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Establishing Foundations for Investigating Inquiry-Oriented Teaching

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Establishing Foundations for Investigating Inquiry-Oriented Teaching

by

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Abstract

The Teaching Abstract Algebra for Understanding (TAAFU) project was centered on an innovative abstract algebra curriculum and was designed to accomplish three main objectives: to produce a set of multi-media support materials for instructors, to understand the challenges faced by mathematicians as they implemented this curriculum, and to study how this curriculum supports student learning of abstract algebra. Throughout the course of the project I took the lead investigating the teaching and learning in classrooms using the TAAFU curriculum. My dissertation is composed of three components of this research. First, I will report on a study that aimed to describe the experiences of mathematicians implementing the curriculum from their perspective. Second. I will describe a study that explores the mathematical work done by teachers as they respond to the mathematical activity of their students. Finally, I will discuss a theoretical paper in which I synthesize aspects of the instructional theory underlying the TAAFU curriculum in order to develop an analytic framework for analyzing student learning. This dissertation will serve as a foundation for my future research focused on the relationship between teachers' mathematical work and the learning of their students.

Dedication

I dedicate this dissertation to my loving, and growing, family.

I could not have done this without the love and support of my husband, Nathan; my parents, Clif and Julie; and my brother, Cerro. And, while I may have been able to finish without the constant threat of my son coming early, I suspect I would have been a bit less motivated.

Thank you all, especially Nate, for getting me through the freak-outs, accepting the sacrifices, and celebrating the successes.

Acknowledgments

I would like to acknowledge that this was hard, so hard that without the support of my husband, family, friends, committee, and advisor it would have been impossible. It certainly took a village, and the mayor of that village was my advisor, Dr. Sean Larsen. Sean provided me with countless opportunities to enjoy personal and academic growth, an effort that took tremendous dedication and strength on his part. Sean, thank you for forcing me to do better than "good enough". I know that I complained. I know that I was stubborn. I know that I was full of sass and attitude. And, I know that the fight was worth it. I look forward to transitioning into the research partner and colleague that you have always treated me as - hopefully with a little less growth on my part.

Table of Contents

List of Tables

List of Figures

Introduction

The *Teaching Abstract Algebra for Understanding* (TAAFU) project was based on an innovative abstract algebra curriculum and was designed to accomplish three main objectives: to produce a set of multi-media curriculum support materials for instructors, to understand the challenges and opportunities of mathematicians as they implemented this curriculum, and to study how this curriculum supports student learning of abstract algebra. Throughout the course of the project I took the lead investigating the challenges and opportunities of mathematicians as they worked to implement the TAAFU curriculum. Specifically, in working with the instructors and watching classroom videos I began to wonder about differences I was seeing in implementation. This inspired the three papers that will compose my dissertation. Before presenting the three papers I will present an overview of the TAAFU project, discuss my research agenda; and provide some background from the mathematics education research literature to help situate my work.

The TAAFU Project

TAAFU Curriculum. The *TAAFU* curriculum is a research based, inquiry-oriented abstract algebra curriculum that actively engages students in developing the fundamental concepts of group theory. This curriculum was developed through a series of design experiments (Design-Based Research Collective, 2003) and has been refined through several iterations of classroom trials. The *TAAFU* curriculum was primarily designed to be used in an upper-division, undergraduate abstract algebra course and is composed of three primary units: groups/subgroups, isomorphism, and quotient groups. Each unit

1

begins with a *reinvention phase* in which students develop concepts based on their intuition, informal strategies, and prior knowledge. The end product of the reinvention phase is a formal definition (or definitions) and a collection of conjectures. The *deductive phase* begins with the formal definitions that are relevant to the concept. During this phase, students work to prove various theorems (often based on conjectures arising during the reinvention phase) using the formal definitions and previously proved results.

Groups and subgroups. The curriculum opens with an instructional sequence for the group concept. The reinvention phase of this sequence begins with an exploration of the symmetries of an equilateral triangle and culminates with the formal definition of group. Students begin by identifying, describing, and symbolizing all of the symmetries of an equilateral triangle. After the students have identified these six symmetries, they are asked to represent them in terms of a flip across the vertical axis, *F*, and a 120° clockwise rotation, *R*. The students are next asked to consider each combination of two symmetries and determine which symmetry is equivalent to it. This task eventually leads to the production of an operation table and a shift to *calculating* the combinations of symmetries. Building on the idea of calculating combinations of symmetries, the students eventually produce a short list of rules (including the group axioms) that are sufficient for calculating all 36 combinations. After the students investigate other systems (e.g., symmetries of a square, integers under addition) to develop similar rules, they develop a definition of group to describe the common structure of these systems. The reinvention phase culminates with this definition of group and a collection of conjectures, typically including the uniqueness of the identity element and the cancellation law.

The deductive phase starts with the formal definition of group. The students prove a number of results including those conjectured during the reinvention phase, such as the cancellation law. Later a subgroup is defined to be a subset of a group that is itself a group under the same operation. Students develop and prove characterization theorems that allow one to verify that a given subset is a subgroup without verifying all four (counting closure) of the group axioms. These typically include 1) the standard theorem that states that a non-empty subset is a subgroup if it is closed under the operation and if each element in the subset has its inverse in the subset, and 2) a second theorem that states that if the subset is non-empty and finite, it is sufficient to check that it is closed under the operation (see Larsen & Zandieh, 2007).

Isomorphism. The reinvention phase of the isomorphism sequence begins with an activity in which the students are given an operation table for a "Mystery Group" of order six. The students are asked whether this mystery table could be an operation table for the group of symmetries of an equilateral triangle, D_6. In an effort to determine if the mystery group is indeed D_6, students focus on relational properties to guide them as they form a 1-1 correspondence between the elements of the two groups. Once the students arrive at a correspondence that they believe will work they use it to compare the tables, often by renaming the elements and/or reordering the elements of one of the tables. When they find that the only real difference between the two tables is the symbols used to represent the elements, they conclude that the mystery table could in fact be an operation table for the symmetries of a triangle.

At this point, the term *isomorphic* is introduced as a way to refer to two groups being equivalent in this sense (that the only real difference is in the notation). The goal becomes to develop a definition for isomorphism. Students typically do not have difficulty realizing that isomorphism can be defined in terms of a one-to-one correspondence between the groups. (The students' initial renaming schemes are often informally expressed as bijective mappings using arrow diagrams.) By working with partial mappings, the students then determine what additional property is needed for a bijective mapping to show that two groups are isomorphic. This property, the homomorphism property, is commonly expressed by the condition that there exists a function φ such that, $\varphi(ab) = \varphi(a)\varphi(b)$ for every a, b in one of the groups.

The final step in the reinvention phase is for the class to formulate a formal definition of isomorphism. As with the group sequence, the reinvention phase of the isomorphism sequence also generates a number of conjectures (e.g., that inverse pairs map to inverse pairs) that are then proved during the deductive phase. These conjectures are obviously true from the perspective that two groups are isomorphic if they are the same except for notation. This fact reinforces the connection between the definitions and the students' informal understanding of isomorphism because the definition can be used to prove theorems that intuitively should be true. For many undergraduate students this connection is missing for some important concepts, including isomorphism (Weber & Alcock, 2004).

Quotient groups. The reinvention phase of the quotient group unit begins with the students considering the parity of the integers and whether there exists a similar partition of the symmetries of a square. The students eventually determine that there are three

ways to partition this group into subsets that interact like the even and odd integers. For example, one can consider the rotations of a square to be "even" and the reflections to be "odd" and produce a table that mirrors the behavior in the EVEN/ODD group. The students are then asked whether these partitions form groups and are engaged in describing the meaning of the operation given by the table. Typically this operation is described as "set multiplication" in the sense that multiplying two of these sets means multiplying (in order) every element of the first set by every element in the second set.

Next students are asked to form a larger group by partitioning D_8 into smaller subsets. As they attempt to do this, a number of conjectures are considered and proved or disproved. These include the fact that the identity subset must be a subgroup. Armed with this fact, the students develop a procedure (coset formation) for partitioning the remaining elements given a subgroup. The students then compare partitions that work with partitions that do not work, enabling them to develop a version of the normality condition (usually the condition that left and right cosets must be equal) by leveraging the commutativity of the identity property.

The quotient group unit transitions from the reinvention phase to a deductive phase as students (with guidance from the instructor) prove that normality is a necessary and sufficient condition for the cosets of a subgroup to form a group under set multiplication. After the closure axiom is verified (by proving that the product of the cosets aH and bH is abH using normality), the operation can be redefined in terms of representatives. This paves the way for a straightforward verification of the other axioms. Finally in the deductive phase, students prove basic related results (e.g., every subgroup of an abelian group is normal) and more significant results (e.g., the First Isomorphism Theorem).

TAAFU Project Overview and Timeline. The TAAFU project comprised three overlapping stages of research and design. The first stage resulted in the initial design of the three core instructional units of the TAAFU curriculum. These initial designs emerged along with local instructional theories (see Larsen, 2013; Larsen & Lockwood, 2013) from a set of small-scale design experiments conducted with pairs of students. The primary goal of the second stage of the research and design process was generalizing from the initial design context (two students in a laboratory setting) to a more authentic context (a full classroom). Two significant products emerged from this second stage, a full group theory curriculum and a set of instructor notes. These notes also represented the beginning of the third stage of research and design. The focus of this third (ongoing) stage is on generalizing from the curriculum designer as instructor to other instructors (especially mathematicians), first with extensive support from the project team and then with only the support of the web-based instructor support materials we are developing (Lockwood, Johnson, & Larsen, 2013). Table 1 provides a timeline of the primary project activities that featured data collection. All small-scale design experiments were video-recorded using one camera and (unless otherwise noted) all classroom activity was recorded using two cameras to capture video of small group and whole class activity.

Year	Stage	Activity	Units
2002	1	Sequence of Three Small-Scale Design Experiments	Group & Isomorphism
2004-2005	2	Experimental Teaching in Group Theory Course w/ Designer as Teacher	Group & Isomorphism
2005	2-3	Teaching Experiment in Algebra Course for K-12 Teachers	Group & Isomorphism
2006	1	Small-Scale Design Experiment	Quotient Groups
2007	2-3	Whole Class Teaching Experiment w/ Mathematician as Teacher	Full Curriculum[a]
2008-2009	3	Implementation by Three Mathematicians w/Limited Support (Instructor Notes, Email Conversations)	Full Curriculum
2011	3	Off-Site Implementation by a Mathematician w/ Only Web-Based ISMs for Support[b]	Full Curriculum
2012	3	Iterative Interviews Focusing on the Experiences of the Mathematicians	

[a]Debriefing/planning video for full curriculum + classroom video for Quotient Groups only
[b]Data includes smart board video capture and audio recordings of class sessions

Table 1. Project activities featuring data collection.

I started working on the TAAFU project as we transitioned into Stage 3 – working with mathematicians as they implemented the TAAFU curriculum. By this time in the project, the majority of data collection efforts were focused on capturing the teaching and learning of the TAAFU curriculum in a whole class setting. While some data collection focused on students (including video recordings of small group work and copies of student work), the majority of this data was focused on the mathematicians. My work on the TAAFU project centered on supporting and investigating the implementation efforts of these mathematicians.

Relevant Data for Investigating Implementation. To understand ways in which instructors and students engage with the inquiry-oriented, abstract algebra curriculum we collected video data from four mathematicians' classrooms. The first instructor, Peter[1], is graph theorist and co-investigator on the TAAFU project. During his first implementation members of the research team held debriefing/planning meeting with Peter and his teaching assistant once a week. These meetings focused on Peter and his teaching assistant summarizing how the lessons played out in class, identifying aspects of the curriculum or the curriculum support materials that either went really well or that needed further development, and discussing what Peter could expect in the coming lessons. In addition to the debriefing meetings, classroom video data was collected in Peter's first implementation of the quotient group unit. During this three-day unit, one video camera captured small group work and another camera focused on Peter while he was directing class and on a different small group otherwise.

In the following year, classroom video data was collected from all regular class sessions of Peter's second implementation of the curriculum and of Mary's first implementation. Mary is an algebraist who volunteered to use the curriculum. Members of the research team again used two cameras to capture both small group work and teacher lead discussions. However, during this round of data collection there were no debriefing meetings. Instead, each mathematician participated in two individual interviews, one at the beginning of the term and one in the middle of the term. The goal of the interview at the beginning of the term was to gain a sense for how the mathematicians viewed the curriculum. The interview at the middle of the term had two

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[1] Instructor names are pseudonyms.

components, the first was a reflection on the challenges faced thus far and the second involved the mathematicians evaluating common student conjectures and then discussing what they would do if these conjectures came up in class.

A third mathematician, Sally, implemented the TAAFU curriculum during an 8-week course in the summer of 2009. Two cameras were used to record Sally's implementation, one camera that focused on small group work and one camera that focused on Sally. Like Peter, Sally participated in semi-weekly debriefing meetings throughout her implementation. The primary goal of these meetings was to investigate the utility of the instructor support materials. However, during each interview Sally would also discuss the previous class sessions and share her plans for future classes.

Finally, a fourth mathematician, Bill, implemented the TAAFU curriculum in the fall of 2011. During his implementation Bill posted video recordings of class for his students. These videos captured the "Smart Board" recordings from times in which Bill was leading whole class discussions. Most clearly recorded on these videos are Bill's contributions to the whole class discussion and the notes he recorded. During implementation Bill did not participate in any interviews.

In total, classroom video data was collected during the implementations of four mathematicians. Three mathematicians, Bill, Peter, and Sally, were then asked to participate in a series of interviews in the winter of 2012. These interviews were designed to investigate and document the teaching experiences of the three mathematicians. These three mathematicians were selected because of the amount of data collected during their implementation of the TAAFU curriculum. Every regular class session of Peter, Sally, and Bill's course was video taped. Additionally, Peter and Sally participated in regular

9

debriefing meetings throughout their implementation. While Bill did not participate in regular debriefing meetings, his implementation of the TAAFU curriculum was the most recent. Therefore, there was reason to believe that Bill would be able to provide detailed and accurate accounts of his experience with the curriculum.

The data set focused on the three mathematicians' experiences implementing the TAAFU curriculum includes written reflections, individual interviews, and stimulated recall interviews (Schepens, Aelterman, & Van Keer, 2007). Initially the mathematicians were asked to write up a personal reflection on their implementation of the TAAFU curriculum. Rounds of interviews/written reflections followed in which the mathematicians were asked to 1) expound on specific aspects of their own reflections, 2) respond to emerging themes present in other mathematicians' reflections, and 3) react to clips of classroom video data, and debriefing data when possible, selected in response to their reflections.

Research Agenda

One of my main roles on the TAAFU project was to lead the investigation into the instructional practices of the mathematicians implementing this curriculum. My research for the *TAAFU* project is focused on the mathematical work done in the moment as mathematicians implement inquiry-oriented curriculum. The goals of this work are to better understand the work carried out by teachers as they implement inquiry-oriented curriculum and to understand the ways in which teachers' mathematical work influences the learning of their students.

In January of 2012 I had an article appear in the Journal of Mathematical Behavior in which I examined the Mathematical Knowledge for Teaching needed by a mathematician in order to productively listen to her students' mathematical contributions. After identifying instances in which the mathematician worked to make sense of her students' mathematical contributions, I looked to debriefing data to understand what kind of knowledge supported these efforts. As a result of my analysis, I was able to identify Knowledge of Content and Students (a subdomain of Mathematical Knowledge for Teaching) as a key factor in supporting productive listening.

Then, last year, Sean Larsen and I were asked to write and edit a special issue of the Journal of Mathematical Behavior chronicling the development, scaling-up, and dissemination of the *TAAFU* curriculum. Two of the papers in this special issue, both focusing on the teaching of the *TAAFU* curriculum, are included as part of my dissertation. One of these papers looks at the experiences of mathematicians implementing the curriculum and another explores the mathematical activity done by teachers as they respond to the mathematical activity of their students. The third dissertation paper is a theoretical paper in which I synthesize the literature on the instructional design theory of Realistic Mathematics Education, in order to consider methodological implications for documenting student learning in the context of inquiry-oriented instruction. The need for this paper became apparent as I worked to understand the ways in which the teachers' decisions, activity, and perspectives were influencing their students' learning.

I see my current body of work, and my dissertation specifically, as a foundation for my future research. These three papers are intended to provide a theoretical foundation

for investigating the relationship between teachers' mathematical work and student learning, particularly in inquiry-oriented settings. Additionally, the two empirical papers have begun to answer questions about how teachers' decisions, activity, and perspectives *might* influence student learning. These relationships, now identified, can be tested. Finally, this work will serve to support the development of workshops for mathematicians who are interested in implementing inquiry-oriented curriculum.

There are a number of ongoing efforts to design and scale-up inquiry-oriented curricula for undergraduate mathematics (including linear algebra and differential equations). These projects create excellent opportunities for me to generalize and expand my research program. For example, by investigating instruction in other topic areas, I can determine to what extent my findings regarding the kinds of knowledge needed to make sense of student contributions generalize to topics beyond abstract algebra. This kind of investigation can further a more general research agenda focused on developing theory related to teachers' mathematical and pedagogical work and discovering strategies for supporting motivated mathematicians in effectively implementing these curricula. Such research has practical implications as well, as it can support the development of materials and strategies for helping mathematicians use reform-oriented curricula more effectively.

Related Literature

The TAAFU curriculum was developed in accordance with the instructional design theory of Realistic Mathematics Education (RME). This theoretical foundation of the TAAFU curriculum significantly contributes to the classroom context of my research.

Here I will provide an overview of RME and discuss the implications of RME for teaching and learning in the TAAFU context.

Realistic Mathematics Education and student learning. The TAAFU curriculum was heavily influenced by two RME design heuristics - *emergent models* and *guided reinvention*. The emergent model design heuristic (Gravemeijer, 1999) is based on the idea that students' informal solution strategies, which first emerge as a *model-of* the students' informal activity, can evolve into a *model-for* more formal reasoning. The guided reinvention design heuristic is typically characterized in terms of the nature of the learning process, where the goal is for "learners to come to regard the knowledge they acquire as their own, personal knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p. 786). In order to achieve this goal, mathematics instruction is designed around mathematical activities intended to progressively expand the students' common sense.

Part of the power of these two design heuristics resides in the fact that they place a dual emphasis on supporting both the students' mathematical activity and the formal mathematics that the curriculum is intended to develop. As a result, the curriculum developer can design curriculum with both student activity and concept development in mind. For instance, these design heuristics promote curriculum developers asking questions like: "What informal strategies and ways of thinking are the students likely to have and how do those anticipate the formal mathematics?"; "What instructional tasks/activities/contexts can be used to evoke these strategies and ways of thinking?";

and, "What instructional activities are going to be useful for supporting students in leveraging these strategies and ways of thinking to develop of the formal mathematics?".

While this focus on both activity and the concept development makes the theory flexible and powerful as an instructional design theory, it can be a confounding factor when trying to carefully articulate some of the fundamental RME constructs. In particular, each design heuristic is routinely discussed both in terms of increasingly general student activity and in terms of concept development. For instance, the term *model* (as in *emergent models*) is defined in the RME research literature both as "student-generated ways of organizing their activity with observable and mental tools" (Zandieh & Rasmussen, 2010, p. 58) and as " an overarching concept" (Gravemeijer, 1999, p. 170).

Additionally, both the guided reinvention and emergent models design heuristics are described in terms of the creation of a *new mathematical reality*. However, what exactly a *new mathematical reality* is remains unclear. For instance, the creation of a new mathematical reality is sometimes discussed as being equivalent to activity. "Defining can function both as an organizing activity (horizontal mathematizing) and as a means for generalizing, formalizing or creating a new mathematical reality (vertical mathematizing)" (Rasmussen, Zandieh, King, & Teppo, 2005, p. 66). While other times the creation of a new mathematical reality is discussed in terms of object reification. "The shift from model-of to model-for and the resulting new mathematical reality is compatible with Sfard's (1991) process of reification" (Rasmussen, & Blumenfeld, 2007, p. 196). As a result, efforts to document the development of a model or the creation of a new mathematical reality (both of which can be viewed as student learning) are not supported by a clear theoretical foundation.

This lack of precision in the descriptions of these theoretical constructs became a significant problem as I tried to investigate the impact of the TAAFU curriculum on student learning. The curriculum was designed to support students in creating a new mathematical reality and developing formal mathematics through a model-of/model-for transition, so it made sense to rely on these constructs to support my investigation. However, without knowing precisely what a *new mathematical reality* is, it is very difficult to argue that one has been established. Similarly, without knowing what a *model* is, it is difficult to document the development of one.

My first paper will articulate RME in a way that supports analytic techniques for documenting student learning. I will first describe two RME design heuristics, guided reinvention and emergent models, and explicate each of these heuristics in terms of related theoretical constructs. Each of these design heuristics will be illustrated using examples from the TAAFU curriculum. I will then clarify my discussion of the design heuristics by framing them separately in terms of both increasingly general student activity and in terms of concept development. Finally, I will consider how the RME design heuristics could inform how one conceptualizes student learning. To do so, I will draw on two metaphors for learning and, by drawing on these two perspectives, propose ways in which the RME design heuristics can inform the analysis of student learning.

Teachers as active participants in the process of Guided Reinvention. While the TAAFU curriculum intends to place the emphasis on the students' developing the mathematics through engaging in mathematical activities, as opposed to the teacher presenting the mathematics, teachers do have an important role when implementing the

curriculum. Namely, the teacher needs to find ways to build on the students' informal ideas in order to help them construct the formal mathematics. As a result, teachers implementing such curricula must be active participants in establishing the mathematical path of the class while at the same time allowing students to retain ownership of the mathematics.

As students progress through the reinvention process, they are actively engaged in mathematical activity. The research literature describes a number of student mathematical activities. For instance, Rasmussen et al.'s (2005) *advancing mathematical activity* includes symbolizing, algorithmatizing, and defining as specific examples of mathematical activity. Further, in order to understand and generate mathematical proofs, students would likely engage in proof related activities, such as evaluating arguments (Selden & Selden, 2003), instantiating concepts (Weber & Alcock, 2004), and proof analysis (Larsen & Zandieh, 2007). Still other mathematical activity that students are likely to engage in as they work to reinvent mathematical concepts includes conjecturing, questioning, and generalizing.

As students engage in such mathematical activity, one would expect that teachers would need to engage in mathematical activity in response. For instance, Speer and Wagner (2009) presented a study in which they sought to account for the difficulties a mathematician was facing while trying to provide analytic scaffolding while implementing an inquiry-oriented differential equations curriculum, where analytic scaffolding is used to "support progress towards the mathematical goals of the discussion" (p. 493). Speer and Wagner identified several components necessary for providing analytic scaffolding, including the ability to recognize and figure out the ideas

expressed by their students and the potential for these ideas to contribute towards the mathematical goals of the lesson. Speer and Wagner went on to state that, "recognizing draws heavily on a teacher's PCK [pedagogical content knowledge], whereas *figuring out requires that a teacher do some mathematical work in the moment* [emphasis added]" (p. 8).

This finding that implementing inquiry-oriented curriculum requires mathematicians to engage in mathematical work in the moment aligns with my earlier investigation (Johnson & Larsen, 2012) into a mathematician's ability to *interpretively and/or generatively listening* to their students' contributions (where interpretive listening involves a teacher's intent of making sense of student contributions and generative listening reflects a readiness for using student contributions to generate new mathematical understanding or instructional activities (Davis, 1997; Yackel, Stephan, Rasmussen, & Underwood, 2003)). In that study we found that, in order to engage in interpretive and/or generative listening, a mathematician may need to interpret a student's imprecise language, generalize a student's statement into a testable mathematical conjecture, or identify counterexamples to a student's claim. Each of these activities would require mathematical work on the part of the teacher.

Both of these research reports suggest that, because of the teacher's role as an active participant in developing of the classroom community's mathematical ideas, a teacher's engagement in mathematical activity may be an important aspect for successfully implementing inquiry-oriented curriculum. My second paper will focus on the mathematical activity that teachers engage in as they implement the TAAFU curriculum. Specifically, I will investigate the nature of the teachers' mathematical activity that is

present in classrooms enacting the TAAFU curriculum, and try to understand the ways in which teacher's mathematical activity may interact with the mathematical activity of their students.

Mathematicians' experiences and perspectives. As curriculum developers, our goal was to provide a curriculum that would support students in reinventing the fundamental concepts of group theory and a set of teacher support materials that would enable teachers to effectively guide this process while ensuring that the students retained ownership of the mathematics. However, even with our best efforts to support mathematicians in implementing the TAAFU curriculum, the adoption of reform-oriented curriculum is a complex issue that warrants further study. For instance, Remillard (2005) calls for research into "characteristics that relate specifically to teachers' interactions with curriculum, such as the teacher's perceptions and stance towards curriculum materials and the teacher's professional identity as it relates to the use of curriculum resources" (p. 235). The need for this kind of research is even more pressing at the undergraduate level, as little research has been carried out to investigate the teaching practices of research mathematicians (Speer, Smith, & Horvath, 2010).

