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AN ABSTRACT OF THE THESIS of Steve William Justice for the Master of Science in Applied Science presented February 1972.

Title: Application of Numerical Analysis to Root Locus Design of Feedback Control Systems.

APPROVED BY MEMBERS OF THE THESIS COMMITTEE:

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Many practical problems in the field of engineering become so complex that they may be effectively solved only with the aid of a computer. An effective solution depends on the use of an efficient algorithm. Plotting root locus diagrams is such a problem. This thesis presents such an algorithm.

Root locus design of feedback control systems is a very powerful tool. Stability of systems under the influence of variables can be easily determined from the root locus diagram. For even moderately complex systems of the type found in practical applications, determination of the locus is extremely difficult if accuracy is required. The difficulty lies in the classical method of graphically determining the location of points on the locus by trial and error. Such a method cannot be efficiently applied to a computer program.

The text presents an original algorithm for plotting the root locus of a general system. The algorithm is derived using the combined methods of complex variable algebra and numerical analysis. For each abscissa desired a polynomial is generated. The real roots of this polynomial are the ordinate values for points on the root locus. Root finding methods from numerical analysis enable the solution of the problem to be one of convergent iteration rather than trial and error.

Among the material presented is a computer program for solution of the general problem, an example of a completely analytic solution, and a table of solutions for more simple systems. The program inputs are the coefficients of the open loop transfer function and the range and increments of the real axis which are to be swept. The output lists the real and imaginary components of all solution points at each increment of the sweep. Also listed are the magnitude and angle components of the solution point and the value of system gain for which this is a solution. For less complex problems, the method can be applied analytically. This may result in an explicit relation between the real and imaginary components of all solution points or even in a single expression which can be analyzed using the methods of analytic geometry.

As with any advance in the theory of problem solving, the ideas presented in the thesis are best applied in conjunction with previous solution methods. Specifically, an idea of the approximate location of the root locus can be obtained using sketching rules which are well

known. The method presented here becomes much more efficient when even a rough approximation is known. Furthermore, the specific locations of system poles and zeros are not required, but can be helpful in planning areas in which to search for solutions.

APPLICATION OF NUMERICAL ANALYSIS TO ROOT
LOCUS DESIGN OF FEEDBACK CONTROL SYSTEMS

by

STEVE WILLIAM JUSTICE

A thesis submitted in partial fulfillment of the
requirements for the degree of

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1972

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CHAPTER I

INTRODUCTION

The root locus method of analysis and synthesis of feedback control systems was introduced in 1949 by W. R. Evans.¹ Since that time it has become very popular. It is not difficult for an experienced person to sketch an approximate root locus diagram. For even moderate accuracy, however, the calculations involved become very time consuming. Computers and special devices can shorten computation time but the inherent inefficiency of Evans' method cannot be corrected without a complete reversal of the approach to the problem.

The purpose of this paper is to present such an alternative approach. This method, developed expressly for digital computers, provides speed of computation and ease of programming. The method has been hinted at by some authors² but the full scope of the analytical approach was not realized. Adaptation of the specific problem to the program presented in the third chapter is generally easier than to Evans' method since the characteristic equation need not be factored.

The method of solution derived in Chapter II also leads to an analytical solution of the problem. Equations can be derived relat-

¹W. R. Evans, "Graphical Analysis of Control Systems," Trans. AIEE, vol. 68 (1949), pp. 765-777.

²Kenneth Steiglitz, "An Analytical Approach to Root Loci," IRE Transactions on Automatic Control, Sept. 1961, pp. 326-332.

ing the ordinates to the abscissas of all solution points. Chapter IV contains an example of this method and a table of solutions for the more simple problems.

The Root Locus Method of Evans

The canonical feedback control system of Figure 1 has an open-

Figure 1. The canonical feedback control system.

loop transfer function which can be represented in factored form as

$$GH = \frac{K \prod_{i=1}^m (s-z_i)}{\prod_{j=1}^n (s-p_j)} \quad (1.1)$$

where K is a constant which is independent of the complex variable $s = \sigma + j\omega$. Each z_i represents a zero of the open-loop transfer function. There are m of these any or all of which may be equal. There are also n poles of equation 1.1 denoted by p_j . It is assumed that complex poles and zeros occur in conjugate pairs.

The closed-loop transfer function is

$$\frac{C}{R} = \frac{G}{1 + GH} \quad (1.2)$$

The stability of the closed-loop system can be determined by the loca-

tions in the s -plane of the poles of the closed-loop system.³ These poles are the solutions to the equation $GH = -1$ or, from equation 1.1,

$$\frac{\prod_{i=1}^m (s-z_i)}{\prod_{j=1}^n (s-p_j)} = \frac{-1}{K} \quad (1.3)$$

As K is varied from zero to infinity, the solutions of equation 1.3 move from the open-loop poles to the open-loop zeros. The path taken in the s -plane by these solutions is called the root locus.

The graphical method of Evans is based on the representation of each term $(s-s_i)$ as a vector in the s -plane. When treated as a vector equation, equation 1.3 may be separated into two equations, one relating the arguments of the vectors or the angle formed by the vectors with respect to the horizontal and one relating the magnitudes of the vectors.

$$\sum_{i=1}^m \arg(s-z_i) - \sum_{j=1}^n \arg(s-p_j) = 180(d) \quad (1.4a)$$

$$\frac{\prod_{i=1}^m |s-z_i|}{\prod_{j=1}^n |s-p_j|} = \frac{1}{|K|} \quad (1.4b)$$

Equation 1.4a is calculated in degrees with d odd if K is greater than zero and even if K is less than or equal to zero. The points which are solutions of equation 1.4a define the root locus. These points

³E. P. Popov, The Dynamics of Automatic Control Systems, 1962, p.241.

are found by a trial and error method. For this reason the number of calculations becomes excessive even for simple loci. Obviously, any computer program using this method must be very inefficient.

Advances in Root Locus Design

To reduce the number of calculations involved, sketching rules are presented in most texts.⁴ Using these rules, and with experience, it is possible to sketch a relatively complex locus with a fair degree of accuracy. Use of a Spirule can reduce the inaccuracy from 10% to about 1%.⁵ The Spirule simplifies computation of equations 1.4 by reducing the number of readings taken and the number of additions necessary.

The ESIAC, an electronic computer, was designed and built exclusively for this and related problems.⁶ Programming was relatively easy and direct. Once programmed the solution could be quickly obtained.⁷ The inaccuracy of the ESIAC as well as the expense of having a separate computer for a specific problem, limited its feasibility.

In the following chapters, methods will be derived for a systematic solution of the general root locus problem. The characteristic equation is manipulated to find the solutions on the imaginary axis.

⁴Robert H. Cannon Jr., Dynamics of Physical Systems, 1967, pp. 653-655.

⁵Cannon, p. 651.

⁶M. L. Morgan and J. C. Looney, "Design of the ESIAC Algebraic Computer," IRE Transactions on Electronic Computers, vol. EC-10 No.3, Sept. 1961, p.524.

⁷Morgan and Looney, p. 529.

Methods for shifting the position of the imaginary axis give the method its generality. Numerical analysis provides the means for replacing the trial and error method with a directly convergent iterative scheme. Time is saved not only by reducing the number of calculations done by the computer but also by making the adaptation of a problem to the computer program very simple.

CHAPTER II

DERIVATION OF THE BASIC EQUATION

Equation 1.3 can be expressed not only as a ratio of factors of the form $(s-s_i)$ but also as a ratio of two polynomials in s . The polynomial form of equation 1.1 is

$$GH = \frac{K \sum_{i=0}^m a_i s^i}{\sum_{h=0}^n b_h s^h} .$$

With this notation equation 1.3 becomes

$$\frac{\sum_{i=0}^m a_i s^i}{\sum_{h=0}^n b_h s^h} = - \frac{1}{K} . \quad (2.1)$$

These polynomials are unique for any given set of factors.⁸ The coefficients can be generated by multiplication of the factors. In the derivation which follows we wish to find solutions of equation 2.1 first for points on the imaginary axis ($s = j\omega$) and then for all points by generalizing the method to $s = \sigma + j\omega$.

