The Non-Vanishing of the Trace of $T_3$

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THE NON-VANISHING OF THE TRACE OF $T_3$

LIUBOMIR CHIRIAC, DAPHNE KURZENHAUSER, AND ERIN WILLIAMS

Abstract. A generalized Lehmer conjecture predicts that, for every positive integer $n$, the trace of the Hecke operator $T_n$ in level one does not vanish, unless the space of cusp forms acted upon is trivial. So far, this has only been established for $n = 2$. In this paper, we use $p$-adic methods to prove the statement for $n = 3$.

1. Introduction

Consider the Delta function

$$\Delta(z) = q \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n)q^n, \quad (q = \exp(2\pi iz))$$

which is the unique normalized cusp form of weight 12 for the full modular group. A famous open question, due to Lehmer [Leh47], asks whether there exists an integer $n \geq 1$ such that $\tau(n)$ is zero. It is widely believed not only that $\Delta$ has no vanishing Fourier coefficients, but also that the same phenomenon occurs whenever the space of cusp forms for $SL_2(\mathbb{Z})$ is one-dimensional, i.e., the weight $2k$ is in the set $\{12, 16, 18, 20, 22, 26\}$. For these values, the $n$-th Fourier coefficient of the corresponding normalized cusp form coincides with the trace of the Hecke operator $T_n$ acting on the space of cusp forms of weight $2k$ and level one, which will be denoted by $\text{Tr} \ T_n(2k)$.

Motivated by the above, Rouse [Rou06] proposed a “Generalized Lehmer Conjecture”, which posits that the trace $\text{Tr} \ T_n(2k)$ does not vanish for all even weights $2k \geq 16$ or $2k = 12$. In fact, [Rou06, Conjecture 1.5] predicts the non-vanishing of the trace of $T_n$, with $n \geq 1$ not a square, in every level $N$ coprime to $n$. As evidence, Rouse proved the case $n = 2$ using a computational algorithm.

In this paper, we make further progress on the Generalized Lehmer Conjecture by proving it for $n = 3$ and level one.

Theorem 1. Suppose that $2k \geq 16$ or $2k = 12$. Then $\text{Tr} \ T_3(2k) \neq 0$.

We mention that the analogous statement for $n = 2$ also follows from a recent paper of Chiriac and Jorza [CJ22], where a stronger result was obtained—namely that $\text{Tr} \ T_2(2k)$ takes no repeated values, except for 0, which occurs only when the space is trivial. Their proof combines 2-adic results from [CJ21] and a new application of classical bounds on linear forms in logarithms in the context of exponential sums with more than two terms. The case $n = 3$ appears to be more difficult because of the presence of additional terms in the exponential sums defining the trace.

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We take our lead from [CJ22] and use the Eichler-Selberg trace formula to express the trace of $T_3$ in terms of linear recurrence sequences. This reduces the problem to proving the non-vanishing of three subsequences, according to some congruence classes of $k$ (see Proposition 3). To this end, we employ a general practical method, based on $p$-adic arguments, developed by Mignotte and Tzanakis [MT91]. This method has found applications in the study of ternary sequences, particularly Berstel’s sequence [MT93]. The idea is that given an equation of the form $u_n = c$ and a conjectured set $\mathcal{M}$ of solutions $n \in \mathbb{Z}$, a suitable choice of primes possessing certain properties can guarantee that $\mathcal{M}$ does indeed include all solutions; we elaborate on the specifics in Section 4.

2. Background

In this section, we briefly review some basic facts from the theory of modular forms for the full modular group $SL_2(\mathbb{Z})$, that is, of level one. For a more comprehensive account, the reader is referred to [Kil15].

Let $\mathfrak{h} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \}$ be the upper half-plane. A modular form of even weight $2k$ and level one is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ with the following properties:

(i) $f(z + 1) = f(z)$ and $f(-1/z) = z^{2k}f(z)$ for all $z \in \mathfrak{h}$;
(ii) $f(z)$ is bounded as $\text{Im}(z) \to \infty$.

