispanic) prisoners is 1240 per 100,000 **Given** Incarcerated population rate of white (and not Hispanic) prisoners is 261 per 100,000		
	Population (percent)	Incarcerated Population (per 100,000)	Rate adjusted to reflect proportionality to population (Assumption Step 3)
	13 % of US population Identifies as Black (and not Hispanic) is	1240 per 100,000 Rate of Black (and not Hispanic) prisoners	If 261 represents 60 % of the prison population then **what number would represent 13 %**
	60 % identifies as white (and not Hispanic)	261 per 100,000 Rate of white (and not Hispanic) prisoners	261 reflects 60 % of the prison population
Step 5: Calculating	**Question:** If 261 prisoners make up 60 %, then what number makes up 13 %? **Calculation:** 261 is 60 % $4.35 \times 13 = 56.55$, thus 56.55 is 13 % The calculation relates to the *Common Core Standards (7. RP.A.3)*: Use proportional relationships to solve multistep ratio and percent problems. Examples: simple interest, tax, markups and markdowns, gratuities and commissions, fees, percent increase and decrease, percent error.		
Step 6: Interpreting and interrogating	With the assumption and the calculation, we understand that 57 Black (and not Hispanic) people should be incarcerated for each 261 white (and not Hispanic) incarcerated people. **However, the real number is 1240 which is 1240/57 = 21.75 times the rate that it should be (if it were proportional).** In general, the relationship between Black (and not Hispanic) incarcerated people and white (and not Hispanic) incarcerated people should be about 1 Black (and not Hispanic) to 5 white (and not Hispanic) because 13/60 is about 1–5. The fact that Black people are incarcerated at such a higher rate is based on a racist, white supremacist system that over polices and underprivileges this community.		

Note that I am using the term African American above to stick with the term Reynolds & Kendi used and I use the term Black below as that is what is currently used on the U.S. Census.

represent the population at large. So that if a group represents 50 % of the population, then we would expect this group to represent 50 % of the prison population. With African Americans currently making up about 13 % of the US population, the underlying expectation would be that African Americans should make up about 13 % of Americans sitting in prisons (Reynolds & Kendi, 2020). However, that is not the case. What does this mean in particular? How much more likely is it for an African American person to be in prison as compared to a white person? How is comparing a group to the whole similar and or different from comparing a group to another group? While this includes all 6 steps of our example it does not necessarily consider the students' culture, interest, or identity. One could imagine the exploration of this sentence as an assignment rather than as something the students are genuinely interested in and want to engage with.

7.3. Considering Frame 3

Mathematics as a verb (not a noun), a human activity, part of one's identity. Applying this frame, one would consider the students as an essential component of engaging with the sentence. How and why a student would engage with this sentence depends on the student's history, criticality, and identity. My daughter and myself for example were wondering when reading this sentence about what this sentence means and thus set about to explore that. In general, when reading or hearing numerical sentences we often wonder about them and engage with them. This engagement with the world through sense making is part of my identity as a mathematics educator and I will often bring things I notice (i.e. in hte news, in a store, etc.) into the classroom. However, a context for a math classroom cannot be chosen without considering the students and their communities and their relation to the context. With respect to our example above that means that it depends on the students, the community, etc. whether an examination of this particular context makes sense for students in the classroom, whether it might need to be modified, or whether a different context might serve them better. A sensemaking stance however can drive a classroom when students are invited to bring in their own selves and topics, they are interested in exploring. Next, I consider the implications of each of the frames on student-centered instruction, context, and culture.

8. Implications of each frame on the student-centered classrooms, context, and culture

Each Frame allows for student-centered instruction. However, the focus of what student-centered means is different within each frame. Frame 1 will have the standard focus on a) student mathematical thinking being made public, b) students engaging with each other's mathematical thinking, and c) student sense-making, conjecturing, and justifying driving instruction (Thanheiser & Melhuish, 2023). Frame 2 will focus on students' *general* (mathematical, political, social, etc.) sense making, conjecturing, and justifying. Finally Frame 3 will focus on the students' identities (see Table 2 for the relationship between student-centered instruction and the three frames).

Mathematics as we encounter it in schools mostly falls into Frame 1. It is mostly formal and abstract as described by the NRC's *mathematical proficiency*. Students are often not seeing the relevance of school mathematics to their own life, regularly asking 'when will we ever need this'? This abstract view of mathematics, or mathematics as a universal language and politically neutral is rooted in Eurocentric and colonial ways of knowing (Patel, 2015) which allows white male students to succeed at higher rates than others (Melhuish et al., 2022; Reinholz et al., 2022; Rodriguez et al., 2022). Considering this, wouldn't it make more sense to broaden mathematics to include teaching mathematics in context (Frame 2) and as related to the students' identities (Frame 3)?