To simplify the notation, let $\sum_{i=a}^{b:c}$ mean the summation as i varies from a to b in increments of c . For example,

$$\begin{array}{l} 14:4 \\ \sum_{i=1} i = 1 + 5 + 9 + 13 = 28 \end{array}$$

⁸R. R. Stoll and E. J. Wong, Linear Algebra, 1968, p. 185.

and

$$\sum_{i=0}^{11:2} i^2 = 0 + 4 + 16 + 36 + 64 + 100 = 220.$$

Also j is used to designate an imaginary number. That is, $j = \sqrt{-1}$, $j^2 = -1$, $j^3 = -\sqrt{-1} = -j$, $j^4 = 1$, $j^5 = j$, etc. Using the notation above

$$\frac{GH}{K} = \frac{\sum_{h=0}^{m:4} a_h s^h + \sum_{i=1}^{m:4} a_i s^i + \sum_{k=2}^{m:4} a_k s^k + \sum_{l=3}^{m:4} a_l s^l}{\sum_{p=0}^{n:4} b_p s^p + \sum_{q=1}^{n:4} b_q s^q + \sum_{r=2}^{n:4} b_r s^r + \sum_{t=3}^{n:4} b_t s^t} \quad (2.2)$$

Since $h = 0, 4, 8, \dots$, if $s = j\omega$, it follows that $s^h = \omega^h$. Also $i = 1, 5, 9, \dots$. Therefore, if $s = j\omega$, $s^i = j\omega^i$. Similarly $s^k = -\omega^k$, $s^l = -j\omega^l$, $s^p = \omega^p$, $s^q = j\omega^q$, $s^r = -\omega^r$, and $s^t = -j\omega^t$. Substituting $j\omega$ for s in equation 2.2 and collecting the real and imaginary parts gives

$$\frac{GH}{K} = \frac{\sum_{h=0}^{m:4} a_h \omega^h - \sum_{k=2}^{m:4} a_k \omega^k + j \sum_{i=1}^{m:4} a_i \omega^i - j \sum_{l=3}^{m:4} a_l \omega^l}{\sum_{p=0}^{n:4} b_p \omega^p - \sum_{r=2}^{n:4} b_r \omega^r + j \sum_{q=1}^{n:4} b_q \omega^q - j \sum_{t=3}^{n:4} b_t \omega^t}$$

To rationalize the denominator, the numerator is multiplied by the complex conjugate of the denominator. The resulting expression for the numerator is

$$\begin{aligned} & \left(\sum_{h=0}^{m:4} a_h \omega^h - \sum_{k=2}^{m:4} a_k \omega^k \right) \left(\sum_{p=0}^{n:4} b_p \omega^p - \sum_{r=2}^{n:4} b_r \omega^r \right) \\ & + \left(\sum_{i=1}^{m:4} a_i \omega^i - \sum_{l=3}^{m:4} a_l \omega^l \right) \left(\sum_{q=1}^{n:4} b_q \omega^q - \sum_{t=3}^{n:4} b_t \omega^t \right) \\ & + j \left(\sum_{i=1}^{m:4} a_i \omega^i - \sum_{l=3}^{m:4} a_l \omega^l \right) \left(\sum_{p=0}^{n:4} b_p \omega^p - \sum_{r=2}^{n:4} b_r \omega^r \right) \\ & - j \left(\sum_{h=0}^{m:4} a_h \omega^h - \sum_{k=2}^{m:4} a_k \omega^k \right) \left(\sum_{q=1}^{n:4} b_q \omega^q - \sum_{t=3}^{n:4} b_t \omega^t \right) \end{aligned}$$

Solutions to equation 2.1 must be real. The rationalized denominator of equation 2.2 must be real. Therefore if a solution is to be found for equation 2.1, the expression for the numerator on the preceding page must be real. This implies that the imaginary part is equal to zero. Expanding the imaginary part of the numerator gives an expression which shall be called the basic equation and notated as "D(ω)" in the text.

$$\begin{aligned}
 D(\omega) = & \sum_{h=0}^{m:4} a_h \omega^h \sum_{t=3}^{n:4} b_t \omega^t - \sum_{h=0}^{m:4} a_h \omega^h \sum_{q=1}^{n:4} b_q \omega^q \\
 & + \sum_{i=1}^{m:4} a_i \omega^i \sum_{p=0}^{n:4} b_p \omega^p - \sum_{i=1}^{m:4} a_i \omega^i \sum_{r=2}^{n:4} b_r \omega^r \\
 & + \sum_{k=2}^{m:4} a_k \omega^k \sum_{q=1}^{n:4} b_q \omega^q - \sum_{k=2}^{m:4} a_k \omega^k \sum_{t=3}^{n:4} b_t \omega^t \\
 & + \sum_{l=3}^{m:4} a_l \omega^l \sum_{r=2}^{n:4} b_r \omega^r - \sum_{l=3}^{m:4} a_l \omega^l \sum_{p=0}^{n:4} b_p \omega^p .
 \end{aligned} \tag{2.3}$$

To simplify the expression the coefficients of like powers of ω are collected. This is done by noting that

$$\begin{aligned}
 \sum_{h=0}^{m:4} a_h \omega^h \sum_{t=3}^{n:4} b_t \omega^t &= a_0 \omega^0 b_3 \omega^3 + a_4 \omega^4 b_3 \omega^3 + a_8 \omega^8 b_3 \omega^3 + \dots \\
 &\quad + a_0 \omega^0 b_7 \omega^7 + a_4 \omega^4 b_7 \omega^7 + \dots \\
 &= a_0 b_3 \omega^3 + (a_4 b_3 + a_0 b_7) \omega^7 + (a_8 b_3 + a_4 b_7 + a_0 b_{11}) \omega^{11} \dots
 \end{aligned}$$

and that for $i > m$, $a_i = 0$ and for $i > n$, $b_i = 0$. These simplifications, when applied to equation 2.3 result in the general form of the basic equation:

$$D(\omega) = (-a_0 b_1 + a_1 b_0) \omega + (a_0 b_3 - a_1 b_2 + a_2 b_1 - a_3 b_0) \omega^3 + \dots \tag{2.4}$$

The last term of equation 2.4 is

$$\begin{aligned} & (-1)^k a_m b_n \omega^{m+n} \text{ if } m+n \text{ is odd or} \\ & (-1)^k (-a_{m-1} b_n + a_m b_{n-1}) \omega^{m+n-1} \text{ if } m+n \text{ is even.} \end{aligned}$$

The value of k in the first case is $(m+n-1)/2$. If $m+n$ is even, k is equal to $(m+n)/2$. It can be verified that if d_k is the coefficient of ω^k in $D(\omega)$ then

$$d_k = \begin{cases} 0 & \text{for } k \text{ even or} \\ \frac{k+1}{(-1)^2} \sum_{i=0}^k (-1)^i a_i b_{k-1} & \text{for } k \text{ odd.} \end{cases} \quad (2.5)$$

Application of $D(\omega)$ to Root Locus Plotting

The polynomial $D(\omega)$ has the general form

$$D(\omega) = d_1 \omega + d_3 \omega^3 + d_5 \omega^5 + \dots + d_p \omega^p \quad (2.6)$$

where, from equation 2.4, p equals $m+n$ or $m+n-1$ whichever is odd. The roots of $D(\omega)$ give the values of ω for which the imaginary part of equation 2.2 is zero and, therefore, the points on the imaginary axis which are solutions of equation 2.1 for some value of the gain constant K .

One of the roots is always zero since the coefficient of ω^0 is zero. After generalization it will be shown that this is true of the polynomial $D(\omega)$ at every point along the real axis. (By restricting the polarity of K , some or all of the real axis may be excluded from the 'root locus'). Extracting the zero root from equation 2.6 leaves

$$D'(\omega) = d_1 + d_3 \omega^2 + d_5 \omega^4 + \dots + d_p \omega^{p-1}. \quad (2.7)$$

Substituting $\omega = \omega^2$ and $p' = (p-1)/2$ into equation 2.7 gives

$$D'(\omega) = d_1 + d_3\omega + d_5\omega^2 + \dots + d_p\omega^{p'}. \quad (2.8)$$

Equation 2.8 is the basis for the solution of equation 2.1 by the computer program to be presented in Chapter III. The advantages are obvious. There are well known methods for finding the roots of polynomials. Only the positive real roots of equation 2.8 need be found since ω must be a real number. The order of the polynomial is always less than $(m+n)/2$. For example, if $G(s)H(s)$ contains five zeros and seven poles, then $D'(\omega)$ would be only a fifth order polynomial.