Every modular form $f$ admits a unique Fourier expansion $f(z) = \sum_{n \geq 0} a(n)q^n$, and if $a(0) = 0$ we call $f$ a cusp form. In addition, a cusp form $f$ is normalized if $a(1) = 1$.

Denote by $S_{2k}$ the set of cusp forms of weight $2k$ and level one. This is a vector space over $\mathbb{C}$ of finite dimension $d$, namely

$$d = \begin{cases} \lfloor k/6 \rfloor - 1 & \text{if } k \equiv 1 \pmod{6} \\ \lfloor k/6 \rfloor & \text{otherwise.} \end{cases}$$

The space $S_{2k}$ is endowed with the action of certain linear transformations, called Hecke operators. These can be defined, for all positive integer $m$, as linear maps $T_m : S_{2k} \to S_{2k}$ given by

$$T_m \left( \sum_{n \geq 1} a(n)q^n \right) = \sum_{n \geq 1} \left( \sum_{d | \gcd(m,n)} d^{2k-1}a(mn/d^2) \right) q^n.$$  

If $p$ is a prime number, the effect of $T_p$ on $f(z) = \sum_{n \geq 1} a(n)q^n$ can be described as

$$T_p f = \sum_{n \geq 1} \left( a(pm) + p^{2k-1}a(n/p) \right) q^n,$$

with the understanding that $a(n/p) = 0$ whenever $p \nmid n$. It turns out that the characteristic polynomial of $T_m$ has integer coefficients. In particular, its trace $\text{Tr} T_m(2k)$ is also an integer.

A common way to compute traces of Hecke operators is using the Eichler-Selberg trace formula. This involves the Hurwitz class number $H(n)$, which counts the weighted number of equivalence classes of positive definite binary quadratic forms of discriminant $-n$. More precisely, the class containing $x^2 + y^2$ is weighted by $1/2$, and the class containing $x^2 + xy + y^2$ is weighted by $1/3$. For instance, $H(3) = 1/3$ and $H(12) = 4/3$, whereas $H(8) = H(11) = 1$. We also set $H(n) = 0$ if $n \equiv 1$ or $2 \pmod{4}$, and $H(0) = -1/12$. 

2
Following Zagier’s Appendix to [Lan76], we recall the following version of the Eichler-Selberg trace formula on \( SL_2(\mathbb{Z}) \): for all integers \( m \geq 1 \) and \( k \geq 2 \) we have

\[
\text{Tr } T_m(2k) = -\frac{1}{2} \sum_{|t| \leq 2\sqrt{m}} P_{2k}(t, m) H(4m - t^2) - \frac{1}{2} \sum_{dd' = m} \min(d, d')^{2k-1},
\]

where \( P_{2k}(t, m) \) is the coefficient of \( x^{2k-2} \) in the power series expansion of \( (1-tx+mx^2)^{-1} \). It is not hard to verify (see, for example, the proof of [CJ22, Lemma 3]) that \( P_{2k}(t, m) \) satisfies the following combinatorial formula:

\[
P_{2k}(t, m) = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} m^j t^{2k-2-2j}.
\] (1)

Since our main interest is in the case \( m = 3 \), we compute

\[
\text{Tr } T_3(2k) = -\frac{1}{2} P_{2k}(0, 3) H(12) - P_{2k}(1, 3) H(11) - P_{2k}(2, 3) H(8) - P_{2k}(3, 3) H(3) - 1
\]

\[
= -\frac{2}{3} P_{2k}(0, 3) - P_{2k}(1, 3) - P_{2k}(2, 3) - \frac{1}{3} P_{2k}(3, 3) - 1,
\]

which combined with (1) gives

\[
\text{Tr } T_3(2k) = -1 - 2 \cdot (-3)^{k-2} - \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j (1 + 2^{2k-2-2j} + 3^{2k-3-2j}).
\] (2)

3. THE TRACE AS A SUM OF RECURRENT SEQUENCES

Our immediate goal is to manipulate identity (2) in order to obtain certain recurrent sequences. The following lemma summarizes the calculations needed.

