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Laurent Series Expansion and its Applications

Anna Sobczyk
Portland State University

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By

Anna Sobczyk

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Dr. Bin Jiang

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Abstract

The Laurent expansion is a well-known topic in complex analysis for its application in obtaining residues of complex functions around their singularities. Computing the Laurent series of a function around its singularities turns out to be an efficient way to determine the residue of the function as well as to compute the integral of the function along any closed curves around its singularities. Based on the theory of the Laurent series, this paper provides several working examples where the Laurent series of a function is determined and then used to calculate the integral of the function along any closed curve around the singularities of the function. A brief description of the Frobenius method in solving ordinary differential equations is also provided.

Section I. Introduction

The method of Laurent series expansions is an important tool in complex analysis. Where a Taylor series can only be used to describe the analytic part of a function, Laurent series allows us to work around the singularities of a complex function. To do this, we need to determine the singularities of the function and can then construct several concentric rings with the same center z_0 based on those singularities and obtain a unique Laurent series of $z - z_0$inside each ring where the function is analytic.

The construction of Laurent series is important because the coefficient corresponding to the $\frac{1}{z-z_0}$ term gives the residue of the function. The calculation of the integral of the function along any closed curve can be done efficiently by using such residue based on the Residue Theorem. Not only does the Laurent series create an efficient method for the integration, it also has many other applications in physics and engineering.

While the residue of the function has been used extensively in calculating both complex and real integration, we seldom investigate the coefficient of the $\frac{1}{z-z_0}$ term that occur in the outer rings of a Laurent series expansion. This paper serves to speak on the significance of this coefficient in the outer rings by providing several working examples of the Laurent series outside of the center annulus and using them to compute the integral of the function along any closed curve outside of the center annulus. This paper also describes the Frobenius method, a method very similar to Laurent series, in solving second-order ordinary differential equations around their singularities.

Section II. Background Theory

First, let us define a Laurent series.[1]

Theorem 1.1 Let $0 \leq r_1 < r_2$ and $z_0 \in \mathbb{C}$. Suppose $f(z)$ is analytic on the region $A = \{z \in \mathbb{C} | \ r_1 < |z - z_0| < r_2\}$ where $r_1 \geq 0$ and $r_2 \leq \infty$. Then we have:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$$

which converge absolutely on A. This series for $f(z)$ is called the Laurent series or Laurent expansion around z_0 in the annulus A.

The coefficients can be determined by:

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \ , \qquad n = 0, 1, 2, \dots$$

$$b_n = \frac{1}{2\pi i} \int_\gamma f(\zeta)(\zeta - z_0)^{n-1} d\zeta \ , \qquad n = 1, 2, \dots$$

Where γ can be any circle with center z_0 and radius $r, r_1 < r < r_2$. Furthermore, $a_n$and $b_n$are unique.

However, the equations for finding the coefficients $a_n$and $b_n$in the Laurent Series are impractical for a given function. Although the formulas exist, they are seldom used as there are always easier tricks to obtain the Laurent expansion. For instance, we can use well-known Taylor expansions of some fundamental function to obtain the Laurent series. Another common trick is to creatively manipulate a function f to fit into the form of a common geometric series. See the working examples for demonstrations of these tricks.

Next, let us introduce the classification of singularities, which are central to constructing a Laurent series. Simply put, a singularity is a point z_0 in which a function is differentiable at points arbitrarily close to but not including z_0. Singularities are not always easily for a complicated function. There are different types and classifications of singularities that we will define now.

Definition 1.1 If a function f is analytic on a region $A = \{z | \ 0 < |z - z_0| < r_2\} \ with \ r_1 = 0$, which is a deleted neighborhood of z_0, then z_0 is called an *isolated singularity*[2] and is expressed as:

$$f(z) = \cdots + \frac{b_n}{(z-z_0)^n} + \cdots + \frac{b_1}{z-z_0} + a_n + a_1(z - z_0) + \cdots \ in \ \{z | \ 0 < |z - z_0| < r_2\}$$

[1] Theorem adapted from Marsden & Hoffman textbook.
[2] Definition taken from Marsden & Hoffman and Apelian & Surace text books.