The root locus is symmetrical with respect to the real axis because for each positive real root of $D'(\omega)$ there are two real roots of $D(\omega)$. That is, for each positive real ω , $\omega = \pm \sqrt{\omega}$.

The roots of $D(\omega)$ give all the points along the imaginary axis which are solutions of equation 2.1 for some value of the gain constant K . To find the solutions on some other vertical line, let $GH = -1/K$ at $s = \sigma + j\omega$. Assume that the coefficients of $G(s+\sigma)H(s+\sigma)$ are known. Then the points on the root locus are the solutions for $G(s+\sigma)H(s+\sigma)$ at $s = j\omega$. These solutions can be found by generating the coefficients of $D(\omega)$ from those of $G(s+\sigma)H(s+\sigma)$. An algorithm for finding the coefficients of $G(s+\sigma)H(s+\sigma)$ from those of $G(s)H(s)$ is given in Chapter III. Graphically this amounts to shifting all the poles and zeros of GH σ units to the right.

Plotting the root locus diagram involves repeating the steps below for a sufficient number of points on the real axis.

1. Find the coefficients of $G(s+\sigma)H(s+\sigma)$ from those of $G(s)H(s)$.
2. Solve for the coefficients of $D(\omega)$ using equation 2.5.

3. Find the real roots of $D(\omega)$ preferably by finding the positive real roots of $D'(\omega')$.

4. (Optional) Solve for the gain constant K at each real root of $D(\omega)$.

When applied in a computer program the method above is systematic and general. Algorithms for each of the steps will be presented in Chapter III. When working by hand or with a desk calculator it is suggested that as much advantage as possible be gained from experience and the sketching rules mentioned in the first chapter. The root finding methods in general require an iterative scheme. Thus, a first guess within 10% is extremely helpful. Also, it is an aid to know in advance approximately where along the real axis to look for branches of the root locus. For more simple problems ($m+n \leq 6$) it is advisable to use the analytic method proposed in Chapter IV. Using this method formulas can be derived to solve equation 2.1 by relating σ to ω in a single expression.

CHAPTER III

ALGORITHMS

The purpose of this chapter is to present algorithms for the operations necessary for the solution of equation 2.1 by the four step procedure presented in Chapter II. Fundamental to these algorithms is synthetic division of a polynomial. Shifting the roots of a polynomial is done by successive synthetic division as shown in section 2. The third algorithm uses the Birge-Vieta method to find the real roots of a polynomial. Synthetic division is expanded to complex numbers to derive a simple method for evaluating $G(s)H(s)$ at the solution points of equation 2.1 in order to find the value of K which gives this solution. The last section of this chapter presents a computer program which combines the algorithms and efficiently calculates the data necessary for plotting root locus diagrams.

Synthetic Division

Synthetic division is a term applied to a shorthand method of dividing a polynomial by a linear factor. Kunz derives the method using a mathematical scheme.⁹ The derivation given here is basically the same except where notation has been altered to conform with the present text.

⁹Kaiser S. Kunz, Numerical Analysis, 1957, pp. 19,20.

Consider the following set of polynomials in x :

$$\begin{aligned}
 P_0(x) &= a_m \\
 P_1(x) &= xP_0(x) + a_{m-1} \\
 &\vdots \\
 P_k(x) &= xP_{k-1}(x) + a_{m-k} \\
 &\vdots \\
 P_m(x) &= xP_{m-1}(x) + a_0 .
 \end{aligned} \tag{3.1}$$

Multiplying each equation by x^{m-k} and adding the resulting $m+1$ equations gives a single polynomial of the form

$$P_m(x) = a_m x^m + a_{m-1} x^{m-1} + \dots + a_0$$

or

$$P_m(x) = \sum_{k=0}^m a_k x^k . \tag{3.2}$$

Since the evaluation of equation 3.2 requires calculation of powers of x , it is generally easier to evaluate $P_m(x)$ using equations 3.1. When calculated in order of increasing values of k , each equation has only one unknown. A convenient scheme for arranging hand calculations is shown below:

<u>$x /$</u>	a_m	a_{m-1}	a_{m-2}	$\dots$	a_1	a_0
	xP_0	xP_1	$\dots$	xP_{m-2}	xP_{m-1}	
	P_0	P_1	P_2	$\dots$	P_{m-1}	$P_m(x)$

Working from left to right, each P_k is calculated by adding the values a_{m-k} and xP_{k-1} above it.

Algorithm 3.1: Synthetic Division

Initial conditions: Let $a_0, a_1, \dots, a_m$ be the coefficients of the polynomial $P_m(x)$ as in equation 3.2. Let x_0 be the value at which $P_m(x)$ is to be evaluated.

A3.1.1. Let $P_0 = a_m$ and $k = 0$.

A3.1.2. Increase k by 1.

A3.1.3. Let $P_k = x_0 P_{k-1} + a_{m-k}$.

A3.1.4. If $k = m$, stop. Otherwise go back to step A3.1.2.

Results: The value of $P_m(x)$ at $x = x_0$ has been found.

Shifting the Roots of a Polynomial

The first step in the method of Chapter II for plotting root locus diagrams is to find the coefficients of $G(s+\sigma)H(s+\sigma)$. Algorithm 3.2 is developed to accomplish this step.

The first derivatives of equations 3.1 are

$$P'_0(x) = 0$$

$$P'_1(x) = P_0(x)$$

$$P'_2(x) = xP'_1(x) + P_1(x)$$

.

.

.

$$P'_k(x) = xP'_{k-1}(x) + P_{k-1}(x)$$

.

.

.

$$P'_m(x) = xP'_{m-1}(x) + P_{m-1}(x).$$

These values can be calculated simply by extending the diagram on page 13. Continuing this process $m + 1$ times and dropping the zero

terms on the right produces a series of synthetic divisions the rightmost terms of which are $P_m(x_0)$, $P'_m(x_0)$, $P''_m(x_0)/2!$, ..., $P^{(m)}_m(x_0)/m!$. Taylor's series for shifting a polynomial is

$$P(x+x_0) = P(x_0) + xP'(x_0) + x^2P''(x_0)/2! + \dots + x^mP^{(m)}(x_0)/m!.^{10}$$

The coefficients are the same as those found by the extended synthetic division. Let $P_3(x) = x^3 - 6x^2 + 8x - 5$. The solution for the coefficients of $P_3(x+3)$ would be calculated as shown below:

$$\begin{array}{r} 3 / \quad 1 \quad -6 \quad 8 \quad -5 \\ \quad \quad 3 \quad -9 \quad -3 \\ \hline 1 \quad -3 \quad -1 \quad -8 \\ \quad \quad 3 \quad 0 \\ \hline 1 \quad 0 \quad -1 \\ \quad \quad 3 \\ \hline 1 \quad 3 \\ \hline 1 \end{array}$$

$$P_3(x+3) = x^3 + 3x^2 - x - 8.$$

Algorithm 3.2: Shifting The Roots of a Polynomial

Initial conditions: Let $P_m(x) = \sum_{k=0}^m a_k x^k$ be the given polynomial.

Let x_0 be the amount by which $P_m(x)$ is to be shifted.

A3.2.1. Let $k = m + 1$, $i = m + 1$, and $a_{m+1} = 0$.

A3.2.2. Decrease i by one.

A3.2.3. Decrease k by one.

A3.2.4. Let $a_k = x_0 a_{k+1} + a_k$.

A3.2.5. If k is greater than $m-i$ go back to A3.2.3. Otherwise continue.

A3.2.6. Let $k = m+1$. If i is positive go back to A3.2.2. Otherwise stop.

Results: The coefficients of $P_m(x-x_0)$ are a_m , a_{m-1} , ..., a_0 .

¹⁰H. P. Westman, ed., Reference Data For Radio Engineers, 4th ed. 1967, p. 1084.

