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Basics of Bessel Functions

by

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requirements for the degree of

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in

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and

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## Abstract

This paper is a deep exploration of the project *Bessel Functions* by Martin Kreh of Pennsylvania State University. We begin with a derivation of the Bessel functions  $J_a(x)$  and  $Y_a(x)$ , which are two solutions to Bessel's differential equation. Next we find the generating function and use it to prove some useful standard results and recurrence relations. We use these recurrence relations to examine the behavior of the Bessel functions at some special values. Then we use contour integration to derive their integral representations, from which we can produce their asymptotic formulae. We also show an alternate method for deriving the first Bessel function using the generating function. Finally, a graph created using Python illustrates the Bessel functions of order 0, 1, 2, 3, and 4.

# 1 Introduction to Bessel Functions

Bessel functions are the standard form of the solutions to Bessel's differential equation,

$$x^2 \frac{\partial^2 y}{\partial x^2} + x \frac{\partial y}{\partial x} + (x^2 - n^2)y = 0, \quad (1)$$

where  $n$  is the *order* of the Bessel equation. It is often obtained by the separation of the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u \quad (2)$$

in cylindric or spherical coordinates. For this reason, the Bessel functions fall under the umbrella of *cylindrical* (or *spherical*) *harmonics* when  $n$  is an integer or half-integer, and we see them appear in the separable solutions to both the Helmholtz equation and Laplace's equation in cylindric or spherical coordinates. Since the Bessel equation is a 2<sup>nd</sup> order differential equation, it has two linearly independent solutions,  $J_n(x)$  and  $Y_n(x)$ .

## 1.1 Bessel Functions of the First Kind

To find the first solution we begin by taking a power series,

$$y(x) = x^n \sum_{k=0}^{\infty} b_k x^k, \quad (3)$$

which we will plug into the Bessel equation (1) and solve for its necessary components. For convenience, the first and second partial derivatives of this power series are:

$$y'(x) = nx^{n-1} \sum_{k=0}^{\infty} b_k x^k + x^n \sum_{k=0}^{\infty} k b_k x^{k-1} \quad (4)$$

and

$$y''(x) = n(n-1)x^{n-2} \sum_{k=0}^{\infty} b_k x^k + 2nx^{n-1} \sum_{k=0}^{\infty} k b_k x^{k-1} + x^n \sum_{k=0}^{\infty} k(k-1)b_k x^{k-2} \quad (5)$$

as given by the multiplication rule. We next multiply equation (4) by  $x$  and equation (5) by  $x^2$  so that we can easily plug them into (1). (Note that the  $x$  or  $x^2$  can be inserted into the front of each term or be distributed into the summation.) We then have

$$xy'(x) = nx^n \sum_{k=0}^{\infty} b_k x^k + x^n \sum_{k=0}^{\infty} kb_k x^k \quad (6)$$

and

$$x^2 y''(x) = n(n-1)x^n \sum_{k=0}^{\infty} b_k x^k + 2nx^n \sum_{k=0}^{\infty} kb_k x^k + x^n \sum_{k=0}^{\infty} k(k-1)x^k. \quad (7)$$

Finally, we will need the term

$$x^2 y(x) = x^n \sum_{k=0}^{\infty} b_k x^{k+2} = x^n \sum_{k=2}^{\infty} b_{k-2} x^k. \quad (8)$$

Note that because we are solving for the appropriate  $b_k$ , we can artificially set  $b_{-2} := b_{-1} := 0$ . This will become useful when simplifying the full Bessel differential equation below. Plugging equations (6), (7), and (8) into (1), we get

$$\begin{aligned} n(n-1)x^n \sum_{k=0}^{\infty} b_k x^k + 2nx^n \sum_{k=0}^{\infty} kb_k x^k + x^n \sum_{k=0}^{\infty} k(k-1)b_k x^k + nx^n \sum_{k=0}^{\infty} b_k x^k \\ + x^n \sum_{k=0}^{\infty} kb_k x^k + x^n \sum_{k=2}^{\infty} b_{k-2} x^k - n^2 x^n \sum_{k=0}^{\infty} b_k x^k = 0, \end{aligned} \quad (9)$$

which can be simplified in the following steps.

**Step One.** Combine the summation terms (we can do this because we defined  $b_{-2}$  and  $b_{-1}$  to be equal to zero, so  $\sum_{k=2}^{\infty} b_{k-2} x^k = \sum_{k=0}^{\infty} b_{k-2} x^k$ ):

$$x^n \sum_{k=0}^{\infty} [n(n-1)b_k + 2nkb_k + k(k-1)b_k + nb_k + kb_k + b_{k-2} - n^2 b_k] x^k = 0. \quad (10)$$

**Step Two.** Cancel like-terms:

$$x^n \sum_{k=0}^{\infty} [2nkb_k + k^2 b_k + b_{k-2}] x^k = 0. \quad (11)$$

**Step Three.** Compare the coefficients to yield:

$$2nkb_k + k^2 b_k + b_{k-2} = 0. \quad (12)$$

We use equation (12) to create the general formula:

$$b_k = \frac{-b_{k-2}}{k(k+2n)}. \quad (13)$$

Since  $b_{-1} = 0$  we can infer that  $b_1 = \frac{-b_{-1}}{1+2n} = 0$ , and continuing in this manner,  $b_{2k-1} = 0 \forall k \in \mathbb{N}$ . What about for even values of  $k$ ? We have no condition on  $b_0$  (since  $k = 0$  puts

equation (13) in indeterminate form). We can choose a convenient value for  $b_0$  as needed. First, let us examine the case where  $-n \notin \mathbb{N}$  (we will take care of the  $-n \in \mathbb{N}$  case later). From equation (13) we have

$$b_{2k} = -\frac{b_{2k-2}}{2k(2k+2n)} = -\frac{b_{2k-2}}{4k(n+k)}. \quad (14)$$

We will perform induction on this equation to find a general formula for  $b_{2k}$ .

**Step One.** Examine the base cases  $k = 1$  and  $k = 2$ .

$$b_{2(1)} = b_2 = (-1) \frac{b_0}{4(1)(n+1)} \quad (15)$$

$$b_{2(2)} = b_4 = (-1) \frac{(-1) \frac{b_0}{4(1)(n+1)}}{4(2)(n+2)} = \frac{(-1)^2 b_0}{4^2(2 \cdot 1)(n+2)(n+1)} \quad (16)$$

**Step Two.** Perform the inductive step. Based on the above cases, suppose that the following formula holds for some positive integer  $l$ :

$$b_{2l} = (-1)^l \frac{b_0}{4^l l! \prod_{n+1}^{n+l} m}. \quad (17)$$

We will show that this holds for the  $l+1$  case as well:

$$\begin{aligned} b_{2(l+1)} &= (-1) \frac{(-1)^l \frac{b_0}{4^l l! \prod_{n+1}^{n+l} m}}{4(l+1)(n+l+1)} \\ &= (-1)^{l+1} \frac{b_0}{4^{l+1} (l+1)! \prod_{n+1}^{n+l+1} m}. \end{aligned} \quad (18)$$

Therefore we know that equation (17) holds in general for all positive integers  $k$ .

Before we can plug equation (17) back into the original power series (3), we need to choose a convenient value for  $b_0$ . We know that the summation in the final power series equation needs to be convergent in order for it to be a solution to the Bessel equation (1). In addition, it would be advantageous to use the factorial  $(n+k)!$  in the denominator of  $b_k$  (instead of having to terminate the product at  $(n+1)$ ). For these reasons, we will choose

$$b_0 = \frac{1}{2^n n!}. \quad (19)$$

Plugging this into equation (17), we have

$$b_{2k} = \frac{(-1)^k}{2^n 4^k k! (n+k)(n+k-1) \dots (n+1) n!}, \quad (20)$$

which can clearly be simplified to

$$b_{2k} = \frac{(-1)^k}{2^n 4^k k! (n+k)!}. \quad (21)$$

Now we are finally ready to plug our formula for  $b_{2k}$  into the full power series (3). We have

$$y(x) = x^n \sum_{k=0}^{\infty} \frac{(-1)^k}{2^n 4^k k! (n+k)!} x^{2k}. \quad (22)$$

Observe that the  $x^{2k}$  term at the end comes from the  $b_{2k}$  in equation (17). I.e., we want the summation term

$$b_0 x^0 + b_2 x^2 + b_4 x^4 + \dots = \sum_{k=0}^{\infty} b_{2k} x^{2k}, \quad (23)$$

so we must ensure that the power of  $x$  is the same as the subscript of  $b$ . Rearranging, we have

$$y(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k! (n+k)!} \left(\frac{x}{2}\right)^{2k}. \quad (24)$$

Clearly, this power series is only meaningful if it is convergent. We will check by the well-known Ratio Test. If it passes, we have found a solution to Bessel's equation. We take

$$\begin{aligned} \rho &= \lim_{k \rightarrow \infty} \frac{b_{k+1}}{b_k} \\ &= \frac{\frac{(-1)^{k+1}}{(k+1)!(n+k+1)!} \left(\frac{x}{2}\right)^{2(k+1)}}{\frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k}} \\ &= \lim_{k \rightarrow \infty} \frac{-1}{(k+1)(n+k+1)} \left(\frac{x}{2}\right)^2 = 0. \end{aligned} \quad (25)$$

Since  $\rho < 1$ , the series converges. We have indeed found a solution to equation (1).

However, we also want to construct a solution for the complex order  $v$ , not just for the order  $n \in \mathbb{N}$ . We must find a continuous function with which we can replace the factorial in equation (24).

The most versatile way to extend the factorial function to non-integer and complex numbers is the Gamma function. The reciprocal of the Gamma function also happens to be holomorphic, meaning that it is infinitely differentiable and equal to its own Taylor series. Since we are applying the Gamma function to the denominator of equation (24), this property of its reciprocal will be especially convenient.