In previous research, my focus has been on analyzing mathematicians' in-the-moment teaching practices and decisions as they work to implement the curriculum. This includes investigating the knowledge needed by mathematicians to productively listen to their students' mathematical contributions (Johnson & Larsen, 2012) and investigating the influence of teachers' mathematical activity on the mathematical development of the classroom community investigated (Johnson, 2013). This work, like that done by Speer

and Wagner (2009), is part of a small body of research that looks to account for challenges mathematicians face while implementing student-centered curriculum. In each of these cases, the focus of analysis is the teaching practices of the mathematicians. However, even with these studies, research on the teaching practices of mathematicians is relatively sparse. Speer, Wagner, and Horvarth (2010) found only five studies that they deemed to be "empirical research on collegiate teaching practices" (p. 105).

In addition to the few studies focused on the teaching practices of mathematicians, the mathematics education research literature also includes studies to determine the extent of student-centered instruction at the collegiate level and to identify institutional policies that may account for the lack of such instruction (McDuffie & Graeber, 2003; Walczyk & Ramsey, 2003; Walczyk, Ramsey, & Zha, 2007). For instance, in a case study of two mathematicians trying to implement reform curriculum McDuffie and Graeber (2003) identified a number of institutional norms and policies that either supported or curtailed the mathematicians' efforts. Some of the norms and policies that curtailed change include: limited time for planning and developing new lessons and activities, and institutional pressures to cover a set syllabus that does not allow time for reform-based approaches.

The third paper in my dissertation will focus not on the mathematicians' teaching practices or the institutional norms and policies, but instead on the teaching experiences of three mathematicians. The primary source of data for this investigation will be post implementation reflections by the three mathematicians. Additionally, each mathematician was asked to react to the written synopses characterizing their views and write a first-person commentary to further expound their views. As a result, the three

mathematicians are co-authors on this paper. Ideally this focus on the mathematicians' teaching experiences will provide an opportunity for the research field to gain insight into the factors and considerations that matter to mathematicians: in particular, mathematicians that are willing and excited to implement student-centered curricula. Further, it may provide new insights into "the teacher's perceptions and stance towards curriculum materials and the teacher's professional identity" as called for by Remillard (2005, p. 235).

Structure of the Remainder of the Dissertation

Each of the remaining three sections of the proposal will focus on one of the research topics. The first section is a theoretical paper focusing on the implications of RME for analyzing student learning. The second section is an empirical paper that looks at the mathematical activity that teachers engage in while implementing the TAAFU curriculum. The third section is an empirical paper that documents the experiences of three mathematicians who have implemented the TAAFU curriculum. I will then conclude by summarizing the contributions made by these three papers and considering some theoretical questions raised by the first paper.

Paper 1: Realistic Mathematics Education Design Heuristics and Implications for Documenting Student Learning

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Abstract

The primary goal of this work is to articulate a theoretical foundation based on Realistic Mathematics Education (RME) that can support the analysis of student learning. To do so, I first describe two RME design heuristics, guided reinvention and emergent models, and explicate each of these heuristics in terms of related theoretical constructs. I then frame both heuristics separately in terms of both increasingly general student activity and in terms of concept development. Finally, I consider how the RME design heuristics could inform how one conceptualizes student learning. To do so, I draw on two metaphors for learning and, by drawing on these two perspectives, propose ways in which the RME design heuristics can inform the analysis of student learning.

Key Words: Realistic Mathematics Education, Learning, Mathematizing, Emergent Models, Analytic methods

Realistic Mathematics Education (RME) is an instructional design theory used to inform the development of inquiry-oriented curriculum. The emergence of such instructional approaches creates a need to investigate student learning in these contexts. However, as it was designed to be an instructional design theory, the current formulations of RME are not articulated in a way that readily supports investigations of student learning. As I worked to investigate the implementation of an inquiry-oriented abstract algebra curriculum (Johnson & Larsen, 2012; Johnson, 2013) I became increasingly aware of the need for a comprehensive approach to making sense of student learning in this context. In particular, because the curriculum was designed based on RME, it was

apparent that analyses of student learning should draw on the design heuristics and other theoretical constructs that compose RME.

Part of the difficulty in using the current formulations of RME to investigate student learning is due to variations in the ways that the RME design heuristics (*guided reinvention* and *emergent models*) are discussed in the research literature. In particular, these design heuristics are routinely discussed both in terms of increasingly general student activity and in terms of concept development. For instance, the emergent model design heuristic is based on the idea that students' informal solution strategies, which first emerge as a *model-of* the students' informal activity, can evolve into a *model-for* more formal reasoning. However, the term *model* is defined in the RME research literature both as "student-generated ways of organizing their activity with observable and mental tools" (Zandieh & Rasmussen, 2010, p. 58) and as " an overarching concept" (Gravemeijer, 1999, p. 170).

Additionally, both the guided reinvention and emergent models design heuristics are described in terms of the creation of a *new mathematical reality*. However, what exactly a new mathematical reality is remains unclear. For instance, the creation of a new mathematical reality is sometimes discussed as being equivalent to activity. "Defining can function both as an organizing activity (horizontal mathematizing) and as a means for generalizing, formalizing or creating a new mathematical reality (vertical mathematizing)" (Rasmussen, Zandieh, King, & Teppo, 2005, p. 66). Other times the creation of a new mathematical reality is discussed in terms of object reification. "The shift from model-of to model-for and the resulting new mathematical reality is compatible with Sfard's (1991) process of reification" (Rasmussen, & Blumenfeld, 2007,

p. 196). As a result, efforts to document the development of a model or the creation of a new mathematical reality (both of which can be viewed as student learning) are not supported by a clear theoretical foundation.

In order to articulate RME in a way that supports analytic techniques for documenting student learning, I will first describe two RME design heuristics, guided reinvention and emergent models, and explicate each of these heuristics in terms of related theoretical constructs. Each of these design heuristics will be illustrated using examples from an instructional design project in abstract algebra. I will then clarify my discussion of the design heuristics by framing them separately in terms of both increasingly general student activity and in terms of concept development. Finally, I will consider how the RME design heuristics could inform how one conceptualizes student learning. To do so, I will draw on two metaphors for learning and, by drawing on these two perspectives, propose ways in which the RME design heuristics can inform the analysis of student learning.

Guided Reinvention and Emergent Models

RME is grounded in the belief that mathematics is "first and foremost an activity, a *human* activity" (Gravemeijer & Terwel, 2000, p. 780). Accordingly, Freudenthal argued that mathematics education should "take its point of departure primarily in *mathematics as an activity,* and not in mathematics as a ready-made-system" (Gravemeijer & Doorman, 1999, p. 116). Within RME there are a number of heuristics that are meant to guide the design of instruction that supports students in developing formal mathematics by engaging them in mathematical activity. For both the *guided reinvention* and the

emergent model heuristics, the nature of this activity becomes more general as the instructional sequence unfolds.

Guided Reinvention

Guided reinvention is typically characterized in terms of the nature of the learning process, where the goal is for "learners to come to regard the knowledge they acquire as their own, personal knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Terwel, 2000, p. 786). In order to achieve this goal, mathematics instruction is designed around mathematical actives intended to progressively expand the students' common sense. As described by Gravemeijer (1999), "what is aimed for is a process of gradual growth in which formal mathematics comes to the fore as a natural extension of the student's experiential reality" (p. 156).

The student's experiential reality includes what the students can access on a "commonsensical level".

> "Real" is not intended here to be understood ontologically (whatever ontology may mean), therefore neither metaphysically (Plato) nor physically (Aristotle); not even, I would even say, psychologically, but instead commonsensically as … meant by the one who uses the term unreflectingly. It is not bound to the space-time world. It includes mental objects and mental activities. (Freudenthal, 1991, p. 17)

Therefore, the problem context that serves as the basis of the reinvention process need not be "real" in the sense that the students would access such scenarios in their everyday life. Instead, the students only need to be able to access the problem context on an intuitive level. In this way a magic carpet can be understood as experientially real, even though it is not physically real. For instance, the movements of a magic carpet may

24

provide a context for the reinvention of the formal mathematics of linear algebra (Wawro, Sweeney, & Rabin, 2011).

Within an experientially real context, the reinvention process progresses through a series of instructional tasks that promote mathematizing the problem context. This activity of mathematizing, "which stands for organizing from a mathematical perspective" (Gravemeijer & Doorman, 1999, p. 116), is viewed as the mechanism through which students reinvent the mathematics.

> In this view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. (Gravemeijer, 1999, p. 158)

It is through this cycle of progressive mathematizing, between mathematizing reality and mathematizing their own mathematical activity, that students reinvent mathematics by expanding their mathematical reality. Because progressive mathematizing has been described in the RME literature as the primary mechanism supporting guided reinvention, my discussion of connections between guided reinvention and student learning will be focused on progressive mathematizing. Progressive mathematizing is typically described as consisting of cycles of horizontal and vertical mathematizing.

Initially, as students mathematize their own experiential reality, they are engaging in horizontal mathematizing. Horizontal mathematizing could include activities such as translating, describing, and organizing aspects of problem context into mathematical terms (Gravemeijer & Doorman, 1999, p. 116-117). It is the nature of the artifact of the activity that provides an indication that horizontal activity has taken place. The artifacts

of horizontal mathematizing, which may include inscriptions, symbols, and procedures, are used by the students to "express, support, and communicate ideas that were more or less already familiar" (Rasmussen et al., 2005, p. 164).

While horizontal mathematizing is a crucial step in the reinvention process, as it serves to mathematize the problem context, reinvention "demands that the students mathematize their own mathematical activity as well" (Gravemeijer & Doorman, 1999, p. 116-177). Vertical mathematizing characterizes activities through which students mathematize their own mathematical activity and could include generalizing, defining, and algorithmatizing (Rasmussen et al., 2005). One indicator that an activity is an example of vertical mathematizing is the way that the students use the artifacts of the activity. Instead of using the artifacts to describe already familiar situations and ideas (as with horizontal mathematizing), artifacts of vertical mathematizing can be used by students in more general settings to describe, express, and create previously unfamiliar mathematical ideas. As described by Rasmussen et al. (2005):

> The creating in vertical mathematizing is therefore unlike the creating in horizontal mathematizing because, as we said earlier, creating the phase line, the procedure, and the definition [all examples of horizontal mathematizing] were done in part to express, support, and communicate ideas that were more or less already familiar, as opposed to creating new mathematical realities. (p. 70).

In this way, vertical mathematizing incorporates artifacts of student activity into the students' expanding mathematical reality.

This process of *progressive mathematizing* describes how engaging in horizontal and vertical mathematizing can support students as they reinvent the intended mathematics by incrementally expanding their mathematical reality. As Gravemeijer and Doorman (1999)

state, "it is in the process of progressive mathematization - which comprises both the horizontal and vertical component - that the students construct (new) mathematics" (p. 116 – 117).

An Example of Progressive Mathematizing in Abstract Algebra

To illustrate such a reinvention process, and the cycle of progressive mathematizing, consider an example from an RME inspired, abstract algebra curriculum – *Teaching Abstract Algebra for Understanding* (TAAFU). The TAAFU curriculum was designed to be used in proof based, introductory group theory courses at the undergraduate level. The TAAFU curriculum includes three main instructional units: groups and subgroups, isomorphism, and quotient groups. Each of these three units begins with a reinvention phase, where the students work on a sequence of tasks designed to help them develop and formalize a concept, drawing on their prior knowledge and informal strategies. The end product of the reinvention phase is a formal definition (or definitions) and a collection of conjectures (for a detailed description of the TAAFU project and associated curriculum see Larsen, Johnson, and Weber, 2013).

The quotient group unit is launched in the context of the symmetries of a square[2]. By this point in the curriculum the students have worked extensively with symmetry groups as they reinvented the concepts of group and isomorphism (Larsen, 2013). As a result, the group of symmetries of a square (and the associated operation table) is experientially real to the students, in that this group is accessible on an intuitive level. Also available within the students' experiential reality is the behavior of the even and odd integers, specifically

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[2] See Larsen and Lockwood (2013) for a full description of the quotient group sequence.

the pattern that *even + even = even, odd + even = odd, even + odd = odd,* and *odd + odd = even*. Given this problem context, the students are asked if they can find anything like the evens and odds in the symmetries of a square. This task represents a horizontal mathematizing activity because the students are being asked to mathematize two already familiar contexts, the symmetries of a square and the even/odd pattern.

One possible artifact of this horizontal mathematizing may be a partition of the symmetries of square. For instance, in Figure 1 we see a student's partitioning in which the symmetries of a square are divided into the flip symmetries and the rotational symmetries. In the TAAFU curriculum, the students are then asked to further mathematize their activity (and the associated artifact) by determining if this partition satisfies the definition of a group. Because the students are now mathematizing their own mathematical activity, determining if such a partition forms a group is an example of vertical mathematizing. In the course of this activity, the students determine that this partition could be viewed as a special type of a group – one in which the two elements of the group are subsets and the operation between any two subsets is determined by combining each element of one subset with each element of the other subset. In this way, the artifact of this vertical mathematizing activity is a new way to think about partitions and a new kind of group.
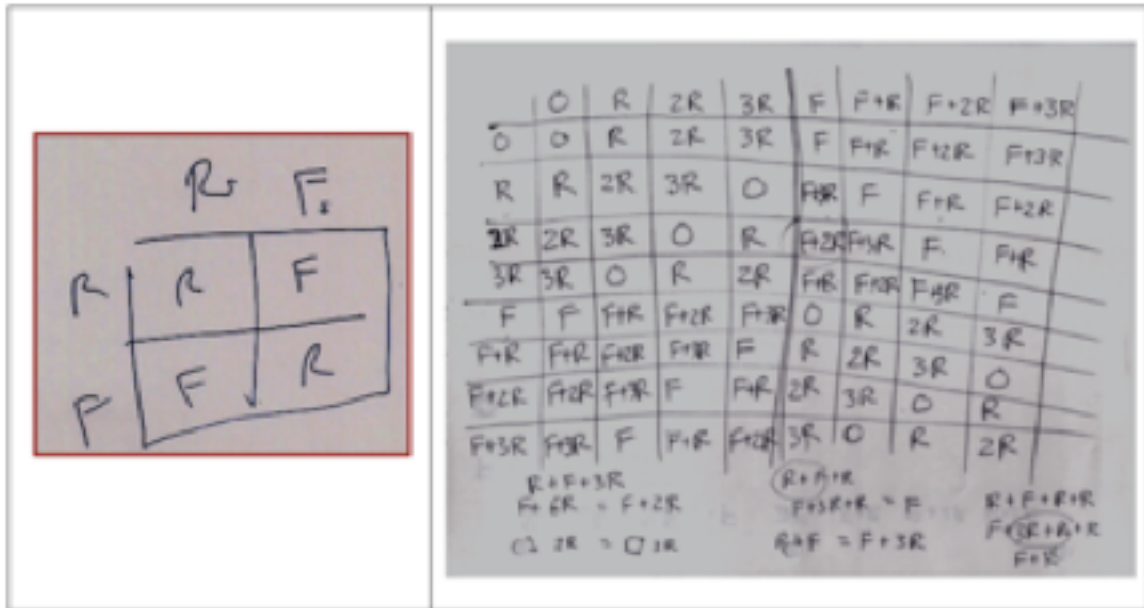
Figure 1. An even/odd partition of the symmetries of a square

The TAAFU curriculum then asks the students if they can make a larger group by breaking the symmetries of a square into four subsets. Notice that this task is not explicitly asking the students to mathematize their previous activity (i.e., their new example of group). Instead, this task is asked from the perspective that the experientially real problem context has been expanded to include this new notion of a partition forming a group. This reflects a perceived shift in the mathematical reality, where this new type of group (an artifact of vertical mathematizing) is now accessible to the students. Therefore, the task posed to the students (to make a larger group by partitioning the symmetries of a square into more subsets) is an example of horizontal mathematizing. The artifact of this task is the symmetries of the square portioned into four subsets.

The students are then again asked to engage in vertical mathematizing, as they determine if this four-element partition forms a group. This round of vertical

mathematizing results in a more generalized view of partitions that form groups, and ultimately a working definition of (and means for constructing) a quotient group - a group of subsets under the operation of "set multiplication". In this way, the TAAFU curriculum guides students in reinventing the concept of quotient group, through a process of progressive mathematizing.

Emergent Models

One approach to supporting students' reinvention of mathematics is to design starting point tasks that can elicit informal student strategies that anticipate the more formal mathematics that is the goal of instruction. The RME emergent models instructional design heuristic (Gravemeijer, 1999) is meant to support this approach to guided reinvention. In the emergent model heuristic, informal and intuitive models of students' mathematical activity transition to models for more formal activity. A model is considered a *model-of* when an expert observer can describe the students' activity in terms of formal mathematics that is the target of the instructional sequence (Larsen & Lockwood, 2013). The model later evolves into a *model for* more formal activity. The model is considered to be a *model-for* when students can use the model to support more general reasoning in new situations.

In describing the progression from a model-of informal activity to a model-for more formal mathematics, Gravemeijer (1999) discusses four layers of activity. Initially student activity is restricted to the *task setting*, where their work is dependent on their understanding of the problem setting. *Referential activity* develops as students construct models that refer to their work in the task setting. *General activity* is reached when these

models are no longer tied to the task setting. Finally, *formal activity* no longer relies on models. In regards to these four levels of activity, the shift from *model-of* to *model-for* is said to occur as students shift from *referential activity* to *general activity*. It is during this transition from referential to general activity that "the model becomes an entity in its own right and serves more as a means for mathematical reasoning than as a way to symbolize mathematical activity grounded in particular settings" (Gravemeijer, 1999, p. 164).

During the instructional sequence the "model manifests itself in various symbolic representations" (Gravemeijer, 1999, p. 170). The *chain of signification* construct provides one way to describe changes in the symbolic representation of the model during as instructional sequence, and ultimately the evolution of the global model. Central to the chain of signification construct is the idea of a sign, which is made up of a signifier (a name or symbol) and the signified (that which the signifier is referencing, such as the students' activity). A "chain of signification" occurs as students' previous signs become the signified in subsequent signs. When this happens, it is said that the initial sign has slid under the subsequent sign. These local shifts in the *form* of the emerging model support the evolution of the global model in a number of ways. As Gravemeijer (1999) notes, "the chain of signification is in a sense the counterpart, on a more specific level, of what the model is on a more general level" (p. 175). When one sign slides under, the new symbol efficiently encapsulates the students' previous activity. In this way, the new sign serves to condense the earlier rounds of activity - placing the most general activity at the forefront of the chain while still allowing students to access their earlier activity if needed. Additionally, as the instructional sequence progresses, the constant revision of the signs ensures that the current sign is the most useful for the students' current activity.

While a chain of signification looks at the development of the model on a local scale by focusing on the form of the model, the transition from a *record-of* to a *tool-for* serves as a way to understand the development of the model on a local scale by focusing on the function of the model. As described by Larsen (2004), an inscription representing students' mathematical activity transitions from a *record-of* to a *tool-for* when the students use the notational record to achieve subsequent mathematical goals. Therefore, instead of focusing on the relationships between the students' emerging symbols and notations (as with chains of signification), the record-of/tool-for[3] construct focuses on changes in how the emerging symbols and notations are used. These local shifts in the *function* of the emerging model support the evolution of the global model in a number of ways. For instance, a local of/for shift may indicate that one aspect of the students' activity has become available to the students for more formal reasoning. The availability of this new tool reflects that certain aspects, or certain representations, of the global model are beginning to transition to a model-for more formal activity.

Chains of signification and the record-of/tool-for construct provide lenses for describing local shifts in the various symbolic representations of the global model. The former attends to changes in the form of these symbolic representations, and the later attends to changes in the function of these symbolic representations. These local changes also support each other. Changes in the form of the representation provide students with more powerful inscriptions that better meet the needs of their current activity. As a result,

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[3] This record-of/tool-for distinction is similar to the idea of *transformational records* - teacher introduced "notations, diagrams, or other graphical representations that are initially used to record student thinking and that are later used by students to solve new problems" (Rasmussen & Marrongelle, 2006, p. 389).

these new inscriptions are more useful as tools. Additionally, as students change the function of the inscriptions to achieve new goals, they may adopt more efficient forms of the representations that highlight aspects that are especially useful. Therefore, the chains of signification and the record-of/tool-for constructs are reflexively related and work together to support changes in the global model.

An Example of Emergent Models in Abstract Algebra

In the first unit of the TAAFU curriculum, the students reinvent the group concept by investigating the symmetries of an equilateral triangle (see Larsen, 2013). For this unit, the model is considered to be the algebraic structure of this particular group (the symmetries of an equilateral triangle). The students begin the unit by physically manipulating an equilateral triangle. Within this initial task setting, students identify, describe, and symbolize all of the symmetries of an equilateral triangle. The students then engage in referential activity when they being to manipulate the symbols that represent the symmetries of the triangle. This referential activity includes working with these symbolic representations in order to create a method for calculating combinations of symmetries. It is this referential student activity that an expert observer can describe in terms of the algebraic structure of this group. For instance, as students calculate certain combinations, they may regroup pairs of symmetries (implicitly using the associativity property) and notice that certain pairs of symmetries undo each other (implicitly using inverses). In this way, the structure of this group can be seen as a *model-of* the students' intuitive and informal activity. As the instructional sequence unfolds, the students' activity progresses to generalized activity as they use the algebraic structure of this group

33

of symmetries to analyze other systems (e.g., the integers under addition) and ultimately develop a formal definition of group. At this point, the concept is considered to be a *model-for,* as the students can use the concept to support more general reasoning in new situations.

In this example the model (i.e., the algebraic structure of the group of symmetries of an equilateral triangle) undergoes a series of local changes in various symbolic representations of the students' activity. These symbolic representations include the list of symmetries, an operation table, and a set of rules for manipulating symbols. The model comprises the collection of these representations, along with the connections between them. As a result, the development of this global model is reflexively related to 1) the development of more powerful forms of the model (as described by the chain of signification construct), and 2) the students' increasing ability to reason with the various forms of the model (as described by the record-of/tool-for transition).

Chain of signification. Initially, the students begin the group unit by physically moving a triangle in order to identify the six symmetries of an equilateral triangle. The students are then asked to represent these six symmetries with a diagram, a written description, and a symbol (see Figure 2). This set of inscriptions can be thought of as a signifier that signifies the students' activity of manipulating the triangle. The students are then asked to generate a new set of symbols, this time representing each symmetry in terms of a vertical flip, F, and a 120° clockwise rotation, R. This new set of symbols represents the next step in the chain of signification, with the earlier sign sliding under this subsequent sign. The original sign, which was composed of both the students' initial

34

signifier (i.e., their initial inscriptions) and the original signified activity (i.e., physically manipulating the triangle), is now signified by this new set of symbols. So as the chain of signification builds, students no longer need to directly consider their original activity of manipulating the triangle. For example, when working with symbols expressed in terms of F and R, students may no longer need to keep in mind that they refer to motions of a triangle.
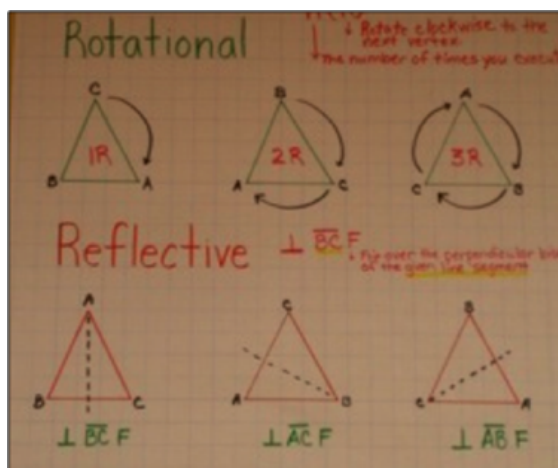
Figure 2. Diagrams and initial symbols for the symmetries of an equilateral triangle

This new set of symbols (in terms of two generators F and R) supports the students in developing a set of rules for calculating the combination of any two symmetries. For instance, when combining a vertical flip (F) and a flip over one of the diagonals ($F + R$) the symbolic expression $F + (F + R)$ invites regrouping the F's and then ignoring them because performing a flip twice is the same as doing nothing[4]. Therefore, it is in the

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[4] With the students' original symbols (Figure 2) this combination would be represented by ($\perp\overline{BC}\ F$) ($\perp\overline{AB}\ F$), which does not suggest the possibility of rule-based calculations the same way that $F + (F + R)$ does. See Larsen (2009, 2013) for a more detailed discussion of the use of compound symbols.

35

students' use of these more efficient symbols (in terms of *F* and *R*) that an expert can begin to describe the students' activity in terms of the model. Specifically, as students begin to develop rules for calculating combinations of these symmetries, an expert would be able to recognize the algebraic structure of the group of symmetries as a *model-of* the students' activity. As a result, the sliding under of the students' initial inscriptions helped to support the development of the students' emerging model by 1) providing them with powerful inscriptions, and 2) transitioning their activity from the initial task setting to referential activity.