The convenience of Laurent series is that we can always find a Laurent expansion centered at an isolated singularity in an annulus that omits that point.[3] The Laurent expansion allows for a series representation in both negative and positive powers of $(z - z_0)$ in a region excluding points where f is not differentiable. If f is differentiable in the entire region, then it is analytic and the Laurent series centered at z_0 will reduce to the Taylor series of the function below:

$$f(z) = \sum_{n=0}^{\infty} \frac{f^k(c)}{k!}(z-c)^k \quad for\ |z-c| < r$$

There are different classifications of isolated singularities as below:

> **Definition 1.2** Let z_0 be an isolated singularity of f. If all but a finite number of the b_n terms are zero, then z_0 is called a *pole* of f. The order of the pole is determined by the highest integer *k* such that $b_k \neq 0$ and is called a *pole of order k*. A pole of order one is commonly referred to as a *simple pole* or *single pole*.
>
> **Definition 1.3** If an infinite number of b_k are nonzero, z_0 is called an *essential singularity*. This z_0 is also sometimes called a pole of infinite order.
>
> **Definition 1.4** We call z_0 a *removable singularity* if all b_k's are zero. A Taylor series expansion always exists for removable singularities.

We focus on the main application of Laurent series: finding the residue of a function. While some complex functions have handy formulas for calculating the residue, it mainly depends on the type of singularity you are dealing with. For instance, there is no efficient way to find the residue of a function with an essential singularity. To find the residue in this case, you must find the Laurent expansion of the function and locate the $\frac{\boldsymbol{b_1}}{z-z_0}$ term:

$$f(z) = \cdots + \frac{b_n}{(z-z_0)^n} + \cdots + \frac{\boldsymbol{b_1}}{z-z_0} + a_n + a_1(z-z_0) + \cdots \quad in\ \{z|\ 0 < |z-z_0| < r_2\}$$

In the Laurent expansion, $\boldsymbol{b_1}$ is the coefficient of the $\frac{1}{z-z_0}$ term in the series. Even if the residue can be found easily through a formula, one might want to look at the Laurent series obtained in the outer annuli associated with the singularities. Once the Laurent expansion is known, we can find the corresponding $\frac{\boldsymbol{b_1}}{z-z_0}$ term. It can help us with calculating the integration of $f(z)$ along any closed curve located within those annuli.

[3] See Apelian & Surace.

Theorem 1.2 Residue Theorem[4]: Let A be a region and let $z_{1,}z_2, \dots, z_n$ be n distinct points in A. Let f be analytic in A except for at the isolated singularities at $z_{1,}z_2, \dots, z_n$. Let γ be a closed curve in A homotopic to a point in A. Assume no z_i lies on γ. Then:

$$\int_\gamma f(z)dz = 2\pi i \sum_{i=1}^{n} [Res(f; z_i)] I(\gamma; z_i)$$

Here $Res(f; z_i)$ is the residue of f at z_i and $I(\gamma; z_i)$ is the index of γ with respect to z_i. For simplicity, we assume $I(\gamma; z_i) = 1$ for this paper.

Simply put, for a clockwise γ, the integral equals $2\pi i$ times the sum of the residues of f inside γ. In complex analysis, residues play a critical role in allowing us to easily compute integrals. Using the Residue Theorem to solve a line integral is significantly more efficient than other methods because it greatly reduces computational time. The Residue Theorem in complex analysis also makes the integration of some real functions feasible without need of numerical approximation.

The examples in this paper focus on obtaining the residue from a Laurent series. The residues obtained from the Laurent series would speed up the complex integration on closed curves.

[4] Theorem adapted from Marsden & Hoffman textbook.

Section III. Working Examples

In the following examples we will refer to a helpful geometric series and some common known Taylor expansions.

Geometric Series

$$\frac{1}{1-z} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} z^n, & |z| < 1 \\ \displaystyle -\sum_{n=1}^{\infty} \frac{1}{z^n}, & |z| > 1 \end{cases}$$

Table 1.1

Known Function Expansions		
Function	*Taylor Series around 0*	*Valid for*
$\sin(z)$	$z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{2n-1}}{(2n-1)!}$	all z
$\cos(z)$	$1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \cdots = \sum_{n=1}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}$	all z
e^z	$1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$	all z

Example 1

Let $f(z) = \frac{1}{2+z}$. Determine the Laurent series around $z = 1$.