The Gamma function is defined as

$$\Gamma(n) = (n-1)! \quad (26)$$

(the factorial function with its argument shifted down by 1) if  $n$  is a positive integer. For complex numbers and non-integers, the Gamma function corresponds to the Mellin transform of the negative exponential function,

$$\Gamma(z) = \{M e^{-x}\}(z), \quad (27)$$

where the Mellin transform is

$$\{Mf\}(s) = \varphi(s) = \int_0^{\infty} x^{s-1} f(x) dx. \quad (28)$$

Making this modification to equation (24), we now have

$$J_v(x) = \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(v+k+1)} \left(\frac{x}{2}\right)^{2k}. \quad (29)$$

This is not valid for  $-n \in \mathbb{N}$ , where the Gamma function is undefined. In this case, we will begin the summation at  $k = n$  to bypass any undefined Gamma terms (since  $k = n$  corresponds to  $\Gamma(-n + n + 1)$ ):

$$\begin{aligned} J_{-n}(x) &= \left(\frac{x}{2}\right)^{-n} \sum_{k=n}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(-n+k+1)} \left(\frac{x}{2}\right)^{2k} \\ &= \left(\frac{x}{2}\right)^{-n} \sum_{k=0}^{\infty} \frac{(-1)^{k+n}}{\Gamma(n+k+1)\Gamma(-n+n+k+1)} \left(\frac{x}{2}\right)^{2(k+n)} \\ &= \left(\frac{x}{2}\right)^{-n} \left(\frac{x}{2}\right)^{2n} \sum_{k=0}^{\infty} \frac{(-1)^k (-1)^n}{\Gamma(n+k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} \\ &= (-1)^n J_n(x). \end{aligned} \quad (30)$$

This will also solve the Bessel differential equation (1). So we have found our first Bessel function,  $J_n(x)$  or  $J_v(x)$ .

## 1.2 Bessel Functions of the Second Kind

We will now determine a second, linearly independent solution. Let us begin by examining the behavior of  $J_v(x)$  as  $x \rightarrow 0$ .

**Step One.** Let  $\Re(v) > 0$ . We have

$$\lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^v \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+v)} \left(\frac{x}{2}\right)^{2k} = 0, \quad (31)$$

since

$$\lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^v = 0. \quad (32)$$

**Step Two.** Let  $\Re(v) = 0$ . We examine

$$\lim_{x \rightarrow 0} \left(\frac{x}{2}\right)^0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = 1. \quad (33)$$

Though it may be difficult to see at first, the limit of this is 1 because of the property that  $0^0 = 1$ , since our summation will become

$$\lim_{x \rightarrow 0} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1)} \left(\frac{x}{2}\right)^{2k} = 1 + 0 + 0 + 0 + \dots \quad (34)$$

**Step Three.** Let  $\Re(v) < 0$ ,  $v \notin \mathbb{Z}$  (since the gamma function is undefined here). Then we have

$$\lim_{x \rightarrow 0} \left(\frac{2}{x}\right)^{-v} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+1+v)} \left(\frac{x}{2}\right)^{2k} = \pm\infty. \quad (35)$$

Summarized, the results from steps one, two, and three are:

$$\lim_{x \rightarrow 0} J_v(x) = \begin{cases} 0, & \Re(v) > 0 \\ 1, & v = 0 \\ \pm\infty, & \Re(v) < 0, v \notin \mathbb{Z} \end{cases}. \quad (36)$$

We can see here that  $J_v(x)$  and  $J_{-v}(x)$  are two linearly independent solutions (i.e. they cannot be expressed as linear combinations of each other) if  $v \notin \mathbb{Z}$ . If  $v \in \mathbb{Z}$ , they are linearly dependent (recall equation (30)). Because of this property (and the homogeneity of Bessel's differential equation) any linear combination of  $J_v$  and  $J_{-v}$ , where  $v \notin \mathbb{Z}$ , is also solution. We build the equation

$$Y_v(x) = \frac{\cos(v\pi)J_v(x) - J_{-v}(x)}{\sin(v\pi)}, \quad (37)$$

for  $v \notin \mathbb{Z}$ . Notice that this vanishes if we have order  $n \in \mathbb{N}_0$ , since  $\cos(n\pi) = (-1)^n$ . For  $n \in \mathbb{Z}$ , we let

$$Y_v(x) := \lim_{v \rightarrow n} Y_v(x). \quad (38)$$

It can be shown by L'Hôpital's rule that this limit exists (the calculation is too lengthy to be included here, but it follows from the properties of the digamma function, which gives the relationship between the gamma function and its derivative).

We need to check that the Wronskian determinant of  $J_v(z)$  and  $Y_v(z)$  does not vanish for any  $v, z \in \mathbb{C}$ . Abel's Theorem says that if  $y_1(z)$  and  $y_2(z)$  are two solutions to the differential equation  $p(z)y'' + q(z)y' + r(z)y = 0$ , then the Wronskian of the two solutions is of the form  $\frac{C}{p(z)}$ , where  $C$  is a constant which does not depend on  $z$ . Notice that we can write Bessel's equation in the form  $zy'' + y' + (z - \frac{v^2}{z})y = 0$ , so that it is self-adjoint. Then the Wronskian determinant of  $J_v(z)$  and  $Y_v(z)$  is of the form  $\frac{A_v}{p(z)}$ , where  $A_v$  does not depend on  $z$ . We omit the detailed calculation here, but the Wronskian determinant of  $J_v(z)$  and  $Y_v(z)$  turns out to be

$$W(J_v(z), Y_v(z)) = J_{v+1}(z)Y_v(z) - J_v(z)Y_{v+1}(z) = \frac{2}{\pi z}, \quad (39)$$

which does not vanish for any  $z \in \mathbb{C}$ . Therefore, they are linearly independent for all  $v \in \mathbb{C}$ . We have successfully found two linearly independent solutions to the Bessel differential equation.



## 2 Properties of Bessel Functions

Now that we have derived the two Bessel functions, we will prove some of their fundamental properties.

### 2.1 The Generating Function

Several properties of the Bessel functions can be proven using their generating function. We will begin this section by introducing the concept of generating functions and showing that one exists for the Bessel functions.

**Definition.** A **power series** is an infinite sum of the form

$$\sum_{i=0}^{\infty} a_i z^i, \quad (40)$$

where the  $a_i$ 's are quantities given by a particular function or rule.

**Definition.** A **generating function** of another function  $a_n$  is the function whose power series has  $a_n$  as the coefficient of  $x^n$ . I.e., the generating function of  $a_n$  is the function  $G(a_n; x)$  where

$$G(a_n; x) = \sum_{n=0}^{\infty} a_n x^n. \quad (41)$$

**Proposition 2.1.** *We have*

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n. \quad (42)$$

*I.e., the function  $e^{\frac{x}{2}(z-z^{-1})}$  is the generating function of the first Bessel function.*

(Note: This power series is actually a *Laurent series*, since it includes terms of a negative degree. This will be important later.)

*Proof.* Recall the power series representation

$$e^x = \sum_{l=0}^{\infty} \frac{x^l}{l!}.$$

We have

$$\begin{aligned} e^{\frac{x}{2}z - \frac{x}{2}\frac{1}{z}} &= e^{\frac{x}{2}z} e^{-\frac{x}{2}\frac{1}{z}} \\ &= \sum_{m=0}^{\infty} \frac{(\frac{x}{2}z)^m}{m!} \sum_{k=0}^{\infty} \frac{(-\frac{x}{2z})^k}{k!} \end{aligned}$$

$$= \sum_{m=0}^{\infty} \frac{(x/2)^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k}.$$

Recall that the multiplication of two infinite power series can be written as a Cauchy Product; i.e., if two power series  $f(z) = \sum_{n=0}^{\infty} a_n z^n$  and  $g(z) = \sum_{n=0}^{\infty} b_n z^n$  each have a radius of convergence of  $R > 0$ , then their product can also be expressed as a power series in the disc  $|z| < R$ :

$$(fg)(z) = \sum_{n=0}^{\infty} c_n z^n,$$

where

$$c_n = \sum_{k=0}^n a_k b_{n-k}.$$

Applying this property, we have:

$$\begin{aligned} \sum_{m=0}^{\infty} \frac{(x/2)^m}{m!} z^m \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!} z^{-k} &= \sum_{n=-\infty}^{\infty} \left( \sum_{\substack{m-k=n \\ m, k \geq 0}} \frac{(-1)^k (\frac{x}{2})^{m+k}}{m!k!} \right) z^{m-k} \\ &= \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{n+k+k} \right) z^n \\ &= \sum_{n=-\infty}^{\infty} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k} \left(\frac{x}{2}\right)^n \right) z^n \\ &= \sum_{n=-\infty}^{\infty} J_n(x) z^n. \end{aligned}$$

□

We will now use the generating function to prove some standard results.

**Lemma 2.1.1.** *We have*

$$\cos(x) = J_0(x) + 2 \sum_{n=1}^{\infty} (-1)^n J_{2n}(x), \quad (43)$$

$$\sin(x) = 2 \sum_{n=0}^{\infty} (-1)^n J_{2n+1}(x), \quad (44)$$

$$1 = J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x). \quad (45)$$

*Proof.* Take  $z = e^{i\phi}$  and set  $i \sin(\phi) = \frac{1}{z} (z - \frac{1}{z})$ . From Euler's formula,

$$\cos(x \sin \phi) + i \sin(x \sin \phi) = e^{ix \sin \phi}$$

$$\begin{aligned}
&= \sum_{n=-\infty}^{\infty} J_n(x) e^{in\phi} \\
&= \sum_{n=-\infty}^{\infty} J_n(x) (\cos(n\phi) + i \sin(n\phi)).
\end{aligned}$$

Separating real and imaginary parts, we get

$$\cos(x \sin \phi) = \sum_{n=-\infty}^{\infty} J_n(x) \cos(n\phi)$$

and

$$\sin(x \sin \phi) = \sum_{n=-\infty}^{\infty} J_n(x) \sin(n\phi).$$

Setting  $\phi = \frac{\pi}{2}$ , we have

$$\cos(x) = \sum_{n=-\infty}^{\infty} J_n(x) \cos\left(\frac{n\pi}{2}\right).$$

Consider the cases.