**Lemma 2.** For any \( u, v \in \mathbb{R} \setminus \{0\} \), consider the sequence \( \{\alpha_n\}_{n \geq 0} \) defined as

\[
\alpha_n := \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} u^j v^{n-j}.
\]

Then for every integer \( n \geq 2 \) we have the recurrence relation

\[
\alpha_n = (v - 2u)\alpha_{n-1} - u^2 \alpha_{n-2},
\]

where \( \alpha_0 = 1 \) and \( \alpha_1 = v - u \).

**Proof.** The generating function of the sum given by the right-hand side is

\[
\sum_{n=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} u^j v^{n-j} x^n.
\]
Setting \( m = n - j \) and changing variables gives us

\[
\sum_{m=0}^{\infty} \sum_{j=0}^{n} (-1)^j \binom{2m+j}{j} u^j v^m x^j x^m = \sum_{m=0}^{\infty} (vx)^m \sum_{j=0}^{\infty} (-1)^j \binom{2m+1+j-1}{j} (ux)^j
\]

\[
= \sum_{m=0}^{\infty} (vx)^m \frac{1}{(1+ux)^{2m+1}}
\]

(3)

\[
= \frac{1}{1+ux} \sum_{m=0}^{\infty} \left( \frac{vx}{(1+ux)^2} \right)^m
\]

\[
= \frac{1}{1+ux} \left( 1 - \frac{vx}{(1+ux)^2} \right)^{-1}
\]

(4)

\[
= \frac{1}{1+ux} \left( \frac{1+2ux+u^2x^2 - vx}{(1+ux)^2} \right)^{-1}
\]

\[
= \frac{1 + ux}{1 + (2u-v)x + u^2x^2}
\]

where in (3) we have used the negative binomial series

\[
(1+x)^{-d} = \sum_{j=0}^{\infty} (-1)^j \binom{d+j-1}{j} x^j,
\]

and in (4) used the formula for the sum of a geometric series. Now, let \( G(x) = \sum_{n=0}^{\infty} \alpha_n x^n \) be the generating function of the sequence \( \{\alpha_n\}_{n \geq 0} \). As

\[
\alpha_n + (2u-v)\alpha_{n-1} + u^2\alpha_{n-2} = 0
\]

for \( n \geq 2 \), it follows that \( G(x) \) satisfies

\[
(G(x) - 1 + (u-v)x) + (2u-v)x(G(x) - 1) + u^2x^2G(x) = 0.
\]

This further shows that

\[
G(x) = \frac{1 + ux}{1 + (2u-v)x + u^2x^2}.
\]

The generating functions of both sequences are the same, so the statement of the lemma follows.

Using Lemma 2 we are now prepared to prove the main result of this section.

**Proposition 3.** Let \( \{a_n\}_{n \geq 0} \) and \( \{b_n\}_{n \geq 0} \) be the sequences given by the recurrences

\[
a_n = -5a_{n-1} - 9a_{n-2} \text{ for } n \geq 2, \quad a_0 = 1, \quad a_1 = -2
\]

and

\[
b_n = -2b_{n-1} - 9b_{n-2} \text{ for } n \geq 2, \quad b_0 = 1, \quad b_1 = 1,
\]

respectively. Then for all integers \( k \geq 2 \), we have that

\[
\text{Tr } T_3(2k) = \begin{cases} 
-1 - a_{k-1} - b_{k-1} & \text{if } k \equiv 0 \pmod{3} \\
-1 - a_{k-1} - b_{k-1} - 3^{k-1} & \text{if } k \equiv 1, 3 \pmod{6} \\
-1 - a_{k-1} - b_{k-1} + 3^{k-1} & \text{if } k \equiv 0, 4 \pmod{6}
\end{cases}
\]
Proof. In view of identity (2), we can write \( \text{Tr} T_3(2k) = -1 - S_1 - S_2 - S_3 \), where

\[
S_1 = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j
\]

\[
S_2 = \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^j 2^{2k-2j-2}
\]

\[
S_3 = 2 \cdot (-3)^{k-2} + \sum_{j=0}^{k-1} (-1)^j \binom{2k-2-j}{j} 3^{2k-3-j}
\]