Tool-of/record-for. once the students develop a common set of symbols using *F* and *R*, they are asked to determine the result of the combination of any two symmetries. As a way to organize the 36 different combinations, some students choose to record their results in an operation table. In this way, an operation table can initially emerge as a *record-of* the students' activity. This *record-of* can later be used by students as a *tool-for* subsequent mathematical activity. For instance, partial operation tables can be used by students to argue that the identity element of a group must be unique. This was the case during a teaching experiment that supported the development of the TAAFU curriculum. In this teaching experiment, a student worked to show that the identity element of a group must be unique. This student employed a partial operation table, with arbitrary elements, in order to construct a proof (see Figure 3). As discussed by Weber and Larsen (2008), the student's modification of the operation table, including the use of arbitrary elements and only including aspects of the table that were needed to support her reasoning,

"suggests that she was using the table as a tool to support her reasoning and not merely as a crutch for recalling the steps of a procedure" (p. 148).
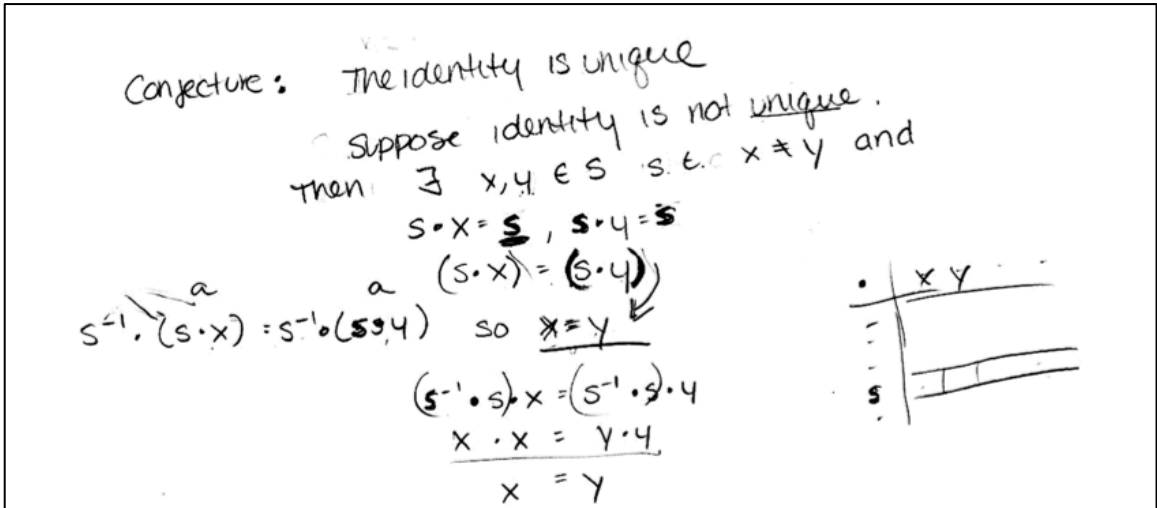
Figure 3. Operation table as s tool-for

In this example, the student was able to use an operation table to prove that the identity of a group must be unique. In this way, the operation table shifted from being a record-of the student's activity to tool-for further mathematizing. This shift in the function of the operation table, from an inscription (of one aspect) of the model to an instrument that can used for justification, can be seen as a local change that supports the more global of/for transition of the model in two important ways. First, the operation table represents one aspect of the global model. Therefore, this local of/for shift in the function of the operation table reflects that one aspect of the global model is now available to the students for more formal reasoning. Second, the student was able to leverage the operation as a tool as she reasoned about the formulation of the identity property. In this way, the operation table served as a tool for supporting the development

of another aspect of the global model – the axioms that characterize the algebraic structure of this group. Therefore, this local shift of the operation table from a record-of activity to tool-for further mathematizing aided in the development of other aspects of the global model, and reflects an increasing ability to reason with the various forms of the model.

RME Design Heuristics in Terms of Activity and Concept Development

Captured within both the guided reinvention and the emergent model heuristics is the duality of engaging in more generalized activity and developing mathematical concepts. By teasing apart these two aspects, two lenses for describing the purpose of these RME design heuristics come into focus. One lens, which considers the guided reinvention and emergent model heuristics in terms of more generalized student activity, places the emphasis on instruction that promotes "socially and culturally situated mathematical practices" (Rasmussen et al., 2005, p. 55). The other lens, which considers the guided reinvention and emergent model heuristics in terms of concept development, places the emphasis on instruction the supports the reification of student activity.

Increasingly General Activity

An emphasis on the students' activity within a given problem context is at the forefront of Rasmussen et al. (2005) discussion of progressive mathematizing. Instead of framing progressive mathematizing in terms of the concepts being developed (e.g., fractions or long division), Rasmussen et al. frame progressive mathematizing in terms of the practices students engage in that promote the evolution of such concepts. As they

explain, "this is a nontrivial modification because it calls for attention to the types of activities in which learners engage for the purpose of building new mathematical ideas and methods for solving problems" (p. 55). This focus on student activity is also captured by Zandieh and Rasmussen's (2010) conceptualization of the emergent models heuristic and their definition of a model, where they define a model as "student-generated ways of organizing their activity with observable and mental tools" (p. 58). However, with either design heuristic the point is not merely to design instructional sequences that engage students in mathematical activity. The point is to design instructional sequences that engage students in mathematical activity that is more and more general.

With the guided reinvention heuristic, instruction can be designed with purpose of supporting student activity through progressive mathematizing. During the process of progressive mathematizing, the students' activity shifts repeatedly from horizontal to vertical mathematizing. This shift in the type of mathematizing corresponds to a shift in the generality of the student activity. Initially, horizontal mathematizing is limited to the specific problem context. As students transition to vertical mathematizing, this specific problem context is no longer the focus of the activity, rather the students mathematize their own mathematical activity to support their reasoning in a different or more general situation. As a result, instructional sequences informed by the guided reinvention design heuristic can be characterized by the shifts in the generality of student activity that the instruction is designed to support.

Similarly, within the emergent models heuristic, there is an intention to progress students from activity situated within a specific task context to referential, general, and formal activity. In particular, the *model-of/model-for* transition is linked to a shift in the

students' activity from referential (where their activity references aspects of the original task setting) to general (where the students activity is no longer tied to the original task setting). The activity that supports the transition between a *model-of* to a *model-for* is an especially significant example of vertical mathematizing. When the students are engaged in referential activity, the model emerges as a result of the students mathematizing the problem context (i.e., horizontal mathematizing). As the students move into general activity, they begin to mathematize aspects of their emerging model. In this way the transition between referential and general activity can be interpreted as the result of vertical mathematizing.

Therefore both the guided reinvention and the emergent models heuristics can be framed in terms of increases in the generality of student activity, either as they progress thorough more general layers of activity (in the emergent models heuristic) or as they engage in progressive mathematizing (in the guided reinvention heuristic). With either heuristic, the student activity ultimately supports the development of a new mathematical reality. In the case of progressive mathematizing, it is vertical mathematizing that results in a new mathematical reality. "Vertical mathematizing activities serve the purpose of creating new mathematical realities for the students. These new mathematical realities can then be the context or ground for further horizontal and/or vertical mathematizing activities" (Rasmussen et al., 2005, p. 54). With the emergent model heurist the model-of/model-for transition is concurrent with the creation of a new mathematical reality, as "formal activity involves students reasoning in ways that reflect the emergence of a new mathematical reality and consequently no longer require support of prior models-for activity" (Zandieh & Rasmussen, 2010, p. 58).

Concept Development

Instead of focusing on the activity in which the students are engaged (and the context in which the student activity is taking place), we could instead focus on the evolution of a mathematical concept. Both the guided reinvention and emergent models heuristics have been connected to reification (Gravemeijer, 1999; Gravemeijer & Doorman, 1999). As described by Sfard (1991) mathematical objects (such as numbers and functions) historically developed through a recurring pattern of reification, in which "various processes had to be converted into compact static wholes" (p. 16). Similarly, one can conceive of the guided reinvention and emergent models heuristics as processes through which student activity becomes reified into mathematical objects.

> The analogy with this process of objectivation, or *reification*, however, begs the question: What is it that is being reified? The analogy with the transition from process to object-like entity, observed by Sfard (1991), illuminates the point that it is not the inscription that is being reified but some activity. (Gravemeijer, 1999, p. 164)

This emphasis on reification offers a lens to describe these two design heuristics in terms of the development of the concept, where aspects of the students' mathematical activity become reified as they engage in more general activity.

The guided reinvention heuristic describes this evolution as an expansion of what is experientially real for the students, where the process of progressive mathematizing supports the construction of new mathematical realities. By engaging in horizontal mathematizing, the students translate aspects of their mathematical reality into mathematical terms. The artifacts of horizontal mathematizing may include inscriptions, symbols, and procedures that represent aspects of an already familiar problem context.

During vertical mathematizing, it is the students' own horizontal mathematizing (and resulting representations/artifacts) that are mathematized. Instead of resulting in representations of an already familiar context (as with the artifacts of horizontal mathematizing), vertical mathematizing results in objects that are now accessible to students on an intuitive level (i.e., these objects are now incorporated into the students' experiential reality). As described by Freudenthal (1971), "the activity on one level is subjected to analysis on the next, the operational matter on one level becomes a subject matter on the next level" (p. 417). It is this shift, from "operational" to "subject matter", that Gravemeijer and Terwel (2000) state is related to reification, where this shift reflects that aspects of the students' activity have evolved "into entities of their own" (p. 787).

Similarly, the shift from model-of to model-for is related to the process of reification (Gravemeijer, 1999). As students shift from referential activity to general activity "the model becomes an entity in its own right and serves more as a means of mathematical reasoning than as a way to symbolize mathematical activity grounded in particular settings" (p. 164). Therefore, the model – which Gravemeijer (1999) describes as "an overarching concept" (p. 170) – transitions from an artifact of the students' mathematical activity to a mathematical object independent of the students' original activity. In this way the shift from model-of to model-for, which can be understood to be the incorporation of this new object into the students' experiential reality, reflects the creation of a new mathematical reality.

Therefore, both the guided reinvention and the emergent models heuristics can be described as processes through which the engagement in progressively more general activity supports the development of mathematical concepts through the reification of

student activity. With the guided reinvention heuristic, these mathematical concepts develop during the process of progressive mathematizing. By engaging in vertical mathematizing, the students' activity becomes a new type of object that is accessible to them on an intuitive level. Similarly, with the emergent models heuristic, as the students transition from referential activity to general activity the model of student activity becomes an object that is available for student reasoning.

Framing RME Design Heuristics as Lenses on Student Learning: Two Metaphors

While RME is primarily an instructional design theory, the design heuristics carry with them a view of what it means to learn mathematics. As described by Cobb (2000), RME contends that 1) mathematics is a creative human activity, 2) learning occurs as students develop effective ways to solve problems, and 3) mathematical development involves the creation of a new mathematical reality (p. 317). This suggests that RME can be used to describe the process through which learning takes place. Specifically, learning happens as students engage in activity that is situated in accessible contexts, where this activity brings forth a new mathematical reality. However, as discussed by Sfard (1998) there is another question that needs to be addressed. "What is this thing called learning?" In order to answer this question, Sfard presented two metaphors for learning, the participation metaphor and the acquisition metaphor. In the following sections I will discuss how these two metaphors can be used to provide insight into how the design heuristics support student learning. Additionally, by considering the implication of these two metaphors for learning, I will present two conceptualizations of the notion of a "new mathematical reality".

Participation Metaphor and the Creation of New Mathematical Realities

Sfard (1998) describes the *participation metaphor* for learning as a view in which "learning" is synonymous with becoming a participant in a community, and "knowledge" is synonymous with aspects of practice/discourse/activity (p. 7). With this view, the emphasis is placed on what the student is *doing*, and the context in which that practice is taking place (as opposed to emphasizing on the mental constructs the students have).

As discussed earlier, both the guided reinvention and the emergent models heuristics can be framed in terms of increases in the generality of student activity. With this framing, the process of progressive mathematizing (guided reinvention) and the progression thorough more general layers of activity (emergent models) are consistent with Rasmussen et al.'s (2005) notion of *advancing mathematical activity* – where advancing mathematical activity is understood as "acts of participation in different mathematical practices" (p. 53). From this perspective:

> The notion of advancing mathematical activity is the building and progression of practices. Participation in these practices, and changes in these practices, is synonymous with learning (Rasmussen et al., 2005, p. 55).

Therefore, one way to conceptualize student learning in a way that is consistent with an RME perspective is to view student learning as participating in situated activity. Continuing with the participation metaphor, one could ask what it means for student activity (i.e., learning) to support the development of a new mathematical reality. I propose that, from a participation perspective on learning, the creation of a new mathematical reality can be understood as the creation of a new context for further activity, and new ways for students to participate in that context.

Acquisition Metaphor and the Creation of New Mathematical Realities

With the *acquisition metaphor*, learning is viewed as the acquisition of knowledge and concepts. This perspective "makes us think about the human mind as a container to be filled with certain materials and about the learner as becoming an owner of these materials" (Sfard, 1998, p. 5). Therefore, this perspective places the emphasis on concept development, where "concepts are to be understood as basic units of knowledge that can be accumulated, gradually refined, and combined to form ever richer cognitive structures" (p. 5).

This perspective comes to the forefront when the guided reinvention and emergent models heuristics are framed in terms of reification. With guided reinvention, mathematical concepts develop as a result of horizontal and vertical mathematizing. During vertical mathematizing, the students mathematize their own horizontal mathematizing (and resulting representations/artifacts). In this way, the students' activity becomes the subject matter for subsequent mathematical activity. This results in the students' activity becoming a new type of object that is accessible to them on an intuitive level. Similarly, the emergent models heuristic described a process through which students' activity emerges first as a *model-of* the students' informal activity and then transitions to being a model-for supporting students' more formal reasoning (and thus an object in the students' mathematical reality). Framing these two heuristics in terms of reification emphasizes the view that learning is synonymous with concept development, which happens as student activity is reified into mathematical objects that are available for subsequent student activity.

Therefore, from an RME perspective, it is also consistent to conceptualize student learning as the acquisition of mathematical concepts. From this perspective, learning not only supports the creation of a new mathematical reality (as it did with the participation metaphor), learning can be viewed as synonymous with the creation of a new mathematical reality. I propose that, from an acquisition perspective on learning, the creation of a new mathematical reality can be conceptualized as the incorporation of new mathematical objects into the students' experiential reality. These new mathematical objects can be understood as concepts that form "ever richer cognitive structures" (Sfard, 1998, p. 5), and the fact that they become incorporated into the students' experiential reality reflects that the students are able to access these concepts on an intuitive level.

Implications for Analyzing Student Learning

The participation and acquisition metaphors offer two perspectives on student learning – one primarily focusing on the activity of the students and the other primarily focusing on the development of the mathematical concepts. These two perspectives can provide theoretical support for analytic techniques designed to document student learning, especially in classrooms with RME inspired curricula materials. For instance, if the curricular materials were designed to encourage student learning by way of an emergent models transition (either because the engagement in such activity *is* learning or because engaging in such activity *supports* learning by developing a mathematical concept), then any attempts to document student learning should explicitly draw on the theoretical constructs related to such a transition. In this section I will consider the implications of these two perspectives for analyzing student learning in cases where the

instructional design is consistent with theses RME heuristics (guided reinvention and/or emergent models). Specifically, I will consider what should count as evidence of student learning by considering examples from the TAAFU curriculum.

Evidence for Student Learning from a Participation Perspective

In order to document student learning from a participation perspective, the focus of analysis needs to be placed on 1) the students' participation in mathematical practices, and 2) changes in the mathematical practices of the students. The RME design heuristics provide a lens for looking at: the nature of the practices that students engage in (mathematizing), the trajectory of the students' activity in terms of generality (progressive mathematizing and layers of generality), and changes in the context and the ways students participate in that new context (new mathematical realities).

Because student learning is taken to be synonymous with participation in practices (and changes in those practices) documenting the mathematical activity of the students is a necessary component to document student learning. Central to the guided reinvention heuristics is the idea of *mathematizing*. Broadly speaking, *mathematizing* is defined as "organizing from a mathematical perspective" (Gravemeijer & Doorman, 1999, p. 116). While a complete taxonomy of such practices has not been compiled, there are several examples of mathematizing practices in the literature. These include: translating, describing, organizing symbolizing, algorithmatizing, defining, and generalizing (Gravemeijer & Doorman, 1999; Rasmussen et al., 2005; Zandieh & Rasmussen, 2010). For instance, in the quotient group unit of the TAAFU curriculum, tasks are designed to support students in *organizing* the operation table of the symmetries of a square into an

even/odd pattern, *proving* that some partitions of groups themselves form groups and *defining* a new category of group. When taking a participation perspective on learning, the documentation of student engagement in such activity provides evidence of learning.

Analysis can go even further if the instructional sequence was designed in accordance with the guided reinvention or emergent models heuristics. Both design heuristics, as used by the curriculum developer, inform a hypothesized trajectory that the instructional sequence is designed to support. These hypothesized trajectories can serve as a guide for analyzing the progression of the students' participation in mathematical activity. With the guided reinvention design heuristic, the instructional sequences are designed to support hypothesized trajectories of progressive mathematizing. With the emergent models design heuristic, the instructional sequences are designed to support hypothesized trajectories of increasingly general activity. Therefore, with either design heuristic, student learning can be documented in relation to a hypothesized trajectory of student activity. For instance, in the group unit of the TAAFU curriculum, instruction was designed to support students in moving from: identifying, describing, and symbolizing all of the symmetries of an equilateral triangle (task setting activity), to developing a calculus for combining symmetries (referential activity), to using the algebraic structure of this particular group of symmetries to analyze other systems and defining *a group* (generalized activity), to leveraging the properties of groups in order to develop the isomorphism concept (formal activity). Using this hypothesized progression as a guide, analysis can be carried out to determine the extent to which the students' activity followed this path. Tracing such a development would be evidence of changes in the students' participation and therefore would be evidence of student learning.

In addition to looking at changes in the students' mathematical practices by analyzing the trajectory of the student activity, it is also possible to look at changes in the mathematical practices by analyzing changes in the mathematical context in which the practices are taking place. The notion of a new mathematical reality (as understood from a participation perspective) provides a lens for describing the development of new mathematical contexts for further activity, and for describing new ways students participate in the new context. For instance, in the quotient group unit of the TAAFU curriculum, the guided reinvention design heuristic supported the development of a progressive mathematizing sequence that culminates with an expanded mathematical reality that includes a working definition of quotient groups. Additionally, this sequence provides students with opportunities to reason within this new context. (As students try to build partitions that form a group, they often try a number of different partitions and begin to develop a process for building quotient groups and an intuition about why some partitions form groups while others do not). In this way, the mathematical reality for the students' activity changes as they engage with the instructional sequence – both in terms of the context in which the activity takes place (an expanded context which includes quotient groups) and in terms of the way that the students interact in the context (in terms of the ways students reason about partitions). Again, documenting such shifts provides evidence of student learning.

Therefore, when analyzing student learning from a participation perspective, the RME design heuristics provide powerful lenses for documenting student practice and changes in these practices. The various mathematizing activities described in the literature provide examples and characterizations of mathematical practices.

Documenting student participation in such practices is a necessary component to documenting student learning. However, it is also necessary to understand changes in the students' practice. The RME design heuristics provide two avenues for analyzing changes in practice. Learning trajectories based on supporting students in progressive mathematizing and/or progressing through layers of generality provide a framework for analyzing how the mathematical practices of the students are changing in regards to the generality of their activity. Additionally, the notion of a new mathematical reality provides a way to discuss both changes in the context of the students' activity and changes in how students participate in this new context.

Evidence for Student Learning from an Acquisition Perspective

The documentation of student learning from an acquisition perspective focuses on the development of the mathematical concepts. With both the emergent models and guided reinvention design heuristics, the mathematical concepts develop as aspects of the students' mathematical activity become reified. Instead of considering the reification of a global concept, here I will consider a smaller grain size of analysis by discussing the documentation of local evidence of student learning. This approach is similar to the one taken by Rasmussen and Marrongelle's (2006), who pointed out that, "connecting the model-of/model-for transition to reification is a strong requirement that typically accompanies extended periods of time" (p. 391). Therefore, Rasmussen and Marrongelle chose to analyze teaching practices on the day-to-day level by focusing on a version of the emergent model heuristic that did not require reification (*transformational records*). Similarly, in order to document student learning on a local level, I will look for evidence

of the development of the global model by identifying and documenting local changes to the model. These local changes can either be 1) related to the form of the model the, as described by the chains of signification construct, or 2) related to the function of the model, as described by the record-of/tool-for construct. In the case of the guided reinvention heuristic, the goal is to find evidence of an expansion in what is experientially real for the students. From an acquisition perspective, this is understood as a creation of a new mathematical reality, where new mathematical objects become incorporated into the students' experiential reality. Documenting changes to the mathematical reality will focus on changes in the objects that arise as artifacts of progressive mathematizing.

The TAAFU curriculum launches in the context of the symmetries of an equilateral triangle. As seen in Figure 2, an early sign that emerges in this context is composed of a signifier (an initial set of inscriptions for the six symmetries) and a signified (the students' activity of physically manipulating an equilateral triangle). The curriculum then prompts students to generate a new set of symbols in terms of F and R, and in doing so supports the progression of the chain of signification. This new set of symbols represents a signifier in the next step in the chain of signification, with the earlier sign sliding under to become the object that that is being signified by these symbols. This shift in the form of the model to one that is more powerful can be seen as a local change that is part of (and supports) the more global transition to a model for. Therefore, one sign sliding under a subsequent sign supports the reification of the global model by supporting shifts in the form that the model takes. As a result, documenting instances in which signs slide under

is a way to capture local shifts in students' concept development and can provide evidence of student learning.

Once the students develop a common set of symbols using *F* and *R*, they are asked to determine the result of the combination of any two symmetries. An operation table initially emerges as a record-of the students' activity. Later, as the students argue that the identity element of a group must be unique, students may draw on the operation table as a tool-for constructing a proof. This shift in the function of a representation of the model, from an inscription to a instrument, can be seen as a local change that is part of (and supports) the more global transition to a model-for. Therefore, an inscription shifting from a record-of to a tool-for supports the reification of the global model by supporting shifts in the function that the model serves. As a result, documenting local instances of such transitions is a way to capture local shifts in students' concept development and can provide evidence of student learning.

The progressive mathematizing in the TAAFU curriculum that results in the quotient group concept becoming part of the students' mathematical reality encompasses several transitions between "the world of life" and "the world of symbols". The quotient group unit is launched with the assumption that both the behavior of the even/odd integers and the operation table for the symmetries of a square are experientially real for the students. From here, the first symbolic artifact is a partition of the symmetries of square into two sets. However, it is not until the students engage in vertical mathematizing (by proving that this partition forms a group) that this partitioning activity becomes a new type of object that is accessible to the students on an intuitive level (i.e., a special type of group with two subsets as elements). This expansion in the students' mathematical reality,

52

which can be understood in terms of new mathematical objects being accessible to students on an intuitive level, represents student learning from an acquisition perspective. Therefore, one way to document learning in this context is to look for evidence that new mathematical objects have become accessible and useful to the students as they work in more general problem contexts. In the example provided here, the students' activity of forming this new type of group (with two subsets as elements) resulted in a new object within the students' experiential reality (where the new object is the new type of group). In order to document such a change in the students' mathematical reality, one could look for evidence that this new object has become available for further progressive mathematizing. This could include students being able to further mathematize this expanded context to move beyond a focus on parity by intentionally forming groups made up of subsets.

So, when analyzing student learning from an acquisition perspective in situations where the learning is designed to be supported through a model of/for transition, we can look for local shifts in the form and function of the emerging model. This includes looking for indications that one sign has slid under a subsequent sign and looking for indications that a record-of student activity is serving as a tool-for subsequent student activity. Both of these local shifts support the reification of the global model (i.e., student learning from an acquisition perspective). Further, when analyzing student learning from an acquisition perspective in situations where the learning is designed to be supported through progressive mathematizing, we can look for incremental additions to the students' mathematical reality. These additions reflect that aspects of the students' activity have become objects that are now accessible for further mathematizing.

Conclusions

RME offers curriculum developers with a powerful theory for instructional design. The emergent models design heuristic supports instructional design efforts by describing a mechanism through which students' informal and intuitive activity can be leveraged to support the development of formal mathematics. The guided reinvention design heuristic provides a description of how, by engaging in mathematical activity, students can expand the mathematical reality that they are able to access on an intuitive level. Part of the power of these two design heuristics resides in the fact that they place a duel emphasis on supporting both the students' mathematical activity and the formal mathematics that the curriculum is intended to develop. As a result, the curriculum developer can design curriculum with both student activity and concept development in mind. For instance, these design heuristics promote curriculum developers asking questions like: "What informal strategies and ways of thinking are the students likely to have and how do those anticipate the formal mathematics?"; "What instructional tasks/activities/contexts can be used to evoke these strategies and ways of thinking?"; and, "What instructional activities are going to be useful for supporting students in leveraging these strategies and ways of thinking to develop of the formal mathematics?".