Solution:

Obviously, we have a simple pole at $z = -2$. Hence, we are dealing with a radius of 3 and want to find the Laurent series for both $|z-1| < 3$ *and* $|z-1| > 3$. The Laurent series will reduce to a Taylor series inside $|z-1| < 3$ where $f(z)$ is analytic.

For $|z-1| < 3$, we refer to the well-known geometric series. We begin by trying to create a $(z-1)$ term in the denominator.

$$f(z) = \frac{1}{2+z} = \frac{1}{2+z-1+1} = \frac{1}{3+(z-1)} = \left(\frac{1}{3}\right)\frac{1}{1-\left(\frac{-(z-1)}{3}\right)}$$

Since $\left|\frac{-(z-1)}{3}\right| < 1$, we can now represent the function as a series:

$$\Rightarrow f(z) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{-(z-1)^n}{3^n} = \sum_{n=0}^{\infty}\frac{(-1)^n(z-1)^n}{3^{n+1}} \text{ for } |z-1| < 3$$

Which is just a Taylor series as the function is analytic inside the region.

For $|z-1| > 3$, we can use $\frac{3}{|z-1|} < 1$ and follow our previous work to obtain:

$$f(z) = \frac{1}{3+(z-1)} = \frac{1}{z-1}\frac{1}{1-\left(\frac{-3}{z-1}\right)} = \frac{1}{z-1}\sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(z-1)^n} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(z-1)^{n+1}}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{(z-1)^{n+1}} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{n-1}}{(z-1)^n}$$

In this case, we have obtained the Laurent expansion. The generalized residue for the outer ring $|z-1| > 3$ is the coefficient of $\frac{1}{z-1}$, that is $b_1 = 1$.

Remarks:

a) Integrating along a closed path inside the inner ring $|z-1| < 3$ means we should use the residue obtained from the Laurent series in $|z-1| < 3$. By *Theorem 1.1*, we can compute

$$\int_{|z-1|=1} f(z)dz = 2\pi i(b_1) = 2\pi i(0) = 0.$$

b) Integrating along a non-simple closed path inside the outer ring $|z-1|>3$ means we must use the generalized residue obtained from the Laurent series in $|z-1|>3$. By *Theorem 1.1*, we can compute

$$\int_{|z-1|=4} f(z)dz = 2\pi i(b_1) = 2\pi i(1) = 2\pi i.$$

c) By combining a) and b), we conclude that the Laurent series computed in each ring, either inner ring or outer ring can be used directly with coefficient $b_1$to compute the integral along any closed curve inside that ring.

Example 2

This example will use the same method as in Example 1 based on the known geometric series to find the Laurent Expansion. However, this is a more complicated example and will require extra techniques—a substitution and partial fraction decomposition. Suppose:

$$f(z) = \frac{1}{z(z-2)(z-5)}$$

By observation, we can easily see that $f(z)$ has simple poles at $z = 0, z = 2\ and\ z = 5$. These poles determine the radii of the annuli produced with the Laurent Series. We have $0 < |z| < 2$ as the center annulus, $2 < |z| < 5$ as the middle annulus, and $5 < |z| < \infty$ as the outermost infinite "ring."

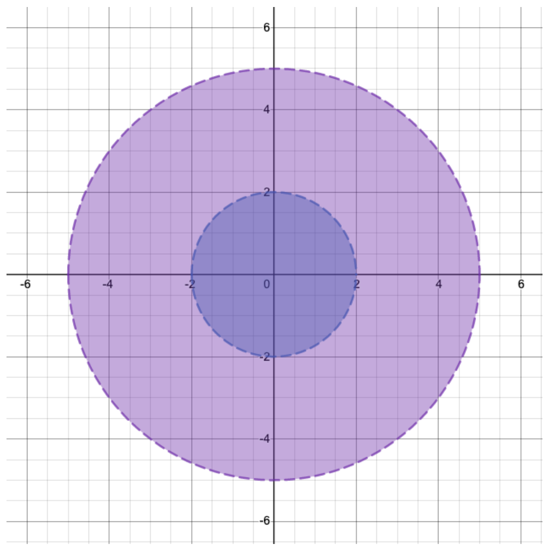

Figure 1.1

In this example, we will find the Laurent expansion valid on $R = \{z \mid |z-3| < 1\}$. This is the circle of radius 1 centered at $z = 3$ where $f(z)$ is analytic.