Setting \( u = 3 \) and \( v = 1 \) in Lemma 2, we see that \( S_1 \) is equal to \( a_{k-1} \). Similarly, \( u = 3 \) and \( v = 4 \) give that \( S_2 \) is equal to \( b_{k-1} \). It remains to show that

\[
S_3 = \begin{cases} 
0 & \text{if } k \equiv 2 \pmod{3} \\
-3^{k-1} & \text{if } k \equiv 1, 3 \pmod{6} \\
3^{k-1} & \text{if } k \equiv 0, 4 \pmod{6}.
\end{cases}
\]

Equivalently, for every \( n \geq 1 \), we must prove that

\[
2(-3)^{n-1} - \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{2n-j-1} = \begin{cases} 
0 & \text{if } n \equiv 1 \pmod{3} \\
-3^n & \text{if } n \equiv 0, 2 \pmod{6} \\
3^n & \text{if } n \equiv 3, 5 \pmod{6}.
\end{cases}
\]

To do this, we introduce an auxiliary sequence defined as

\[
d_n := \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j}.
\]

Applying Lemma 2 with \( u = 1 \) and \( v = 3 \), we see that \( \{d_n\} \) is a sequence with initial values \( d_0 = 1 \) and \( d_1 = 2 \), and \( d_n = d_{n-1} - d_{n-2} \) for \( n \geq 2 \). Induction verifies that \( \{d_n\} \) is a periodic sequence of period 6, and

\[
\sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j} = \begin{cases} 
2 & \text{if } n \equiv 1 \pmod{6} \\
-2 & \text{if } n \equiv 4 \pmod{6} \\
1 & \text{if } n \equiv 0, 2 \pmod{6} \\
-1 & \text{if } n \equiv 3, 5 \pmod{6}.
\end{cases}
\]

Adding \( 2(-1)^n \) to both sides yields

\[
2(-1)^n + \sum_{j=0}^{n} (-1)^j \binom{2n-j}{j} 3^{n-j} = \begin{cases} 
0 & \text{if } n \equiv 1, 4 \pmod{6} \\
3 & \text{if } n \equiv 0, 2 \pmod{6} \\
-3 & \text{if } n \equiv 3, 5 \pmod{6}.
\end{cases}
\]

and multiplying by \( -3^{n-1} \) gives the desired result.

\[ \square \]

As \( \text{Tr} T_3 \) is the sum of recurrent sequences, it is also a recurrent sequence. The following corollary makes this observation explicit.
Corollary 4. Let \( \{t_k\}_{k \geq 2} \) be the sequence defined as \( t_k := \text{Tr} \ T_3(2k) \). Then \( \{t_k\} \) has initial values given by the table below

\[
\begin{array}{c|cccccccc}
 k & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
 t_k & 0 & 0 & 0 & 252 & 0 & -3348 & -4284 \\
\end{array}
\]

and for every \( k \geq 10 \) it satisfies the recurrence relation

\[
t_k = -6t_{k-1} - 21t_{k-2} - 62t_{k-3} - 180t_{k-4} - 486t_{k-5} - 945t_{k-6} - 486k_{k-7} + 2187t_{k-8}.
\]

Proof. It is easy to compute the initial values directly from Proposition 3. The characteristic polynomial of \( \{t_k\} \), denoted by \( t(x) \), is the product of the characteristic polynomials of the individual sequences that comprise \( \text{Tr} \ T_3(2k) \).

Obviously, one can regard the constant \(-1\) as a trivial recurrent sequence with characteristic polynomial \( x - 1 \). The characteristic polynomials of \( \{a_n\} \) and \( \{b_n\} \) are \( x^2 + 5x + 9 \) and \( x^2 + 2x + 9 \), respectively. Finally, let \( \{c_n\} \) be the sequence satisfying \( c_n = -27c_{n-3} \) for \( n \geq 3 \) with the initial conditions \( c_0 = -1, c_1 = 0, \) and \( c_2 = -9 \); its characteristic polynomial is \( x^3 + 27 \). Using induction, one can show that

\[
c_n = \begin{cases} 
0 & \text{if } n \equiv 1 \pmod{3} \\
-3^n & \text{if } n \equiv 0, 2 \pmod{6} \\
3^n & \text{if } n \equiv 3, 5 \pmod{6}
\end{cases}
\]

for all \( n \geq 0 \). We obtain that

\[
t(x) = (x - 1)(x^2 + 5x + 9)(x^2 + 2x + 9)(x^3 + 27)
\]

\[
= x^8 + 6x^7 + 21x^6 + 62x^5 + 180x^4 + 486x^3 + 945x^2 + 486x - 2187
\]

and the conclusion follows.