**Case One.** Suppose  $n = 1 \pmod{4}$ . Then  $\cos(\frac{n\pi}{2}) = 0$  and  $\cos(\frac{-n\pi}{2}) = 0$ , so the summation between  $n$  and  $-n$  terms vanishes. The same occurs for  $n = 3 \pmod{4}$  (or any odd  $n$ ).

**Case Two.** Suppose  $n = 2 \pmod{4}$ . Then  $\cos(\frac{n\pi}{2}) = -1$  and  $\cos(\frac{-n\pi}{2}) = -1$ . Recalling equation (30), the summation between  $n$  and  $-n$  terms becomes

$$\begin{aligned}
J_n(x) \cos\left(\frac{n\pi}{2}\right) + J_{-n}(x) \cos\left(\frac{-n\pi}{2}\right) &= J_n(x)(-1) + J_n(x)(-1) \\
&= -2J_n(x).
\end{aligned}$$

**Case Three.** Suppose  $n = 0 \pmod{4}$ . Then  $\cos(\frac{n\pi}{2}) = 1$  and  $\cos(\frac{-n\pi}{2}) = 1$ . So the summation becomes

$$\begin{aligned}
J_n(x) \cos\left(\frac{n\pi}{2}\right) + J_{-n}(x) \cos\left(\frac{-n\pi}{2}\right) &= J_n(x)(1) + J_n(x)(1) \\
&= 2J_n(x).
\end{aligned}$$

Based on these cases, we can rewrite the equation as

$$\cos(x) = 2 \sum_{n=1}^{\infty} J_{2n}(-1)^n.$$

Plugging  $\phi = \frac{\pi}{2}$  into  $\sin(x \sin \phi)$ , we get

$$\sin(x) = \sum_{n=-\infty}^{\infty} J_n(x) \sin\left(\frac{n\pi}{2}\right).$$

We will consider cases once again.

**Case One.** Suppose  $n = 1 \pmod 4$ . Then  $-n = 3 \pmod 4$ , so  $\sin(\frac{n\pi}{2}) = 1$  and  $\sin(\frac{-n\pi}{2}) = -1$ . So the summation between the terms for  $-n$  and  $n$  becomes

$$\begin{aligned} J_n(x) \sin(\frac{n\pi}{2}) + J_{-n}(x) \sin(\frac{-n\pi}{2}) &= J_n(x)(1) + J_{-n}(x)(-1) \\ &= J_n(x) + (-1)^2 J_n(x) \\ &= 2J_n(x). \end{aligned}$$

**Case Two.** Suppose  $n = 2 \pmod 4$ . Then  $\sin(\frac{n\pi}{2}) = 0$ , so we no longer have the term in the summation. The same applies for the case  $n = 0 \pmod 4$ .

**Case Three.** Suppose  $n = 3 \pmod 4$ . Then  $\sin(\frac{n\pi}{2}) = -1$  and  $\sin(\frac{-n\pi}{2}) = 1$ . So the summation between  $n$  and  $-n$  terms becomes

$$\begin{aligned} J_n(x) \sin(\frac{n\pi}{2}) + J_{-n}(x) \sin(\frac{-n\pi}{2}) &= J_n(x)(-1) + J_{-n}(x)(1) \\ &= J_n(x)(-1) + (-1)J_n(x) \\ &= -2J_n(x). \end{aligned}$$

Based on these cases, we can clearly rewrite this summation as

$$\sin(x) = 2 \sum_{n=0}^{\infty} J_{2n+1}(x)(-1)^n.$$

Finally, setting  $\phi = 2\pi$ , we have

$$\begin{aligned} \cos(\sin(2\pi)) &= \cos(0) = 1 = \sum_{n=-\infty}^{\infty} J_n(x) \cos(2\pi n) \\ &= \sum_{n=-\infty}^{\infty} J_n(x) \\ &= J_0(x) + \sum_{n=-\infty}^{-1} J_n(x) + \sum_{n=1}^{\infty} J_n(x) \\ &= J_0(x) + 2 \sum_{n=1}^{\infty} J_{2n}(x) \end{aligned}$$

since odd values of  $n$  cause the terms to cancel, and even values sum to  $2J_n(x)$ . This is the third result.

□

**Lemma 2.1.2.** *We have*

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x) \tag{46}$$

$\forall n \in \mathbb{Z}$ .

*Proof.* We make the change of variables  $x \rightarrow -x$  and  $z \rightarrow z^{-1}$  and insert into the generating function:

$$\begin{aligned}
\sum_{n=-\infty}^{\infty} J_n(-x)z^{-n} &= e^{\frac{-x}{2z}}(z^{-1}-z) \\
&= e^{\frac{-x}{2z}} e^{\frac{x}{2}z} \\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{-x}{2z}\right)^m}{m!} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}z\right)^k}{k!} \\
&= \sum_{m=0}^{\infty} \frac{\left(\frac{x}{2}\right)^m (-1)^m}{m!} z^{-m} \sum_{k=0}^{\infty} \frac{\left(\frac{x}{2}\right)^k}{k!} z^k \\
&= \sum_{n=-\infty}^{\infty} \left( \sum_{\substack{m-k=n \\ m,k \geq 0}} \frac{(-1)^m \left(\frac{x}{2}\right)^{m+k}}{m!k!} \right) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} \left( \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(m-n)!} \left(\frac{x}{2}\right)^{2m} \left(\frac{x}{2}\right)^{-n} \right) z^{-n} \\
&= \sum_{n=-\infty}^{\infty} J_{-n}(x)z^{-n}
\end{aligned}$$

So we have,

$$\sum_{n=-\infty}^{\infty} J_n(-x)z^{-n} = \sum_{n=-\infty}^{\infty} J_{-n}(x)z^{-n}.$$

Comparing the coefficients, we get

$$J_n(-x) = J_{-n}(x),$$

and from equation (30),

$$J_n(-x) = J_{-n}(x) = (-1)^n J_n(x).$$

□

**Proposition 2.2.** For any  $n \in \mathbb{Z}$ ,

$$\frac{d}{dx} \left( x^{-n} J_n(x) \right) = -x^{-n} J_{n+1}(x) \quad (47)$$

and

$$\frac{d}{dx} \left( x^n J_n(x) \right) = x^n J_{n-1}(x). \quad (48)$$

*Proof.* Multiply the series representation of  $J_n(x)$  by  $x^{-n}$  and differentiate.

$$\begin{aligned}
\frac{d}{dx} \left( x^{-n} J_n(x) \right) &= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left(\frac{x}{2}\right)^{2k+n} (x)^{-n} \right) \\
&= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \frac{x^{2k}}{2^{2k+n}} \right)
\end{aligned}$$

$$\begin{aligned}
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left( 2k \frac{x^{2k-1}}{2^{2k+n}} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left( k \frac{x^{2k-1}}{2^{2k+n-1}} \right)
\end{aligned}$$

Observe that when  $k = 0$  the summation term is

$$\frac{(-1)^0}{0!n!} \left( (0) \frac{x^{-1}}{2^{n-1}} \right) = 0,$$

so we can rewrite the equation starting with the index  $k = 1$ :

$$\begin{aligned}
\frac{d}{dx} \left( x^{-n} J_n(x) \right) &= \sum_{k=1}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left( k \frac{x^{2k-1}}{2^{2k+n-1}} \right) \\
&= \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!(n+k)!} \left( \frac{x^{2k-1}}{2^{2k-1+n}} \right) \\
&= x^{-n} \sum_{k=1}^{\infty} \frac{(-1)^k}{(k-1)!(n+k)!} \left( \frac{x}{2} \right)^{2k-1+n}.
\end{aligned}$$

Let  $j = k - 1$ . Then  $k = j + 1$ , and we have:

$$\begin{aligned}
\frac{d}{dx} \left( x^{-n} J_n(x) \right) &= x^{-n} \sum_{j=0}^{\infty} \frac{(-1)^{j+1}}{j!(n+j+1)!} \left( \frac{x}{2} \right)^{2j+n+1} \\
&= x^{-n} \sum_{j=0}^{\infty} (-1) \frac{(-1)^j}{j!(n+j+1)!} \left( \frac{x}{2} \right)^{2j+(n+1)} \\
&= -x^{-n} J_{n+1}(x).
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{d}{dx} \left( x^n J_n(x) \right) &= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left( \frac{x}{2} \right)^{2k+n} x^n \right) \\
&= \frac{d}{dx} \left( \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n+k)!} \left( \frac{x^{2k+2n}}{2^{2k+n}} \right) \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k (2k+2n)}{k!(n+k)!} \left( \frac{x^{2k+2n-1}}{2^{2k+2n}} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(n-1+k)!} \left( \frac{x^{2k+2n-1}}{2^{2k+n-1}} \right) \\
&= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(k+n-1)!} \left( \frac{x}{2} \right)^{2k+n-1} x^n
\end{aligned}$$

$$= x^n J_{n-1}(x).$$

□

**Lemma 2.2.1.** For any  $n \in \mathbb{Z}$ ,

$$2 \frac{d}{dx} (J_n(x)) = J_{n-1}(x) - J_{n+1}(x), \quad (49)$$

and

$$\frac{2n}{x} J_n(x) = J_{n+1}(x) + J_{n-1}(x). \quad (50)$$

*Proof.* We take

$$\frac{d}{dx} (J_n(x)) = \frac{d}{dx} (x^n (x^{-n} J_n(x)))$$

Applying the results from Proposition 2.2 and the product rule, we have

$$\begin{aligned} \frac{d}{dx} (J_n(x)) &= nx^{n-1} (x^n J_n(x)) + x^n \frac{d}{dx} (x^{-n} J_n(x)) \\ &= nx^{-1} J_n(x) + x^n (-x^{-n} J_{n+1}(x)) \\ &= nx^{-1} J_n(x) - J_{n+1}(x). \end{aligned}$$

Similarly, we take

$$\begin{aligned} \frac{d}{dx} (J_n(x)) &= \frac{d}{dx} (x^{-n} (x^n J_n(x))) \\ &= -nx^{-n-1} (x^n J_n(x)) + x^{-n} \frac{d}{dx} (x^n J_n(x)) \\ &= -nx^{-1} J_n(x) + x^{-n} (x^n J_{n-1}(x)) \\ &= -nx^{-1} J_n(x) + J_{n-1}(x). \end{aligned}$$

Adding these two expressions, we get the first result from the Lemma:

$$2 \frac{d}{dx} (J_n(x)) = J_{n-1}(x) - J_{n+1}(x).$$

Subtracting the same expressions, we get the second result:

$$2nx^{-1} J_n(x) = J_{n-1}(x) + J_{n+1}(x).$$

□

**Remark 2.2.1.** Lemma 2.2.1 can be proved similarly by differentiating the generating function of  $J_n(x)$  with respect to  $x$  and  $z$ , one at a time, and comparing the coefficients. Adding the resulting relations and multiplying by  $\frac{x^n}{2}$  will produce the second result from Proposition 2.2.