Step 1 In order to construct the Laurent series of this function, we first use a substitution such that $w = z - 3$. Then we can center w around $w = 0$ within $|w| < 1$.

$$f(z) = \frac{1}{(w+3)(w+1)(w-2)}$$

Step 2 We now use partial fraction decomposition:

$$f(w) = \frac{1}{(w+3)(w+1)(w-2)} = \frac{A}{w+3} + \frac{B}{w+1} + \frac{C}{w-2}$$

$$\Rightarrow A(w^2 - w - 2) + B(w^2 + w - 6) + C(w^2 + 4w + 3) = 1$$

$$A + B + C = 0$$
$$-A + B + 4C = 0$$
$$-2A - 6B + 3C = 1$$

From the above equations, we obtain the following coefficients:

$A = \frac{1}{10}$; $B = -\frac{1}{6}$; $C = \frac{1}{15}$.

We will denote the partial fraction components of $f(z)$ as:

$$f_1(w) = \frac{1}{10(w+3)},\ f_2(w) = \frac{-1}{6(w+1)},\ f_3(w) = \frac{1}{15(w-2)}.$$

Step 3 Determine the Laurent expansion valid for $|w| < 1$ based on the above partial fractions.

Let us start with $f_1(w) = \frac{1}{10(w+3)}$. For $|w| < 1$:

$$\frac{1}{10(w+3)} = \frac{1}{10}\frac{1}{3-(-w)} = \frac{1}{10}\frac{1}{3}\frac{1}{1-\left(\frac{-w}{3}\right)} = \frac{1}{10}\sum_{n=0}^{\infty}\frac{(-1)^n w^n}{3^{n+1}}$$

We obtain the following geometric series and then substitute $w = z - 3$.

$$f_1(w) = \sum_{n=0}^{\infty}\frac{(-1)^n w^n}{10(3^{n+1})},\ \ |w| < 1 \Rightarrow f_1(z) = \sum_{n=0}^{\infty}\frac{(-1)^n (z-3)^n}{10(3^{n+1})},\ \ |z-3| < 1.$$

For $f_2(w) = \frac{-1}{6(w+1)}$ on $|w| < 1$:

$$\frac{-1}{6(w+1)} = -\frac{1}{6}\frac{1}{1-(-w)} = -\frac{1}{6}\sum_{n=0}^{\infty}(-1)^n w^n$$

We obtain the following geometric series and substitute back in for $w = z - 3$.

$$f_2(w) = -\sum_{n=0}^{\infty}\frac{(-1)^n w^n}{6},\ \ |w| < 1 \Rightarrow f_2(z) = -\sum_{n=0}^{\infty}\frac{(-1)^n (z-3)^n}{6},\quad |z-3| < 1$$

For $f_3(w) = \frac{1}{15(w-2)}$ on $|w| < 1$:

$$\frac{1}{15(w-2)} = \frac{1}{15(-2-(-w))} = \frac{1}{15}\frac{-1}{2}\frac{1}{\left(1-\frac{w}{2}\right)} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{w^n}{2^{n+1}}$$

We obtain the following geometric series and substitute back in for $w = z - 3$.

$$f_3(w) = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{w^n}{2^{n+1}} \quad |w| < 1 \Rightarrow f_3(z) = -\sum_{n=0}^{\infty}\frac{(z-3)^n}{15(2^{n+1})},\ \ |z-3| < 1.$$

Step 4 Determine which parts are relevant for the Laurent series. In this case, we only want $|w| < 1$, or $|z - 3| < 1$. We combine the above geometric series and obtain:

$$f(z) = \sum_{n=0}^{\infty}\frac{(-1)^n(z-3)^n}{10(3^{n+1})} - \sum_{n=0}^{\infty}\frac{(-1)^n(z-3)^n}{6} - \sum_{n=0}^{\infty}\frac{(z-3)^n}{15(2^{n+1})}$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n(z-3)^n}{10(3^{n+1})} - \frac{(-1)^n(z-3)^n}{6} - \frac{(z-3)^n}{15(2^{n+1})}\right)$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\left(\frac{(-1)^n}{10(3^{n+1})} - \frac{(-1)^n}{6} - \frac{1}{15(2^{n+1})}\right)(z-3)^n$$

$$\Rightarrow f(z) = \sum_{n=0}^{\infty}\frac{\{(-1)^n(2^{n+1})[3 - 5(3^{n+1})] - 2(3^{n+1})\}}{30(6^{n+1})}(z-3)^n$$

Note that this results in the analytic part of the Laurent expansion without principal part. This means our Laurent series has reduced to a Taylor series as in Example 1.