While this focus on both activity and the concept development makes the theory flexible and powerful as an instructional design theory, it can be a confounding factor when trying to carefully articulate some of the fundamental RME constructs. For example, the idea of a *new mathematical reality* is left undefined although it is used often in the literature in order to describe the results of students' mathematical activity. Similarly, in the RME literature, the term model tends to be discussed in terms of both

student activity and concept development. This lack of precision in the descriptions of these theoretical constructs became a significant problem as I tried to investigate the impact of the TAAFU curriculum on student learning. The curriculum was designed to support students in creating a new mathematical reality and developing formal mathematics through a model-of/model-for transition, so it made sense to rely on these constructs to support my investigation. However, without knowing precisely what a *new mathematical reality* is, it is very difficult to argue that one has been established. Similarly, without knowing what a *model* is, it is difficult to document the development of one.

In an effort to address such difficulties, this paper was written to explore the implications of RME for documenting student learning. I set out to first coordinate the RME theory related to the emergent model and guided reinvention design heuristics. Both of these heuristics support the development of new mathematical realities by engaging students in increasingly generalized activity, and both can be described either in terms of more generalized activity or in terms of concept development. By focusing independently on these two aspects of the design heuristics, I was able to draw on Sfard's (1998) participation and acquisition metaphors for learning in order to discuss how these design heuristics support student learning.

Considering the design heuristics in light of these two perspectives on learning afforded a powerful lens for making sense of the idea of a new mathematical reality and for discussing what could be considered as evidence for student learning. I propose that, from a participation perspective, the creation of a new mathematical reality can be understood as the creation of a new context for further activity, and new ways for

students to participate in that context. From a participation perspective, the RME design heuristics suggest a number of ways to document student learning. This includes: documenting the mathematizing activities that students are engaged in, documenting how the mathematical practices of the students are changing in terms of the generality of their activity, and documenting changes in the students mathematical reality – both in terms of the context of the students' activity and in terms of how students participate in this new context. From an acquisition perspective on learning, I propose that the creation of a new mathematical reality can be conceptualized as the incorporation of new mathematical objects into the students' experiential reality. The incorporation of these new objects reflects that they have become accessible to students on an intuitive level. Again, the RME design heuristics suggest a number of ways to document student learning from an acquisition perspective. This includes: documenting when one sign has slid under a subsequent sign, documenting when a record-of student activity is serving as a tool-for subsequent student activity, and documenting incremental additions to the students' mathematical reality.

Paper 2: Teachers' Mathematical Activity in Inquiry-Oriented Instruction

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Abstract

This work investigates the relationship between teachers' mathematical activity and the mathematical activity of their students. By analyzing the classroom video data of mathematicians implementing an inquiry-oriented abstract algebra curriculum I was able to identify a variety of ways in which teachers engaged in mathematical activity in response to the mathematical activity of their students. Further, my analysis considered the interactions between teachers' mathematical activity and the mathematical activity of their students. This analysis suggests that teachers' mathematical activity can play a significant role in supporting students' mathematical development, in that it has the potential to both support students' mathematical activity and influence the mathematical discourse of the classroom community.

*Key Words***:** Teaching, Mathematical Activity, Abstract Algebra, Realistic Mathematics Education

Teachers' successful implementation of inquiry-oriented curricula requires a focus on students – both on their ways of understanding and their mathematical practices. This is especially true with curriculum based on the Realistic Mathematics Education (RME) notion of *guided reinvention* (Freudenthal, 1991), in which "the idea is to allow learners to come to regard the knowledge they acquire as their own private knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999, p. 116). One way in which RME based curricula promote such ownership is by promoting the evolution of formal mathematics from students' informal understandings (Gravemeijer, 1999). In this way, the concepts first emerge from the students' informal activity and then

develop into more formal ways of reasoning. The teacher has a crucial role in this transition. Namely, the teacher needs to find ways to build on the students' informal ideas in order to help them construct the formal mathematics. As a result, teachers implementing such curricula must be active participants in establishing the mathematical path of the classroom community while at the same time allowing students to retain ownership of the mathematics. The goal of this research is to better understand what it is that teachers *do* while teaching in order to achieve these instructional goals.

The research presented in this paper is part of a larger research agenda designed to understand the challenges and opportunities that emerge as different faculty implement an RME-based, inquiry-oriented abstract algebra curriculum. In order to develop instructor support materials for the *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum (Lockwood, Johnson, & Larsen, 2013), it became important to 1) identify the mathematical activities that teachers implementing the TAAFU curriculum engage in during classroom teaching in response to the mathematical activity of their students, and 2) investigate the ways in which teacher' mathematical activity interacts with students' mathematical activity. This paper will address the complexity of teaching with an RME based, inquiry-oriented curriculum by analyzing teachers' mathematical activity as they work to support the mathematical activity of their students. To this end, classroom video data from two implementations of an inquiry-oriented abstract algebra course were analyzed to understand the kinds of mathematical activity that teachers engaged in when responding to and supporting the mathematical activity of their students.

1. Theoretical Perspective

RME was developed to be consistent with the idea that, "mathematics education should take its point of departure primarily in *mathematics as an activity*" (Gravemeijer & Doorman, 1999, p. 116). This view of mathematics, elaborated by Freudenthal (1991) below, rejects the view of mathematics as a ready-made-system to be memorized by students.

> Mathematics as an activity is a point of view quite distinct from mathematics as printed in books and imprinted in minds…That is mathematics as an activity of discovery and organising in an interplay of content and form (p. 16-17)

Consistent with this perspective of mathematics as an activity, the research literature describes a number of student mathematical activities. For instance, Rasmussen et al.'s (2005) *advancing mathematical activity* includes symbolizing, algorithmatizing, and defining as specific examples of mathematical activity. Further, in order to understand and generate mathematical proofs, students would likely engage in proof related activities, such as evaluating arguments (Selden & Selden, 2003), instantiating concepts (Weber & Alcock, 2004), and proof analysis (Larsen & Zandieh, 2007). Still other mathematical activity that students are likely to engage in as they work to reinvent mathematical concepts includes conjecturing, questioning, and generalizing.

As students engage in such mathematical activity, one would expect that teachers would need to engage in mathematical activity in response. For instance when faced with a novel student-generated proof a teacher may need to evaluate the proof to determine the validity of the argument and possible advantages/disadvantages of this new approach, both in terms of the current task and in terms of the goals for the lesson. As part of

59

evaluating a student's proof, the teacher may engage in proof analysis (Larsen & Zandieh, 2007), such as searching for hidden assumptions. Further, the teacher may need to identify connections between the student's proof technique and other mathematical justifications the students would be likely to encounter later in the course (Johnson & Larsen, 2012).

The research literature on teachers' implementations of reform curriculum provides a number of examples of mathematical work done by teachers in response to their students' mathematical activity. At the undergraduate level, Speer and Wagner (2009) presented a study in which they sought to account for the difficulties a mathematician was facing while trying to provided analytic scaffolding during whole class discussions, where analytic scaffolding is used to "support progress towards the mathematical goals of the discussion" (p. 493). Speer and Wagner identified several skills necessary for providing analytic scaffolding, including the ability to recognize and figure out both the ideas expressed by their students and the potential for these ideas to contribute to the mathematical goals of the lesson. Speer and Wagner went on to state that, "recognizing draws heavily on a teacher's PCK (pedagogical content knowledge), whereas *figuring out requires that a teacher do some mathematical work in the moment* [emphasis added]" (p. 8).

Johnson and Larsen (2012) investigated a mathematician's ability to interpretively and/or generatively listen to her students' contributions. Interpretive listening involves a teacher's intent to make sense of student contributions and generative listening reflects a readiness for using student contributions to generate new mathematical understanding or instructional activities (Davis, 1997; Yackel, Stephan, Rasmussen, & Underwood, 2003).

60

In order to engage in interpretive and/or generative listening, a mathematician may need to interpret a student's imprecise language, generalize a student's statement into a testable mathematical conjecture, or identify counterexamples to a student's claim (see Johnson & Larsen, 2012), all of which require mathematical work on the part of the teacher. Therefore, interpretive and generative listening are examples of teaching practices that are supported by a teacher's mathematical work.

Examples of teaching practices that are likely to require mathematical work are not limited to research on mathematicians teaching undergraduate mathematics. Studies focused on in-service and pre-service elementary teachers have also identified analyzing student work, interpreting student explanations, and building on student contributions as important instructional activities needed for teaching mathematics (Charalambous 2008, 2010; Hill et al., 2008). Each of these tasks requires teachers to engage in mathematical activity in response to the mathematical activity of their students. Figure 4 lists some of the kinds of mathematical activity that have been identified in the research literature as activities that teachers engage in during classroom teaching in response to the mathematical activity of their students.

Teacher Mathematical Activity

Interpreting students' mathematical reasoning and contributions (Charalambous, 2008; Hill et al., 2008; Johnson & Larsen, 2012; Speer & Wagner, 2009)
- Interpreting students' imprecise language
- Interpreting student explanations
- Making sense of student errors

Analyzing student work, contributions, and proofs (Charalambous, 2008; Johnson & Larsen, 2012)
- Assessing the validity of student strategies
- Generalizing student statements into testable conjectures
- Engaging in proof analysis
- Identifying counterexamples in order to evaluate student conjectures

Figuring out the potential of student ideas to advance the mathematical agenda (Charalambous, 2010; Speer & Wagner, 2009)
- Co-constructing explanations that highlight the meaning underlying procedures
- Building on student contributions to support the students in meaning-making

Leveraging student ideas to advance the mathematical agenda (Rasmussen & Marrongelle, 2006)
- Generating a *transformational record*
- Introducing a *generative alternative*

Figure 4. Teacher mathematical activity

It is important to note that in all of these examples, the teachers' mathematical activity is 1) in response to student mathematical activity, and 2) connected to pedagogical considerations, such as advancing the mathematical agenda or assessing student work. Given the context for this mathematical work (teaching), it seems likely that a teacher's mathematical activity may support students' mathematical activity indirectly in the sense that teachers' mathematical activity would inform their *pedagogical activity.* For instance, providing counterexamples, stating the formal mathematical version of a student contribution for a class discussion, and exhibiting a proof for the class could all be examples of pedagogical activity that was informed by a teacher's mathematical activity.

In each of these examples the teacher's pedagogical activity introduces new mathematics into the classroom discourse. The teacher's contribution serves to alter, test, refine, or expand the mathematical ideas under development. Thus, pedagogical activity that introduces new mathematics into the classroom discourse is likely an indicator of teacher mathematical activity. Such teacher activity and corresponding mathematical contributions are of particular interest because, once introduced into the mathematical discourse of the classroom community, these contributions have the potential to support and direct the development of the mathematical ideas. Therefore one of the goals of my analysis was to identify instances in which students engaged with the teacher's mathematical contributions by enlarging, generalizing, refining, and structuring these new mathematical ideas.

Motivated by the needs of the TAAFU project (e.g., the need to develop instructor support materials) and inspired by the theoretical ideas described above, the following research questions were investigated.

1) What teacher mathematical activity is present in classrooms enacting the TAAFU curriculum?

2) In what ways does teachers' mathematical activity interact with students' mathematical activity?

2. Methods

To understand the ways that instructors engage with the TAAFU curriculum we have collected data from the classrooms of four mathematicians over the course of four years. During these four years, there have been five implementations of the curriculum – taking

place at two different, urban, comprehensive universities. Typical students in these courses were junior and senior math majors, with a significant percentage of students planning on becoming high school mathematics teachers. Class sizes ranged from 22 to 35 students, and each of the four mathematicians volunteered to use the TAAFU curriculum.

Two mathematicians, Dr. James and Dr. Bond, are the focus of the analysis presented here. These two mathematicians were selected because every regular class session of their implementation was videotaped. Additionally Dr. James and Dr. Bond participated in debriefing interviews and stimulated recall interviews (Schepens, Aelterman, & Van Keer, 2007), both of which focused on their experiences implementing the TAAFU curriculum. While these debriefing sessions and stimulated recall interviews were not held with the above research questions specifically in mind, they occasionally offered supporting or disconfirming evidence for emerging conjectures.

Because of the exploratory nature of my research questions, I first sought to identify classroom episodes in which the teachers appeared to be engaging in responsive mathematical activity. Specifically, I sought to identify episodes in which the teacher was engaging in activities that were mathematically interesting in response to their students' mathematical activity. The teacher's activity was determined to be mathematical if 1) the teacher was explicitly engaged in an observable mathematical practice, such as proving a student conjecture or analyzing a student proof, in response to students' mathematical activity, or 2) the teacher's actions introduced new mathematics to the classroom discourse, such as the introduction of new mathematical tools, new questions for students to consider, or new mathematical objects and processes. I then looked at this subset of

episodes in order to identify ones that could serve as fruitful cases for exploring the possible connections between the students' and teachers' mathematical activity. After assessing the students' initial mathematical activity and the responsive teacher activity, I looked for changes (and opportunities for changes) in the students' mathematical activity. In this way, four episodes were selected that illustrate the complex the relationship between teachers' mathematical activity and the mathematical activity of their students.

3. Results

Here I will present four episodes that were selected to illustrate the complex relationship between a teacher's mathematical activity and their students' mathematical activity. In the first episode, I will re-examine an episode analyzed by Johnson and Larsen (2012) to explore connections between teacher listening (Davis, 1997) and teacher knowledge. In this re-analysis, teacher listening is considered to be an example of a teacher's mathematical activity. The primary purpose of this new analysis is to consider the impact of this kind of teacher mathematical activity on students' mathematical activity. This episode is also particularly interesting because it provides an example in which a teacher's mathematical activity actually seems to have a limiting effect on subsequent student activity. The other three episodes will illustrate varying degrees to which teachers' mathematical activity 1) provided opportunities for subsequent student mathematical activity, and 2) influenced the mathematical discourse of the classroom community.

3.1. Episode 1 – Coset Formation Algorithm

As part of the quotient group unit of the TAAFU curriculum, the students reinvent the notion of coset by considering how they would need to partition a group in order to form a quotient group (see Larsen & Lockwood, 2013). At this point in Dr. Bond's course the students had been forming quotient groups[5] by breaking dihedral groups into subsets, where those subsets act as group elements with respect to set multiplication ($A * B = \{ab|$ $a \in A, b \in B\}$). The class had already proved that the identity subset must be a subgroup. In this task students were given an identity subset and asked how to determine what the other subsets must be. While working on this task a couple of groups of students had noticed that, in each of the quotient groups they had constructed thus far, any time two elements from the same subset were combined the result was an element of the identity subset.

> *Mark*: I just observed that the one that we worked, in the beginning, the combination of the two elements always gives you an element of the identity set. I have no idea if that actually leads somewhere.

What Mark is observing here is the fact that in one of the earlier examples (and in fact for each example where the class formed a quotient group by partitioning D_8,[6]) the subsets all had a certain property. If you combined any two elements in any subset the result would be an element of the identity subset. While this was true for all of the classroom-generated examples, this is a property that does not generalize to other

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[5] As a result, the students had a working definition of quotient groups that did not include the formal notion of coset.

[6] See Johnson and Larsen (2012) for a more detailed descriprion of how Mark's observation emerged as part of the larger lesson.

quotient groups. As Dr. Bond will realize in her investigation of Mark's observation, this property is actually related to the fact that all the elements of the quotient groups were of order two.

In response to Mark's observation, Dr. Bond led the students through a joint exploration of this idea in Z_4. Dr. Bond began by listing the elements of Z_4, the sets [0], [1], [2], and [3]. She then considered one of these elements, [1], and led the class in considering what happens when two elements of [1] are added together.

Dr. Bond: So what happens when I take two elements of the [1] subset and I add them together? What do I get? One plus five gives me? Six, and where does six live?

Class: In [2]

Dr. Bond: Uh, negative three plus five gives me?

Mark: Ah two.

Dr. Bond: In fact I think you find that you always end up in the same place. Now, I think, why are we always ending up in [2]? … It makes sense that, now that we think about it, that one plus five shouldn't get me back to the identity. It also makes sense that R times R^3 should get me back here[7]. Because what do we know about this set? What's the order of it? Who's its inverse?

Class: Itself

Dr. Bond: It's its own inverse. I just had that ah-ha. Because I was thinking about it, I actually thought it might work in Z_n. I hadn't worked it out yet. But it just kind of occurred to me when these all started ending up in [2]. It was like, oh well-duh, because one plus one is two. And that's what's going on there. Those elements were all of order two, so when you multiply them with themselves you're supposed to get the identity back. So that is a pattern that we noticed, but unfortunately it isn't going to help us.

Mark: It's just for elements of order two.

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[7] Here R and R^3 represent elements in D_8 (and $R\,R^3$ is the identity). These two elements made up one of the quotient group subsets that Mark drew on when making his observation.

We see that during this discussion both Dr. Bond and her students were considering Mark's observation in this new context. Initially Dr. Bond appeared to be surprised that the sums of both pairs of elements from [1] were elements of [2]. One explanation for Dr. Bond's surprise is that, up to this point, the quotient group operation had been defined in terms of sets as opposed to representatives (see Larsen & Lockwood, 2013). Given the class's current view of this operation, the sum of [1] and [1] is defined as the set obtained when each element of [1] is added to each element of [1]. This is contrasted with an operation defined by representatives, where the sum of [1] and [1] can be determined by taking any representative of [1] and adding that to any representative of [1]. Reconciling this unexpected result appears to have led to Dr. Bond's "ah-ha", as we see with her comment "it just kind of occurred to me when these all started ending up in [2]. It was like, oh well-duh, because one plus one is two". Mark initially noticed that every pair of elements of each subset combined to produce an element of the identity subset. As Dr. Bond investigated this observation, she eventually noticed that it was equivalent to each subset being a self-inverse in the quotient group.

Therefore, in this episode we see students[8] engaging in conjecturing and generalizing. In response to these student mathematical activities, Dr. Bond led the students through an investigation of the mathematics and in doing so was able to figure out a connection that she did not initially recognize. During the exploration of this observation in Z_4, it is clear that Dr. Bond was testing a mathematical observation to check for validity. Further, in

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[8] Throughout the paper I will make claims about *students'* mathematical activity as opposed to the mathematical activity of a few students. While Mark's is the primary student voice presented in this episode, the whole class did similar work in their small groups and contributed to whole class discussion. For clarity, the voice most central to the episode being discussed was presented.

order to lead this investigation, Dr. Bond had to first translate Mark's original

observation, "the combination of the two elements always gives you an element of the

identity set", into a new context, Z_4. While the students had been introduced to Z_n in their

current homework assignment, this was still a relatively new group for the class (and not

one they had worked with as a quotient group). Therefore, both the Z_4 context and the

translation of Mark's conjecture represent a teacher-induced change in the mathematical

discourse of the classroom.

Following this interaction the students' mathematical activity underwent a substantial

change. Once it was shown that Mark's observation failed to generalize, Dr. Bond asked

for other ideas. This time Mark volunteered the identity property to help determine which

element of D_8 would need to be paired with R (in order to form a subset that would serve

as an element of a quotient group), if the identity element of the quotient group was $\{I,$

$R^2\}$. Dr. Bond took this suggestion as an opportunity to introduce a coset formation

algorithm.

Dr. Bond: So okay, we are back to the drawing board. So we know that we are going to use this as the identity [underlines $\{I, R^2\}$] and we know that we are going to have some other subset out there, I'm calling it A, and I am going to say R is in it. How can I figure out who can be paired with R? What else do we got going on here? What do we know how to do?

Mark: If we multiply it by the identity two different ways it has to get back to itself.

Dr. Bond; We know that A times I has to equal A and so does I times A. Right? [writes down $A*I = A =I*A\}$]. So let's see if we can make that help. So I am looking at A times I. I am going to have R times I and R times R^2 and then blank times I and blank times R^2 [see Figure 5]. So I'm going to get R back, so that's good. And I am going to get R^3. So I have to write that down right? And then what am I hoping at this point? I still have to fill in this blank but in a way I did, right? In this case I am only supposed to have a two element set. So right now I am conjecturing that R^3 is going to be R's

buddy. So if I put R^3 here [in the blank] do I get R and R^3 back? Well, R^3 and I is R^3 and R^3 and R^2 is R.

After working through this method for determining which element must be paired with R,

Dr. Bond then modeled this procedure in two additional examples before defining cosets.
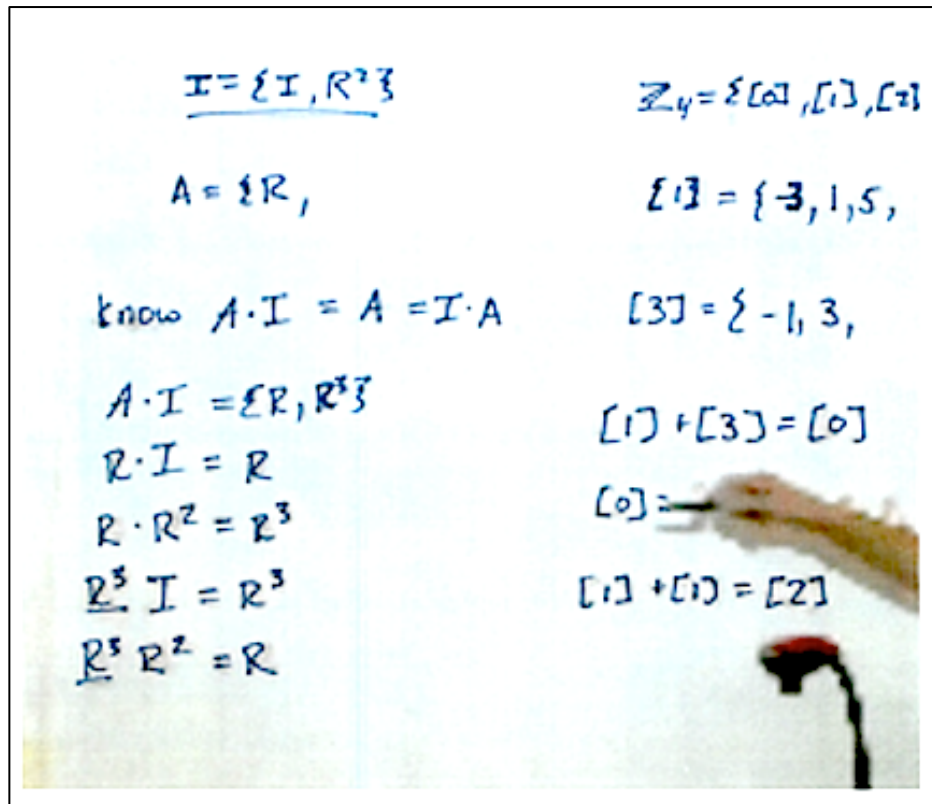
Figure 5. Dr. Bond's coset formation

Notice that this episode began with students conjecturing and generalizing and concluded with Dr. Bond presenting the students with a new algorithm. Therefore, by the end of the episode, the students' mathematical activity was reduced to the "mathematical activity of reproducing what is being offered" (Freudenthal, 1991, p. 15). While this transition from active to passive student participation may appear as a negative

consequence of Dr. Bond's mathematical and/or pedagogical activity, it is important to note that when implementing an inquiry-oriented curriculum there is an inevitable tension between remaining faithful to the students' mathematics while still advancing the mathematical agenda.

> There will always be tension between a bottom-up approach that capitalizes on the inventions of the students and the need, (a) to reach certain given educational goals, and (b) to plan instructional activities in advance… this boils down to striving to keep the gap between 'where the students are' and what is being introduced as small as possible. (Gravemeijer & Doorman, 1999, p. 124-125)

While Dr. Bond's introduction of a coset formation algorithm was in response to Mark's suggestion to utilize the identity property, the TAAFU design team felt that the gap between where the students were and what was being introduced could have been decreased. As a result, a new task designed to support students in building on Mark's idea was developed and added to the instructor support materials (for a discussion of how this episode informed the instructor support materials see Lockwood, Johnson, & Larsen, 2013).

Throughout this episode Dr. Bond's mathematical activity was responsive to her students' mathematical activity. First Dr. Bond responded to Mark's observation ("the combination of the two elements always gives you an element of the identity set") by exploring this observation in a new setting. Then, Dr. Bond used Mark's contribution ("if we multiply it by the identity two different ways it has to get back to itself") as a way to introduce a coset formation algorithm. However, while Dr. Bond's activity was responsive to her students' activity and contributed to the mathematical discourse of the classroom, it did not provide instructional space for the students to engage in new

mathematical activity because Dr. Bond carried out the mathematical investigations herself.