**Remark 2.2.2.** Note that the relation shown in equation (51) also holds for all  $v \in \mathbb{R}$ , not just for  $n \in \mathbb{N}$ . This will become useful in the next section during our discussion of Lommel polynomials.

**Lemma 2.2.2.** For any  $n \in \mathbb{Z}$  we have

$$\int x^{n+1} J_n(x) dx = x^{n+1} J_{n+1}(x) + C \quad (51)$$

and

$$\int x^{-n+1} J_n(x) dx = -x^{-n+1} J_{n-1}(x) + D. \quad (52)$$

where  $C$  and  $D$  are arbitrary constants.

*Proof.* We take the integral of equation (48) from Proposition 2.2:

$$\begin{aligned} \int x^{n+1} J_n(x) dx &= \int \frac{d}{dx} (x^{n+1} J_{n+1}(x)) dx \\ &= x^{n+1} J_{n+1}(x) + C. \end{aligned}$$

To obtain the second equation, recall from Lemma 2.1.2 that  $J_{-n}(x) = (-1)^n J_n(x)$ , or equivalently,  $J_n(x) = (-1)^n J_{-n}(x)$ . We apply this to the equation below:

$$\begin{aligned} \int x^{-n+1} J_n(x) dx &= \int x^{-n+1} (-1)^n J_{-n}(x) dx \\ &= (-1)^n \int x^{-n+1} J_{-n}(x) dx \\ &= (-1)^n x^{-n+1} J_{-n+1}(x) + D \end{aligned}$$

by the first part of the lemma. Then we have

$$\begin{aligned} \int x^{-n+1} J_n(x) dx &= (-1)^n x^{-n+1} J_{-n+1}(x) + D \\ &= (-1)^{-n+2} x^{-n+1} J_{-n+1}(x) + D \\ &= -x^{-n+1} (-1)^{-n+1} J_{-n+1}(x) + D. \end{aligned}$$

Applying Lemma 2.1.2,

$$\int x^{-n+1} J_n(x) dx = -x^{-n+1} J_{n-1}(x) + D.$$

□

**Lemma 2.2.3.** For any  $m \neq 0$ ,

$$J_0^2(x) + 2 \sum_{n=1}^{\infty} J_n^2(x) = 1 \quad (53)$$

and

$$\sum_{n=-\infty}^{\infty} J_{n+m}(x) J_n(x) = 0. \quad (54)$$



*Proof.* We have

$$e^{\frac{x}{2}(z-z^{-1})}e^{\frac{x}{2}(z^{-1}-z)} = e^0 = 1$$

and by Proposition 2.1,

$$1 = \sum_{k=-\infty}^{\infty} J_k(x)z^k \sum_{n=-\infty}^{\infty} J_n(x)z^{-n}.$$

Letting  $m = k - n$  we write

$$\begin{aligned} 1 &= \sum_{m=-\infty}^{\infty} \left( \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) \right) z^m \\ &= \sum_{n=-\infty}^{\infty} J_n^2(x)z^0 + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \left( \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) \right) z^m \\ &= \sum_{n=-\infty}^{\infty} J_n^2(x) + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \left( \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) \right) z^m. \end{aligned}$$

Because of the case where  $z = 0$ , we have

$$1 = \sum_{n=-\infty}^{\infty} J_n^2(x).$$

In addition,

$$1 = 1 + \sum_{\substack{m \in \mathbb{Z} \\ m \neq 0}} \left( \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x) \right) z^m,$$

which implies that

$$0 = \sum_{n=-\infty}^{\infty} J_{n+m}(x)J_n(x)$$

for all  $m \neq 0$ .

We will now examine the equation  $1 = \sum_{n=-\infty}^{\infty} J_n^2(x)$  to find an equivalent form. We can write this summation as  $1 = J_0^2(x) + \sum_{n=-\infty}^{-1} J_n^2(x) + \sum_{n=1}^{\infty} J_n^2(x)$ .

**Step One.** Fix any even  $n \in \mathbb{Z}$ . Lemma 2.1.2 gives that  $J_n(x) = J_{-n}(x)$ . Squaring both sides, we get that  $J_n^2(x) = J_{-n}^2(x)$ . Then the sum of the  $n$  and  $-n$  terms is  $2J_n^2(x)$ .

**Step Two.** Fix any odd  $n \in \mathbb{Z}$ . Then we have that  $J_n(x) = (-1)J_{-n}(x)$ . Squaring both sides, we now have that  $J_n^2(x) = J_{-n}^2(x)$ . Then  $J_n^2(x) + J_{-n}^2(x) = 2J_n^2(x)$ .

Then our equivalent equation is

$$1 = J_0^2(x) + 2 \sum_{n=1}^{\infty} J_n^2(x).$$

□

**Lemma 2.2.4.**

$$\sum_{n \in \mathbb{Z}} J_n(x) = 1. \quad (55)$$

*Proof.* Set  $z = 1$  in the generating function. Then we have

$$\begin{aligned} e^{\frac{x}{2}(1-1^{-1})} &= 1 = \sum_{n=-\infty}^{\infty} J_n(x)(1)^n \\ &= \sum_{n \in \mathbb{Z}} J_n(x). \end{aligned}$$

□

**Lemma 2.2.5.**  $\forall n \in \mathbb{Z}$ , we have

$$J_n(x+y) = \sum_{k \in \mathbb{Z}} J_k(x) J_{n-k}(y). \quad (56)$$

*Proof.* Observe that

$$\begin{aligned} \sum_{n \in \mathbb{Z}} J_n(x+y)t^n &= e^{\frac{1}{2}(x+y)(t-t^{-1})} \\ &= e^{\frac{1}{2}x(t-t^{-1})} e^{\frac{1}{2}y(t-t^{-1})} \\ &= \sum_{k \in \mathbb{Z}} J_k(x)t^k \sum_{m \in \mathbb{Z}} J_m(y)t^m \\ &= \sum_{n \in \mathbb{Z}} \left( \sum_{k \in \mathbb{Z}} J_k(x) J_{n-k}(y) \right) t^n. \end{aligned}$$

Comparing the coefficients yields the result. □

## 2.2 Some Special Values

In this section we will examine what the Bessel functions look like when some particular values are chosen.

**Lemma 2.2.6.** If  $v = \frac{1}{2}$  or  $-\frac{1}{2}$  we have

$$J_{\frac{1}{2}}(x) = Y_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \sin(x) \quad (57)$$

and

$$J_{-\frac{1}{2}}(x) = -Y_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos(x). \quad (58)$$

*Proof.* Observe the series representation of  $J_\nu$  and apply the properties of the gamma function. We begin with

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{3}{2})} \left(\frac{x}{2}\right)^{2k}.$$

Since  $\Gamma\left(k+\frac{3}{2}\right) = \frac{(2k+1)!\sqrt{\pi}}{k!2^{2k+1}}$ , we have

$$\begin{aligned} J_{\frac{1}{2}}(x) &= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k! \frac{(2k+1)!\sqrt{\pi}}{k!2^{2k+1}}} \left(\frac{x}{2}\right)^{2k} \\ &= \sqrt{\frac{x}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1-\frac{1}{2}} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k 2^{2k+1}}{(2k+1)!\sqrt{\pi}} \left(\frac{x}{2}\right)^{2k+1} \\ &= \sqrt{\frac{2}{x\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}. \end{aligned}$$

Since we know the Taylor series

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1},$$

we can conclude that

$$J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}} \sin(x).$$

We also have that

$$\begin{aligned} Y_{-\frac{1}{2}} &= \frac{\cos(-\frac{\pi}{2})J_{-\frac{1}{2}}(x) - J_{\frac{1}{2}}(x)}{\sin(-\frac{\pi}{2})} \\ &= \frac{(0)J_{-\frac{1}{2}} - J_{\frac{1}{2}}(x)}{-1} \\ &= J_{\frac{1}{2}}(x). \end{aligned}$$

Thus we have proven the first result. The second result follows similarly, considering that  $\Gamma(k+\frac{1}{2}) = \frac{(2k)!\sqrt{\pi}}{2^{2k}k!}$  and  $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$ . We have

$$\begin{aligned} J_{-\frac{1}{2}}(x) &= \left(\frac{x}{2}\right)^{-\frac{1}{2}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\frac{1}{2})} \left(\frac{x}{2}\right)^{2k} \\ &= \sqrt{\frac{2}{x}} \sum_{k=0}^{\infty} \frac{(-1)^k}{\frac{(2k)!\sqrt{\pi}}{2^{2k}}} \left(\frac{x}{2}\right)^{2k} \end{aligned}$$

$$\begin{aligned}
&= \sqrt{\frac{2}{x\pi}} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} \\
&= \sqrt{\frac{2}{x\pi}} \cos(x).
\end{aligned}$$