Example 3

Let us find the Laurent series of $f(z) = \frac{1}{z(z-2)(z-5)}$ valid in the region $2 < |z| < 5$ around $z = 0$.

Step 1 we start with partial fraction decomposition.

$$f(z) = \frac{1}{z(z-2)(z-5)} = \frac{A}{z} + \frac{B}{z-2} + \frac{C}{z-5}$$

$$\begin{aligned}
&\Rightarrow A(z^2 - 7z + 10) + B(z^2 - 5z) + C(z^2 - 2z) = 1 \\
&A + B + C = 0 \\
&-7A - 5B - 2C = 0 \\
&10A = 1
\end{aligned}$$

From the above equations, we obtain the following coefficients:

$A = \frac{1}{10}$; $B = -\frac{1}{6}$; $C = \frac{1}{15}$.

We will denote the partial fraction components of $f(z)$ as:

$f_1(z) = \frac{1}{10z}$, $f_2(z) = \frac{-1}{6(z-2)}$, $f_3(z) = \frac{1}{15(z-5)}$.

Step 2 Determining the geometric series to each corresponding partial fraction.

We consider the region $2 < |z| < 5$. In order to use our geometric series, we must represent the function in the region in terms of $z\ or\ \frac{1}{z}$ based on the fact that

$$2 < |z| \Rightarrow |\frac{2}{z}| < 1 \ and\ |z| < 5 \Rightarrow |\frac{z}{5}| < 1.$$

The Laurent series representation is found by looking for $\frac{2}{z}$ for $f_2(z)$ and $\frac{z}{5}$ for $f_3(z)$:

a. $f_2(z) = \frac{-1}{6(z-2)} = \frac{-1}{6z}\frac{1}{\left(1-\frac{2}{z}\right)} = -\frac{1}{6z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} = -\frac{1}{6}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}}$

b. $f_3(z) = \frac{1}{15(z-5)} = \frac{1}{15}\frac{1}{5}\left(\frac{1}{-1+\frac{z}{5}}\right) = -\frac{1}{15}\frac{1}{5}\sum_{n=0}^{\infty}\frac{z^n}{5^n} = -\frac{1}{15}\sum_{n=0}^{\infty}\frac{z^n}{5^{n+1}}$

Note that $f_1(z) = \frac{1}{10z} = \frac{1}{10}\frac{1}{z}$ is the form of series representation.

Step 3 Build the Laurent expansion from the above geometric series.

$$f(z) = \frac{1}{10}\frac{1}{z} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{2^n}{z^{n+1}} - \frac{1}{15}\sum_{n=0}^{\infty}\frac{z^n}{5^{n+1}}$$

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{15(5^{n+1})} + \frac{1}{10}\frac{1}{z} - \sum_{n=0}^{\infty}\frac{2^n}{6(z^{n+1})}$$

$$\Rightarrow f(z) = -\sum_{n=0}^{\infty}\frac{z^n}{15(5^{n+1})} - \frac{1}{15}\frac{1}{z} - \sum_{n=2}^{\infty}\frac{2^{n-1}}{6}\frac{1}{z^n}$$

Step 4 Find the generalized residue. We only need to collect the $\frac{b_1}{z}$ term.

Therefore, the generalized residue of $f(z)$ is $-\frac{1}{15}$ in the region $2 < |z| < 5$.

Example 4

To demonstrate application of how the generalized residue can be used to solve complex function integration, we will use $f(z)$ from Example 3. Suppose we want to integrate $f(z)$ along the path $|z| = 4$.

We can solve this problem using the Residue Theorem with two methods:

a. Finding the residues using known formulas and apply the Residue Theorem.
b. Use our previously constructed Laurent series and apply Theorem 1.1 directly.