3.2. Episode 2 – Proof Based on Non-Equality

In this episode Dr. Bond allowed student questions about the validity of a student-generated proof to guide the trajectory of the lesson. As a result, Dr. Bond was able to gain insight into the student's argument and provide the class with a new tool for argumentation. This episode took place during the deductive phase of the group unit of the TAAFU curriculum as the students worked to prove that, given group elements a and b, if the order of b is 4 and $ab = b^3a$ then $ab^2 = b^2a$. After the students were given a chance to work alone, Dr. Bond asked for volunteers to share their proofs. A student, Tyler, presented a proof by contradiction to the class. In this proof Tyler assumed that $ab^2 \neq b^2a$ and was able to deduce that $ab \neq b^3a$. However, this relied on the fact that if two expressions that are not equal ($b^3ab \neq b^2a$) and they are both multiplied on the left by the same element, then the resulting expressions are still not equal ($bb^3ab \neq bb^2a$). Dr. Bond's inscription of Tyler's proof is shown in Figure 6.
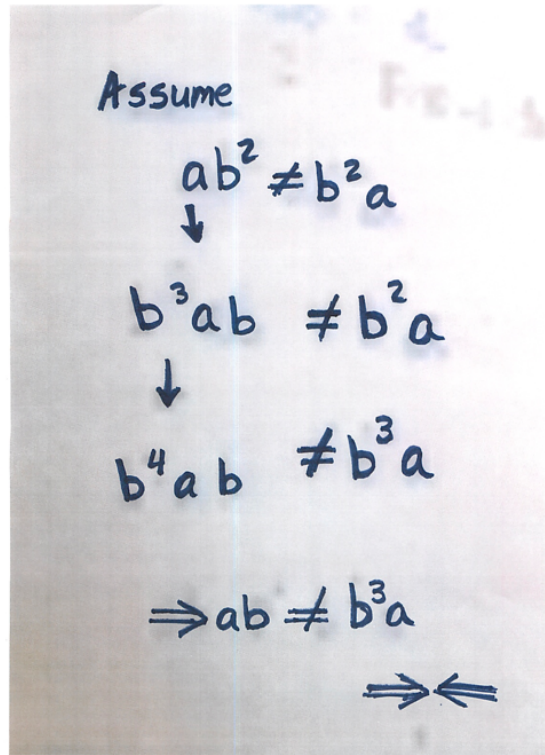
Figure 6. Dr. Bond's inscription of Tyler's proof

The class session ended shortly after Tyler's proof, but many students questioned the validity of his proof in a written reflection Dr. Bond collected at the end of class. In these "exit cards" students questioned whether it was valid to assume $bb^3ab \neq bb^2a$ based on the fact that $b^3ab \neq b^2a$. The students were right to question the validity of such a step because this does not work outside of the context of a group. For instance, we cannot assume that $4 \times 0 \neq 2 \times 0$ based on the fact that $4 \neq 2$.

Using these concerns to guide the trajectory of the course, and therefore engaging in *generative listening* (Yackel et al., 2003; Johnson & Larsen, 2012), Dr. Bond began the next class by addressing this concern.

> *Dr. Bond*: The other comment that came in, and I had the same thought after this was up on the board but we were kind of out of time so I made a choice not to look into it,

73

but someone made a comment about the second proof that we did. We had a proof by contradiction and someone raised the question that, in this step… someone asked the question, if we take two things that are not equal and we multiply them on the same side by something that is the same, do we know that we are still not equal?

Keith: No

Dr. Bond: Yeah, we don't. … Anyway, it is something. I don't know, we don't necessarily need to figure it out, the answer to the question. But it is an important question to ask.

Notice that initially Dr. Bond stated that she was unsure if this was a valid step and that the question did not need to be resolved; instead she just wanted to make sure that the students were aware that "this is an important question to ask". During the debriefing meeting following this class, Dr. Bond admitted that she planned on leaving this question unresolved and that she, "hadn't decided if it was valid or not … I really hadn't thought it through yet".

However, as she was sharing this concern with the class, Dr. Bond gained insight into the justification of the step in question by connecting the student's proof to a previously established result, if $ab = ac$ then $b = c$.

Dr. Bond: We don't know if our same rules apply. So when you do something like that you need to take the moment to stop and think, if I know these two things are not equal and I multiply them on the left by the same thing, do I know that the not equal is preserved? And my gut at the moment is, we'll let's think about it for a second. What is our, our cancelation property says that if ab equals ac then b equals c, right? And what was the contrapositive to this?

Rachel: If b does not equal c

Dr. Bond: [Writes on the board if $b \neq c$ then $ab \neq ac$] Does that help? When we're at this step, we're at b does not equal c. And then down here we have this a times b, we have something times what we already had, equals a times c. So, is that a valid conclusion to make? That this is still not equal? Yeah, I think it is. I think it does hold.

74

Having made the connection between the steps in Tyler's proof and the cancelation law, Dr. Bond was then able to verify the steps of the student's proof with one minor contribution from Rachel. Given Dr. Bond's debriefing statement, it is clear that this result was not knowledge that she carried with her into class. Instead, Dr. Bond's decision to present this hidden assumption to the class provided an opportunity for Dr. Bond to gain insight into its justification. As a result, Dr. Bond was able to spontaneously justify the troublesome step in Tyler's proof.

When considered in terms of mathematical activity, this episode began with students proving and, given their reflections following class, engaging in proof analysis (in which students analyzed Tyler's proof and identified a hidden assumption). Dr. Bond responded to these two forms of student mathematical activity by sharing the students' concerns with the class. As she explained these concerns, Dr. Bond became aware of a connection between Tyler's proof and the cancellation law. In that way, we see Dr. Bond's pedagogical decision to share the students' concerns as *generative* (Yackel et al., 2003) in that it allowed for new opportunities for mathematical activity.

Further, while Dr. Bond's verification of Tyler's proof effectively concluded the classroom's proof analysis activity, her activity did provide an opportunity for students to subsequently engage in questioning. Following the proof verification, Dr. Bond reiterated that the form of Tyler's proof (i.e., a proof by contradiction in which non-equality is the basis for argumentation) was different than typical proofs that rely on statements about equality. As Dr. Bond explained, "there is a pivotal difference… instead of dealing with equality, which we know is a balance and we keep balanced, we are now dealing with not equal which is a different way to compare." By addressing her students' concerns over

Tyler's proof and highlighting the differences between Tyler's approach and more typical proof techniques, Dr. Bond introduced a new technique for mathematical argumentation into the classroom's mathematical discourse. In response to Dr. Bond's contribution, her students began to negotiate the use of this new technique as they engaged in questioning and example generation. For instance, one student, Ted, asked if it would be valid to only multiply one side of an non-equality by a group element, "If we're using a non-equal sign, can we just do, could we just multiply on one side if we wanted?" Ted's proposed scenario prompted Tyler to generate an example in which it would be valid. "If you knew you were in the triangle and you had R on one side and R^2 on the other, then could you do F to one side?"[9]

Dr. Bond's mathematical activity both responded to the mathematical activity of her students and introduced a new argumentation technique into the classroom's mathematical discourse. By validating Tyler's proof technique and addressing the ways in which Tyler's argument is different from the class's previous work, Dr. Bond introduced a new tool for mathematical argumentation: the idea of maintaining non-equality as a way to construct a proof by contradiction. In response to the introduction of this new tool, Dr. Bond's students began to negotiate its use by generating examples and questioning when it was valid.

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[9] Of course, in general it is not true that non-equality will be preserved if only one side of a non-equality is multiplied by a group element. For instance, $2 \neq 6$ but $2 \times 3 = 6$.

3.3. Episode 3 – Identity and Inverses in a Subgroup

In this episode Dr. James tracked the development of his students' emerging understanding of identities and inverses during a whole class discussion. By analyzing his students' contributions, Dr. James was able to identify a key issue to be addressed and, guided by his own mathematical activity, tasked the students with addressing this issue. This episode began with Dr. James introducing the idea of subgroup by proving that *5Z* under addition is a group in which every element is a member of a larger group, *Z*, under addition.

After working through the proof that *5Z* under addition is indeed a subgroup, Dr. James' class then started to conjecture about the minimal list of criteria needed to ensure that a subset of a group is a subgroup. After giving the students time to work in their small groups, Dr. James asked the groups to report observations or conjectures. One student, Sam, observed that the identity of the group would still act as the identity of the subset. Thus, it would be sufficient to show that the identity of the group is present in the subset, as opposed to proving that the identity of the group satisfies the identity property in the subset. Similarly, Sam observed that the inverse of any element in the group would still be an inverse of that element in the subset. Thus, it is sufficient to show these inverses are present in the subset, as opposed to showing that these elements satisfy the inverse property. At this point, neither of these observations were proved, instead Dr. James was focused on collecting conjectures and observations that the groups would later be given the opportunity to prove.

Shortly after Sam's observation, another student conjectured that it was not necessary to check if the identity was in the subset. Bryan conjectured that, in order to know that a

subset of a group is a subgroup, it is enough to check that: 1) for every element in the subset, the inverse of that element is also in the subset, and 2) that the subset is closed under the operation of the group. Bryan argued that, if the subset contains the inverse element of every member, then closure will guarantee that the identity element is also a member of the subset (since when you combine any element with its inverse you will get the identity).

> *Dr. James*: So, any element with its inverse, when multiplied together gives you the identity, yeah. That's from closure and from the definition of inverses.
>
> *Bryan*: And if the identity isn't in your subgroup, then closure would be wrong or if you get a different identity then your group would be wrong.
>
> *Dr. James*: So that's kind of the general thinking. It's not a full-blown proof. But, if you just check closure and you just check the existence of inverses then it seems like maybe that will be enough to get that the identity to be in the subgroup. So, that's an interesting idea that may in fact save us an axiom.

Even though this is not the standard subgroup theorem, it is a common and true[10] conjecture that students come up with while working through this curriculum (see Larsen & Zandieh, 2007). However, following this conjecture, the classroom discourse again turned to the identity of the subgroup.

> *Billy*: How do you check inverses without knowing the identity?
>
> *Eric*: He's saying check the original.
>
> *Billy*: I'm not sure.
>
> *Dr. James*: So, talking about checking inverses for… ok, so it's a good point, the identity of the whole group verse the identity of the subgroup.

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[10] Technically, of course, one must also assume the subset is non-empty.

Bryan: I said that the identity of the subgroup has to be the identity of the group, because if they differ then you have two identities in the main group[11].

Dr. James: Ok, this group was harping on these two things as well. This is probably something we should talk about, clear the air on, and then forevermore be happy about. If you have a subgroup, is it or isn't it, let's actually detour for just a moment. Let's take a few minutes and try to write, everyone, some sort of a proof or something, that if you have a subgroup inside of a group, true or false, the identity of the subgroup has to equal the identity of the group. And, if you can crank that out then consider the second question. Does the inverse of an element in *H* then have to be equal to the inverse of the same element in *G* [where *H* is a subgroup of *G*]?

In this exchange we see Billy questioning another student's justification for this conjecture – that if the subset is closed and contains the inverse of each element then the subset will contain the identity element. Billy's challenge is based on the fact that inverses are defined in relation to an identity element. So, without knowing that the subset contains an identity element how could you know that the subset contains the inverse of each element. To this challenge Bryan provided additional support, stating that the identity of the subgroup must be the same as the identity of the group.

In terms of student mathematical activity, this episode includes conjecturing, justifying, and evaluating arguments. In response to his students' mathematical activity and contributions, Dr. James presented the class with two questions to be proved or disproved: 1) Does the identity of a subgroup have to equal the identity of the group that contains it?, and 2) Does the inverse of an element in a subgroup have to be equal to that element's inverse in the group? On the surface, these tasks appear to be a natural extension of Bryan's comment that the identity of the subgroup has to be the identity of

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[11] This is incorrect since, by itself, the statement that an element is the identity of a subgroup does not include a claim that it acts as the identity for elements outside the subgroup.

the group and of Sam's observations that the identity and inverses from the group will act as an identity and inverses within the subgroup. However, during a stimulated recall interview (Schepens, Aelterman, & Van Keer, 2007) with Dr. James in which he was asked to watch the classroom video data and respond to what he saw, the mathematical activity that supported the assignment of these two tasks came into focus.

> Estrella: They were concerned about identity. Why did you assign identity and inverse? Do you have any sense for that now looking back?
>
> Dr. James: Um, okay so, it's probably been on my mind since Sam's initial contribution to the conversation because the two original suggestions… I did pick up on Billy's concern about is the identity for the whole group the same as the identity for the subgroup… So, maybe just having this resolved for both for the identity and for the inverses would allow us to go on. I'm hoping that if there are lingering doubts like Billy's out there, this would be good for them to deal with first. So yeah, I think I inserted the inverse thing.
>
> Estrella: So my other question is, so you asked them to prove that the identity of the subgroup is the same as the identity of the whole group. Do you find that statement to be mathematically equivalent to what Sam originally said, that you only have to check that the group's identity is in there?
>
> Dr. James: No, no I think that it is stronger and you could get his [Sam's] as a consequence of this. So I think that this is a slightly more general observation. That, yeah I am just granting and hoping, with knowledge that this discussion will move forward in a more efficient way. Like you could then deduce that Sam's contribution was correct, you could probably state the point that this student [Bryan] was making a little more succinctly, etcetera. It just seemed like maybe a good time to kind of address that issue.

Therefore, the assignment of these two tasks can be understood as an artifact of Dr. James's mathematical activity. As the classroom discussion unfolded, Dr. James exhibited a sophisticated and nuanced understanding of identities and inverses in order to track the development of his students' ideas throughout the classroom discussion. Initially, Sam's contribution merely stated that the identity and inverses of a group would

still function as an identity and inverses within a subset of the group. This discussion about identities and inverses then evolved with Bryan's argument that closure and inverses would guarantee the existence of the identity within a subset and Billy's challenge that claims about inverses are invalid without first identifying the identity element. Dr. James was able to follow this mathematical progression and determine that all three of these student contributions were related to, and could be addressed by, the fact that the identity and inverses elements in a subgroup must be the same as the identity and inverse elements of the group. Specifically, Dr. James was able to determine the key bit of mathematical knowledge that would allow his students to 1) deduce Sam's observation, 2) state Bryan's conjecture without ambiguity, and 3) resolve Billy's concern about inverse elements only being defined in terms of a known identity element.

By assigning the two tasks to the class, Dr. James was able to provide structure to the emerging mathematical understanding and the ideas that his students were raising. From the students' perspective, Sam's observation, Bryan's conjecture, and Billy's concern were somewhat disconnected, as they did not have the mathematical background to place these ideas in a broader mathematical landscape. Dr. James's mathematical activity, and subsequent pedagogical decision to pose these tasks to the students, represented a significant contribution to the mathematical discourse of the community and provided his students with an opportunity to engage further in proving. As a result, Dr. James's mathematical activity, and corresponding contribution to the mathematical discourse, supported subsequent student mathematical activity.

3.4. Episode 4 – Uniqueness of the Identity

In this final episode Dr. Bond co-constructed a proof with her class, and then provided her students the opportunity to compare proofs. This episode took place as part of the deductive phase of the group unit, in which Dr. Bond's students were asked to prove or disprove that the identity of a group is unique. As students worked on this task, Dr. Bond circulated the room and briefly interacted with most of the groups. The majority of Dr. Bond's interactions with these groups were primarily evaluative in nature. Dr. Bond quickly read or listened to student arguments and responded by saying "good". There were two exceptions. With one group of students Dr. Bond discussed what it means to be equal when working with groups that have equivalence classes as elements. With another group of students Dr. Bond determined that the group had a valid proof and then asked them to consider if the inverse of a group element was unique.

Group work came to a conclusion with Dr. Bond asking a student, Amos, to share his proof with the class. As Amos described his steps, Dr. Bond interjected repeatedly. In this way Dr. Bond used Amos's proof to structure an interactive lecture. This interactive lecture served as a way for the class to collaboratively construct a proof and to highlight standard proof conventions. The first part of the interactive lecture focused on the logic of Amos's argument and the second half addressed student questions about the format of the proof.

> *Dr. Bond*: All right, let's come together. I think most, if not all the groups, have come up with at least one way to prove this. I have seen two ways as I have gone around the classroom so let's try to share these arguments. Let's see, and um, Amos, can we start with yours? What you just shared with me.

> *Amos*: Suppose there exists two identities, I_1 and I_2.

82

Dr. Bond: And so, what do we know about them if they are identities? What does it mean when he says, "suppose there exists two identities, I_1 and I_2"? What is the one thing that we know? [Pause] What does it mean to be an identity?

Sarah: For any element a, multiplied by the identity

Dr. Bond: [As she writes $I_1a = aI_1 = a$ and $I_2a = aI_2 = a$] So for every a in G, I_1 times a equal a times I_1 equals a, right? And, I_2 times a equals a times I_2. You know, I'm writing this out just because, this turns out to be a fairly simple proof. Right, it comes pretty fast. Most of the groups got it pretty fast. But when you do something longer it's these kinds of steps that can really help you see. So, like we know what it means to be an identity. But this method of, okay I say it's an identity and then writing out what it means in that formal language can sometimes help you to see what to do next. Okay, so Amos, what did you do?

Amos: Also, I don't know if it would be necessary, but I wrote that I_1 and I_2 don't equal each other.

Dr. Bond: Oh yeah, I should put that up here. So let's assume that they are different.

Amos: And then, using the definition, I went ahead and let a equal I_2.

Dr. Bond: [As she writes $I_1I_2 = I_2$] So he is saying, so take this one [referring to the equation $I_1a = aI_1 = a$] and he is going to plug in I_2 for a.

Amos: And then just do the same for I_2.

Here we see that, initially, Dr. Bond was acting to mediate between the class and Amos's small group. Accordingly, Dr. Bond spaced out the steps of Amos's proof in order to provide additional information and checked to ensure that other students were following his logic. Amos's modified proof is shown in Figure 7.
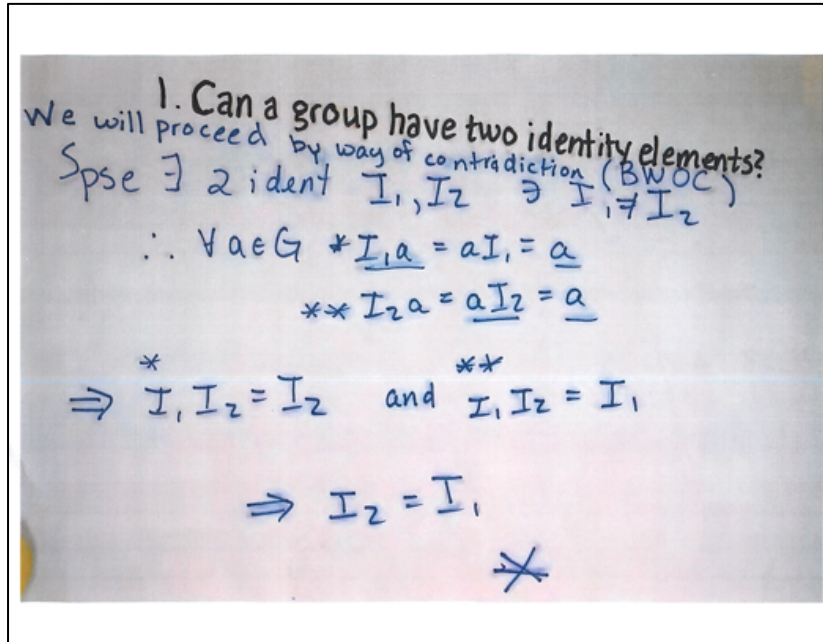
Figure 7. Amos's modified proof

Following the construction of this proof, students began to question the form of the argument and the assumptions made. Responding to a students' observation, Dr. Bond used this as an opportunity to talk about different approaches to this proof.

Richard: Now do we have to include the top there, where we say that I_1 is not equal to I_2?
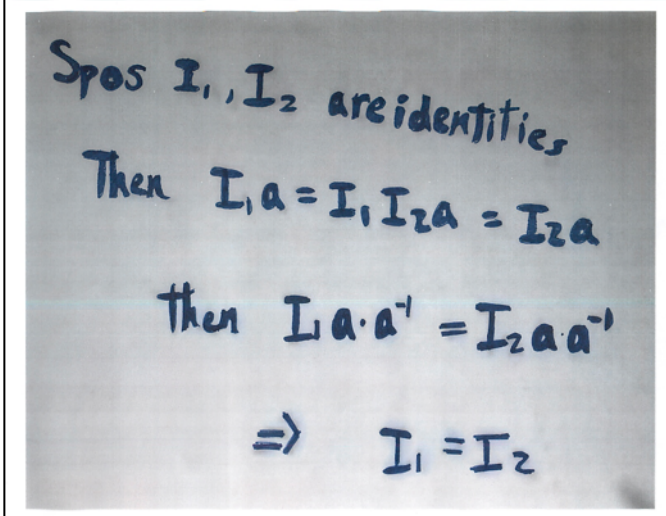
Keith: That is a contradiction, a proof by contradiction.

Dr. Bond: The way it is right now, Keith is right, you set it up to be a proof by contradiction. And so, often when you prove by contradiction, you assume the thing that you are trying to prove is false. We are trying to prove that the identity is unique so we assume that it is not… A lot of times in the beginning of the proof you will say something like "we will proceed by way of contradiction"… You don't have to use contradiction. But when you add this "such that I_1 doesn't equal I_2" you set it up to be a contradiction proof. And then right here you would say "contradiction"… there is a symbol in mathematics that means contradiction. And then you would say, "therefore I_1 equals I_2" because that's what you contradicted. But you do not have to use contradiction in this proof. Erase this first line and cover this one up. Suppose there are two identities I_1 and I_2, blah blah blah blah blah, we find out that they are equal to each other. That in itself proves that it is unique. Adam?

Adam: Is there something wrong with not introducing *a* and just pitting I_1 and I_2 against each other?

Dr. Bond: No, and he [Amos] didn't introduce *a*. I put it there to emphasize, all I am doing here is restating my assumption that I have identities. So, the *a* is not something I am using in my proof, but I put it there to remind myself what it meant to be an identity… There were other ways to do this. I know I saw, did you guys do something different?

Dr. Bond then solicited alternate proofs from the class. Two additional proofs were shared with the class, one by a student and one by Dr. Bond. Both proofs relied on multiplying a group element by its inverse. The proof Dr. Bond presented is shown in Figure 8. Dr. Bond commented that both of these proofs would need additional justification because if there were two identity elements then the product of an element and its inverse would not be well defined.

Figure 8. Dr. Bond's unique identity proof

Dr. Bond: You run into that same kind of glitch… because I know a times a^{-1} equals an identity but I don't know which one.

Amos: Can you do it by cases?

Dr. Bond: You probably could. You could, I mean that would be a way to really nail it down. But then it starts to make you appreciate, I didn't really have a preference for proofs, but you start to appreciate this one [Amos's modified proof] because we never had to use inverses. This one was kind of nice and slick so we could avoid that issue all together. But I do think that that issue could be dealt with.

Keith: Operations are one-to-one right?...Then there could only be one output for a times a^{-1}.

Dr. Bond: That's not one-to-one. They're functions. So, a times a^{-1}

Keith: Only has one output and it's the identity. Therefore, in order for it to be one-to-one there can only be one identity.

Dr. Bond: It is the identity, otherwise it would be more complicated. It would be an identity. It couldn't go to both, so it would go to a specific one. And then Amos is right, we wouldn't have to know which one it went to, because we could do a case-by-case… There are two cases right, so it is either going to go to I_1 or I_2. We could do two cases and make our argument. The thing is, is that once you, so say it goes to I_2. Once you recognize that I_1 times I_2 is still I_1 you are kind of back to this argument [Amos's modified proof].

Dr. Bond initiated the discussion of these two additional proofs by correctly observing that they shared a problematic aspect; namely that if one assumes that a group has more than one identity then it is unclear what the result will be when one multiplies an element by its inverse. Amos suggested that this issue could be addressed by a proof by cases. Dr. Bond did not notice the flaw in this approach. (Namely that the proof actually begins with the assumption that there are at least two identities, not the assumption that there are exactly two. So one cannot proceed by cases because it is unknown how many identity elements there are if there are more than one.) However, Dr. Bond did notice that this approach would still require the same kind of argument that

appeared in Amos's modified proof (the fact that the product of two identity elements can be taken to be either of those elements). Additionally, Dr. Bond noticed the error in Keith's reference to the fact that operations are one-to-one. It is really well-definedness that guarantees that there can only be one product of a and a^{-1}. Note that Keith does not seem to notice her observation and the issue was not discussed further.

In terms of activity, we can see that this episode began with students engaging in proving. Given her role in the classroom community, Dr. Bond was tasked with responsively utilizing the students' mathematical activity as a way to advance the mathematical agenda. As the class worked in small groups to prove that the identity element of a group is unique, Dr. Bond circulated the room quickly evaluating the mathematical arguments of her students. Then, after assessing and selecting Amos's proof, Dr. Bond drew on Amos's proof to co-construct a proof with the classroom community. Finally, Dr. Bond solicited other proofs from students, engaged in proof analysis (Larsen & Zandieh, 2007) to identify a hidden assumption, and contrasted these proofs with the proof that the class co-constructed. Thus, it is evident that in this episode Dr. Bond responded to the students' mathematical activity with her own mathematical practices. Dr. Bond's observable mathematical activity included: 1) evaluating students' proofs as they worked in small groups, 2) co-constructing a proof with the class, and 3) soliciting, analyzing, contrasting, and connecting alternative student proofs.