Root Finding Methods

The third step of the method outlined in Chapter II calls for finding the roots of $D'(\omega')$. Many good methods for extracting the roots of a general polynomial may be found in the literature. Only the positive real roots need be found. This allows for a great deal of simplification since more complex methods used for extracting complex roots need not be used. Descarte's rule of signs¹¹ will determine whether or not positive roots exist. If the order of $D'(\omega')$ is four or less, general equations may be employed to find the roots.^{12,13} In general, an iterative scheme must be used. The algorithm of this section is based on the Birge-Vieta method.¹⁴

Let $P_m(x)$ be a polynomial of order m . If x_1 is a sufficiently close approximation to a root of $P(x)$ then

$$x_{1+1} = x_1 - \frac{P_m(x_1)}{P'_m(x_1)} \quad (3.3)$$

is a closer approximation. Using synthetic division as prescribed in the previous section, $P_m(x)$ and $P'_m(x)$ may be calculated and substituted into equation 3.3. When the change in x from one iteration to another is sufficiently small the method may be terminated. If no real roots exist, or under unusual circumstances, the method can continue

¹¹Ralph H. Pennington, Introductory Computer Methods and Numerical Analysis Second Edition, 1970, p. 299.

¹²Kaj L. Nielson, College Mathematics, 1958, pp. 102,164.

¹³Kunz, p. 33.

¹⁴Kunz, p. 22.

endlessly without converging. Thus some upper bound on the number of iterations must be set.

Algorithm 3.2: Finding the Roots of a Polynomial

Initial conditions: Let $P_m(x) = \sum_{k=0}^m a_k x^k$ be the given polynomial.

Let y be an approximation to a root of $P_m(x)$.

A3.3.1. If $a_0 = 0$ then stop since 0 is a root. Otherwise continue.

A3.3.2. If $m = 2$ then $y_{1,2} = (-a_1 \pm \sqrt{a_1^2 - 4a_0a_2})/(2a_2)$ are roots.

A3.3.3. Let $e_0 = f_0 = a_0$, and $j = t = 0$.

A3.3.4. Increase j by one. Let $e_j = a_j + ye_{j-1}$ and let $f_j = e_j + yf_{j-1}$.

A3.3.5. If $j = m$ continue. Otherwise go back to A3.3.4.

A3.3.6. If e_m is sufficiently small, stop. If not, let $y = y - \Delta y$

Where $\Delta y = e_m/f_{m-1}$. Increase t by one.

A3.3.7. If t is excessively large, stop. Otherwise go to A3.3.4.

Results: a. Zero was a root. The remaining roots may be found by applying the algorithm to the polynomial after removing a_0 and reducing the power of each term of the original polynomial.

b. Both roots of the second order polynomial were found.

c. One root was found. The remaining roots may be found by applying the algorithm to the polynomial with coefficients e_k .

d. No root was found. The roots may all be complex.

Evaluating GH to Find K

It is necessary in order to solve equation 2.1 completely to determine the value of the gain constant K at the points which have been found as solutions using the method presented in Chapter II.

From equation 2.1

$$K = - \frac{1}{G(s)H(s)} = - \frac{1}{G(\sigma+j\omega)H(\sigma+j\omega)} \quad (3.3)$$

It is possible to extend the method of synthetic division to complex numbers. However, since the coefficients of $G(s+σ)H(s+σ)$ are known, it is easier to consider only pure imaginary numbers. That is, evaluation of $G(\sigma+j\omega)H(\sigma+j\omega)$ can be done by evaluating $G(s+σ)H(s+σ)$ at $s = j\omega$.

Equations 3.1 are listed below with x replaced by $j\omega$

$$\begin{aligned} P_0 &= a_m \\ P_1 &= j\omega P_0 + a_{m-1} \\ P_2 &= j\omega P_1 + a_{m-2} \\ &\vdots \\ P_k &= j\omega P_{k-1} + a_{m-k} \\ &\vdots \\ P_m &= j\omega P_{m-1} + a_0 \end{aligned}$$

Substituting P_0 into P_1 , P_1 into P_2 etc. gives

$$\begin{aligned} P_0 &= a_m \\ P_1 &= j\omega a_m + a_{m-1} \\ P_2 &= j\omega(j\omega a_m + a_{m-1}) + a_{m-2} \\ &= a_{m-2} - \omega^2 a_m + j\omega a_{m-1} \\ P_3 &= j\omega(a_{m-2} - \omega^2 a_m + j\omega a_{m-1}) + a_{m-3} \\ &= a_{m-3} - \omega^2 a_{m-1} + j\omega(a_{m-2} - \omega^2 a_m) \\ &\vdots \end{aligned}$$

Separating the real and imaginary parts of P_k , denoted as $\text{Re}(P_k)$ and $\text{Im}(P_k)$ respectively, gives the necessary recurrence relations for the synthetic division of a polynomial by a purely imaginary number.

$$\begin{aligned}
 \text{Re}(P_0) &= a_m & \text{Im}(P_0) &= 0 \\
 \text{Re}(P_1) &= a_{m-1} & \text{Im}(P_1) &= \omega \text{Re}(P_0) \\
 \text{Re}(P_2) &= a_{m-2} - \omega^2 \text{Re}(P_0) & \text{Im}(P_2) &= \omega \text{Re}(P_1) \\
 &\vdots & &\vdots \\
 \text{Re}(P_k) &= a_{m-k} - \omega^2 \text{Re}(P_{k-2}) & \text{Im}(P_k) &= \omega \text{Re}(P_{k-1}) \\
 &\vdots & &\vdots \\
 \text{Re}(P_m) &= a_0 - \omega^2 \text{Re}(P_{m-2}) & \text{Im}(P_m) &= \omega \text{Re}(P_{m-1})
 \end{aligned} \tag{3.4}$$

The transfer function in general is a ratio of two polynomials. Therefore, two polynomials must be evaluated using the relations above. The result is

$$GH = \frac{\text{Re}(P_m) + j \text{Im}(P_m)}{\text{Re}(P_n) + j \text{Im}(P_n)} \tag{3.5}$$

The subscripts m and n are used to denote the last terms of the synthetic division operation on the numerator and denominator polynomials respectively. The magnitude of GH denoted $|GH|$ is

$$\begin{aligned}
 |GH| &= \sqrt{\frac{[\text{Re}(P_m)]^2 + [\text{Im}(P_m)]^2}{[\text{Re}(P_n)]^2 + [\text{Im}(P_n)]^2}} \\
 &= \sqrt{\frac{[\text{Re}(P_m)]^2 + \omega^2 [\text{Re}(P_{m-1})]^2}{[\text{Re}(P_n)]^2 + \omega^2 [\text{Re}(P_{n-1})]^2}} \tag{3.6}
 \end{aligned}$$

Note that in the recurrence relations 3.4 above and from the values

needed to solve equation 3.6, the imaginary parts of the P_k in equations 3.4 need not be calculated.

Finally, it must be determined whether GH is positive or negative. It has been determined that GH is real. Considering P_m and P_n as vectors in equation 3.5 leads to the deduction that either vectors P_m and P_n have the same direction or they are in exactly opposite directions. In the first case the angle of the vector GH is zero and GH is positive. In the second case the angle of GH is 180 degrees and GH is negative. In the first case $\text{Re}(P_m)$ and $\text{Re}(P_n)$ have the same sign and in the second case they have opposite signs. Define the following function:

$$\text{sgn}(y) = \begin{cases} +1 & \text{if } y \text{ is positive or zero.} \\ -1 & \text{if } y \text{ is negative.} \end{cases}$$

Then the following relation holds and is the equation solved by algorithm 3.4:

$$-\frac{1}{K} = GH = \text{sgn}[\text{Re}(P_m)\text{Re}(P_n)] \sqrt{\frac{[\text{Re}(P_m)]^2 - \omega^2 [\text{Re}(P_{m-1})]^2}{[\text{Re}(P_n)]^2 - \omega^2 [\text{Re}(P_{n-1})]^2}}.$$

Algorithm 3.4: Evaluating K

Initial conditions: Let $a_m, a_{m-1}, \dots, a_0$ be the numerator coefficients and $b_n, b_{n-1}, \dots, b_0$ be the denominator coefficients of GH. Let ω be a real root of $D(\omega)$.

A3.4.1. Let $P_0 = a_m, P_1 = a_{m-1}$, and $i = 1$.

A3.4.2. Increase i by one.

A3.4.3. Let $P_i = a_{m-i} - \omega^2 P_{i-2}$.

A3.4.4. If i is less than m go back to A3.4.2. Otherwise go to A3.4.5.

A3.4.5. Let $u = \omega^2 P_{m-1}$.

A3.3.6. Let $K = \text{sgn}(P_m) \sqrt{u + P_m^2}$.

A3.4.7. Let $i = 1$, $P_0 = b_n$, $P_1 = b_{n-1}$.