For (a), we can find the residue of a pole using the formula $Res(f, z_0) = \lim_{z \to z_0} (z - z_0) f(z)$.[5]
We will only need to find the residue of the poles $z = 0$ and $z = 2$ because they are the only two simple poles that lie within the closed path. The other simple pole $z_0 = 5$ lies outside the given path, and will not be considered for this integral.

$$Res(f, 0) = \lim_{z \to 0} (z - 0) f(z) = \frac{1}{10}$$
$$Res(f, 2) = \lim_{z \to 2} (z - 2) f(z) = -\frac{1}{6}$$

Applying the Residue Theorem, we have $\int_{|z|=4} f(z) dz = 2\pi i \left(\frac{1}{10} - \frac{1}{6}\right) = -\frac{2\pi i}{15}$.

For (b), we can use Theorem 1.1 directly. This means we can use the generalized residue found in our Laurent expansion obtained in Example 3 because the Laurent series was constructed in the ring containing $|z| = 4$. We simply take the generalized residue, and multiply it by $2\pi i$:

$$\int_{|z|=4} f(z) dz = 2\pi i \left(-\frac{1}{15}\right) = -\frac{2\pi i}{15}$$

Remarks

If we wanted to integrate along any path $|z| > 5$, then we need the Laurent expansion for the outmost ring $|z| > 5$. The Laurent series is the only way to obtain the generalized residue associated with this region. We skip the details as its approach is similar to Example 3.

[5] See Marsden & Hoffman textbook for this formula and others to calculate residues.

Example 5

The following two functions are further examples on how to use a well-known series expansion to find the Laurent series. Each of these functions contains essential singularities. Notice that they have infinitely many b_n terms.

1. $f(z) = z^2 e^{\frac{1}{z}} = z^2\left(1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots\right) = z^2 + z + \frac{1}{2} + \frac{1}{3!z} + \frac{1}{4!z^2} \ldots \;\; for\ |z| > 0$
2. $g(z) = \frac{sinz}{z^2} = \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots\right) = \frac{1}{z} - \frac{z^3}{3!} + \frac{z^5}{5!} - + \cdots$

For $f(z)$, you simply multiply a z^2 term through the Taylor Series of $e^{\frac{1}{z}}$. In order to find the residue, you need to find the constant associated with the $\frac{1}{z}$ term. Clearly, that term for $f(z)$ is $\frac{1}{3!z}$. Therefore $b_1 = \frac{1}{6}$ is the residue of $f(z)$ around $z = 0$.

For $g(z)$, we divide by z^2 the Taylor Series of sin (z). We immediately get our residue from the first term $\frac{1}{z}$. Then $b_1 = 1$ is the residue of $g(z)$ around $z = 0$.

Section IV. The Frobenius Method

The Frobenius method is used to solve linear differential equations with variable coefficients.

> **Theorem 1.3**[6] Any differential equation that has the form:
>
> $$(\mathbf{1}) \qquad y'' + \frac{p(x)}{x} y' + \frac{q(x)}{x^2} y = 0$$
>
> Here the functions $p(x)$ and $q(x)$ are analytic at $x = 0$ will have at least one solution that can be represented by:
>
> $$(\mathbf{2}) \qquad y(x) = x^r \sum_{n=0}^{\infty} a_m x^m = x^r \left(a_0 + a_1 x + a_2 x^2 + \cdots\right)$$
>
> The exponent *r* can be any real or complex number and is chosen so that $a_0 \neq 0$.

Solving **(1)** involves expanding $p(x)$ and $q(x)$ in power series and differentiating **(2)** term by term. The process results in a very important quadratic equation called the *indicial equation*.

We begin by expanding *y(x)* as defined in **(2)** along with its first and second derivatives:

i. $y(x) = a_0 x^r + a_1 x^{r+1} + a_2 x^{r+2} + \cdots$
ii. $y'(x) = r a_0 x^{r-1} + (r+1) a_1 x^r + \cdots$
iii. $y''(x) = a_0 r(r-1) x^{r-2} + a_1 r(r+1) x^{r-1} + \cdots$

If we multiply **(1)** by x^2, we get $x^2 y'' + x p(x) y' + q(x) y = 0$.

Now plug the expansions of *y(x)* from (i), (ii), and (iii) into our simplified **(1)** and let $x \to 0$ then $p(x) \to p_0 and\ q(x) \to q_0$. From this, we get the equation:

$$a_0 r(r-1) x^r + p_0 a_0 r x^r + q_0 a_0 x^r = 0$$

Here $a_0 r(r-1) x^{r-2}$ of $y''(x)$, $a_0 r x^{r-1}$ of $y'(x)$, and $a_0 x^r$ of *y(x)* are the major terms.