Further, as Dr. Bond collaboratively re-constructed Amos's proof, she modified and expanded his argument. Dr. Bond's modification is evidenced by her statement, "no, and he [Amos] didn't introduce a. I put it there to emphasize, all I am doing here is restating my assumption that I have identities". By co-constructing a modified version of Amos's

proof with the class, Dr. Bond served as a mediator between Amos's small group and the classroom community. As a result, Dr. Bond contributed an alteration to the classroom community's mathematical discourse – an alteration adopted and expressed by Amos in his comment "And then, using the definition, I went ahead and let a equal I_2". Additionally, Dr. Bond's students both contributed to the proof and questioned its form and assumptions. Therefore, Dr. Bond's students actively responded to her contributions to the classroom's mathematical discourse.

Dr. Bond's mathematical activity supported her students' subsequent mathematical activity. By co-constructing a modified version of Amos's proof with the class, as opposed to simply presenting a preplanned proof, Dr. Bond provided an opportunity for students to continue to engage in proving. Further, Dr. Bond's major modification to Amos's proof, explicating the role of the identity property in his argument, highlighted the fact that group theory is entirely based on the definition and axioms of a group. Dr. Bond explicitly addressed this fact during the whole class discussion.

> Dr. Bond: The reason I put this here is to emphasize why I can write these two things down... I do want to see that connection there. I want you to tie it to the rule. Because we don't have very many rules and I really want to emphasize how we are going to build these theorems up from our rules. Like the properties of a group, that's all we got.

Additionally, the co-construction of this proof provided an opportunity for Dr. Bond's students to engage in questioning and proof comparison. This is evident in both Richard's question, "Now do we have to include the top there, where we say that I_1 is not equal to I_2?" and Adam's question, "Is there something wrong with not introducing a and just pitting I_1 and I_2 against each other?" Finally, by soliciting and presenting alternative

proofs, Dr. Bond was able to highlight an ambiguity in the arguments (i.e., without knowing that a group has a unique identity there is an ambiguity regarding the product of inverses). Drawing explicit attention to this flaw allowed an opportunity for Dr. Bond's students to explore possible avenues to resolve the problematic issue.

4. Conclusions

As stated, these four episodes were selected to 1) illustrate some of the complex mathematical activity that mathematicians engaged in while implementing the TAAFU curriculum, and 2) identify ways in which teachers' mathematical activity may interact with and influence students' mathematical activity. Here I will discuss the four episodes in terms of these two research goals.

4.1. Teachers' Mathematical Activity

As Dr. Bond and Dr. James implemented the TAAFU curriculum, and responded to their students' mathematical activity, they engaged in various forms of mathematical activity themselves. In the four episodes presented here, there is evidence of teachers: interpreting students' mathematical reasoning and contributions; analyzing and evaluating students' mathematical contributions, conjectures, and arguments; and identifying mathematical connections, both between multiple student contributions and between student contributions and known mathematical results.

In each of the episodes, the teachers' were able to successfully interpret and make sense of their students' contributions. As such, the teachers engaged in interpretive listening, in which their intent was to "make sense of the sense they [the students] are

making" (Davis, 1997, p. 356). In fact, this was the first mathematical task carried out by the teacher in each of the four episodes. For instance, in order to test Mark's contribution in the first episode, Dr. Bond first had to interpret his imprecise language. Additionally, in the third episode, in order to determine the key bit of mathematical knowledge needed to resolve his students' concerns, Dr. James first needed to make sense of what they were saying. While interpretive listening may seem like a routine task for teachers, Johnson and Larsen (2012) found that it can be quite challenging and that a mathematician's ability to make sense of student contributions appears to be tied to their Knowledge of Content and Students (Ball et al., 2008).

Another mathematical activity featured in each episode involved the mathematicians analyzing and evaluating student conjectures and contributions. As trained mathematicians, the teachers considered here certainly had expertise in analyzing and evaluating mathematical statements. It is likely that this expertise influenced the mathematical activity they engaged in while teaching. For instance, in Episode 1, Dr. Bond chose to translate Mark's contribution into Z_4. While this may not have been a fruitful context for the students (as they may not have been thinking of Z_4 as a quotient group at this point in the curriculum), it was a prudent choice mathematically for evaluating this contribution. Mark's contribution arose as he worked to generalize from a specific four-element quotient group, one in which each element was its own inverse. By selecting Z_4 to test the contribution, Dr. Bond selected a group that was similar to the one Mark was working with (a four-element quotient group) while also being structurally different (Z_4 is cyclic with only two self-inverses).

Finally, in each episode the mathematicians engaged in mathematical activity that resulted in the identification of mathematical connections. During the first episode, Dr. Bond was able to connect Mark's contribution to what it meant for a quotient group element to be a self-inverse. In the second episode Dr. Bond became aware of a connection between the argument used in Tyler's proof and the cancellation law. In the third episode, Dr. James was able to determine the key bit of mathematical knowledge that connected (and resolved) Sam's observation, Bryan's conjecture, and Billy's concern. Then, in the forth episode Dr. Bond solicited alternate student proofs in order to compare and contrast alternative arguments with her students. Interestingly, it is only in the last episode that finding such connections was clearly the intention of the teacher's mathematical activity. In the other three episodes it appears that the connections were the result of evaluating a student contribution, engaging in proof analysis, and interpreting student contributions respectively. This suggests that the explicit discussion of mathematical connections may be good evidence for other kinds of teachers' mathematical activity.

Given that these episodes were selected specifically because of the complex mathematical work being done on the part of the teacher, more research certainly needs to be done before claims can be made about how often teachers need to engage in this kind of mathematical work. However, these episodes begin to shed light the variety of mathematical activities that teachers may need to engage in as they respond to the mathematical activity of their students.

4.2. Influence of Teachers' Mathematical Activity

The second goal of this research was to identify ways in which teachers' mathematical activity may interact with and influence students' mathematical activity. My analysis focused on two ways in which the teachers' activity may impact the students' activity. First, I assessed the extent to which teachers' mathematical activity constrained or supported subsequent student mathematical activity. Second, I assessed the extent to which the artifacts of teachers' mathematical activity (such as new mathematical tools, objects, or processes) contributed to the classroom community's mathematical discourse.

4.2.1. Subsequent Student Activity

If one takes *mathematics as an activity* as the primary point of departure for mathematics education, it becomes of utmost importance for teachers to support their students' mathematical activity. My analysis suggests that teachers' mathematical activity generates and supports shifts in student mathematical activity. However, not all of the episodes described here reflect the same level of support for students' subsequent mathematical activity.

In the second episode, Dr. Bond's realization that the preservation of non-equality under multiplication followed from the cancellation law (by contraposition) did not promote subsequent mathematical activity on the part of her students. This was also true of her realization that Mark's observation (described in the first episode) was related to the order of the elements in the quotient group. In each case, this can be explained by the fact that this activity unfolded publicly in real time so that Dr. Bond ended up doing the

mathematics for the students. In both of these episodes Dr. Bond's mathematical activity, while responsive to her students' mathematics, did not leave instructional space for her students to engage in new mathematical activity. However, it was not necessarily Dr. Bond's mathematical activity in these episodes that constrained subsequent student mathematical activity. The same mathematical activity, if enacted differently pedagogically, could have had a different impact on subsequent student mathematical activity. For example, in the first episode, Dr. Bond could have had the students construct Z_4 in the way they had constructed the partitions of D_8. The students could have then been asked to consider Mark's observations in the new context. It is worth noting that there can be good reasons to decide not to open up a new avenue for student investigation. Perhaps more pertinently, there is one very positive aspect of Dr. Bond's mathematical and pedagogical activity in these two instances. In each case, she can be seen as an active participant in the mathematical investigation, thereby positioning herself as a co-investigator with the students – a role that one would expect to have a positive impact on the classroom culture.

In the third and fourth episodes, the teachers' mathematical activity served to generate new opportunities for student mathematical activity. In third episode there is a direct connection between Dr. James's mathematical activity and subsequent student mathematical activity, in that Dr. James's activity resulted in the assignment of new tasks that explicitly required the students to engage in proof. Additionally, in the forth episode, Dr. Bond's co-construction of a modified version of Amos's proof provided an opportunity for students to continue to engage in proving. Then, in responding to this newly constructed proof, Dr. Bond's students engaged in questioning and proof

comparison. Finally, after being presented with alternative proofs that involved an ambiguous product of an element and its inverse, Dr. Bond's students explored possible avenues to resolve this problematic issue. In this way, Dr. Bond's mathematical activity supported and promoted student mathematical activity.

The disparate effects of teacher mathematical activity on their students' mathematical activity raise further questions for research. For example, one could investigate how a teacher's mathematical and pedagogical activity collectively support or constrain subsequent student mathematical activity. Such an investigation could provide important insights into the complexity of inquiry-oriented instruction.

4.2.2. Contributions to the Mathematical Discourse

One focus of analysis was the extent to which the teacher's mathematical activity contributed to the classroom community's mathematical discourse. Such contributions include the introduction of new mathematical objects or processes, new mathematical tools, or refined questions to be investigated. These contributions to the classroom discourse are of particular interest because, if adopted by the students, they have the potential to shape the mathematics available to the classroom community. Here I will discuss contributions made by the teacher, and the extent to which the students latched onto these contributions.

In the first episode Dr. Bond's mathematical activity resulted in a contribution to the mathematical discourse in the form of a new example to consider. However, the students' reaction to this contribution was rather limited. In this episode Dr. Bond's decision to explore Mark's conjecture in Z_4, a relatively new context for the students, had the

94

potential to expand the classroom's quotient group example space (Watson & Mason, 2005) if the students had been allotted time to consider whether Z_4 is in fact a quotient group. Further, Dr. Bond's introduction of a coset formation algorithm provided the students with a new mathematical tool for determining how to partition group elements in order to form a quotient group. However, following these teacher-generated advances, the students' mathematical activity transitioned to that of passive reproduction. Therefore, while Dr. Bond's mathematical activity resulted in contributions to the mathematical discourse, the level of student engagement with these new ideas appears to be quite limited. As a result, it appears that Dr. Bond's contributions to the mathematical discourse may have had little impact on the students' emerging mathematics.

In contrast, the teachers' mathematical activity in each of the other three episodes all appeared to have a more substantial impact on the classroom community's discourse. In the second episode, Dr. Bond's activity supported the introduction of a new technique for mathematical argumentation into the classroom community's discourse– the idea of maintaining non-equality as a way to construct a proof by contradiction. In response to the introduction of this new technique, Dr. Bond's students began to negotiate its use by generating examples and questioning the conditions under which this new technique is valid. Then, in the third episode, Dr. James identified a critical piece of knowledge needed for the classroom's mathematical development. By assigning instructional tasks informed by his mathematical activity, Dr. James effectively introduced this missing piece of knowledge into the classroom community. The new conjecture proposed by Dr. James, and the ensuing student mathematical arguments, represent a significant impact on the mathematical discourse. Finally, in the fourth episode, Dr. Bond co-constructed a

proof with the class. This co-construction, and the subsequent discussion, allowed an opportunity for Dr. Bond's students to both contribute to the proof and questioned its form and assumptions. As such, the students actively engaged in the construction and negotiation of these new mathematical ideas.

Taken as a whole, these four episodes suggest ways in which the teacher's mathematical activity can support contributions to the mathematical discourse of the classroom community. Given the nature of the curriculum, and the teacher's role in helping to establish the mathematical path of the classroom, such contributions are particularly important. Introducing new mathematics to the classroom discourse serves as a powerful avenue for the teacher to direct the development of the mathematical ideas and advance the mathematical agenda. However, the differing degrees to which the teacher's contributions had a substantive influence on the classroom discourse suggests that more research needs to be carried out. For instance, one could explore what kinds of factors act to curtail or support the potential for a teacher's contributions to have a substantive impact on the mathematical discourse of the classroom community.

4.3. Summary

My analysis of classroom video data resulted in the identification of a variety of instances in which teachers engaged in observable mathematical activity in response to the mathematical activity of their students. For example, in the first episode Dr. Bond could be seen actively investigating the generality of a student's observation. Additionally, evidence for teacher mathematical activity that was not readily observable was found by identifying teachers' contributions to the classroom's mathematical

discourse. This was the case with Dr. James's development and assignment of the two additional subgroup tasks.

Further, my analysis considered the interactions between teachers' mathematical activity and the mathematical activity of their students. This analysis suggests that teachers' mathematical activity can be a significant component in supporting students' mathematical development. Indeed, in each episode, teachers' mathematical activity had the potential to support students' mathematical activity and influence the mathematical discourse of the classroom community.

**Paper 3: Implementing Inquiry-Oriented Curriculum:
From the Mathematicians' Perspective**

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Abstract

As part of an effort to scale up an instructional innovation in abstract algebra, several mathematicians have implemented an inquiry-oriented, group theory curriculum. Three of those mathematicians (co-authors here) also participated in iterative rounds of interviews designed to document and investigate their experiences as they worked to implement this curriculum. Analyses of these interviews uncovered three themes that were important to these mathematicians: coverage, goals for student learning, and the role of the teacher. Here, by drawing on interview data, classroom data, and first person commentaries, we will present and discuss each teacher's perspective on these three themes.

Key Words: Teaching, Mathematicians, Realistic mathematics education, Abstract algebra

The increase in the development and adoption of reform-oriented curriculum has spurred an increase in research aimed at understanding how teachers interact with such curricula. While this area of research has been quite fertile, there remain several gaps in the research literature. For instance, Remillard (2005) calls for research into "characteristics that relate specifically to teachers' interactions with curriculum, such as the teacher's perceptions and stance towards curriculum materials and the teacher's professional identity as it relates to the use of curriculum resources" (p. 235). The need

for this kind of research is even more pressing at the undergraduate level, as little research has been carried out to investigate the teaching practices of research mathematicians (Speer, Smith, & Horvath, 2010).

Research into teachers' interactions with reform curriculum became of particular importance as the *Teaching Abstract Algebra for Understanding* (TAAFU) design team worked to develop, test, and disseminate an inquiry-oriented abstract algebra curriculum. The TAAFU curriculum is a result of design-based research (The Design-Based Research Collective, 2003). Because design-based research is typically initiated with small-scale design experiments, there has been an increasing call for design-based research to include scaling-up considerations. For instance, Clements (2008) calls for field-testing in which "several classrooms involving teachers not closely connected to the developers are observed to learn about the effectiveness and usability of the curriculum" (p. 416). Accordingly, members of the TAAFU curriculum design team have been working with a number of mathematicians as they implement the TAAFU curriculum. The larger goal of this work has been to understand the challenges and opportunities that emerge as mathematicians work to implement the TAAFU curriculum.

In general, the research on the teaching practices of mathematicians is relatively sparse. Speer, Wagner, and Horvarth (2010) found only five studies that they deemed to be "empirical research on collegiate teaching practices" (p. 105). While such empirical studies on mathematicians' teaching practices are scarce, there have been studies to determine the extent of student-centered instruction at the collegiate level and to identify institutional policies that may account for the lack of such instruction (McDuffie & Graeber, 2003; Walczyk & Ramsey, 2003; Walczyk, Ramsey, & Zha, 2007). For

instance, in a case study of two mathematicians trying to implement reform curriculum McDuffie and Graeber (2003) identified a number of institutional norms and policies that either supported or curtailed the mathematicians' efforts. Some of the norms and policies that curtailed change include: limited time for planning and developing new lessons and activities, and institutional pressures to cover a set syllabus that does not allow time for reform-based approaches.

In previous research, the TAAFU project team has worked to add to the research literature by analyzing mathematicians' in-the-moment teaching practices and decisions as they work to implement the curriculum. Johnson & Larsen (2012) investigated the knowledge needed by mathematicians to productively listen to their students' mathematical contributions. Johnson (*this special issue*) investigated the influence of teachers' mathematical activity on the mathematical development of the classroom community. We see this work as related to a small, but growing, body of research that looks to account for challenges mathematicians face while implementing student-centered curriculum. For instance, Speer and Wagner (2009) presented a study in which they sought to account for the difficulties a mathematician was facing while trying to provide analytic scaffolding during whole class discussions. Notice that, in each of these cases, the focus of analysis is the teaching practices of the mathematicians.

In this paper we look not to analyze mathematicians' teaching practices or to investigate institutional norms and policies, but instead to investigate and document the teaching experiences of three mathematicians (and co-authors): John, Julie, and Lee. The primary source of data for this investigation will be post implementation reflections by the three mathematicians. Wagner, Speer, and Rossa (2007) also drew on personal

teaching reflections in order to understand the challenges faced by a mathematician implementing an inquiry-oriented curriculum. For Wagner, Speer, and Rossa, the main function of these reflections was as a data source for identifying areas in which "issues of knowledge might be present" (p. 253). In this way, Wagner, Speer and Rossa connected struggles identified by the researchers to personal reflections provided by the mathematician. In our work here, the mathematicians' teaching experiences (as captured by their post reflections), and not their teaching practices, will be the focus of analysis.

Our work was carried out to accomplish two goals. First, this paper will present the mathematicians' experiences from their own perspective. This provides an opportunity for the research field to gain insight into the factors and considerations that matter to mathematicians; in particular, mathematicians that are willing and excited to implement student-centered curricula. Second, because this paper is written to capture the mathematicians' perspective, it will provide new insights into "the teacher's perceptions and stance towards curriculum materials and the teacher's professional identity" as called for by Remillard (2005, p. 235). Because of the unique source of these insights (teachers' reflections rather than analyses of practice) our findings provide both a useful confirmation of some existing findings and new ideas to be explored via careful analyses of classroom practice.

1. Description of Curriculum and Instructor Support Materials

The TAAFU curriculum was developed in accordance with the Realistic Mathematics Education (RME) notion of *guided reinvention* (Freudenthal, 1991), in which "the idea is to allow learners to come to regard the knowledge they acquire as their own private

knowledge, knowledge for which they themselves are responsible" (Gravemeijer & Doorman, 1999, p. 116). In the TAAFU curriculum, the process of guided reinvention is supported through the *emergent models* design heuristic. The emergent models heuristic is meant to promote the evolution of formal mathematics from students' informal understandings (Gravemeijer, 1999). In this way, the concepts first emerge from the students' informal activity and then develop into more formal ways of reasoning. The teacher has a crucial role in this transition. Namely, the teacher needs to finds ways to build on the students' informal ideas in order to help them construct the formal mathematics. As curriculum developers, our goal was to provide a curriculum that would support students in reinventing the fundamental concepts of group theory, and a set of teacher support materials that would enable teachers to effectively guide this process while ensuring that the students retained ownership of the mathematics.

1.1. The TAAFU Curriculum

The TAAFU curriculum includes three instructional units: groups and subgroups, isomorphism, and quotient groups. Each of these three units includes both a reinvention phase and a deductive phase. During the reinvention phase, students work on a sequence of tasks designed to help them develop and formalize a concept, drawing on their prior knowledge and informal strategies. The sequencing of these tasks is designed so that launching activities will evoke student strategies and ways of thinking that anticipate the formal concepts that we want them to learn. Then follow-up activities, and teacher guidance, are supposed to leverage these ideas to develop the formal concepts. The end product of the reinvention phase is a formal definition (or definitions) and a collection of

conjectures. The deductive phase begins with the formal definitions that are relevant to the concept. During this phase, students work to prove various theorems (often based on conjectures arising during the reinvention phase) using the formal definitions and previously proved results. (For a detailed description of the curriculum see Larsen, Johnson, & Bartlo, 2013.)

1.2. The Instructor Support Materials

The larger goal of the TAAFU project was to develop a curriculum that could be implemented by a broad audience. Accordingly, a set of instructor support materials was developed in tandem with the development and field-testing of the curriculum (see Lockwood, Johnson, & Larsen, 2013). These instructor support materials were designed to provide the critical information needed in order to successfully implement the TAAFU curriculum. Through the course of the project, these instructor support materials have gone through a number of revisions. The three mathematicians featured in this report were provided with three different iterations of the instructor support materials.

Initially John was provided with a set of "notes" that explained goals for the lessons, provided implementation suggestions, and included additional information about tasks that had been identified as being challenging to implement during the design experiments (for an example see Lockwood, Johnson, & Larsen, 2013). John also met with members of the research team regularly throughout his first implementation of the TAAFU curriculum. The purpose of these meetings was to discuss the previous class sessions and to collaboratively plan upcoming sessions.

Based on lessons learned from John's implementation, and further analysis of student learning from the design experiments, a new set of expanded instructor support materials was developed and provided to Julie for her implementation. These lesson sheets 1) described the goal of each task within the larger unit, 2) offered implementation suggestions, and 3) described common types of student thinking (including both difficulties and solution strategies). For more information on this iteration of the instructor support materials, see Lockwood, Johnson, & Larsen (2013). While implementing the curriculum, Julie met twice a week with the first author to debrief and to discuss the usefulness of the expanded instructor support materials.

The next round of improvements to the instructor support materials was again guided by our ongoing research into teaching and learning in the context of the TAAFU curriculum, and by our analysis of Julie's implementation of the curriculum. This iteration of the instructor support materials was in the form of an interactive website, which Lee used for his implementation. This website was designed to avoid overwhelming the instructor with information while still making available any information they might find useful. The default setting displays only the instructional tasks. For any task, the instructor can then choose to expand any (or all) of three expandable text boxes: one for the rationale of the task, one for information about student thinking, and one for implementation suggestions and issues. When appropriate, these textboxes contain relevant media, including classroom video and images of student solutions. Unlike the previous field testers, Lee did not meet with members of the research team to debrief or plan. However, Lee did communicate occasionally with the lead curriculum developer to get advice on lessons he found to be especially challenging.

2. Teaching Biographies of the Three Mathematicians

All three of the teachers featured, John, Julie, and Lee, hold a PhD in mathematics and all volunteered to implement the TAAFU curriculum. Here we will share some of the mathematicians' background experiences that they identified as influential to their implementation of the TAAFU curriculum in particular and their teaching in general.

2.1. John's Teaching Biography

John, a professor and research mathematician specializing in combinatorics and graph theory, earned his PhD from the University of Wisconsin in 1998. John describes his mathematics education as "very traditional". All the courses he took were lecture based. When John began teaching as a graduate student he often found himself teaching courses that were highly structured and, while these courses were mostly lecture based, a college algebra course he taught was structured to include "Problem Days" one day a week. These days, during which "students were given a fairly difficult, contextualized, no-clear-roadmap-to-simple-answer type of problem to be worked on in groups", were John's first experience with group work.

John stated that the 'Problem Days' helped him appreciate group work and the opportunities it offered for students to "think and communicate quantitatively". Years later, John participated in a large NSF-funded professional development project that emphasized mathematical discourse (Oregon Mathematics Leadership Institutes – OMLI)[12]. During this experience John team-taught with master teachers (ranging from elementary math teachers to university professors). For John, this experience resonated

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[12] NSF HER-0412553

with his early 'Problem Days', and John "learned quite a bit about different kinds of discourse in the classroom and some protocols for measuring these". Coming out of the OMLI project, John "was still interested to learn more (post-OMLI) about ways to implement group work in the college classroom, and so helping out on this research seemed a great way to experiment and learn".

However, even with these experiences, John describes his typical teaching (outside of implementing the TAAFU curriculum) as follows:

> My teaching style is almost entirely lecture. I do try hard to solicit student questions and input and I make it my goal to create a comfortable environment for meaningful mathematical discussion for at least part of each class period. A few of my upper-level courses made significant use of student projects (done in small groups), for which a few class periods were available.

It is important to note that John views his teaching practice as a work in progress, where he envisions this practice "evolving toward a fairly hybrid approach of content delivery". One component of this approach would be "a (hopefully) judicious use of group work at a few key points in courses where it can be used to maximum effect". Through his experience with OMLI and the TAAFU project, John is "starting to learn where those points are, and how to pull it off successfully".

2.2. Julie's Teaching Biography

Julie earned her PhD from Oregon State University in 2000, specializing in Topology. As a graduate student, and later as a professor, Julie taught a number of different mathematics courses. These courses included: mathematics for elementary teachers, introductory/business statistics, calculus, introduction to proofs, linear algebra, discrete mathematics, elementary analysis, abstract algebra, and topology. Like John, Julie

participated in the OMLI project. As part of the OMLI project, Julie received training in "Best Practices" for teaching mathematics[13]. This training focused on "ways to increase the quality and quantity of student discourse, how to facilitate productive group work, etc." This was Julie's "first 'formal' introduction to the skills I needed to teach an inquiry based [course]".

Julie's experiences with the OMLI project marked a transition in her teaching practice. Prior to OMLI and the Best Practices course, Julie had used some group work activities in her courses. However, she felt that the problems she gave groups to consider were not always well thought out. "Before OMLI I just separated them into groups, made them share, I didn't think anything about it. After OMLI I thought really carefully about like…Does this need a group to work on it or is this really just a homework problem and I am just giving it to them in groups so I can say I do group work?" During OMLI, Julie was able to gain confidence with teaching a course based on developing key ideas through the use of problem posing and drawing out student thinking. Although Julie was sold on this approach to teaching, she found it difficult to develop materials. "I mean, that's the big hang-up. That's why I'm so excited about Sean's curriculum… that's not something you normally have time to do."