Gutstein (2006) argues that it is through the connections among critical, community, and classical knowledge, that teaching for social justice expands students' mathematical identities and adjusts their views of mathematics, from "seeing it as a series of disconnected, rote rules to be memorized and regurgitated [Frame 1], to a powerful and relevant tool for understanding complicated, real-world phenomena [Frame 2 and possibly Frame 3]" (p. 30).

In critical mathematics education there is a call for rehumanizing mathematics (Gutierrez, 2018). The implication is that abstracting mathematics [Frame 1] dehumanizes it. And now there is a need to rehumanize it (sociopolitical turn) to get back to what mathematics *could* be. There is a call to re-envision (Yeh et al., 2021) mathematics to connect to context and culture [Frame 2 and Frame 3].

9. Returning to the example

In our rapidly changing world, it is important that we understand the context questions are asked in (D'ignazio & Klein, 2020) to be able to understand the information presented to us. Our example comes from a sentence in Reynolds and Kendi (2020) book *Stamped: Racism, antiracism, and you: A remix of the National Book Award-winning Stamped from the beginning.* Before that sentence the authors state that they would expect the prison population to be representative of the larger population (Step 3), however, that is not the case. They further explain that while African Americans make up about 13 % of the population, they make up about 40 % of the incarcerated population (Step 2). Let's assume a person reads this sentence, I would state that reading this sentence constitutes part of doing mathematics (Frame 2 and Frame 3) which many people who hold Frame 1 would disagree with. However, I believe that the interrogation of the sentence and its meaning is where most of the mathematical work lies (Frame 2 and Frame 3). Thus, part of doing mathematics is to engage with the sentence (Frame 3) and think through what this means (Frame 2).

For example, when reading this sentence, one might thing that this seems unfair because the prison population has about 3 times as many African Americans than the general population (40 % vs 13 %). Thus, one engages in comparing percentages. However, the acknowledgement of this unfairness is only the first step of engaging with the sentence and doing mathematics. Further interrogation of the sentence is needed. The examination and recognition of the fact that this is not the whole story is also part of mathematics

Table 2
Aspects of student-centered instruction, context and culture in each Frame.

Aspect of student-centered instruction	Frame 1 includes	Frame 2 includes	Frame 3 includes
(a) student mathematical thinking is made public (b) students engage with each other's mathematical thinking (c) student mathematical sense-making, conjecturing, and justifying drive the instruction.	Step 5 in the Opening Example – Table 1		
(a) student general (mathematical, political, social, etc.) thinking is made public (b) students engage with each other's general (mathematical, political, social, etc.) thinking (c) student general (mathematical, political, social, etc.) sense-making, conjecturing, and justifying drive the instruction.		Steps 1 – 6 in the Opening Example – Table 1	
Student (mathematical, political, social, etc.) identities drive the instruction			Potentially Steps 1 – 6 in the Opening Example –Table 1 as related to the students' and ones own identity.

(Frame 2 and Frame 3). To further inquire: How unfair is this really? How does the number compare to other groups? is an essential aspect of engaging with the sentence. To do this a student must see themselves as a mathematical sense maker of the world (Frame 3). All of this, in my eyes, is part of mathematics and lives in the center of Fig. 1 where all three Frames intersect. To live in today's world with the onslaught of information one needs to understand how the information presented connects to one's own life, community etc. to be able to determine its impact. Critical mathematics literacy (Wager, Thanheiser, Koestler, Yeh, & Jessup, 2022) is an essential component of mathematics and of daily life. This example also relates to our study I introduced above (Melhuish et al., 2022). If we had not interrogated the data, we would not have come up with the results that the current way of doing math in those classrooms did not serve all students. Not paying attention to this can have detrimental effects for those students who are not being served.

By teaching K-12 mathematics using Frame 1 without Frame 2 and/or Frame 3 students are not learning how to make sense of the world around them and might not develop a mathematical identity that leads them see themselves in and to investigate the world around them. If we change the frame for school mathematics from exclusive focus on Frame 1 to include Frame 2 and/or Frame 3 all students have a better chance at gaining a real understanding of the world that will be useful in their daily life as democratic mathematically literate citizens of the world.