We factor out and divide by the common term to arrive at the indicial equation:

$$(r(r-1) + p_0 r + q_0)(a_0 x^r) = 0$$

$$\Rightarrow r(r-1) + p_0 r + q_0 = 0$$

[6] Theorem adapted from Kreyszig's *Theorem 1* (p. 216).

Definition 1.5[7] The *indicial equation* is a quadratic equation defined that takes on the form:

$$r(r-1)+p_0 r+q_0=0, \text{ where}$$

$$p(x)=\sum_{k=0}^{\infty} p_k x^k, \; q(x)=\sum_{k=0}^{\infty} q_k x^k.$$

The roots r_1 and r_2 of the indicial equation can present in three different cases:

1. Distinct roots not differing by an integer.
2. Double root $r_1=r_2=r$.
3. Roots differing by an integer, which may or may not have a logarithmic term.

Let us show the framework of the method of Frobenius.[8] Suppose we have the equation:

$$x^2y''+xp(x)y'+q(x)y=0$$

Here $x=0$ is a regular singular point. Then *p(x)* and *q(x)* are analytic at the origin and have power series expansions

$$p(x)=\sum_{k=0}^{\infty} p_k x^k, \qquad q(x)=\sum_{k=0}^{\infty} q_k x^k, \qquad |x|<\rho$$

Those are convergent for some $\rho>0$. Let r_1, r_2 be the roots of the indicial equation

$$F(r)=r(r-1)+p_0 r+q_o=0.$$

The power series given in each of the following solution forms are convergent at least in the interval $|x|<\rho$.

Case 1: Distinct roots not differing by an integer (i.e. $r_1-r_2\neq n, n\in\mathbb{Z}$).

The solutions will have the form:

$$y_1(x)=x^{r_1}\sum_{k=0}^{\infty} a_k(r_1)x^k$$
$$y_2(x)=x^{r_2}\sum_{k=0}^{\infty} b_k(r_2)x^k$$

Where $a_k(r_1)$ and $b_k(r_2)$ are determined by substituting $y_1(x)$ or $y_2(x)$ into the original ODE.

[7] Indicial equation and form of solutions taken from Kreyszig.
[8] Framework adapted from Phinney's lecture notes

Case 2: Repeated root (i.e. $r_1 = r_2$).

The solutions will have the form:

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k(r_1) x^k$$

$$y_2(x) = y_1(x) logx + x^{r_1} \sum_{k=0}^{\infty} b_k(r_1) x^k$$

Let us show that the formula for $y_2(x)$ is fesible by demonstrating why the term $y_1(x)logx$ is needed, based on the fact that $y_1(x)$ and $y_2(x)$ need to be linearly independent.

Let $E_2(x) = x^{r_1} \sum_{k=0}^{\infty} b_k(r_1) x^k$ denote the second term of $y_2(x)$.

Begin by differentiating:

$$y_2'(x) = y_1'(x) logx + y_1(x) \frac{1}{x} + E_2{}'(x)$$
$$y_2''(x) = y_1''(x) logx + 2y_1'(x) \frac{1}{x} - y_1(x) \frac{1}{x^2} + E_2''(x)$$

Recall:

$$x^2 y_2{}'' + xp(x) y_2{}' + q(x) y_2 = 0$$

Plugging in for $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ we get:

$$x^2 \left[y_1''(x) logx + \frac{2y_1(x)}{x} - \frac{y_1(x)}{x^2} + E_2{}''(x) \right] + xp(x) \left[y_1'(x) logx + \frac{y_1(x)}{x} + E_2'(x) \right]$$
$$+ q(x)[y_1(x) logx + E_2(x)] = 0$$

$$\Rightarrow logx[x^2 y_1''(x) + xp(x) y_1'(x) + q(x) y_1(x)] + 2xy_1'(x) - y_1(x) + p(x) y_1(x)$$
$$+ \left(x^2 E_2''(x) + xp(x) E_2'(x) + q(x) \right) = 0$$

Since $y_1(x)$ is already a solution, the first term above is zero. The second term $E_2(x)$ of $y_2(x)$ satisfies:

$$x^2 E_2''(x) + xp(x) E_2'(x) + q(x) = 1 - p(x) y_1(x) - 2xy_1'(x)$$

This is nonhomogeneous Frobenius equation for $E_2(x)$. Since $y_1(x)$ is already expressed in a power series, a similar method can be used to solve $E_2(x)$. Adding $E_2(x)$ back into $y_1(x)logx$ will ensure $y_2(x)$ is linearly independent with $y_1(x)$.