2.3. Lee's Teaching Biography

Lee is currently a professor and received his PhD in 2005 from Cornell University, specializing in probability theory. Lee describes himself as coming from a long line of

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[13] John did not become involved in the OMLI project during its first year and so did not participate in this training.

Kentucky educators and feels that "our job as mathematicians is to use our deeper knowledge of the mathematical concepts to construct approaches to the material that students can be active with in such a way that they can get these things". Generally speaking, Lee believes that "almost anything you do is going to work at least as well as lecturing". However, without the proper training, Lee thinks that it can be very difficult to successfully implement "active learning".

Accordingly, Lee actively seeks out professional development opportunities. This includes seminars offered through his university's teaching center, educational blogs, and events like MathFest. However, even with this training and an experience in graduate school working on a project designed to develop 'good questions' for calculus, Lee struggles to find/develop curriculum materials that would allow his students to actively engage in the mathematics. As Lee stated, "it is surprisingly difficult in the mathematics community to find, you know, good scientific approaches to thinking about teaching. People don't seem to have a very clear notion". While Lee is active in the mathematics education blogger community, he feels that there is a gap between mathematicians and mathematics educators. "We are talking about mathematicians who don't know anything about educational theory. I had some meetings with math ed people but they were talking about some pretty deep educational theory… I couldn't connect with it."

As Lee was preparing to teach his first abstract algebra course, he happened to attend a MathFest mini-course on a RME based linear algebra curriculum (Wawro, Sweeney, Zandieh, & Larson, 2011). This experience resulted in Lee approaching the developer of the TAAFU curriculum and asking to implement the materials in his upcoming abstract algebra course. As Lee describes it, "it was kind of serendipitous I would say. I was just

heading into the right place in the right moment when I found out about the project, and it was a pretty obvious decision."

3. Data/Methods

In order to synthesize the experiences of the three mathematicians, and identify commonalities and differences in their experiences, the first author conducted a series of iterative interviews. Initially, all three mathematicians were asked to provide a written response to the prompt: When you reflect on your experience teaching with the TAAFU curriculum, what stands out? The first round of individual interviews was designed to elaborate these reflections. For instance, John's initial written reflection mentioned both concerns about coverage and an effort to "keep a sensible narrative moving forward." Therefore, during John's first interview, he was asked to elaborate on these two points and discuss possible relationships between these ideas.

In addition to elaborating on their own reflections, each mathematician was also asked to discuss and reflect on the statements of the other two mathematicians. For example, following John's written reflection and first interview, it was clear that he was both concerned with covering material and maintaining a classroom narrative. These two areas then became a focus of interview questions for both Lee and Julie. Julie and Lee were first asked broad questions on the topic and then asked to respond to direct quotes from John's interview. This spiral pattern of interviewing, analyzing interviews for possible themes, and developing new interview questions based on those emerging themes characterizes the data generation process. The result of this process is a rich and

complex data set in which each mathematician volunteered personal reflections and commented on the reflections of others.

Each of the three mathematicians also participated in individual stimulated recall interviews (Schepens, Aelterman, & Van Keer, 2007). In these interviews the mathematicians were asked to view video clips from their class. These clips were selected based on the interviews and written reflections. Specifically, clips were selected because 1) they appeared to exemplify a point raised by the mathematician in the interviews and written reflections, 2) they appeared to contradict a point made by the mathematician, 3) they appeared to be indicative of differences in the mathematicians' teaching orientation, or 4) they stood out as important based on the field notes and/or debriefing meetings held during implementation (this data source was only available for John and Julie). As these videos played, the mathematicians were asked to stop the video any time they could remember why an instructional decision had been made, felt like something happened in the video that was significant, or saw something that was particularly representative of their teaching. Further, specific interview questions were prepared and asked at relevant times during the clip, such as "what prompted you to assign that task?" Finally, as differences and themes began to stabilize, additional written reflections were requested. For instance, all three mathematicians were asked to describe their course in terms of process and content. Table 2 summarizes the data set.

Teacher	TAAFU Implementation	Data Collected During Implementation	Reflection Data (Collected Winter 2012)
John	Fall 2007	• Classroom Video Data of Quotient Group Unit • Field Notes of Each Class Session • Weekly Debriefing Meetings	• Initial Written Reflection (John WR #1) • Interview Based on WR#1 (John Int #1) • Interview Based on Lee's Interviews (John Int #2) • Written Reflection on Process/Content and the Role of Group Work (John WR #2)
	Fall 2008	• Classroom Video Data of Each Regular Class Session • Field Notes of Each Class Session • Pre and Post Term Interviews	• Stimulated Recall Interview (John SR #1) • Written Reflection on Preliminary Analysis (John WR #3)
Lee	Fall 2011	• Classroom Video Data of Each Regular Class Session	• Initial Written Reflection (Lee WR #1) • Interview Based on WR#1 (Lee Int #1) • Interview Based on John Int #1 (Lee Int #2) • Stimulated Recall Interview (Lee SR #1) • Written Reflection on goals for course (Lee WR #2) • Written Reflection on Process/Content and the Role of Group Work (Lee WR #3) • Written Reflection on Teacher Moves (Lee WR #4)
Julie	Summer 2009	• Classroom Video Data of Each Regular Class Session • Field Notes of Each Class Session • Semi-weekly Debriefing Meetings	• Initial Written Reflection (Julie WR #1) • Interview Based on WR #1 (Julie Int #1) • Interview Based on John's Int #1 and Int #2 (Julie Int #2) • Interview Based on Lee's Int #1 and Lee's Int #2 (Julie Int #3) • Written Reflection on Process/Content and the Role of Group Work (Julie WR #2) • Stimulated Recall Interview (Julie SR #1) • Written Reflection on Preliminary Analysis (Julie WR #3)

Table 2. Data set

The nature of the data collection allowed the first author to identify possible themes early in the interview process and then test and refine these themes in subsequent

interviews. Interview and reflection data related to these themes were compiled for each mathematician. Based on analyses of these data, a preliminary report was prepared for each teacher on each theme. The mathematicians were asked to react to the written synopses characterizing their views relative to each theme. These reactions allowed an additional opportunity for data collection, in that the mathematicians provided additional exposition in light of the preliminary analysis[14]. As part of these reactions, the mathematicians were asked to comment on the accuracy of the analysis and the relative importance of each theme. Finally, based on the relative salience of each theme in relation to their teaching experience, each mathematician was assigned a specific theme and was asked to write a first-person commentary to further expound their views on this theme in relation to their implementation of the TAAFU curriculum.

The questions asked in these interviews and written reflections were informed by the first author's knowledge of the TAAFU curriculum, and thus were guided by RME theoretical constructs, including the notion of guided reinvention and the emergent models heuristic (Gravemeijer, 1999). For instance, because the curriculum was designed with the intent of students maintaining ownership of the mathematics, questions about what student success looked like when implementing the TAAFU curriculum were asked. The purpose of these questions was to explore how closely the mathematicians' notions of student success were aligned to the principle of guided reinvention. Additionally, because the curriculum developers believed that the teachers needed to support students in formalizing their mathematical understandings, questions about the teachers' views on instruction were asked. These included questions about how they viewed the role of

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[14] Lee did not provide a reaction to the preliminary analysis.

group work and whole class discussions, and questions about how the teachers would describe their role while implementing this curriculum. These questions were asked to probe whether the teachers' views of their roles were consistent with the emergent models heuristic.

The first author's theoretical perspective also informed the analysis of the mathematicians' responses. While the first author worked closely with the mathematicians to ensure their perspectives were being presented accurately, analysis of the interviews and reflections were guided by the design principles underlying the curriculum. For example, teachers' statements about their role in the classroom were understood relative to the notion of guided reinvention and the emergent models heuristic. Special attention was given to statements related to students maintaining ownership of the mathematics and to the process of building on the students' informal ideas to develop the formal concepts. Special care was taken to not attribute the ideas of RME to the mathematicians but to instead to understand the relationship between their ideas (as they expressed them) and these principles.

4. Themes: Coverage, Goals for Student Learning, & Role of the Teacher

As the iterative interviews progressed, and the mathematicians began to reflect on and respond to each other's comments, three themes emerged as salient features of the mathematicians' experiences implementing the TAAFU curriculum: coverage; goals for student learning; and the role of the teacher. Here we will present the mathematicians' perspectives on each of these three themes.

4.1. Coverage

Given the inquiry-oriented nature of the TAAFU curriculum, it is not unreasonable to speculate that 1) the TAAFU curriculum would include less content than a traditional introductory abstract algebra course, and 2) the pace of the course maybe be slowed as the students work in groups to reinvent that mathematics. As John began implementing the TAAFU curriculum, these two issues were at the forefront of his mind. Coverage remained a major concern for John throughout his implementation of the curriculum. In the following section we will present John's perspective on the coverage issues he encountered, along with two alternate views held by Lee and Julie.

John

Prior to implementation, John's predominant concerns about teaching with the curriculum were related to coverage. In fact, John stated his concern about covering enough material in a group work setting was his "biggest barrier going in". For John the coverage issues raised by the TAAFU curriculum were related to two features of the curriculum: the reinventive nature of the curriculum, which John felt slowed the pace of the course; and the use of group work, which John felt compounded the effect.

> I would say the coverage issue, in my mind, is the main concern I would have with group work.… I don't think it has anything to do with whether or not the curriculum is innovative or just sort of produced with pedagogical aims in mind or anything like that. Just specifically, when you have the material reinvented, in my mind it is going to slow down things. And so, I was worried about that.
>
> (John Int #1)

Two main concerns arose for John related to coverage. First, John was concerned with the pace at which the class could move through each lesson, worrying that they would fall

114

behind and not be able to finish the curriculum.

> I would worry about that each day. If I go slow here and don't even get to the point I'm supposed to, or if the students aren't tuned into what we are doing, then it's going to be an even bigger problem later.
>
> <div align="right">(John Int #1)</div>

Second, John was concerned that the class would not have time to consider all of the material that would typically be included in an introductory group theory course. "I was worried that students wouldn't come away from the course having seen stuff that I would maybe want them to know in a later course."[15] In summary, John felt that 1) the curriculum was a bit modest in topic coverage compared to what he would typically aim for, and 2) even when he accepted the more focused topic coverage of the TAAFU curriculum, John was concerned that the class would have a hard time achieving the content goals of the curriculum.

John's concerns over coverage also seemed to stem from his beliefs about the role of group theory courses in regards to a broader mathematics education. Interestingly, the mathematical significance of group theory was not a driving factor. Instead, "the point is to learn a topic and the basic results about it… With a mathematical topic like group theory, you take a simple structure and you develop it to a certain degree". Therefore, one of the sources for John's concerns over coverage was ensuring that the students go far enough into the curriculum to achieve "a certain level of sophistication". For group

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[15] For the most part this concern is unwarranted because most of the material covered in a standard course on group theory is included in the TAAFU curriculum. However, some concessions are made to accommodate the deeper treatment to the key concepts. For example, permutation groups tend to not be explicitly treated in detail.

theory this meant addressing, and understanding the significance of, the first isomorphism theorem.

While John was definitely concerned with coverage throughout the course, he also felt like teaching this course helped him to understand another side of the coverage argument. Initially John was focused on the idea that instruction based on group work would limit the amount of material that could be covered. However, in teaching this class John repeatedly observed that the topics that were covered received a deeper treatment than they would have in a traditional course.

> If you can cover more with just lecturing then the cost of that is maybe the students don't really understand it as deeply as you think… It's a tradeoff. They are going to come out of this [course] knowing things deeper than they would otherwise. And so I am totally willing to explore that. And that's the reason I wasn't disappointed. I definitely felt like in class things got explored a lot more to the level that I feel like I get to when I read a book on my own as a mathematician.
>
> (John Int #1)

For John this tradeoff was only viable if the students actually developed a rich understanding of group theory. Therefore, as much as he wanted to make sure they got through the planned lessons for each day, John also wanted to make sure that the students understood the material at a level that justified the slower pace of the course. These two factors became a tension for John throughout his implementation of the TAAFU curriculum.

This tension was somewhat exacerbated because the curriculum materials brought to light the students' current level of understanding of the material, which was not something John typically saw when he lectured. "I definitely was in denial about the level

of sophistication many students bring into the course and the curriculum brings that right out into broad daylight." As a result, John felt very aware of where his students were and what they were struggling with. This, coupled with John's natural inclination "to slow down and go over it until everyone gets it", made it difficult for John to move through the course at a pace that would have alleviated his concerns about finishing the curriculum. As John explained, "that kind of makes my performance in a group work driven course slow down even more. Which is my greatest fear with implementing the curriculum. It's a dangerous thing."

This tension, between making sure his students understood the concepts while maintaining a pace that would ensure that they finished the material, was often reflected in John's pedagogical decisions. For instance, after providing a definition of subgroup and a quick example, the TAAFU curriculum prompts students to determine a minimal set of conditions needed to ensure that a subset of a group is itself a group[16]. After the students had a chance to work on this task in groups, John led a whole class discussion to gather conjectures and informal justifications. During this discussion, a number of students questioned the relationship between the identity of a group and the identity of a subgroup. In response, John slowed down the lesson and provided a new task for the students to work on.

> John: Let's actually detour for just a moment. Let's take a few minutes and try to write, everyone, some sort of a proof or something, that if you have a subgroup inside of a group, true or false, the identity of the subgroup has to equal the identity of the group.

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[16] For a detailed analysis of this episode see Johnson (2013).

However, while John was happy to diverge from the planned lesson to address this student concern, there were a number of contributions and conjectures that John did not choose to follow. For example, at one point during this lesson, John responded to a student's contribution by saying, "on the grounds of vagueness, we'll table that for now".

John's judicious filtering reflected both of his perspectives on the content issue. In his reflections and in his actual implementation, we see that John balanced his concerns with coverage and his concerns for student understanding. While teaching, John would often continue working with students on a task until he believed the students understood the material. However, John seemed hesitant to follow up on student comments that did not immediately seem to lead towards the mathematical goals of the lesson. Therefore, we see that coverage was a guiding factor in John's implementation of the curriculum that was somewhat mediated by his focus on student understanding.

**Topics and Skills - A Perspective on Coverage
and the TAAFU Curriculum**
John Caughman

I believe mathematics is concerned with patterns and structures. Although we cannot teach every structure students might ever need, many structures are similar. Facility with one can, by analogy, be useful in gaining facility with others. A group theory course chooses just one of these, and, by way of studying that example, teaches topics that have counterparts in many other structures. By learning group theory, students learn to study math.

A course in group theory should therefore cover a certain list of core topics. But it should also teach skills that bridge to advanced mathematics, where mathematical sophistication is assumed. These two aims are mutually attainable and largely consistent, but not identical. The TAAFU curriculum uses guided reinvention to simultaneously target topics and skills, allowing abstract topics to be motivated by issues that arise for students as they make sense of well-chosen tasks. In doing so, it highlights some tough choices a teacher must face concerning coverage, and wherever possible, the TAAFU curriculum seems to favor quality over quantity.

The topics to be expected of a group theory course are fairly standard: the axioms,

basic properties, orders of elements, subgroups, examples, permutations, isomorphisms, homomorphisms, cosets, normal subgroups, quotient groups, and the isomorphism theorems. The TAAFU curriculum manages to sacrifice none of them. In my experience, though, it is a challenge to reach the latter topics. I feel it is critical to reach them, however, since they most readily transfer to other structures, and they represent a secondary level of abstraction (functions between structures) that is otherwise absent. But where topic coverage may be a bit compromised, TAAFU offers some substantial gains related to developing mathematical skills.

Which skills do I want from the TAAFU curriculum? At this level, I think students should no longer take teachers at their word, but develop a skepticism that demands rigorous answers. They must communicate precisely in all modes and they need some specific proof techniques. Students should conjecture, refute, create examples, and discover things. They must formulate good questions and learn how to go about answering them. How does the TAAFU curriculum address these goals? To facilitate reinvention, it largely replaces traditional lecture with in-class group work. Student practice at questioning, clarifying, justifying, and listening are direct benefits of group work. Constant real-time access to student thinking allows many pedagogical choices to be better-informed. The primary benefit of the TAAFU group-work, as I see it, is the direct reinforcement of the goals of group theory related to mathematical skills.

What coverage challenges arose for me in the classroom? The TAAFU curriculum challenged my ability as a facilitator of group work. First, different students sometimes required different amounts of time to resolve issues arising in open-ended tasks. This pace, even when averaged out, sometimes felt too slow to allow complete coverage of all of the topics. Second, my students proved quite creative in resolving issues that arose in reinvention tasks, and wide divergences from convention were not uncommon. Pursuing these could prove fruitful to a single student, but detrimental to others. And finally, a mathematical statement handed to groups gets reflected off of so many mirrors, and a room of novices can generate an abundance of decent questions leading to a multitude of perspectives and confusions. I found it challenging to reconcile student contributions into a coherent narrative, and this refocusing took real minutes in the classroom. So for me, the many benefits of group work were sometimes being won at the cost of intensifying the coverage challenge.

Are these coverage issues resolvable by improved ability with facilitating group work? I suspect so. After a few repetitions of using the TAAFU curriculum, I certainly felt I gained in efficiency and confidence. I cannot deny the profit of having reinvention and group work experiences in the course. To afford more content, perhaps some of this can also happen outside class. Going forward, I hope to improve, both by increasing my skill at group work facilitation, but also by prioritizing which key topics should be reinvented for maximum effect.

Lee

Unlike John, Lee found concerns over coverage to be "nonsense" and did not consider coverage to be a factor for his implementation of the TAAFU curriculum. Lee does not believe that the amount of material presented to students matches the amount of material learned by students.

> Yeah there are a lot more things you could say out loud in a semester, if you don't let students work through the ideas, and they don't learn any of them. So you know, you are really learning a lot less if you just say everything out loud in the classroom.
>
> (Lee Int #2)

As a result, Lee felt that the curriculum provided students with an opportunity to learn more than they would in a traditional lecture-based course, even if it covered fewer topics or if the class did not finish the materials. Additionally, Lee believes that the TAAFU curriculum "is a more effective way for the students to learn the ideas that are contained in this class… [The students] might actually learn a significant proportion of the content, even if we don't have the opportunity to treat every traditional topic."

There were courses in which Lee was more concerned with covering a set amount of material. For instance, when teaching calculus Lee felt that "there are some bottom line things that are difficult to ignore", especially since the students need to be prepared for the next course in the sequence. In Lee's group theory course many of the students were seniors whose only upper division sequence is abstract algebra (Lee was scheduled to teach both semesters of the sequence). As a result, Lee did not feel pressured to cover any set material and instead treated this course as if it were a senior capstone.

Julie

Julie stated that she believes it is common to feel a tension between coverage and student understanding, regardless of the curriculum or the mode of instruction. However, like Lee, Julie was not overly concerned with the amount of material present in the TAAFU curriculum. For one, Julie felt that the curriculum included the topics found in a traditional group theory course. Also, Julie believes that students do not retain material unless they have an opportunity to interact with concepts, wrestle with the ideas, and construct their own meaning.

> My belief is that most of the students don't retain or understand no matter how good you are at teaching and explaining … When you don't give your students the opportunity to interact with the math on their own terms and their own thinking and move forward from that spot I think you just leave out a huge chunk of students.
>
> (Julie Int #3)

Additionally, Julie believes that the depth at which material is covered can actually reduce the amount of material that the students need to be exposed to. "Yeah, that one activity might take two or three class periods, but you might not have to do the other two or three lessons that you were going to have to do if you taught the other way." This belief is based on the idea that, if the students develop a deep understanding of the foundational concepts, they will be able to apply this understanding to new material. "So even if you cover a little bit less, if you cover it deeper the students will be able to figure out the parts you didn't get to." For instance, Julie believes that if the students are able to construct a rich understanding of groups then they will be able to make sense of

permutation groups quickly in future courses, even if they did not receive direct instruction on that during the course.

Summary: Coverage

The research literature provides some evidence that coverage of content can be a concern when teachers are asked to implement student-centered curriculum. For example, Christou et al. (2004) found that elementary teachers were concerned about "covering content of mathematics in the set time limits" (p. 168). The situation here is of course much different because of the nature of the mathematicians' professional preparation and because the mathematicians volunteered to implement this curriculum. However, some research has found that mathematicians may also feel pressure to cover a certain amount of material in a course and that this could impact if or how they implement a student-centered curriculum (McDuffie & Graeber, 2003; Wu, 1999). John, Julie, and Lee exhibited perspectives ranging from viewing coverage as "nonsense" to viewing coverage as "the biggest barrier going in". However, regardless of their perspectives on coverage, each of the mathematicians countered concerns over coverage with an emphasis on student learning. John came to view deep student understanding as another side of the coverage issue. Lee rejected the idea that the amount of material covered accurately captures the amount of material learned by students. Julie expressed a belief that treating foundational topics in depth reduces the amount of material that needs to be covered in class.

The mathematicians' perspectives speak to the potential instructional benefits of an RME inspired curriculum. Specifically, the mathematicians all point to the limited value

of simply covering concepts and emphasize the benefits of engaging students in guided reinvention. As argued by Freudenthal (1991), "knowledge and ability, when acquired by one's own activity, stick better and are more readily available then when imposed by others" (p. 47). The mathematicians all seemed to embrace this idea that learning through reinvention can result in deeper conceptual understanding.

4.2. Goals for Student Learning

The instructional theory underpinning the TAAFU curriculum, RME, takes "its point of departure primarily in *mathematics as an activity*" (Gravemeijer & Doorman, 1999, p. 116). Therefore, the curriculum was designed with the intention that students engage in mathematical activity as a means to learn mathematical content. All three mathematicians seemed to value the idea of actively engaging students in doing mathematics. However, their perspectives were somewhat diverse in terms of the extent to which they saw this as a goal in itself, and in terms of how much they worried about students' learning and understanding of specific concepts.

Lee

As previously discussed, Lee was minimally concerned with covering a set amount of abstract algebra content. In fact, Lee stated that having the students fully engage with the mathematics involved in each task was more important to him than linking the tasks together to present a unified picture of group theory. "I was more interested in the students appreciating the ideas that they were getting than I was about fitting those things into the grander scheme."

This prioritization of doing mathematics over covering group theory content was consistent with Lee's view of the TAAFU curriculum as a guide for students to "rediscover the content" – where the process of rediscovery and the content that gets rediscovered should not be separated. In fact, Lee believes that these two components are so interconnected that teachers should not try to separate them by predetermining the content that is to be rediscovered.

> If by 'focus on the content' we mean making decisions to short-circuit the discovery process in the interests of exposing the students to the entirety of the standard content body for a course of this nature, then no, I didn't focus on the content.
>
> (Lee WR #3)

Given that Lee was not trying to get the students to arrive at predetermined mathematical content, it is not surprising that specific group theory content was absent from his goals for instruction and his discussion of what it meant to learn group theory. Instead, when asked what it meant to for his students to learn group theory, Lee discussed examples, properties, arguments, and applications.

> I think that it means that the students should be exposed to the information and the examples that make up the beginnings of the field. And that they should interact with those examples to discover their properties and to form arguments regarding the generality of those properties. I think they should consider the value of some arguments, and evaluate them for accuracy and clarity. They should come to recognize the applications of the ideas to various aspects of both the world at large and of the world of mathematics, to use their knowledge of the structure of a mathematical system to glean insight regarding such systems as they might observe either in other courses or outside of their formal education.
>
> (Lee WR #2)

Similarly, when asked about his goals for the group theory course, Lee responded with a focus on positive student experiences and a development of an understanding of

mathematics as a subject.

> I wanted as many of my students as possible to have a successful and pleasant experience learning in this course. I wanted them to gain an appreciation for the power of the subject and to grow in their understanding of what math is and how mathematical investigations are carried out.
>
> (Lee WR #2)

It was only when Lee was asked what student success looks like that he made any direct reference to definitions and theorems.

> Minimally, success requires a learning gain adequate to allow the student to regurgitate the definitions, theorems, etc., to accomplish basic computational tasks and to provide proofs of statements that are either identical or nearly so to statements that they have either proved or seen proved during the course.
>
> (Lee WR #2)

This ability to regurgitate, compute, and mimic is contrasted with a more sophisticated understanding of the mathematics that would be indicative of excellence and mastery. Lee described excellence as the "ability to approach a new definition, create examples that satisfy and fail to satisfy the definition, and then use the definition to prove basic facts regarding the class of objects described by the definition". Finally, "true mastery would also include the ability to determine which facts regarding the class might be proved or disproved without actually being told". However, even with these references to some set of definitions and theorems, there was no reference to specific group theoretic concepts, such as groups, isomorphism, or quotient groups. Instead of knowledge of specific topics, Lee wanted his students to develop the tools needed to satisfy their mathematical curiosity.

> I want them to understand the difference between the understanding that they get from figuring it out themselves and what they are able to get from going and reading something out of a book. … So by giving them some skills, some ways of thinking and exploring it starts to be okay to wonder about whether something is true… So, to me, that's what I really want to accomplish in this class… In terms of engaging in realistic mathematics activity I think this is one of the only chances I have to get that.
>
> (Lee WR #2)

Lee's focus on mathematical exploration over formal content was exemplified during his reflection on a lesson from the isomorphism unit. The isomorphism unit is launched with a task to determine if a "Mystery Table" could be an operation table for the symmetries of an equilateral triangle (Larsen, 2013). When given this task, Lee's students generated arrow mappings that assigned each element of D_6 to an element of the mystery table - where this assignment guaranteed that if two elements were multiplied and then mapped the result would be the same as if two elements were first mapped and then multiplied.