10. Note: mathematics as barrier

Frame 1 is often used to *'weed out'* students. Many of the readers of this paper may remember "weeding out language" in the context of their own mathematics education. I remember being told in my first mathematics lecture at university that less than half of us would still be here in a year or two. Of course, that does not inspire confidence, nor does it communicate the belief that everyone can learn mathematics. If our goal is to ensure that all students have equitable access to higher education, shouldn't we focus on how to ensure that all students can learn math? Today we know that calculus and algebra are both major barriers (Ellis, Fosdick, & Rasmussen, 2016; Moses & Cobb, 2001) to enter STEM fields in college. What are we doing about this? Broadening teaching mathematics from Frame 1 to Frames 2 and 3 allows more students access to mathematics and allows students to see how and why learning mathematics help them understand and act on their world.

Historically mathematics was seen as the "hallmark of the educated person. Its study was seen as bringing the discipline of logical thinking to the apprentice scholar" (Kilpatrick et al., 2001, p. 15). The notion of using math (Frame 1) as a measure of education and smartness and therefore as a gateway or barrier is cause for worry, especially when it's teaching clearly privileges some over others. Is success in math what is expected from all students or just from some students (Freire & Macedo, 1987; Yeh, Ellis, & Mahmood, 2020)? And is success/failure in this system of math attributed to students or the system (Frank, 2020)? These are essential question because if not all students are expected to learn math, then mathematics is a barrier for all those students.

11. Conclusion: communicating across frames

As stated above, the goal of this paper is to highlight three very different frames that show how people may use the word mathematics differently.

1. Frame 1: Math as an abstract body of knowledge/ideas, the organization of that into systems and structures, and a set of methods for reaching conclusions.
2. Frame 2: Math as contextual, ever present, as a lens or language to make sense of the world
3. Frame 3: Mathematics as a verb (not a noun), a human activity, part of one's identity

There are different guiding questions underlying each frame (see Table 3).

Returning to my own frames of mathematics, early in my career my focus was on Frame 1. This meant attending to the students' mathematical thinking and developing that but staying in the realm of abstract mathematics. My progress can be seen as moving from abstract mathematics (Frame 1) to include contexts to learn *about* social justice issues (Frame 2). Wager (2008) defines *Teaching Mathematics about Social Justice* as "refers to the context of the lessons that explore critical (and oftentimes controversial) social issues using mathematics." (Stinson, Wager, & Leonard, 2012, p. 6). While this includes all 6 steps of the example it does not necessarily consider the students in the classroom in relation to the context.

As I learned more, I realized that I needed to pay attention to the students in the classroom in relation to the context and include attending to the students' identity (Frame 3). This view of mathematics aligns with *Teaching Mathematics with Social Justice,* which "refers to the pedagogical practices that encourage a co-created classroom (Stinson et al., 2012, p. 6). As well as *Mathematics for Social Justice,* which "is the underlying belief that mathematics can and should be taught in a way that supports students in using mathematics

Table 3
Sample Guiding Questions for each frame.

Frame 1	Sample Guiding Question: How do students learn to make sense of a mathematical topic?
Frame 2	Sample Guiding Question: How do students make sense of the world with mathematics?
Frame 3	Sample Guiding Question: Do students see themselves as (mathematical) sense makers in the world? What is the impact of this? What can be done to work towards all students taking on an empowering mathematical identity? How do students see themselves inside and outside the mathematics classroom?

to challenge the injustices of the status quo as they learn to read and *re*write their world (Freire, 1970/2000)" (Stinson et al., 2012, p. 6). With respect to the example this would imply learning about the students in the classroom, who they are, and what they care about to select or develop context to examine and act upon.

The frames I presented in this paper are three example frames for the word mathematics. Many more frames exist. One thing that became clear to me when talking with other mathematics educators including my co-editor for this special issue is that it is important to be able to explicate one's own frame in discussions to allow for easier communication within and across frames. This will alleviate the frustrating question "Where is the math?" as mathematics may be defined to include the things that some might see outside its frame.

If we take a critical literacy stance, we acknowledge that teaching mathematics and teaching mathematics teachers is a political activity (Felton-Koestler & Koestler, 2017; Frankenstein, 1990; Gutiérrez, 2009, 2015; Gutiérrez, Irving, Gerardo, & Vargas, 2013; Koestler, 2012) and as such neither is nor can be neutral. We also know that activity in the math classroom must be relevant to the students' and their families' lives. Learning opportunities must be connected to students' experiences in their homes and communities (González, Moll, & Amanti, 2006). As a field we are expanding algebra as a civil rights issue (Bob Moses) to mathematical literacy as the civil rights issue of our day (Berry, 2022). As such Frames 2 and 3 are essential aspects of mathematics.

Declarations of interest

None.

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