Case 3: Roots differing by an integer (i.e. $r_1 - r_2 = n, n \in \mathbb{Z}^+$).

The solutions will have the form:

$$y_1(x) = x^{r_1} \sum_{k=0}^{\infty} a_k(r_1)x^k$$

$$y_2(x) = cy_1(x)logx + x^{r_2} \sum_{k=0}^{\infty} b_k(r_2)x^k$$

And *c* may be the value zero. The constant $b_n(r_2)$ is arbitrary and may be set to zero.

Let us justify the feasibility of second solution $y_2(x)$:

Let $E_2(x) = x^{r_2} \sum_{k=0}^{\infty} b_k(r_2)x^k$ denote the second term of $y_2(x)$.

Begin by differentiating:

$$y_2'(x) = c\left(y_1'(x)logx + y_1(x)\frac{1}{x}\right) + E_2{}'(x)$$
$$y_2''(x) = c\left(y_1''(x)logx + 2y_1'(x)\frac{1}{x} - y_1(x)\frac{1}{x^2}\right) + E_2''(x)$$

Recall:

$$x^2 y_2{}'' + xp(x)y_2{}' + q(x)y_2 = 0$$

Plugging in for $y_2(x)$, $y_2'(x)$, and $y_2''(x)$ we get:

$$x^2\left[c\left(y_1''(x)logx + \frac{2y_1{}'(x)}{x} - \frac{y_1(x)}{x^2}\right) + E_2{}''(x)\right] + xp(x)\left[c\left(y_1'(x)logx + \frac{y_1(x)}{x}\right) + E_2'(x)\right] + q(x)[cy_1(x)logx + E_2(x)] = 0$$

$$\Longrightarrow logx[cx^2y_1''(x) + cxp(x)y_1'(x) + cq(x)y_1(x)] + 2cxy_1'(x) - cy_1(x) + cp(x)y_1(x) + \left(x^2E_2''(x) + xp(x)E_2'(x) + E_2(x)q(x)\right) = 0$$

Since $y_1(x)$ is already a solution, the first term above is zero. The second term $E_2(x)$ of $y_2(x)$ satisfies:

$$x^2 E_2''(x) + xp(x)E_2'(x) + q(x)E_2(x) = c(y_1(x) - p(x)y_1(x) - 2xy_1'(x))$$

From here, we must explore two scenarios with the value of c. First, assume $c = 0$, and then solve the above homogeneous ODE to see if there exist two linearly independent solutions $y_1(x)$ and $y_2(x) = E_2(x)$. Linear dependence occurs if $y_2(x)$ turns out to be a multiple of $y_1(x)$. Then we must choose $c = 1$ to solve for $y_2(x)$.

For example, assume $r_1 = 3.5$ and $r_2 = 1.5$. If $y_1(x)$ is already solved, then:

$$y_1(x) = x^{3.5}(a_0 + a_1 x + a_2 x^2 + \cdots)$$

Assume for $c = 0$, we get

$$y_2(x) = x^{1.5}(b_0 + b_1 x + b_2 x^2 + \cdots)$$

If $b_0 \neq 0 \; or \; b_1 \neq 0$, then $y_1(x)$ and $y_2(x)$ are surely linearly independent and we are done. Otherwise we have $y_2(x) = x^{1.5}(b_2 x^2 + b_3 x^3 \ldots) = x^{3.5}(b_2 + b_3 x \ldots) = y_1(x)$. Then we need to choose $c = 1$ to solve for $y_2(x)$ as in Case 2 with the repeated root.

Section V. Conclusion

The Laurent series are central to complex analysis and its applications. Although the residue of some functions can be found easily with well-known formulas, this is not true in general. For example, for a function with an essential singularity—where no such easy formula exists—the Laurent expansion is the only way to determine the function residue at such a singularity. The residues found through the Laurent series will greatly simplify complex integration, which can be applied in many fields outside of pure mathematics. A similar series expansion called the Frobenius method is also useful in solving second-order ordinary different equations with singularities.

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