Also,

$$\begin{aligned}
Y_{\frac{1}{2}} &= \frac{\cos(\frac{\pi}{2})J_{\frac{1}{2}}(x) - J_{-\frac{1}{2}}(x)}{\sin(\frac{\pi}{2})} \\
&= \frac{(0)J_{\frac{1}{2}} - J_{-\frac{1}{2}}(x)}{1} \\
&= -J_{-\frac{1}{2}}(x).
\end{aligned}$$

This proves the second result. □

**Lemma 2.2.7.** *If  $v \in \frac{1}{2} + \mathbb{Z}$ , there are polynomials  $P_n(x)$  and  $Q_n(x)$  with*

$$\begin{aligned}
\deg P_n &= \deg Q_n = n \\
P_n(-x) &= (-1)^n P_n(x) \\
Q_n(-x) &= (-1)^n Q_n(x)
\end{aligned}$$

such that for any  $k \in \mathbb{N}_0$

$$J_{k+\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}} \left( P_k\left(\frac{1}{x}\right) \sin(x) - Q_{k-1}\left(\frac{1}{x}\right) \cos(x) \right) \quad (59)$$

and

$$J_{-k-\frac{1}{2}}(x) = (-1)^k \sqrt{\frac{2}{x\pi}} \left( P_k\left(\frac{1}{x}\right) \cos(x) + Q_{k-1}\left(\frac{1}{x}\right) \sin(x) \right). \quad (60)$$

*Proof.* From Lemma 2.2.1, equation (51), we had the recurrence formula

$$J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x).$$

We will use this formula and show by induction that

$$J_{k+v}(x) = P_n\left(\frac{1}{x}\right) J_v(x) - Q_{n-1}\left(\frac{1}{x}\right) J_{v-1}(x)$$

for the  $P_n$  and  $Q_n$  defined in the lemma. We begin with the base case  $J_{v+2}(x)$ :

$$\begin{aligned}
J_{v+2}(x) &= \frac{2v}{x} J_{v+1}(x) - J_v(x) \\
&= \frac{2v}{x} \left( \frac{2v}{x} J_v(x) - J_{v-1}(x) \right) - J_v(x) \\
&= \left( \left( \frac{2v}{x} \right)^2 - 1 \right) J_v(x) - \frac{2v}{x} J_{v-1}(x).
\end{aligned}$$

We will also show the case  $J_{v+3}(x)$ :

$$\begin{aligned}
J_{v+3}(x) &= \frac{2v}{x} J_{v+2}(x) - J_{v+1}(x) \\
&= \frac{2v}{x} \left( \left( \left( \frac{2v}{x} \right)^2 - 1 \right) J_v(x) - \frac{2v}{x} J_{v-1}(x) \right) - J_{v+1}(x) \\
&= \left( \left( \frac{2v}{x} \right)^3 - \frac{2v}{x} \right) J_v(x) - \left( \frac{2v}{x} \right)^2 J_{v-1}(x) - \left( \frac{2v}{x} J_v(x) - J_{v-1}(x) \right) \\
&= \left( \left( \frac{2v}{x} \right)^3 - \frac{4v}{x} \right) J_v(x) - \left( \left( \frac{2v}{x} \right)^2 + 1 \right) J_{v-1}(x).
\end{aligned}$$

Then we know that the formula is true for some  $n \in \mathbb{N}$ . We will show that it must hold for the  $n+1$  case. We have

$$\begin{aligned}
J_{n+1+v}(x) &= P_n\left(\frac{1}{x}\right) J_{v+1}(x) - Q_{n-1}\left(\frac{1}{x}\right) J_v(x) \\
&= P_n\left(\frac{1}{x}\right) \left( \frac{2v}{x} J_v(x) - J_{v-1}(x) \right) - Q_{n-1}\left(\frac{1}{x}\right) J_v(x) \\
&= \left( \frac{2v}{x} P_n\left(\frac{1}{x}\right) \right) J_v(x) - P_n\left(\frac{1}{x}\right) J_{v-1}(x) - Q_{n-1}\left(\frac{1}{x}\right) J_v(x) \\
&= \left( \left( \frac{2v}{x} P_n\left(\frac{1}{x}\right) \right) - Q_{n-1}\left(\frac{1}{x}\right) \right) J_v(x) - P_n\left(\frac{1}{x}\right) J_{v-1}(x).
\end{aligned}$$

Letting  $P_{n+1}\left(\frac{1}{x}\right) = \frac{2v}{x} P_n\left(\frac{1}{x}\right) - Q_{n-1}\left(\frac{1}{x}\right)$  and  $Q_n\left(\frac{1}{x}\right) = P_n\left(\frac{1}{x}\right)$ , we now have

$$J_{n+1+v}(x) = P_{n+1}\left(\frac{1}{x}\right) J_v(x) - Q_n\left(\frac{1}{x}\right) J_{v-1}(x).$$

By the principle of mathematical induction, the formula holds for all  $k \in \mathbb{N}$ .

Now let  $v = \frac{1}{2}$ . We have

$$J_{k+\frac{1}{2}}(x) = P_k\left(\frac{1}{x}\right) J_{\frac{1}{2}}(x) - Q_{k-1}\left(\frac{1}{x}\right) J_{-\frac{1}{2}}(x).$$

Plugging in equations (58) and (59) from lemma 2.2.6, we get the first result:

$$J_{k+\frac{1}{2}}(x) = \sqrt{\frac{2}{x\pi}} \left( P_k\left(\frac{1}{x}\right) \sin(x) - Q_{k-1}\left(\frac{1}{x}\right) \cos(x) \right).$$

To achieve the second result, we can rewrite our recurrence relation as

$$J_{-v-1}(x) = \frac{-2v}{x} J_{-v}(x) - J_{-v+1}(x).$$

We then have

$$J_{v-2}(x) = \left( \left( \frac{2v}{x} \right)^2 - 1 \right) J_{-v}(x) + J_{-v+1}(x)$$

and

$$J_{v-3}(x) = \left( - \left( \frac{2v}{x} \right)^3 + \frac{2v}{x} - 1 \right) J_{-v}(x) - \left( \left( \frac{2v}{x} \right)^2 - 1 \right) J_{-v+1}(x).$$

Continuing to iterate in the same manner as before, we get the formula

$$J_{-v-k}(x) = (-1)^k \left( P_k \left( \frac{1}{x} \right) J_{-v}(x) + Q_{k-1} \left( \frac{1}{x} \right) J_{-v+1}(x) \right).$$

Letting  $v = \frac{1}{2}$ , we have the second result:

$$J_{-k-\frac{1}{2}}(x) = (-1)^k \sqrt{\frac{2}{x\pi}} \left( P_k \left( \frac{1}{x} \right) \cos(x) + Q_{k-1} \left( \frac{1}{x} \right) \sin(x) \right).$$

□

**Remark 2.2.3.** The polynomials  $P_n$  and  $Q_n$  in Lemma 2.2.7 are called Lommel polynomials and were introduced by the physicist Eugen von Lommel (1837-1899). They solve the recurrence relation

$$J_{m+v}(z) = J_v(z)R_{m,v}(z) - J_{v-1}(z)R_{m-1,v+1}(z)$$

and are given by the formula

$$R_{m,v} = \sum_{n=0}^{\lfloor \frac{m}{2} \rfloor} \frac{(-1)^n (m-n)! \gamma(v+m-n)}{n! (m-2n)! \gamma(v+n)} \left( \frac{z}{2} \right)^{2n-m}.$$

**Lemma 2.2.8.** For  $k \in \mathbb{N}$ ,

$$Y_{-k-\frac{1}{2}} = (-1)^k J_{k+\frac{1}{2}}(x) \tag{61}$$

and

$$Y_{k+\frac{1}{2}}(x) = (-1)^{k-1} J_{-k-\frac{1}{2}}(x). \tag{62}$$

*Proof.* We use the formula to obtain

$$\begin{aligned} Y_{-k-\frac{1}{2}}(x) &= \frac{\cos(-k\pi - \frac{\pi}{2}) J_{-k-\frac{1}{2}}(x) - J_{k+\frac{1}{2}}(x)}{\sin(-k\pi - \frac{\pi}{2})} \\ &= \frac{-J_{k+\frac{1}{2}}(x)}{(-1)^{k+1}} \\ &= (-1)^k J_{k+\frac{1}{2}}(x). \end{aligned}$$

And similarly,

$$\begin{aligned} Y_{k+\frac{1}{2}}(x) &= \frac{\cos(k\pi + \frac{\pi}{2}) J_{k+\frac{1}{2}}(x) - J_{-k-\frac{1}{2}}(x)}{\sin(k\pi + \frac{\pi}{2})} \\ &= \frac{-J_{-k-\frac{1}{2}}(x)}{(-1)^k} \\ &= (-1)^{k-1} J_{-k-\frac{1}{2}}(x). \end{aligned}$$

□

## 2.3 Integral Representations

The purpose of this section is to give the integral representations of each of our two Bessel functions. These will aid us later on in our discussion of asymptotics.

**Theorem 2.3.** For all  $n, x \in \mathbb{C}$  we have

$$J_\nu(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin t - \nu t) dt - \frac{\sin(\pi \nu)}{\pi} \int_0^\infty e^{-x \sinh(t) - \nu t} dt, \quad (63)$$

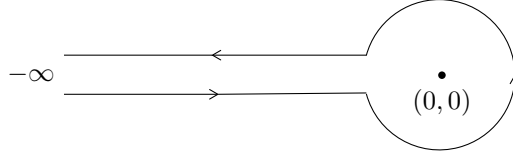
and

$$Y_\nu(x) = \frac{1}{\pi} \int_0^\pi \sin(x \sin t - \nu t) dt - \frac{1}{\pi} \int_0^\infty e^{-x \sinh(t)} (e^{\nu t} + \cos(\pi \nu) e^{-\nu t}) dt. \quad (64)$$

*Proof.* A representation of the Gamma function extended to the complex plane (given to us by the mathematician Hermann Hankel) is

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i} \int_{\gamma_1} t^{-z} e^t dt$$

where  $\gamma_1$  is some contour in the complex plane coming from  $-\infty$ , turning upwards around 0, and heading back towards  $-\infty$ .