A3.4.8. Increase i by 1.

A3.4.9. Let $P_i = a_{n-i} - \omega^2 P_{i-2}$.

A3.4.10. If i is less than n go to A3.4.8. Otherwise continue.

A3.4.11. Let $u = \omega^2 P_{n-1}^2$.

A3.4.12. Let $K = -\text{sgn}(P_n) \sqrt{u + P_n^2} / K$

Results: The value of K at $s = \sigma + j\omega$ which solves equation 2.1 has been found.

The Computer Program

To complete the chapter on algorithms a computer program is presented. The program is complete and operational. It is set up to do a linear sweep or a series of linear sweeps along the real axis in the s -plane. The output consists of the real and imaginary parts of each solution point, its magnitude and direction from the positive real axis, and the value of K which gives this solution. For a given abscissa all ordinates which solve equation 2.1 are found. Solutions for K less than zero are omitted by convention.

Figure 2 is a listing of the program. The following breakdown will aid in understanding the order in which the problem is solved.

<u>Lines</u>	<u>Purpose</u>
100-185	Preliminary operations including reading data and printing column headings for the output.
190-200	
580-585	Control of the sweep along the real axis.
705-710	
205-280	Algorithm 3.2.

```

100 DIM A(60),B(60),D(60),E(60),F(60),G(60),H(60)
105 READ M
110 FOR I= 1 TO M+1
115 READ G(I)
120 NEXT I
125 READ N
130 LET D = M+N+1
135 LET L1=INT((D+1)/2)
140 LET L2=L1-D/2
145 FOR I=1 TO N+1
150 READ H(I)
155 NEXT I
160 PRINT
165 PRINT
170 PRINT"THE FOLLOWING ARE POINTS ON THE ROOT LOCUS:"
175 PRINT
180 PRINT"REAL","IMAGINARY","MAGNITUDE","ANGLE","    K"
185 PRINT
190 READ X1,X2,X3
195 IF X3=0 THEN 710
200 FOR X = X1 TO X2 STEP X3
205 FOR I = 1 TO M+1
210 LET A(I)=G(I)
215 NEXT I
220 FOR I = 1 TO N+1
225 LET B(I)=H(I)
230 NEXT I
235 FOR I = 1 TO M
240 FOR J = 1 TO M+2-I
245 LET A(J)=A(J)+X*B(J-1)
250 NEXT J
255 NEXT I
260 FOR I = 1 TO N
265 FOR J = 1 TO N+2-I
270 LET B(J)=B(J)+X*B(J-1)
275 NEXT J
280 NEXT I
285 FOR I = 1 TO L1
290 LET D(I)=0
295 LET K1=2*I+2*L2
300 FOR J = 1 TO M+1
305 LET K1=K1-1
310 IF K1<1 THEN 325
315 IF K1>N+1 THEN 325
320 LET D(I)=D(I)+(-1)**A(J)*B(K1)
325 NEXT J
330 NEXT I
340 LET L=0

```

Figure 2. The program listing.

```

345 LET I=INT(D/2+2)
350 IF A(M+1)=0 THEN 370
355 IF B(N+1)/A(M+1)>0 THEN 370
360 LET A=90*(1-SGN(X))
365 PRINT X,Z,ABS(X),A,-B(N+1)/A(M+1)
370 LET I=I-1
375 LET T=0
380 IF L=0 THEN 405
385 LET L=0
390 FOR J = 1 TO I
395 LET D(J)=E(J)
400 NEXT J
405 IF ABS(D(I))<1E-8 THEN 565
410 IF I=1 THEN 580
415 IF I/2-INT(I/2)=0 THEN 480
420 IF I<>3 THEN 480
425 LET R=D(2)/D(1)
430 LET I=I-1
435 LET S=.25*R**2-D(3)/D(1)
440 IF S<0 THEN 370
445 LET Y= -R/2+SQR(S)
450 IF Y<0 THEN 370
455 GO SUB 590
460 LET Y=-R/2-SQR(S)
465 IF Y<0 THEN 370
470 GO SUB 590
475 GO TO 370
480 LET M1=0
485 FOR J = 1 TO I-1
490 LET M1=M1+SGN(D(J))*SGN(D(J+1))
495 NEXT J
500 IF M1=I-1 THEN 580
505 LET L=1
510 FOR J = 1 TO I
515 LET E(J)=D(J)+Y*E(J-1)
520 LET F(J)=E(J)+Y*F(J-1)
525 NEXT J
530 IF ABS(E(J))<ABS(D(J))*1E-7 THEN 465
535 LET J=E(I)/F(I-1)
540 IF ABS(J)<1E-20*ABS(Y) THEN 465
545 LET Y=Y-J
550 LET T=T+1
555 IF T>100 THEN 580
560 GO TO 510
565 LET I=I-1
570 IF I>0 THEN 405
575 PRINT"VERTICAL LOCUS AT"; X
580 NEXT X
585 GO TO 190

```

Figure 2. (continued).


```
590 LET F(1)=A(1)
595 FOR J = 2 TO M+1
600 LET F(J)=A(J)-F(J-2)*Y
605 NEXT J
610 LET K=Y*F(M)**2
615 LET K=SQR(K+F(M+1)**2)
620 LET K1=SGN(F(M+1))
625 LET F(1)=B(1)
630 FOR J = 1 TO N+1
635 LET F(J)=B(J)-F(J-2)*Y
640 NEXT J
645 LET W=Y*F(N)**2
650 LET K=SQR(W+F(J)**2)/K
655 LET K1=K1*SGN(F(J))
656 IF K1<>0 THEN 660
657 LET K1=-1
660 IF K1>0 THEN 700
665 LET M1=SQR(X**2+Y)
670 LET A=0
675 IF X=0 THEN 695
680 LET A=ATN(SQR(Y)/X)*180/3.1415926
685 IF X>0 THEN 695
690 LET A=A+180
695 PRINT X, SQRT(Y), M1, A, -K*K1
700 RETURN
705 DATA 0,0,0
710 END
```

Figure 2. (continued).

<u>Lines</u>	<u>Purpose</u>
285-330	Finding the coefficients of $D(\omega)$.
340-365	Determining K for the point on the axis.
370-575	Finding the roots of $D(\omega)$.
480-500	Eliminating polynomials with no positive roots by Decarte's Rule.
505-560	Algorithm 3.3.
590-700	Algorithm 3.4.

The input to the program is as follows:

10 DATA $m, a_m, a_{m-1}, \dots, a_0$

20 DATA $n, b_n, b_{n-1}, \dots, b_0$

30 DATA $x_{11}, x_{21}, x_{31}, x_{12}, x_{22}, x_{32}, \dots, x_{1k}, x_{2k}, x_{3k}$

where the root locus for the transfer function

$$GH = \frac{K \sum_{i=0}^m a_i s^i}{\sum_{h=0}^n b_h s^h}$$

is to be found between x_{11} and x_{21} in steps of x_{31} and from x_{12} to x_{22} in steps of x_{32} etc.

Table I is a listing of the output of this program for the transfer function

$$GH = \frac{12K(s + 3)}{s(s^2 + 6s + 18)(s + 2)} = \frac{K(12s + 36)}{s^4 + 8s^3 + 30s^2 + 36s}.$$

Using the sketching rules, a rough idea of the location of the locus can be determined. This is shown in figure 3. It is seen that the sweep should go from approximately -5 to +1. For good results in one run, a step of 0.1 is chosen. The input to the program is

TABLE I

PROGRAM INPUT AND OUTPUT FOR $GH = 12(s+3)/(s^4+8s^3+30s^2+36s)$

10 DATA 1,12,36
 20 DATA 4,1,8,30,36,0
 30 DATA -5,1,0.1
 RUN

THE FOLLOWING ARE POINTS ON THE ROOT LOCUS:

REAL	IMAGINARY	MAGNITUDE	ANGLE	K
-5	0	5	180	8.125
-4.9	0	4.9	180	7.85913
-4.8	0	4.8	180	7.616
-4.7	0	4.7	180	7.39628
-4.6	0	4.6	180	7.20092
-4.5	0	4.5	180	7.03125
-4.4	0	4.4	180	6.88914
-4.3	0	4.3	180	6.77719
-4.2	0	4.2	180	6.699
-4.1	0	4.1	180	6.6597
-4.	0	4.	180	6.66667
-4.	0.42819	4.02285	173.89	6.2111
-3.9	0	3.9	180	6.73075
-3.9	0.682448	3.95926	170.074	5.55803
-3.8	0	3.8	180	6.8685
-3.8	0.859581	3.89601	167.254	4.96734
-3.7	0	3.7	180	7.1062
-3.7	1.00228	3.83335	164.843	4.43508
-3.6	0	3.6	180	7.488
-3.6	1.12474	3.77161	162.65	3.95731
-3.5	0	3.5	180	8.09375
-3.5	1.23418	3.71122	160.576	3.53007
-3.4	0	3.4	180	9.08367
-3.4	1.33523	3.65279	158.559	3.14934
-3.3	0	3.3	180	10.8323
-3.3	1.43156	3.59713	156.549	2.81095
-3.2	0	3.2	180	14.464
-3.2	1.52664	3.54551	154.495	2.51041
-3.1	0	3.1	180	25.6034
-3.1	1.62467	3.49993	152.342	2.2425
-3.	3.	4.24264	135.	8.34702 E-7
-3.	1.73205	3.4641	150.	2.
-2.9	2.78408	4.02009	136.168	0.484331
-2.9	1.86249	3.44657	147.29	1.769
-2.8	2.4854	3.74396	138.406	0.990073
-2.8	2.07431	3.48465	143.468	1.48993

TABLE I
(continued)

-2.	0	2.	180	-.94729 E-8
-1.9	0	1.9	180	0.146962
-1.8	0	1.8	180	0.261
-1.7	0	1.7	180	0.349481
-1.6	0	1.6	180	0.417524
-1.5	0	1.5	180	0.46875
-1.2	0	1.2	180	0.544
-1.2	1.72547	2.10172	124.817	1.48993
-1.2	1.05013	1.59461	138.811	0.990072
-1.1	0	1.1	180	0.547539
-1.1	2.03743	2.31541	118.364	1.769
-1.	0	1.	180	0.541667
-1.	2.23607	2.44949	114.095	2.
-0.9	0	0.9	180	0.526821
-0.9	2.40426	2.56719	110.523	2.2425
-0.8	0	0.8	180	0.503273
-0.8	2.55917	2.6813	107.359	2.51041
-0.7	0	0.7	180	0.471156
-0.7	2.70752	2.79654	104.496	2.81095
-0.6	0	0.6	180	0.4305
-0.6	2.85257	2.91499	101.878	3.14934
-0.5	0	0.5	180	0.38125
-0.5	2.99613	3.03757	99.4743	3.53007
-0.4	0	0.4	180	0.323282
-0.4	3.13926	3.16464	97.2614	3.95731
-0.3	0	0.3	180	0.256417
-0.3	3.2826	3.29628	95.2218	4.43508
-0.2	0	0.2	180	0.180429
-0.2	3.42653	3.43236	93.3405	4.96734
-0.1	0	0.1	180	9.50546 E-2
-0.1	3.57131	3.57271	91.6039	5.55803
3.53903 E-8	3.71708	3.71708	90.	6.2111
0.1	3.8639	3.8652	88.5175	6.93049
0.2	4.01182	4.01681	87.146	7.72014
0.3	4.16085	4.17165	85.8761	8.58403
0.4	4.31095	4.32947	84.6989	9.52611
0.5	4.46211	4.49004	83.6064	10.5504
0.6	4.6143	4.65314	82.5914	11.6608
0.7	4.7645	4.81857	81.647	12.8614
0.8	4.92155	4.98614	80.7673	14.1562
0.9	5.07653	5.15569	79.9467	15.5491

RUN TIME: 3.32 SECS.

READY

10 DATA 1,12,36

20 DATA 4,1,8,30,36,0

30 DATA -5,1,0.1

The output of the program, given in TABLE I, is used to draw the complete root locus diagram shown in figure 4.

Figure 3. An approximate root locus for the transfer function

$$GH = \frac{12K(s+3)}{s(s^2+6s+18)(s+2)} .$$

Figure 4. The root locus for $GH = \frac{12K(s+3)}{s(s+2)(s^2+6s+18)}$.

CHAPTER IV

AN ANALYTIC METHOD

It is possible for cases in which the number of poles and zeros is small (six or less) to determine analytically an explicit equation which defines the root locus. The process derives directly from the method of CHAPTER II by treating σ as a variable. The quadratic formula can be used to find the roots of $D(\omega)$ when $m+n$ is equal to five or six. For lesser order problems, analytic geometry will give an insight into the solution. An example of a fifth order problem is given in this chapter. A table of solutions for most problems of order four and less is given with an example of the methods used in deriving the table. Some examples are given to illustrate the use of the table in solving problems.

An Example of the Analytic Method

Consider the transfer function $GH = K(s+3)/(s^4+8s^3+37s^2+50s)$. Let σ be replaced by x and let x become a variable. To determine the root locus for this transfer function, follow the steps of the method given in CHAPTER II as follows:

Step 1: Find the coefficients of $G(s+x)H(s+x)$. The numerator is shifted as follows:

$$\begin{array}{r} \underline{x} / \quad \quad \quad 1 \quad \quad 3 \\ \quad \quad \quad \quad \quad \quad \quad x \\ \quad \quad \quad \quad \quad \quad \quad \hline \quad \quad \quad 1 \quad \quad x+3 \\ \quad \quad \quad \hline \quad \quad \quad 1 \end{array}$$

Thus $a_0 = x+3$ and $a_1 = 1$. The denominator is handled similarly:

<u>x /</u>	1	8	37	50	0
	x	x ² + 8x	x ³ + 8x ² +37x	x ⁴ +8x ³ +37x ² +50x	
1	x+8	x ² + 8x+37	x ³ + 8x ² +37x+50	x ⁴ +8x ³ +37x ² +50x	
	x	2x ² + 8x	3x ³ +16x ² +37x		
1	2x+8	3x ² +16x+37	4x ³ +24x ² +74x+50		
	x	3x ² + 8x			
1	3x+8	6x ² +24x+37			
	x				
1	4x+8				
	1				
1					

Thus $b_0 = x^4+8x^3+37x^2+50x$, $b_1 = 4x^3+24x^2+74x+50$, $b_2 = 6x^2+24x+37$,
 $b_3 = 4x+8$ and $b_4 = 1$.

Step 2: Find the coefficients of $D(\omega)$. From equation 2.4,

$$d_1 = -(x+3)(4x^3+24x^2+74x+50)+(x^4+8x^3+37x^2+50x)$$

$$= -(3x^4+28x^3+109x^2+222x+150) \text{ and}$$

$$d_3 = (x+3)(4x+8)-(6x^2+24x+37) = -(2x^2+4x+13). \text{ Also } d_5 = 1.$$

Step 3: Solve for the roots of $D(\omega)$ by setting $D(\omega) = 0$.

$$\omega^5-(2x^2+4x+13)\omega^3-(3x^4+28x^3+109x^2+222x+150)\omega = 0 \quad (4.1)$$

Removing the zero root and using the quadratic formula gives

$$\omega = 0 \text{ and}$$

$$\omega^2 = (x^2+2x+6.5) \pm 2\sqrt{x^4+8x^3+31.5x^2+62x+48.0625} \quad (4.2)$$

Equation 4.2 gives an explicit relation between ω and x for all solutions of equation 2.1. Note that if the discriminant of equation 4.2 is negative, there are no solutions. If the discriminant is positive, there may be either two or four solutions depending on the

relative magnitudes of the elements of the equation. Setting $\omega = 0$ in equation 4.1 gives a polynomial the roots of which are the points where the locus crosses the real axis. In this case they are -4.20588 and -1.17395. The root locus diagram for this problem is shown in figure 5.

The General Analytic Solution

For problems having less than five poles and zeros a solution can be found such that the root locus diagram is completely defined in terms of the system parameters. Combining analytic geometry with the methods above is the key.