\[
\square
\]

4. The non-vanishing of the trace

With \( a_n \) and \( b_n \) as they appear in Proposition 3 let \( u_n = a_n + b_n \). We know that

\[
\text{Tr} \ T_3(2k) = \begin{cases} 
-1 - u_{k-1} & \text{if } k \equiv 2 \pmod{3} \\
-1 - u_{k-1} - 3^{k-1} & \text{if } k \equiv 1, 3 \pmod{6} \\
-1 - u_{k-1} + 3^{k-1} & \text{if } k \equiv 0, 4 \pmod{6}
\end{cases}
\]

To establish Theorem 1 we must prove that \( \text{Tr} \ T_3(2k) = 0 \) if and only if \( k \in \{2, 3, 4, 5, 7\} \). In fact, we will completely determine when the sequences \( \{u_n\} \) and \( \{u_n \pm 3^n\} \) take the value \(-1\). Our main tool is the previously mentioned result of Mignotte and Tzanakis. Their setup works for any \( k \)-th degree recurrence \( \{u_n\} \) with integer coefficients, as long as its characteristic polynomial \( g(x) \) has \( k \) distinct complex roots \( \omega_1, \ldots, \omega_k \).

Given a fixed integer \( c \), to solve the equation \( u_n = c \) (for \( n \)), we choose an odd prime \( p \), not dividing the discriminant or any of the coefficients of \( g(x) \), such that all the roots \( \omega_i \) are \( p \)-adic units. We then search for positive integers \( S \) such that all the numbers \( \omega_i^S \) are congruent to some common integer \( A \) modulo \( p \). If \( A \) has the same order \( R \) both modulo \( p \) and modulo \( p^2 \), then the following result \( [\text{MT91} \ Theorem 1] \) holds.\[1\]

\[1\]There is a typo in the statement of \( [\text{MT91} \ Theorem 1] \), namely the part “if \( n \not\in \mathcal{P} \)” in (ii) is omitted. The correct version appears in \( [\text{MT93} \ Theorem 1] \).
Proposition 5 (Mignotte and Tzanakis). Suppose that $\mathcal{M}$ is a finite set of solutions $m \in \mathbb{Z}$ to the equation $u_m = c$, where either $c \not\equiv 0 \pmod{p}$ or $c = 0$. Let $\mathcal{P}$ be a complete system of residues modulo $S$ such that $\mathcal{M} \subseteq \mathcal{P}$ and which satisfies the following conditions:

(i) $u_m = c$ for each $m \in \mathcal{M}$;
(ii) if $n \in \mathcal{P}$ and $u_n \equiv cA^r \pmod{p}$ for some $r \in \{0, 1, \ldots, R - 1\}$, then $n \in \mathcal{M}$;
(iii) $u_{m+S} \neq Au_m \pmod{p^2}$ for every $m \in \mathcal{M}$.

Then $u_n = c$ implies $n \in \mathcal{M}$.

It is important to emphasize that $\mathcal{M}$ contains all integer solutions, not just positive integers. To extend the definition of $u_n$ to negative integers is straightforward. Indeed, since the characteristic polynomial of $\{u_n\}$ is assumed to have distinct roots, the general term is of the form

$$u_n = \alpha_1 \omega_1^n + \ldots + \alpha_k \omega_k^n,$$

with $\alpha_i \in \mathbb{Q}(\omega_1, \ldots, \omega_k)$. Therefore, it makes sense to talk about $u_n$ for every integer $n$. This will play a role in the proof of Proposition 6 below.