**Figure 1:** The contour  $\gamma_1$ .

Then we have

$$J_\nu(x) = \frac{(\frac{x}{2})^\nu}{2\pi i} \int_{\gamma_1} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k} t^{-\nu-k-1}}{k!} e^t dt$$

since

$$\frac{1}{\Gamma(\nu + k + 1)} = \frac{1}{2\pi i} \int_{\gamma_1} t^{-\nu-k-1} e^t dt.$$

Recall the power series representation

$$e^{-\frac{x^2}{4t}} = \sum_{k=0}^{\infty} \frac{4^{-k} (\frac{-x^2}{t})^k}{k!} = \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^{2k} t^{-k}}{k!}.$$

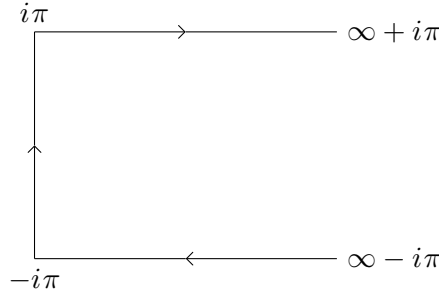
Then  $J_v(x)$  becomes

$$J_v(x) = \frac{\left(\frac{x}{2}\right)^v}{2\pi i} \int_{\gamma_1} t^{-v-1} e^{t-\frac{x^2}{4t}} dt.$$

We will apply u-substitution. Let  $t = \frac{x}{2}u$ . Then

$$\begin{aligned} J_v(x) &= \frac{\left(\frac{x}{2}\right)^v}{2\pi i} \int_{\gamma_2} \left(\frac{x}{2}u\right)^{-v-1} e^{\left(\frac{x}{2}u\right) - \left(\frac{x}{2}\right)^2 \frac{1}{\left(\frac{x}{2}u\right)}} \left(\frac{x}{2}\right) du \\ &= \frac{1}{2\pi i} \int_{\gamma_2} \left(\frac{x}{2}\right)^{v+1} \left(\frac{x}{2}\right)^{-v-1} u^{-v-1} e^{\frac{x}{2}\left(u-\frac{1}{u}\right)} du \\ &= \frac{1}{2\pi i} \int_{\gamma_2} u^{-v-1} e^{\frac{x}{2}\left(u-\frac{1}{u}\right)} du \end{aligned}$$

for some complex contour  $\gamma_2$  of the same type. Next we will perform another u substitution. Let  $u = e^w$ . We will have a new contour, since the exponential function is always positive. So our new contour will now originate from  $+\infty$ , turn around 0 (positively oriented), and head back to  $+\infty$ . A suitable contour following this path is the rectangle with complex vertices  $\infty - i\pi$ ,  $-i\pi$ ,  $i\pi$ , and  $\infty + i\pi$ . This will be our new contour,  $\gamma$ .



**Figure 2:** The contour  $\gamma$ .

Making this substitution, we have

$$\begin{aligned} J_v(x) &= \frac{1}{2\pi i} \int_{\gamma} (e^w)^{-v-1} e^{\left(\frac{x}{2}\right)(e^w - e^{-w})} e^w dw \\ &= \frac{1}{2\pi i} \int_{\gamma} (e^w)^{-v} e^{\frac{x}{2}(e^w - e^{-w})} dw \\ &= \frac{1}{2\pi i} \int_{\gamma} e^{-vw} e^{x \sinh(w)} dw. \end{aligned}$$

This integral in the rectangular  $\gamma$  can be split into three parts: the integral along the left vertical edge, the integral along the top edge, and the negative of the integral along the bottom edge. We can write this:

$$J_v(x) = \frac{1}{2\pi i} (P_1 + P_2 - P_3)$$



with

$$\begin{aligned} P_1 &= \int_{-\pi}^{\pi} e^{-ivt} e^{x \sinh(it)} i \, dt, \\ P_2 &= \int_0^{\infty} e^{-v(i\pi+t)} e^{x \sinh(i\pi+t)} dt, \\ P_3 &= \int_0^{\infty} e^{-v(-i\pi+t)} e^{x \sinh(-i\pi+t)} dt. \end{aligned}$$

**Step One.** Let us examine the first part,  $P_1$ . We have

$$\begin{aligned} \frac{1}{2\pi i} P_1 &= \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{-ivt} e^{x \sinh(it)} i \, dt \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \sinh(it) - ivt} dt. \end{aligned}$$

Recall the formula for the hyperbolic sine which says that  $\sinh(it) = i \sin(t)$ . We use this to get

$$\begin{aligned} \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x \sinh(it) - ivt} dt &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{i(x \sin(t) - vt)} dt \\ &= \frac{1}{2\pi} \left( \int_0^{\pi} e^{i(x \sin(t) - vt)} dt + \int_{-\pi}^0 e^{i(x \sin(t) - vt)} dt \right) \\ &= \frac{1}{2\pi} \left( \int_0^{\pi} e^{i(x \sin(t) - vt)} + e^{-i(x \sin(t) - vt)} dt \right) \end{aligned}$$

By Euler's formula, this becomes

$$\begin{aligned} \frac{1}{2\pi i} P_1 &= \frac{1}{2\pi} \int_0^{\pi} (\cos(x \sin(t) - vt) + i \sin(x \sin(t) - vt) \\ &\quad + \cos(-x \sin(t) + vt) + i \sin(-x \sin(t) + vt)) dt. \end{aligned}$$

Since sine is an odd function,  $i \sin(x \sin(t) - vt) = -i \sin(-x \sin(t) + vt)$ , and since cosine is an even function,  $\cos(x \sin(t) - vt) = \cos(-x \sin(t) + vt)$ . Therefore, we have

$$\frac{1}{2\pi i} P_1 = \frac{1}{\pi} \int_0^{\pi} \cos(x \sin(t) - vt) dt.$$

This gives the first half of the first result.

**Step Two.** We will examine

$$\begin{aligned} \frac{1}{2\pi i} (P_2 - P_3) &= \frac{1}{2\pi i} \left( \int_0^{\infty} e^{-v(-i\pi+t)} e^{x \sinh(i\pi+t)} dt - \int_0^{\infty} e^{-v(-i\pi+t)} e^{x \sinh(-i\pi+t)} dt \right) \\ &= \frac{1}{2\pi i} \int_0^{\infty} (e^{i\pi v} e^{-vt} e^{x \sinh(i\pi+t)} - e^{i\pi v} e^{-vt} e^{x \sinh(-i\pi+t)}) dt. \end{aligned}$$

We will use the property of the hyperbolic sine function which says that  $\sinh(-i\pi + t) = -\sinh(t)$  and  $\sinh(i\pi + t) = \sinh(t)$ . Then we have

$$\frac{1}{2\pi i}(P_2 - P_3) = \frac{1}{2\pi i} \int_0^\infty e^{-x \sinh(t) - vt} (e^{-i\pi v} - e^{i\pi v}) dt$$

Recall the identity  $\sin(x) = \frac{e^{ix} - e^{-ix}}{2i}$  which follows as a result of Euler's formula. Utilizing this, we have

$$\frac{1}{2\pi i}(P_2 - P_3) = \frac{-\sin(\pi v)}{\pi} \int_0^\infty e^{-x \sinh(t) - vt} dt.$$

This proves this 2nd half of the result for  $J_\nu(x)$ .

Next we will find the result for  $Y_\nu(x)$ . Rearranging equation (37), we have that

$$\begin{aligned} \sin(v\pi)Y_\nu(x) &= \cos(v\pi)J_\nu(x) - J_{-\nu}(x) \\ &= \frac{\cos(v\pi)}{\pi} \int_0^\pi \cos(x \sin(t) - vt) dt - \frac{\cos(v\pi) \sin(v\pi)}{\pi} \int_0^\infty e^{-x \sinh(t)} e^{-vt} dt \\ &\quad - \frac{1}{\pi} \int_0^\pi \cos(x \sin(t) + vt) dt + \frac{\sin(-v\pi)}{\pi} \int_0^\infty e^{-x \sinh(t)} e^{vt} dt \\ &= \frac{1}{\pi} \int_0^\pi \cos(v\pi) \cos(x \sin(t) - vt) dt - \frac{1}{\pi} \int_0^\pi \cos(x \sin(t) + vt) dt \\ &\quad - \frac{\sin(v\pi)}{\pi} \int_0^\infty (e^{-x \sinh(t)} \cos(v\pi) e^{-vt} + e^{vt}) dt \\ &= \frac{L_1}{\pi} - \frac{\sin(v\pi)}{\pi} L_2. \end{aligned}$$

Then

$$L_1 = \int_0^\pi \cos(v\pi) \cos(x \sin(t) - vt) dt - \int_0^\pi \cos(x \sin(t) + vt) dt$$

and

$$L_2 = \int_0^\infty (e^{-x \sinh(t)} \cos(v\pi) e^{-vt} + e^{vt}) dt.$$

We will use some rules of trigonometric products to rewrite  $L_1$ . Recall that  $\cos(a) \cos(b) = \frac{1}{2}(\cos(a+b) + \cos(a-b))$  and  $\sin(a) \sin(b) = \frac{1}{2}(\cos(a-b) - \cos(a+b))$ . We have

$$\begin{aligned} \cos(v\pi) \cos(x \sin(t) - vt) &= \frac{1}{2}(\cos(x \sin(t) - vt + v\pi) + \cos(x \sin(t) - vt - v\pi)) \\ &= \left( \cos(x \sin(t) - vt + v\pi) - \frac{1}{2}(\cos(x \sin(t) - vt + v\pi)) \right) \\ &\quad + \frac{1}{2}(\cos(x \sin(t) - vt - v\pi)) \end{aligned}$$

$$\begin{aligned}
&= \cos(x \sin(t) - vt + v\pi) + \sin(v\pi) \sin(x \sin(t) - vt) \\
&= \cos(x \sin(t) + v(\pi - t)) + \sin(v\pi) \sin(x \sin(t) - vt).
\end{aligned}$$