Consider the case where GH has a zero at $s = c$ and a pair of complex conjugate poles at $s = u + jv$ and $s = u - jv$. The expression for the transfer function is

$$GH = \frac{K(s-c)}{s^2 - 2us + u^2 + v^2} \quad (4.3)$$

Following the procedure as before, $a_1 = 1$ and $a_0 = x - c$. Also $b_2 = 1$, $b_1 = 2(x - u)$ and $b_0 = x^2 - 2ux + u^2 + v^2$. Then $D(\omega) = \omega^3 + (-x^2 + 2cx - 2uc + u^2 + v^2)\omega$. Setting $D(\omega) = 0$ leads to the solutions

$$\omega = 0$$

$$\text{and} \quad \omega^2 + (x - c)^2 = (u - c)^2 + v^2. \quad (4.4)$$

From analytic geometry this is recognized as the equation for a circle with its center at c and a radius equal to the square root of $(u - c)^2 + v^2$. Note that the circle passes through the points $u + jv$ and $u - jv$ as expected. The circle is also symmetrical with respect to the real axis.

The value of K can be determined most easily by substitution into

Figure 5. The root locus plot for $GH = \frac{K(s+3)}{s(s+2)(s^2+6s+25)}$.

equation 4.3. That is, for $\omega = 0$, $s = x$ and from equation 2.1,

$$GH = \frac{K(x-c)}{x^2 - 2ux + u^2 + v^2} = -1$$

or $K = (x^2 - 2ux + u^2 + v^2)/(c - x)$. For points on the circle $s = x+j\omega$ where x and ω are related by equation 4.4. Then

$$\begin{aligned} K &= \frac{(x + j\omega)^2 - 2u(x + j\omega) + u^2 + v^2}{c - x - j\omega} \\ &= \frac{x^2 + 2jx\omega - \omega^2 - 2ux - 2ju\omega + u^2 + v^2}{c - x - j\omega} \\ &= \frac{[(x - u)^2 + v^2 - \omega^2 + 2j(x - u)\omega](c - x + j\omega)}{(c - x - j\omega)(c - x + j\omega)} \\ &= \frac{[(x - u)^2 + v^2 - \omega^2](c - x) - 2\omega^2(x - u)}{(c - x)^2 + \omega^2} \quad (4.5) \end{aligned}$$

The imaginary part is equal to zero by the derivation of equation 4.4.

The denominator is equal to the square of the radius of the circle.

By substituting equation 4.4 into equation 4.5,

$$\begin{aligned} K &= \frac{[(x - u)^2 + x^2 - 2cx + 2uc - u^2](c - x) - 2\omega^2(x - u)}{r^2} \\ &= \frac{2(x^2 - ux - cx + uc)(c - x) - 2\omega^2(x - u)}{r^2} \\ &= \frac{-2(x - u)(x - c)^2 - 2\omega^2(x - u)}{r^2} \\ K &= -2(x - u). \quad (4.6) \end{aligned}$$

This is the equation for the gain constant for points on the circle.

The equation is symmetrical with respect to the real axis since the same relation obtains when the substitution $s = x-j\omega$ is made.

Table II is presented as a summary of the solutions for root loci when $m + n$ is less than five. The table was developed using the

techniques above. The entries are coded for easy reference. The first number is the total number of poles and zeros or $m + n$. The second number is the number of complex roots of the transfer function. The third number is the total number of poles. The code for the example on page 33 is 3.2.2. In the diagrams dashed lines will be used to denote portions of the locus for which K is negative. Solid lines will be used where K is positive. The arrows point in the direction of increasing values of K . The real part of solution points will be denoted by σ since the equations now will refer to a general relationship between σ and ω for points in the s -plane.

The table contains entries for which the number of poles is greater than the number of zeros. If the transfer function has an equal number of poles and zeros, and any of the roots are complex, they are listed as poles. This does not limit the generality of the table, however. For example, let $GH = K(s^2 - 2us + u^2 + v^2)$. For this transfer function m is greater than n . The root locus can be found by using the dual of entry 2.2.2. The dual is defined as follows: Replace all zeros by poles and poles by zeros; invert all equations concerning the gain constant K ; reverse the direction of all arrows. Figure 6a is the root locus for the transfer function above.

As a second example let $GH = K(s^2 + 6s + 34)/s(s + 1)$. Since the zeros are complex the solution must be found from the dual of entry 4.2.2. The solution is shown in figure 6b. On the axis $K = -\sigma(\sigma + 1)/(\sigma^2 + 6\sigma + 34)$. On the circle $K = -(2\sigma + 1)/2(\sigma + 3) = -(\sigma + .5)/(\sigma + 3)$.

Figure 6. a. The root locus for $GH = K(s^2 - 2us + u^2 + v^2)$.
 b. The root locus for $GH = K(s^2 + 6s + 34)/s(s + 1)$.

TABLE II
ROOT LOCI FOR LOW ORDER SYSTEMS

CODE: 1.0.1.

TRANSFER FUNCTION: $GH = K/(s-a)$

$D(\omega): \omega = 0$

REMARKS: For all σ the locus is the real axis. $K = -(\sigma-a)$.

Thus K is positive to the left of a and K is negative to the right of a .

DIAGRAM:

CODE: 2.0.1.

TRANSFER FUNCTION: $GH = K(s-a)/(s-b)$

$D(\omega): \omega(a-b) = 0$

REMARKS: For all σ the locus is on the real axis. K is positive for $b < \sigma < a$ or for $a < \sigma < b$ since $K = (b-\sigma)/(\sigma-a)$.

DIAGRAM:

TABLE II (Continued)

CODE: 2.0.2.

TRANSFER FUNCTION: $GH = K/(s-a)(s-b)$ $D(\omega): \omega(a+b-2\sigma) = 0$ REMARKS: If $\sigma = (a+b)/2$, ω can assume any value. If not, $\omega = 0$.

This means that the locus includes the real axis and a vertical line which intersects the real axis at $(a+b)/2$. On the axis $K = (a-\sigma)(\sigma-b)$. On the vertical line $K = \frac{(a-b)^2}{4} + \omega^2$. K is positive on the vertical line and for $a < \sigma < b$ or $b < \sigma < a$.

DIAGRAM:

CODE: 2.2.2.

TRANSFER FUNCTION: $GH = K/(s^2 - 2\sigma s + \sigma^2 + v^2)$ $D(\omega): 2(\sigma - u)\omega = 0$

REMARKS: If $\sigma = u$, ω can assume any value. If not, $\omega = 0$. The locus includes the real axis and a vertical line which intersects the real axis at u . On the axis $K = -(\sigma^2 - 2\sigma u + u^2 + v^2)$ which is always negative. For the vertical line $K = \omega^2 - v^2$ which is positive for $\omega^2 > v^2$.

DIAGRAM:

TABLE II (Continued)

CODE: 3.0.2.

TRANSFER FUNCTION: $GH = K(s-c)/(s-a)(s-b)$

$$D(\omega): \omega^3 + (\sigma^2 - 2c\sigma - ab + ac + bc)\omega = 0$$

$$1). \quad \omega = 0$$

$$2). \quad \omega^2 + \sigma^2 - 2c\sigma - ab + ac + bc = 0$$

REMARKS: The first equation defines the real axis. The second equation defines a circle with its center at c and a radius which is the square root of $c^2 + ab - ac - bc$. The polarity of K depends on the relative positions of the poles and zero. On the axis $K = (\sigma - a)(\sigma - b)/(c - \sigma)$. On the circle $K = a + b - 2\sigma$.

DIAGRAM: 1). $c < b < a$ 2). $b < c < a$ 3). $a < b < c$

TABLE II (Continued)

CODE: 3.0.3.

TRANSFER FUNCTION: $GH = K/(s-a)(s-b)(s-c)$ where $a < b < c$

$$D(\omega): -\omega^3 + [3\sigma^2 - 2\sigma(a+b+c) + ab+ac+bc]\omega = 0$$

$$1). \quad \omega = 0$$

$$2). \quad 3(\sigma-h)^2 - \omega^2 = r^2$$

$$\text{where } h = (a+b+c)/3 \text{ and } r^2 = 3h^2 - (ab+ac+bc)$$

REMARKS: The first equation defines the real axis. The second equation defines an hyperbola with its center at $(h,0)$ and with asymptotes at sixty degrees from the real axis. On the real axis $K = -(\sigma-a)(\sigma-b)(\sigma-c)$ which is positive for $\sigma < a$ and for $b < \sigma < c$. For points on the hyperbola it is found that $K = 8\sigma^3 - 24h\sigma^2 + 2(12h^2 - r^2)\sigma + abc - 3h(3h^2 - r^2)$ which is positive for $\sigma > h$ and negative for $\sigma < h$.

DIAGRAM:

TABLE II (Continued)

CODE: 3.2.2.