While directly tackling the sequence from Corollary 4 with this method is certainly possible, we take a more gradual approach. This has the advantage of better illustrating the “dead-ends” one can run into when searching for appropriate choices of $p$ and $S$. To perform this search, we used a combination of SageMath and Pari/GP.

Proposition 6. Let $u_n = a_n + b_n$ where $\{a_n\}$ and $\{b_n\}$ are the sequences given in Proposition 3. Extend the definition of $\{u_n\}$ to all $n \in \mathbb{Z}$ as described above. Then

(a) $u_n = -1$ if and only if $n \in \{1, 4\}$;
(b) $u_n + 3^n = -1$ if and only if $n \in \{2, 6\}$;
(c) $u_n - 3^n = -1$ if and only if $n \in \{-1, 3\}$.

Proof. We begin by including a table of the first few values of the relevant sequences, with the occurrences of the value $-1$ circled:

<table>
<thead>
<tr>
<th>$n$</th>
<th>$-1$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_n$</td>
<td>$-2/3$</td>
<td>2</td>
<td>(-1)</td>
<td>-10</td>
<td>26</td>
<td>(-1)</td>
<td>-10</td>
<td>-730</td>
</tr>
<tr>
<td>$u_n + 3^n$</td>
<td>-1/3</td>
<td>3</td>
<td>2</td>
<td>(-1)</td>
<td>53</td>
<td>80</td>
<td>233</td>
<td>(-1)</td>
</tr>
<tr>
<td>$u_n - 3^n$</td>
<td>(-1)</td>
<td>1</td>
<td>-4</td>
<td>-19</td>
<td>(-1)</td>
<td>-82</td>
<td>-253</td>
<td>-1459</td>
</tr>
</tbody>
</table>

We also note that the characteristic polynomial of the sequence $\{u_n\}$ is

$$g(x) = (x^2 + 2x + 9)(x^2 + 5x + 9) = x^4 + 7x^3 + 28x^2 + 63x + 81$$

and its discriminant is $2^5 \cdot 3^8 \cdot 11$.

(a) We choose $p = 59$. The roots $\omega_1$ and $\omega_2$ of $x^2 + 2x + 9$, written 59-adically, are

$$12 + 43 \cdot 59 + 28 \cdot 59^2 + O(59^3)$$

and

$$45 + 15 \cdot 59 + 30 \cdot 59^2 + O(59^3).$$

Since $\left(\frac{12}{59}\right) = \left(\frac{45}{59}\right) = 1$, we see that $\omega_1^{59} \equiv \omega_2^{59} \equiv 1 \pmod{59}$.

Similarly, the roots $\omega_3$ and $\omega_4$ of $x^2 + 5x + 9$, written 59-adically, are $5 + 55 \cdot 59 + 57 \cdot 59^2 + O(59^3)$ and $49 + 3 \cdot 59 + 59^2 + O(59^3)$. As before, $\omega_3^{59} \equiv \omega_4^{59} \equiv 1 \pmod{59}$. Thus, all the roots of $g(x)$ satisfy $\omega_i^{59} \equiv 1 \pmod{59}$. 


We now apply Proposition 5 with \( p = 59, S = 29, A = 1 \) (so \( R = 1 \), \( c = -1 \), \( \mathcal{M} = \{1, 4\} \) and \( \mathcal{P} = \{0, \ldots, 28\} \). Condition (i) is clear. Next, a simple computer check shows that the only \( n \) in the range \( 0 \leq n \leq 28 \) for which \( u_n \equiv -1 \pmod{59} \) are \( n = 1 \) and \( n = 4 \).

For requirement (iii), we compute

\[
\begin{align*}
    u_{1+S} & \equiv 707 \not\equiv u_1 \pmod{59^2} \\
    u_{4+S} & \equiv 766 \not\equiv u_4 \pmod{59^2}.
\end{align*}
\]

In conclusion, the elements of \( \mathcal{M} \) are the only integers \( n \) such that \( u_n = -1 \).

(b) For convenience, let \( u'_n := u_n + 3^n \). We claim that the set of solutions to \( u'_n = -1 \) is \( \mathcal{M} = \{2, 6\} \).