**Special Step.** Before we continue, we need to show that the following relation is true:

$$\int_0^\pi \cos(x \sin(t) + v(\pi - t)) dt = \int_0^\pi \cos(x \sin(t) + vt) dt.$$

First we will simplify the statement.

$$\begin{aligned}
\cos(x \sin(t) + v\pi - vt) &= \cos(x \sin(t)) \cos(v\pi - vt) - \sin(x \sin(t)) \sin(v\pi - vt) \\
&= \cos(x \sin(t)) (-\cos(vt)) - \sin(x \sin(t)) \sin(vt) \\
&= -(\cos(x \sin(t)) \cos(vt) + \sin(x \sin(t)) \sin(vt)) \\
&= -\cos(x \sin(t) - vt).
\end{aligned}$$

Then we need to show that

$$\int_0^\pi \cos(x \sin(t) + vt) dt = \int_0^\pi -\cos(x \sin(t) - vt) dt.$$

We have

$$\cos(x \sin(t) + vt) = \cos(x \sin(t)) \cos(vt) - \sin(x \sin(t)) \sin(vt)$$

so the integral is

$$\begin{aligned}
\int_0^\pi \cos(x \sin(t) + vt) dt &= \int_0^\pi \cos(x \sin(t)) \cos(vt) dt - \int_0^\pi \sin(x \sin(t)) \sin(vt) dt \\
&= 0 - \int_0^\pi \sin(x \sin(t)) \sin(vt) dt.
\end{aligned}$$

In addition,

$$-\cos(x \sin(t) - vt) = -\cos(x \sin(t)) \cos(vt) - \sin(x \sin(t)) \sin(vt),$$

and the integral becomes

$$\begin{aligned}
\int_0^\pi -\cos(x \sin(t) - vt) dt &= \int_0^\pi -\cos(x \sin(t)) \cos(vt) dt - \int_0^\pi \sin(x \sin(t)) \sin(vt) dt \\
&= 0 - \int_0^\pi \sin(x \sin(t)) \sin(vt) dt.
\end{aligned}$$

And thus, we have shown that the relation holds. We will use it to simplify  $L_1$  as follows:

$$\begin{aligned}
L_1 &= \int_0^\pi \left( \cos(x \sin(t) + v(\pi - t)) + \sin(v\pi) \sin(x \sin(t) - vt) \right. \\
&\quad \left. - \cos(x \sin(t) + vt) \right) dt \\
&= \sin(v\pi) \int_0^\pi \sin(x \sin(t) - vt) dt.
\end{aligned}$$

This proves the result for  $Y_v(x)$ . □

## 2.4 Using the Generating Function to Derive $J_n(x)$

The previous section gave the integral representation of  $J_\nu(x)$  for all  $\nu \in \mathbb{C}$ . Notice that for all  $n \in \mathbb{N}$ ,  $\sin(n\pi) = 0$ . We see this in the following theorem, which gives the integral representation for orders in the natural numbers.

**Theorem 2.4.** *Let the functions  $y_n(x)$  be defined by the Laurent series*

$$e^{\frac{x}{2}(z-z^{-1})} = \sum_{n=-\infty}^{\infty} y_n(x)z^n. \quad (65)$$

Then

$$y_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k} \quad (66)$$

and

$$y_n(x) = \frac{1}{\pi} \int_0^\pi \cos(x \sin \phi - n\phi) d\phi. \quad (67)$$

*Proof.* Cauchy's integral formula for a Laurent series gives us:

$$y_n(x) = \frac{1}{2\pi i} \oint_{\gamma} \frac{e^{\frac{x}{2}(t-t^{-1})}}{t^{n+1}} dt$$

for any simply closed contour  $\gamma$  centered around 0.

**Step One.** Perform the  $u$ -substitution  $t = \frac{2u}{x}$  and recall the series expansion  $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ . Then we have

$$\begin{aligned} y_n(x) &= \frac{1}{2\pi i} \oint_{\gamma'} \frac{e^{\frac{x}{2}(\frac{2u}{x} - (\frac{2u}{x})^{-1})}}{(\frac{2u}{x})^{n+1}} \frac{2}{x} du \\ &= \frac{1}{2\pi i} \oint_{\gamma'} \frac{e^{u - \frac{x^2}{4u}}}{(\frac{2}{x})^{n+1} u^{n+1}} \frac{2}{x} du \\ &= \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \oint_{\gamma'} e^{u - \frac{x^2}{4u}} u^{-n-1} du \\ &= \frac{1}{2\pi i} \left(\frac{x}{2}\right)^n \oint_{\gamma'} \sum_{m=0}^{\infty} \frac{u^m}{m!} \sum_{k=0}^{\infty} \frac{(\frac{-x}{2})^{2k}}{k!} u^{-k} u^{-n-1} du \\ &= \left(\frac{x}{2}\right)^n \sum_{m=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-1)^k}{m!k!} \left(\frac{x}{2}\right)^{2k} \frac{1}{2\pi i} \oint_{\gamma'} u^{m-k-n-1} du \end{aligned}$$

Choosing  $\gamma'$  to be a circle  $C_R$  of radius  $R$ , Cauchy's formula gives us

$$\oint_{C_R} u^l du = 2\pi i$$

for  $l = -1$ . Otherwise, the integral equals 0. Thus, the only summation term remaining will correspond to  $m = n + k$ , and the equation is

$$y_n(x) = \left(\frac{x}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{(n+k)!k!} \left(\frac{x}{2}\right)^{2k}.$$

This proves the first result.

**Step Two.** We can choose the contour  $t = e^{i\phi}$  and integrate from 0 to  $2\pi$ . Then we have

$$\begin{aligned} y_n(x) &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\phi} - e^{-i\phi})}}{(e^{i\phi})^{n+1}} e^{-i\phi} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\frac{x}{2}(e^{i\phi} - e^{-i\phi})}}{e^{in\phi}} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\frac{x}{2}((\cos\phi + i\sin\phi) - (\cos(-\phi) + i\sin(-\phi)))}}{e^{in\phi}} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\frac{x}{2}(\cos\phi + i\sin\phi - \cos\phi + i\sin\phi)}}{e^{in\phi}} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{xi\sin\phi}}{e^{in\phi}} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ix\sin\phi - in\phi} d\phi \\ &= \frac{1}{2\pi} \int_0^{2\pi} (\cos(x\sin\phi - n\phi) + i\sin(x\sin\phi - n\phi)) d\phi \\ &= \frac{1}{\pi} \int_0^{\pi} \cos(x\sin\phi - n\phi) d\phi. \end{aligned}$$

This proves the second result. □

## 2.5 Asymptotic Analysis

In this section we will use the proven integral representations to derive some asymptotic formulae.

**Theorem 2.5.** For  $x \in \mathbb{R}$ , as  $x \rightarrow \infty$  we have

$$J_v(x) \sim \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right), \quad (68)$$

and

$$Y_v(x) \sim \sqrt{\frac{2}{\pi x}} \sin\left(x - \frac{\pi}{4} - \frac{v\pi}{2}\right). \quad (69)$$

*Proof.* The second integrals in both of the integral representations go to 0 exponentially as  $x$  gets large. So as  $x \rightarrow \infty$ , we use Euler's formula to write

$$J_v(x) + iY_v(x) = \frac{1}{\pi} \int_0^{\pi} e^{i(x\sin(t) - vt)} dt + \mathcal{O}(x^{-A})$$

for all  $A$ . Let  $u = t - \frac{\pi}{2}$ . Then we have

$$\begin{aligned}
J_v(x) + iY_v(x) &= \frac{1}{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(x \sin(u + \frac{\pi}{2}) - v(u + \frac{\pi}{2}))} du + \mathcal{O}(x^{-A}) \\
&= \frac{2}{\pi} \int_0^{\frac{\pi}{2}} e^{ix \cos(u)} e^{-ivu} e^{-\frac{iv\pi}{2}} du + \mathcal{O}(x^{-A}) \\
&= \frac{2e^{-\frac{iv\pi}{2}}}{\pi} \int_0^{\frac{\pi}{2}} e^{ix \cos(u)} (\cos(vu) - i \sin(vu)) du + \mathcal{O}(x^{-A}) \\
&= \frac{2e^{-\frac{iv\pi}{2}}}{\pi} \left( \int_0^{\frac{\pi}{3}} e^{ix \cos(u)} (\cos(vu) - i \sin(vu)) du \right. \\
&\quad \left. + \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} e^{ix \cos(u)} (\cos(vu) - i \sin(vu)) du \right) + \mathcal{O}(x^{-A}) \\
&= \frac{2e^{-\frac{iv\pi}{2}}}{\pi} (P_1 + P_2) + \mathcal{O}(x^{-A})
\end{aligned}$$

where  $P_1 = \int_0^{\frac{\pi}{3}} e^{ix \cos(u)} (\cos(vu) - i \sin(vu)) du$  and  $P_2 = \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} e^{ix \cos(u)} (\cos(vu) - i \sin(vu)) du$ .