TRANSFER FUNCTION: $GH = \frac{K(s-c)}{s^2-2us+u^2+v^2}$ $D(\omega): \omega^3 + (\sigma^2 - 2\sigma u + 2uc - u^2 - v^2)\omega = 0$

1). $\omega = 0$

2). $\omega^2 + (\sigma - c)^2 = (u - c)^2 + v^2$

REMARKS: The first equation defines the real axis. The second equation defines a circle with its center at $(\sigma - c, 0)$ and a radius squared of $(u - c)^2 + v^2$. On the circle $K = -2(\sigma - u)$ which is positive for $\sigma < u$. $K = -(\sigma^2 - 2\sigma u + u^2 + v^2) / (\sigma - c)$ on the axis which is positive for $\sigma < c$.

DIAGRAM:

TABLE II (Continued)

CODE: 3.2.3.

TRANSFER FUNCTION: $GH = \frac{K}{(s-a)(s^2-2us+u^2+v^2)}$

$D(\omega): -\omega^3 + [3\sigma^2 - 2\sigma(2u-a) + 2ua + u^2 + v^2]\omega = 0$

1). $\omega = 0$

2). $3(\sigma-h)^2 - \omega^2 = r^2$

where $h = (2u+a)/3$ and $r^2 = [(a-u)^2/3] - v^2$

REMARKS: The first equation defines the real axis. The second equation defines an hyperbola with its center at h and asymptotes at sixty degrees from the horizontal. On the axis $K = -(\sigma-a)(\sigma^2-2u\sigma+u^2+v^2)$ which is positive for $\sigma < a$. On the hyperbola $K = 2(\sigma-u)[(2\sigma-a)^2 + (a-u)^2 + v^2 - a^2]$ which is positive for $\sigma > u$.

DIAGRAM:

1). $r^2 < 0$

2). $r^2 > 0$

3). $r^2 = 0$

TABLE II (Continued)

CODE: 4.0.2.

TRANSFER FUNCTION: $GH = \frac{(s-c)(s-d)}{(s-a)(s-b)}$ where $a > b$ and $c > d$ $D(\omega): (a+b-c-d)\omega^3 + [(a+b-c-d)\sigma^2 - 2(ab-cd)\sigma + ab(c+d) - cd(a+b)]\omega = 0$

1). $\omega = 0$

2). $\omega^2 + (\sigma - h)^2 = r^2$

where $h = (ab - cd) / (a + b - c - d)$

and $r^2 = h^2 + \frac{cd(a+b) - ab(c+d)}{a+b-c-d}$

REMARKS: The first equation defines the real axis. The second equation defines a circle with its center at $(h, 0)$ and radius r . On the real axis $K = - \frac{(\sigma - a)(\sigma - b)}{(\sigma - c)(\sigma - d)}$ which may be positive or negative depending upon the relative positions of the poles and zeros. On the circle $K = - \frac{2\sigma - (a+b)}{2\sigma - (c+d)}$.

DIAGRAM:

1). $d < c < b < a$

2). $d < b < c < a$

3). $b < d < c < a$

TABLE II (Continued)

CODE: 4.2.2.

TRANSFER FUNCTION: $GH = \frac{K(s-c)(s-d)}{s^2-2us+u^2+v^2}$ where $c < d$ $D(\omega): (2u-c-d)\omega^3 + [(2u-c-d)\sigma^2 - 2(u^2+v^2-cd)\sigma + (u^2+v^2)(c+d) - 2ucd]\omega = 0$

1). $\omega = 0$

2). $\omega^2 + (\sigma - h)^2 = r^2$

where $h = \frac{u^2+v^2-cd}{2u-c-d}$

and $r^2 = h^2 + \frac{2ucd - (u^2+v^2)(c+d)}{2u-c-d}$

REMARKS: The first equation defines the real axis. The second equation defines a circle with its center at $(h, 0)$ and radius r .On the real axis $K = -\frac{\sigma^2 - 2u\sigma + u^2 + v^2}{(\sigma - c)(\sigma - d)}$ and on the circle $K = -\frac{2(\sigma - u)}{2\sigma - c - d}$. Note that if $\sigma = (c+d)/2$ the radius is infinite. In this case the locus is a vertical line at $\sigma = u$ and

$K = \frac{4(v^2 - \omega^2)}{4\omega^2 + (d-c)^2}.$

DIAGRAM:

1).

2). $\sigma = \frac{(c+d)}{2}$

TABLE II (Continued)

CODE: 4.4.2.

TRANSFER FUNCTION: $GH = (s^2 - 2us + u^2 + v^2) / (s^2 - 2ts + t^2 + y^2)$ $D(\omega): (u-t)\omega^3 + [(u-t)\sigma^2 - (u^2 + v^2 - t^2 - y^2)\sigma - t(u^2 + v^2) - u(t^2 + y^2)]\omega = 0$

1). $\omega = 0$

2). $\omega^2 + (\sigma - h)^2 = r^2$

where $h = \frac{u^2 + v^2 - t^2 - y^2}{2(u-t)}$

and $r^2 = h^2 + \frac{u(t^2 + y^2) - t(u^2 + v^2)}{u-t}$

REMARKS: The first equation defines the real axis. The second equation defines a circle with its center at $(h, 0)$ and radius r .

On the real axis $K = -\frac{\sigma^2 - 2t\sigma + t^2 + y^2}{\sigma^2 - 2u\sigma + u^2 + v^2}$. On the circle

$K = -\frac{\sigma - u}{\sigma - t}$. When $u = t$ the radius is infinite. The locus

is a vertical line at $\sigma = u = t$. On the vertical line

$$K = -(y^2 - \omega^2) / (v^2 - \omega^2).$$

DIAGRAM:

1). $t \neq u$ 2). $t = u$

Conclusion

It is hoped that the methods developed herein will be of significance to the field of automatic control theory. The procedures have been developed along new lines in order to avoid the inherent disadvantages of classical methods. Some classical procedures such as rules for sketching loci may be used to advantage. The method developed here does not rely upon these procedures but may be enhanced by them.

The major intent has been to develop an algorithm for solution by digital computers. Previous attempts in this direction involved scanning the entire s -plane by solving an equation which changed sign as a branch of the root locus was crossed.¹⁵ This required a coarse matrix of points in the s -plane with subsequent reduction of the size of the matrix with a corresponding increase in density near areas where the sign changes were noted. This type of attack has many disadvantages. The original matrix must be sufficiently dense to minimize the possibility of skipping over a solution. The accuracy of the method is limited by the final density of the matrix. To improve the accuracy many more computations must be made. Furthermore, the value of the system gain varies greatly in the vicinity of the locus. In many cases this could become a significant factor.

The computer program listed in CHAPTER III does not have these disadvantages. Although the method requires finding a point where a function passes through zero, the function is of one variable and

¹⁵A. M. Krall, "Algorithm for Generating Root Locus Diagrams," Assn. Computing Machinery, May, 1967, pp. 186-188.

lends itself to iterative techniques. The accuracy generally depends on the limits of the computer and the accuracy of the input data rather than the algorithm. Storage of solution points is unnecessary. As each point is found, it may be printed and removed from consideration.

The accuracy and speed with which problems can be solved gives this method preference over the use of tools or special machines for plotting root loci. The example of Figure 4 was solved not only with the digital computer but also using a Spirule and an algebraic computer. The Spirule solution required approximately three hours and proved to be about 1% reliable. The algebraic computer solution took one hour with five percent accuracy. The solution using the digital program required five minutes on a teletype terminal and forty-five minutes of preparation and plotting time.

A further advantage is that the factors of the transfer function need not be known. The graphical methods require that the location of each pole and zero be known. This would require that the transfer function be in factored form. Generally the factors are not known and must be determined. This involves either a separate computer program or a preliminary root locus to find these roots. The program developed in Chapter III does not require that the function be factored. With respect to this also, systems with time delays (denoted by terms of the form e^{-Ts}) are approximated by the equivalent series $1 - Ts + \frac{T^2 s^2}{2!} - \dots$.¹⁶ The necessity of factoring an approximate polynomial of this type would

¹⁶Westman, p. 1084.

limit its value. The program can handle this easily without factoring. The number of terms used would not have a major effect on the time taken to solve the problem.

The method has far-reaching possibilities and the program is fast and reliable. With the aid of plotting routines and oscilloscope interfacing, it may be possible to design systems in a fraction of the time required using other methods.

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