The characteristic polynomial of \( u'_n \) is \( g(x)(x - 3) = x^5 + 4x^4 + 7x^3 - 21x^2 - 108x - 243 \), which has discriminant \( 2^{11} \cdot 3^{12} \cdot 11^3 \). While \( 3^{29} \equiv 1 \pmod{59} \), we note that the choice from part (a): \((p, S, A) = (59, 29, 1)\), will not work. Indeed, requirement (ii) is not satisfied, for

\[ u'_{24} = 326954692403 \equiv -1 \pmod{59} \]

even though \( 24 \not\in \mathcal{M} \).

Fortunately, there are other relatively small values of \( p \) which will work. More precisely, we take \( p = 251 \). The roots of \( g(x)(x - 3) \) reduced modulo 251 are 3, 45, 68, 181, and 201, and

\[ 3^{125} \equiv 45^{125} \equiv 68^{125} \equiv 181^{125} \equiv 201^{125} \equiv 1 \pmod{251} \]

Therefore \((p, S, A) = (251, 125, 1)\) is a valid triple. Once again, one can use software to verify that requirement (ii) is met for the choice \( \mathcal{P} = \{0, \ldots, 124\} \). For (iii), we find that

\[ u'_{2+S} \equiv 24597 \not\equiv u'_2 \pmod{251^2} \]
\[ u'_{6+S} \equiv 34386 \not\equiv u'_6 \pmod{251^2} \]

Proposition 5 tells us that \( u'_n = u_n + 3^n = -1 \) only when \( n \in \{2, 6\} \).

(c) Now let \( u''_n := u_n - 3^n \). This case is different from the previous ones because it is the first time that we encounter a negative solution, namely

\[ u''_{-1} = u_{-1} - 3^{-1} = (-2/3) - (1/3) = -1. \]

As a result, we take \( \mathcal{M} = \{-1, 3\} \). To accommodate for the negative value in \( \mathcal{M} \), we let \( \mathcal{P} = \{-1, \ldots, 27\} \).

The characteristic polynomial of \( u''_n \) is also \( g(x)(x - 3) \), so the choice \((p, S, A) = (59, 29, 1)\) passes the root requirement. As before, software verifies requirement (ii), and we see that

\[ u''_{-1+S} \equiv 2418 \not\equiv u''_{-1} \pmod{59^2} \]
\[ u''_{3+S} \equiv 3303 \not\equiv u''_3 \pmod{59^2} \]

Applying Proposition 5, we obtain that \( u''_n = -1 \) only when \( n \in \{-1, 3\} \).
5. Concluding Remarks

The methods used in this paper are amenable to generalization for larger values of $n$, as well as other congruence subgroups. For instance, one can similarly establish the non-vanishing of the trace of $T_3$ in level 2, denoted by $\text{Tr } T_3(2k, \Gamma_0(2))$. Indeed, work by Frechette, Ono, and Papanikolas [FOP04, Theorem 2.3] gives that for all $k \geq 2$
\[
\text{Tr } T_3(2k, \Gamma_0(2)) = -2 - b_{k-1} - (-3)^{k-1},
\]
where $\{b_n\}$ is the sequence from Proposition 3, namely $b_0 = b_1 = 1$ and $b_n = -2b_{n-1} - 9b_{n-2}$ for $n \geq 2$. Applying Proposition 5 with $u_n = b_n \pm 3^n$ and $(p, S, A) = (11, 5, 1)$ we find that the only zeros occur for $2k \in \{4, 6\}$, which is precisely when the space of weight-2$k$ cusp forms on $\Gamma_0(2)$ (of dimension $\lfloor k/2 \rfloor - 1$) is trivial.

We also remark that in the case of level 4 or level 8, the situation is even easier, for
\[
\text{Tr } T_3(2k, \Gamma_0(4)) = -3 - (-3)^{k-1}
\]
and
\[
\text{Tr } T_3(2k, \Gamma_0(8)) = -4,
\]
as can be seen from [FOP04, Proposition 2.1].

References


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