**Step One.** We take  $P_2$  and make the substitution  $\cos(u) = z$ . Then  $u = \cos^{-1}(z)$  and  $du = -\sqrt{1-z^2} dz$ . So we have

$$\begin{aligned}
P_2 &= \int_{\frac{1}{2}}^0 e^{ixz} \frac{\cos(v \cos^{-1}(z)) - i \sin(v \cos^{-1}(z))}{-\sqrt{1-z^2}} dz \\
&= \int_0^{\frac{1}{2}} e^{ixz} \frac{\cos(v \cos^{-1}(z)) - i \sin(v \cos^{-1}(z))}{\sqrt{1-z^2}} dz \\
&= \int_0^{\frac{1}{2}} e^{ixz} \phi(z) dz
\end{aligned}$$

where

$$\phi(z) = \frac{\cos(v \cos^{-1}(z)) - i \sin(v \cos^{-1}(z))}{\sqrt{1-z^2}}.$$

Integrating by parts, we have

$$\begin{aligned}
\int_0^{\frac{1}{2}} e^{ixz} \phi(z) dz &= \frac{e^{ixz}}{ix} \phi(z) \Big|_0^{\frac{1}{2}} - \frac{1}{ix} \int_0^{\frac{1}{2}} e^{ixz} \phi'(z) dz \\
&= \left( \frac{\cos(xz) + i \sin(xz)}{ix} \right) \phi(z) \Big|_0^{\frac{1}{2}} - \frac{1}{ix} \int_0^{\frac{1}{2}} e^{ixz} \phi'(z) dz \\
&= \left( \frac{\sin(xz)}{x} \phi(z) + \frac{\cos(xz)}{ix} \phi(z) \right) \Big|_0^{\frac{1}{2}} - \frac{1}{ix} \int_0^{\frac{1}{2}} e^{ixz} \phi'(z) dz \\
&= \mathcal{O}(x^{-1}),
\end{aligned}$$

since  $z \in [0, \frac{1}{2}]$  avoids any singularities of  $\phi$  and  $\phi'$ . (Notice that  $\phi$  is not a function of  $x$  and is composed of cyclical functions sine and cosine. We are only interested in the behavior as  $x$  approaches a very large number, where  $\phi$  will have a negligible effect.)

**Step Two.** We take  $P_1$  and substitute  $t = \sqrt{2x} \sin(\frac{u}{2})$ . Then we have  $du = \frac{\sqrt{2}}{\sqrt{x}} \frac{dt}{\sqrt{1 - \frac{t^2}{2x}}}$  and  $\cos(u) = 1 - \frac{t^2}{x}$ . The equation becomes:

$$\begin{aligned} P_2(x) &= \frac{\sqrt{2}}{\sqrt{x}} \int_0^{\sqrt{\frac{x}{2}}} e^{ix(1 - \frac{t^2}{x})} \left( \cos \left( 2v \sin^{-1} \left( \frac{t}{\sqrt{2x}} \right) \right) - i \sin \left( 2v \sin^{-1} \left( \frac{t}{\sqrt{2x}} \right) \right) \right) \frac{dt}{\sqrt{1 - \frac{t^2}{2x}}} \\ &= \frac{\sqrt{2}}{\sqrt{x}} e^{ix} \int_0^{\sqrt{\frac{x}{2}}} e^{-it^2} \left( \cos \left( 2v \sin^{-1} \left( \frac{t}{\sqrt{2x}} \right) \right) - i \sin \left( 2v \sin^{-1} \left( \frac{t}{\sqrt{2x}} \right) \right) \right) \frac{dt}{\sqrt{1 - \frac{t^2}{2x}}}. \end{aligned}$$

As  $x \rightarrow \infty$ , we get

$$P_1(x) \sim \frac{\sqrt{2}}{\sqrt{x}} e^{ix} \int_0^\infty e^{-it^2} dt.$$

Since  $\int_0^\infty e^{-it^2} dt = \frac{\sqrt{\pi}}{2} e^{-\frac{i\pi}{4}}$ , we finally have

$$\begin{aligned} J_v(x) + iY_v(x) &\sim \frac{2e^{-\frac{iv\pi}{2}}}{\pi} \left( \frac{\sqrt{2}}{\sqrt{x}} e^{ix} \right) \left( \frac{\sqrt{\pi}}{2} e^{-\frac{i\pi}{4}} \right) \\ &\sim \sqrt{\frac{2}{\pi x}} e^{i(x - \frac{\pi}{4} - \frac{v\pi}{2})}. \end{aligned}$$

Applying Euler's formula leads to the result. □

## 3 Graphs

The following code was implemented in Python to create graphs of the Bessel functions of orders  $n = 0, 1, 2, 3, 4$ .

### 3.1 Graph of First Bessel Function

```
import matplotlib as matplotlib
import numpy as np
import matplotlib.pyplot as plt
import scipy.integrate as integrate

def f(x,n):
    return integrate.quad(lambda t: 1/np.pi * np.cos(x*np.sin(t) - n*t),
    0, np.pi)

X = np.arange(0.0, 30.0, 0.01)

plt.figure(1, figsize=(10,8))
plt.plot(X, [f(x,0)[0] for x in X], '--', linewidth = 1.7, label = 'n=0')
plt.plot(X, [f(x,1)[0] for x in X], linewidth = 1.5, label = 'n=1')
plt.plot(X, [f(x,2)[0] for x in X], '--',linewidth = 1.25, label = 'n=2')
plt.plot(X, [f(x,3)[0] for x in X], linewidth = 1, label = 'n=3')
plt.plot(X, [f(x,4)[0] for x in X], '--',linewidth = 0.75, label = 'n=4')

legend = plt.legend(loc='upper right', shadow=True)
frame = legend.get_frame()
frame.set_facecolor('0.90')

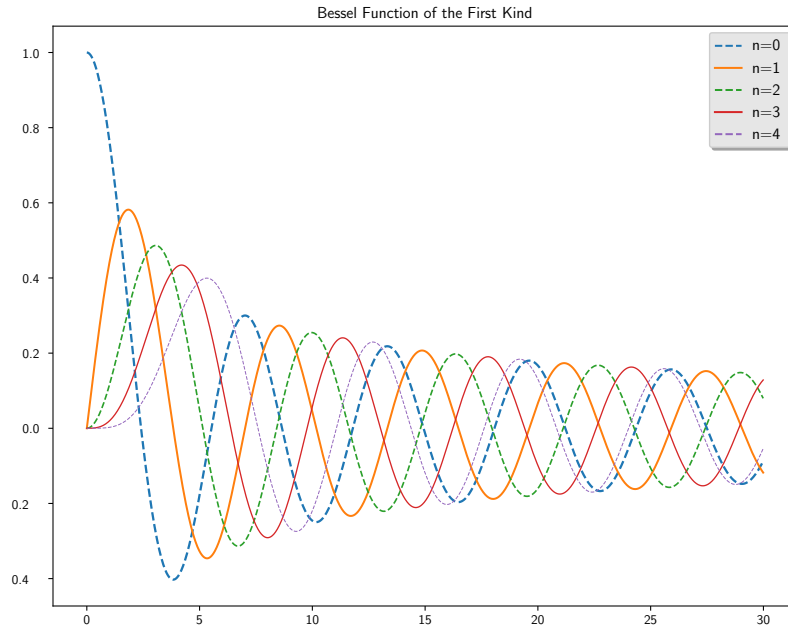
for label in legend.get_texts():
    label.set_fontsize('large')

for label in legend.get_lines():
    label.set_linewidth(1.5)

plt.title('Bessel Function of the First Kind')

plt.show()
```





### 3.2 Graph of Second Bessel Function

```

import matplotlib as matplotlib
import numpy as np
import matplotlib.pyplot as plt
import scipy.integrate as integrate

def f(x,n):
return integrate.quad(lambda t: 1/np.pi * np.sin(x*np.sin(t) - n*t), 0, np.pi)

def g(x,n):
return integrate.quad(lambda t: -1/np.pi * np.exp(-x*np.sinh(t))
*(np.exp(n*t)+np.cos(n*np.pi)*np.exp(-n*t)), 0, 150)

X = np.arange(0.1, 30.0, 0.01)

plt.figure(1, figsize=(10,8))
plt.plot(X, [f(x,0)[0]+g(x,0)[0] for x in X], '--',linewidth = 1.75, label =
'n=0')
plt.plot(X, [f(x,1)[0]+g(x,1)[0] for x in X], linewidth = 1.50, label = 'n=1')
plt.plot(X, [f(x,2)[0]+g(x,2)[0] for x in X], '--',linewidth = 1.25, label =
'n=2')

```

```

plt.plot(X, [f(x,3)[0]+g(x,3)[0] for x in X], linewidth = 1.00, label = 'n=3')
plt.plot(X, [f(x,4)[0]+g(x,4)[0] for x in X], '--',linewidth = 0.75, label =
'n=4')

legend = plt.legend(loc='upper right', shadow=True)
frame = legend.get_frame()
frame.set_facecolor('0.90')

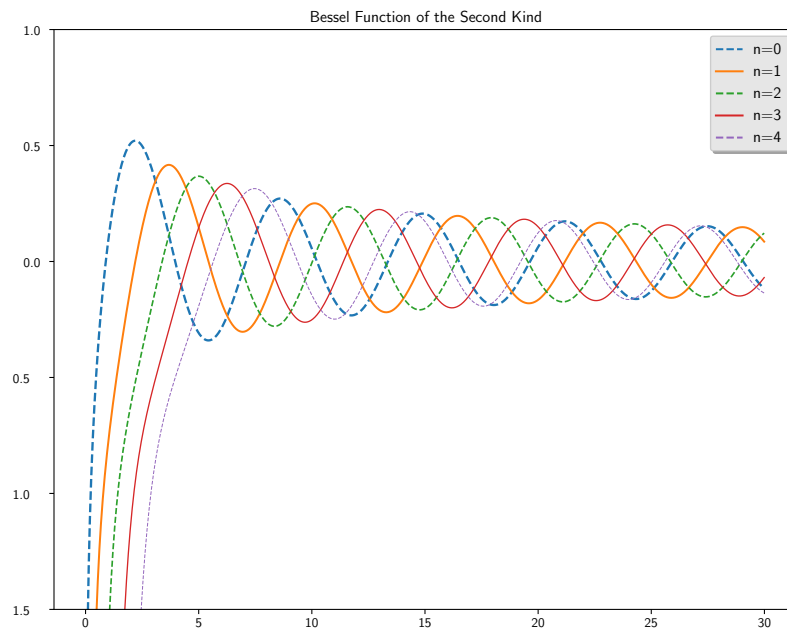
for label in legend.get_texts():
    label.set_fontsize('large')

for label in legend.get_lines():
    label.set_linewidth(1.5)

plt.title('Bessel Function of the Second Kind')
plt.ylim([-1.5,1.0])

plt.show()

```



## References

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