Informally the students had identified the two criteria for an isomorphism, a one-to-one correspondence (as seen with their arrow mappings in this finite case) and the condition that the mapping preserves the operation. From here, Lee began to push the students to formalize this idea by introducing function notation. It soon became clear that his students were not comfortable with function notation, and Lee decided to take a leadership role in the classroom and translated the students' mapping notation and operation preserving rule into standard function notation. When reflecting on this decision to push the class towards the formal definition of isomorphism, Lee expressed dissatisfaction with this lesson.

126

I think it's broken. I think that at this point we are not trying to do the right mathematical activity. We are not trying to define what it means for these things to be related in the correct way. We are trying to present it in a formal language that involves functions and that is a different kind of activity for these guys… This was not a good day for me, and in reflecting back I think it is because I was just too focused on the goal. I needed to just get out of the way and let it go where it wanted to go, and if it wasn't functions that's okay… The effect was that they didn't own the definition.

(Lee SR #1)

Lee generally believed that his own mathematical agenda, or a set list of content goals, should not curtail his students' opportunities to engage in authentic mathematical activity. This belief was exemplified during Lee's reflection on his implementation of the isomorphism lesson, in which Lee stated that his decision to step in and translate the students' ideas into formal function notation privileged the mathematical goals of the lesson over the mathematical activity of his students. The other two mathematicians shared Lee's emphasis on students engaging in mathematical activity and processes, however both Julie on John expressed a greater emphasis on group theoretic content.

Goals for Student Learning
Lee Gibson

I'm pretty sure my mother didn't finish her math degree because of the struggles that she experienced in an upper division abstract algebra course. For her, as for so many math students, the patterns of study behavior learned in earlier courses are inadequate to address the paradigm shift that the material in this course can represent. For the first time, listening attentively in class and struggling alone or in a small group of other strugglers to decipher mysterious homework problems doesn't succeed in creating the necessary level of understanding. The teaching strategies in common use are not widely effective for meeting the standard student learning goals.

By bringing the TAAFU materials to bear on my course, I hope to provide strategies for learning that are broadly sufficient to meet the course goals for student learning. But the power inherent in these strategies makes the standard goal - to have students cover a set amount of material - seem somehow pedestrian. There is a real potential to improve

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both the quality and the usefulness of what students learn here by shifting the focus toward the nature of mathematical discovery.

The mathematics major has the perversely unique quality of producing graduates who have not, at any point, been asked to engage in activities representative of mathematics research. The TAAFU materials provide an opportunity to at once address both this incredible oversight and the myriad of student difficulties that commonly occur. In this context, then, a list of my goals for student learning in this course would include that the students would obtain

- *An appreciation for the power of the subject and growth in their understanding of what math is and how mathematical investigations are carried out*
- *A sense of the additional value that the kind of understanding that they gain from figuring things out for themselves has over the understanding that they get from being told the answers*
- *As much mastery of the standard course content as is feasible within the new classroom parameters*
- *The intellectual courage to take on a new challenge that seems alien to their previous experience*

The last of these was added rather later, when I encountered my students again in a subsequent course. I asked them to explore a definition together in teams, and they just … did it. I found myself marveling at the difference between their response and the way they floundered around at the beginning of the previous course. When we don't commonly provide students with the tools and the opportunities to explore new things on their own, we don't even realize how little intellectual courage and curiosity they possess. They don't understand that creating mathematics is both something that they are capable of doing and represents the core of the joy of doing mathematics. Shame on us for hiding the heart of our discipline from our disciples.

Julie

When Julie reflected on what it means to learn group theory, she focused on the core concepts of the field. Primarily, Julie believes that her students "need to understand that algebra is the study of structures of mathematical objects". Accordingly, Julie believed that students in an introductory group theory course should:

> Understand what a group is and the structure of a group. And they should see different examples. They should understand the limitations of it. I mean, I think so much of that course is just unpacking what a group is.

> (Julie Int #3)

Julie sees understanding the structure of groups as being related to understanding homomorphism and isomorphism. Julie sees a homomorphism as a type of function that "respects and interacts with the group structure", and is therefore an important type of function in the group settings. She sees isomorphism as defining "the idea of sameness that is important… It is still really unpacking the idea of groups". Finally, for Julie, studying quotient groups adds an additional opportunity for students to wrestle with the definition of a group, this time "through the lens of a way to construct new groups from an existing group".

In addition to these algebraic concepts, Julie also believes that in group theory students should develop more general forms of mathematical thinking – including proving, constructing examples, and unpacking definitions. Julie believes one of the difficult transitions for math majors is moving from the specific to the general, and then once that transition takes place student sometimes struggle to make connections back to specific examples.

> I think the TAAFU curriculum really helps students learn to go back and forth between specific examples and the general theorems, properties and definitions of group theory. Since the big ideas are all developed through student exploration of examples, students are constantly asked to go back and forth between specific and general contexts. In a more traditional group theory course, you often teach the students the general context and then give a few examples or vice versa, but the students can get by without relating the two contexts (even though that is usually the intent of the instructor).
>
> (Julie WR #3)

Thus, for Julie, it is important that students learn both "how to prove something… the logic behind proof, what it means to prove something" and (for instance) how to

construct "an example of a group with order eight with certain properties". Julie believes the TAAFU curriculum helps to support and develop this connection between the abstract and the concrete.

John

Consistent with his concerns with coverage, John views group theoretic concepts as a major component of what it means to learn group theory. This list of content includes:

> The definition of a group, properties of, there's a nest of elementary properties of group – orders of elements, homomorphisms, isomorphisms. Things about the definition like equivalent statements of the axioms. Honestly I don't really expect them to know what an axiomatic, algebraic structure is. You know, maybe this is their first example of that, so just the idea that there is something, a mathematical object. Theorem-wise, I think the main theorems are things about like cosets, partitions of group elements, order of elements divides the size of the group, Lagrange. Content wise I feel like that first course, it would be nice if it gets up through kind of up to what the curriculum covers. Up through homomorphism and isomorphism.
>
> (John Int #2)

In relation to this list of mathematical content, John considers a successful student as someone who can "come out of the class and summarize what they've learned with a certain degree of sophistication that you would not expect from a student who just looked up the definition online". However, content is just one facet of John's view of what it means to learn group theory.

In addition to learning the major concepts and theorems of introductory group theory, John also described two other components related to learning group theory: proof and other aspects of advanced mathematical thinking. For John, learning proof includes "both understanding a proof when it's given but also creating a proof, understanding the need

130

for proof, some of the conventions that happen when proofs are written, what rigor looks like". In that way, John sees group theory as a course where students "get some practice and some skill at, and [are] indoctrinated to really, to *doing math* as opposed to just applying a given formula". Therefore, in John's description of what it means to learn group theory we see both group theoretic content and mathematical processes.

Summary: Goals for Student Learning

In discussing the TAAFU curriculum, Lee raised an interesting question: "the focus of the curriculum is for the students to 'rediscover the content'. So is the focus on the rediscovery, or on what is rediscovered?" Lee's view, that the process of reinvention and the mathematics should not be seen as distinct, mirrors aspects of the RME perspective. As Gravemeijer (1999) noted:

> In [Freudenthal's] view, students should learn mathematics by mathematizing: both subject matter from reality and their own mathematical activity. Via a process of progressive mathematization, the students should be given the opportunity to reinvent mathematics. In this manner, Freudenthal transcended the apparent dichotomy between mathematic as an activity and mathematics as a body of knowledge.
>
> (Gravemeijer, 1999, p. 158)

Similar to Lee, John and Julie both clearly cared about the reinvention process (as we will see in their discussion of the role of the teacher in the following section). However, while Lee did not separate the process of rediscovery from what is rediscovered, John and Julie expressed explicit concern over what it was that the students reinvented.

4.3. Role of the Teacher

Due to the RME influence on the development of the TAAFU curriculum, the curriculum developers had certain expectations for the instructors implementing the curriculum. Specifically, the curriculum was designed with the intention that teachers 1) build on the students' informal ideas to help them develop more formal ways of reasoning, and 2) ensure that the students maintain ownership of the emerging mathematical ideas. All three mathematicians were very reflective about their role when implementing the TAAFU curriculum, and each appeared to attend to the development of mathematical ideas while staying true to the students' contributions. However, each teacher seemed to place a different emphasis on these two goals.

Julie

For Julie it does not matter if teachers are lecturing or using student ideas to guide the course, "everybody, regardless of how they teach, are doing what they are doing because they want their students to understand". In fact, Julie saw her role as a teacher implementing the TAAFU curriculum and the role of a lecturer to be similar in many regards. In both cases, Julie sees the teacher as being responsible for getting students to understand the core math ideas for the lesson.

> You have this core math idea. You have this goal. And now you have all their student thinking. You have where they are at and you are trying to get them to where you want them to be. You know, you are trying to think – okay these are the tools they have provided me. How can I fit them together to get them to this place? You have to figure that out mathematically… You're still doing what you did when you lecture. You know, when you thought about how you were going to present these ideas.
>
> (Julie Int #3)

Julie sees the difference between lecturing and teaching with the TAAFU curriculum as being related to whose ideas lead the class in reaching the mathematical goals of each lesson. "Instead of you getting to share your ideas and how you think about it you have to go out there and pull their ideas and figure out how to make it work". In order to do so, Julie believes that teachers must both be able to anticipate their students' ways of thinking and be able to respond to unanticipated contributions. Julie believes this way of teaching requires a "leap of faith that the things I've anticipated, the things I need, are going to come out. And I have to be ready to deal with the things that I didn't anticipate and putting things together in a way that I didn't anticipate".

Because the teacher is not the only one telling the story, "there is a much higher chance that you are going to screw up. That you are going to make a mistake. And that's scary for teachers." However, while Julie could understand why it could be scary for teachers to have to do mathematical work in on the fly in order to respond to students' (sometimes unanticipated) contributions, Julie stated that, "that is one of the things that I love about it. That's fun for me, figuring out what is going to come next". For Julie, this is a central and enjoyable task when implementing the TAAFU curriculum – pulling out student ideas and figuring out how to use them to build towards the mathematic goal of the lesson. "It is why I like it too, as [a] mathematician, it's like doing a proof… And once you figure out the mathematics of it then you think, 'okay how can I facilitate the students to figure it out?'"

Julie strived to guide the students from their current level of understanding to the intended mathematics, utilizing both finesse and intentionally. In describing her role, Julie stated:

> You are just kind of being sneaky about it. You are tricking them into thinking they are doing it. And the difference is that they are doing it this time. Where when you lectured they might have been…Now you have more evidence that they did and they all kind of have to.
>
> (Julie Int #3)

Because of the subtlety Julie described, she believes that there may be some misconceptions about the teacher's role. "You are still the teacher. The students might not see your teaching. But you are still in control." However, the nature and degree of control is different in this setting. Instead of controlling the exact content that gets stated in a lecture, the teacher's responsibility is to monitor, select, and sequence student ideas.

> You know what the idea is you want your students to learn about that day, to monitor what they are thinking about, and to figure out how to orchestrate a discussion that will present their ideas in a way that they will be able to make connections and move forward in their thinking.
>
> (Julie Int #3)

It was important to Julie that making connections between her students' work and her mathematical goals was not left entirely up to the students. Instead Julie saw it as her job to make explicit the connections between the tasks the students were working on and the overarching mathematics concepts the curriculum was building towards.

> I think you have to facilitate… there is this risk that you can pose the problem and then you can have five groups share how they did it and then you can go to the next problem [without any additional discussion of the groups' ideas]. And you can assume that the students will make the connections, and some of them will

and some of them won't. I think to really be effective you have to push yourself further than that. That you have to think about what those connections are and you have to make sure that they explicitly come out. Otherwise you don't know who got it and who didn't. You are right back to where you were when you taught the old way.

(Julie Int #3)

Therefore, in order to help ensure that her students were making the intended connections, Julie strived to bring intentionality to each task and lesson.

I think you need to know what your core idea is for that day and you need to make all of your decisions based off of that… I think that is one of the big misconceptions about inquiry-based instruction… that we just pose these great problems and magic happens… You should be really intentional about managing the ideas that come up and you should be making choices. But I also think that that is the really hard work that is part of teaching this way.

(Julie Int #3)

Even with this intentionality and focus on the mathematical goals for each lesson, Julie worked hard to teach in a way that allowed her students to maintain ownership of the mathematical ideas. Julie's believes that, ideally, "the students see the whole class centered around them and their ideas and they don't see you teaching". This allows students to, in addition to understanding the material, develop an ability to "learn to think about and create mathematics, to have that ownership of it."

This balance between leading the class towards her own mathematical goals for the lesson while maintaining student ownership of the mathematics was reflected in the way that Julie led class discussions. For instance, while working to develop the formal definition of a group, Julie's class struggled to explicate that a group consists of a set of objects along with an operation. By soliciting student ideas and contributions, Julie was able to determine that the major obstacle facing her class was confusion about what type

of objects the symmetries of an equilateral triangle were. Roughly half the class favored a view that the symmetries of a triangle were the various ending configurations resulting from the application of a motion acting on the triangle, and the other half of the class favored a view that the symmetries of a triangle were the motions that acted on the triangle.

Even though Julie had evidence that some of her students held the mathematical understanding she was trying to evoke, Julie did not resolve or conclude this discussion by telling the students "the right answer". Instead, Julie generated debate and introduced new questions to try to leverage the students' previous work with the symmetries of an equilateral triangle. For instance, Julie asked students to share why they were learning towards position or motion, Julie recapped how the class had determined how many symmetries there were, and Julie asked student to think about what it meant when they combined two symmetries. Julie then had the class consider other examples of groups, such as the real numbers under addition, in order to help students recognize the structure of the system (i.e., there is a set of objects and an operation that acts on those objects). Julie then worked with the students to formally define a binary operation by drawing on addition of real numbers as a generic example.

Julie's multiple interventions ultimately resulted in the classroom gaining insight into the nuances of the symmetries of an equilateral triangle. However, even when a student expressed the exact reasoning Julie had intended her students to develop, Julie still provided an additional opportunity for other students to grapple with the mathematics.

> Sarah: So would you say that the objects are like the flips? I don't want to say the operations because that's a different category. But the objects are things and the

operation is, you're adding together the two *F*'s and *R*'s. So, *F* is an object and *R* is an object and the operation is that you are combining them?

Julie: What do you guys think?

In explaining her decision to ask the class this question, as opposed to concluding the discussion now that the "right answer" was clearly expressed by a student, Julie's beliefs about student learning come into focus.

> I just think it's important. I think it just goes back to what I believe about how students learn math. We know that if we sit and tell people the way it works, even if we are really good explainers, … they will hear my explanation and somehow their interpretation of it won't be mathematically correct or valid. And the only way I know how they are thinking about what was just said, whether it is something I said or whether it is something another student said, is to hear them make sense of it. So, I mean, I feel pretty strongly that that's why you have to throw that back to the students.
>
> (Julie SR #1)

It is clear that Julie takes an active role in supporting students' mathematical development. However, Julie strives to ensure that her active participation does not come at the expense of her students' ability to meaningfully engage with the mathematics. As a result, Julie's role when implementing the TAAFU curriculum can be seen as a careful balance between intentional guidance and respect for students' emerging mathematical understanding.

The Role of the Teacher when Implementing
the TAAFU Curriculum
Julie Fredericks

To me, the primary role of the teacher in any course is to help students develop and deepen their mathematical understanding of the core mathematical ideas of the course, regardless of the curriculum used or the style of teaching. The biggest difference I see between a class using the TAAFU curriculum and a course using a more traditional book combined with lectures is where the mathematical ideas that drive the class originate.

When I plan a lesson for a course where I am using a more traditional textbook, I think about the ideas I want students to encounter as a result of this lesson and then select the mathematical examples, definitions, questions, theorems, etc. that I want to introduce in order to help the students process the core math ideas of the lesson and develop their mathematical understanding. In this course, I am the one introducing the mathematical ideas and controlling how and when students interact with them.

In contrast, when planning a lesson using the TAAFU curriculum, the course is centered on student ideas. When I plan a TAAFU lesson, I anticipate the mathematical ideas students will bring up as they wrestle with the task(s) for that day. Then I have to consider which of these student ideas will help the class develop their collective understanding of the core math ideas. Which student mathematical ideas should I focus the class on in order to prepare the students for future lessons? Are there any student ideas from previous lessons that students need to build upon and/or extend? Is there any mathematical information (e.g., mathematical definitions, additional examples, prior theorems, etc.) that I need to introduce and what options do I have for connecting my interjections to their ideas?

In my facilitation of a TAAFU lesson, I am still controlling which ideas the class focuses on, but the ideas we are discussing all originate and are centered on the students' thinking about the tasks. Ideally, the students don't even notice my "teaching" and see themselves as the center of the class and the creators of the understanding. They believe the class discussions just come together naturally.

In lecture courses based on more traditional texts, I can re-use lectures year after year with minimal changes. With a TAAFU course, although you can reuse the thinking you did to prepare for the lesson, each time you teach a TAAFU lesson it is unique. No matter how much you anticipate, you are always faced with having to identify which student ideas have surfaced in that class on that day and figure out how you are going to use those ideas as building blocks for the class' discussion so that the students can process the core math ideas of the lesson and develop their mathematical understanding. The mathematical knowledge and creativity required to do facilitate these courses is what makes them so much fun for me to teach.

John

John saw "maintaining the narrative" as his main role when implementing the TAAFU curriculum. As John states, "my main effort in each day seemed to be trying to keep a sensible narrative moving forward in concert with the tasks, while also trying to incorporate student contributions from class." In order to do so, John found himself constantly evaluating his students' current level of understanding in relation to where the curriculum was headed next.

> You definitely aren't the only person telling the story anymore. Where are the students at [mathematically]? Where are the questions? What is that question based on? How can I get them from what they are asking now to what I would really like to be talking about next? That was where the effort was… That was my role of the teacher, was to try, as much as possible, to get the next thing out of what we were working on and what they were experiencing. That's the narrative.

> (John Int #1)

Notice in his discussion, John was not solely focused on the mathematical agenda. Instead, John expresses an emphasis on bridging student ideas and the intended mathematical goals of each lesson.

Indeed, John felt that he was fulfilling his role as teacher best when he was able to build on his students' understanding in order to set up the next instructional goal.

> And it feels good when it works, and it brings out something new and interesting and by the way, this is kind of what we were going to be doing next anyway, and it came from them. You know, that is kind of the ultimate pat on the back. Oh that went really well. You know they kind of, naturally for themselves, got us to the next spot. So anyway that was kind of my, that was my job.
> (John Int #1)

This monitoring of student ideas in anticipation of advancing the mathematical agenda for the lesson echoes aspects of Julie's perspective.

Lee

Given Lee's focus on the rediscovery process, as opposed to covering specific content, it is not surprising that Lee held a different view of his role as teacher. Instead of trying to get the students to the next step in the reinvention process or move them towards some predetermined mathematical goal, Lee wanted to make sure that his students really understood the task that they were currently working on.

> My primary role is for us to get this, and if that requires us to think about some side item aspects that don't carry us in the right direction then I think I have a responsibility to try to evaluate the value of those side items and allow as many of them as we have opportunity for that are of value to the students.
>
> (Lee Int #2)

For Lee, it was the curriculum developer's job to make sure the tasks fit together into a cohesive whole that leads students through the reinvention process. In fact, Lee believes guiding the students towards a mathematical agenda is actually at odds with the reinvention process. When asked to respond to John's concerns about "maintaining a narrative", Lee expressed some caution about the extent to which the narrative should guide the class.

> The other question is, you still have to decide to what extent the ribbon of narrative is more important than the meaning that the students are in the process of constructing.
>
> (Lee Int #2)

Lee later elaborated his view when discussing what he saw as his role as the teacher.

> If too much of the energy is distributed toward moving the process forward then an insufficient amount of energy will be dedicated to the aspect at hand. And part of dedicating the amount of energy you need to the aspect at hand is allowing that the freedom to go the wrong way.
>
> (Lee Int #2)

Therefore, Lee saw it as the curriculum developers' job to keep the students moving towards an overarching goal, and the teacher's job "to support whatever it is they [the students] are coming up with to the greatest extent you can without completely losing the focus of the class". This support included pressing students on, "points where either it wasn't clear or where I felt like it wasn't clear to the students".

Summary: The Role of the Teacher

All three mathematicians appear to be very sensitive to the competing goals of advancing the mathematical agenda and developing deep understanding on the part of their students through authentic engagement with the mathematical tasks. This is consistent with Gravemeijer and Doorman's (1999) discussion of implementing RME inspired curriculum.

> There will always be tension between a bottom-up approach that capitalizes on the inventions of the students and the need, (a) to reach certain given educational goals, and (b) to plan instructional activities in advance… this boils down to striving to keep the gap between 'where the students are' and what is being introduced as small as possible.
>
> (Gravemeijer & Doorman, 1999, p. 124-125)

141

However, we see in their reflections that each of the three mathematicians seemed to have a slightly different opinion as to the width of the acceptable gap between where the students are and the mathematical goals. The spectrum ranged from Lee's attempt "to support whatever it is they [the students] are coming up with to the greatest extent you can without completely losing the focus of the class" to John's attempt "to try, as much as possible, to get the next thing out of what we were working on". Julie's view of her role seems to fall between these two perspectives, and is consistent with the emergent models heuristic that informed the development of the TAAFU curriculum. In particular, Julie worked intentionally to elicit her students' informal reasoning and then leverage this understanding to develop more formal mathematics.

5. Conclusions

The discussion above makes it clear that the three mathematicians were very reflective on their teaching practice and balanced a variety of concerns and objectives as they implemented the TAAFU curriculum. However, considering each teacher across his or her interviews and reflections we see remarkable consistency. Specifically, for each teacher one of the three themes we described above was a particularly salient aspect of their experience.

After implementing the curriculum, John came away with two new insights about the coverage issue: 1) that he had previously "been in denial about the level of sophistication many students bring into the course", and 2) the curriculum developed a depth of understanding that he did not see when he taught the course through lecture. Interestingly, these two factors exacerbated John's initial concerns about coverage

142

because, when he implemented the TAAFU curriculum, he was more aware of his students' struggles and more concerned about students coming away with a deep understanding of the material. As a result, John had a tendency to actually slow the pace of the course. Throughout John's reflections we see he was concerned both with the depth of his students' understanding and his coverage of the core concepts. In John's description of his role as the teacher, we see him balancing these two priorities. This is best exemplified by the questions John asked himself while teaching: "How can I get them from what they are asking now to what I would really like to be talking about next?" Ultimately, John appeared to be satisfied with this balance and his implementation of the curriculum. "They are going to come out of this [course] knowing things deeper than they would otherwise… And that's the reason I wasn't disappointed."

Julie's reflections consistently captured her focus on identifying and utilizing student thinking as a way to advance the mathematical agenda. In order to meet this objective, Julie carefully considered both the mathematical goals of the lesson and her students' current understanding. These considerations allowed Julie to 1) anticipate her students' ways of thinking and take a "leap of faith" that the students will generate the ideas that the curriculum intended, and 2) sneakily maintain a level of control that allowed her to carry out her role as the teacher. While implementing the TAAFU curriculum, Julie saw her job as: knowing the key mathematical point for each day, monitoring student thinking, and supporting students in making connections that advanced their understanding. As a result, Julie was confident that her students were developing a deep mathematical understanding that would support them as they worked with new material. Because of this, Julie was not overly concerned with coverage. "So even if you cover a

143

little bit less, if you cover it deeper the students will be able to figure out the parts you didn't get to."

In Lee's reflections we see a consistent emphasis on students engaging in authentic mathematical activity. Accordingly, Lee believed that his own mathematical agenda, or a set list of content goals, should not curtail his students' opportunities to engage in authentic mathematical activity. As a result, Lee was unconcerned with coverage issues. Instead Lee's efforts were focused on providing students with an opportunity to engage in activities that encourage skills and experiences that would support them in further mathematical explorations. Through such activity, Lee believed that his students would come out of this class with a mathematical curiosity and a solid understanding of group theory. Therefore, Lee felt that his role in the classroom was "to support whatever it is they [the students] are coming up with to the greatest extent you can without completely losing the focus of the class".

Interestingly, even with their different perspectives, all three mathematicians reported success and enjoyment with their implementation of the TAAFU curriculum. The success of each of the mathematicians seems to have been supported by their beliefs, which while different from each other, seemed to be aligned with RME and the intent of the curriculum. For instance, the reflections of all three teachers mentioned the importance of: valuing the idea that reinvention can result in deeper conceptual understanding; supporting students in developing formal mathematics from their informal understandings; and balancing advancing the mathematical agenda with the students' development of deep understanding. This is consistent with research at the elementary

level that reports that teachers' beliefs are an integral factor in curriculum enactment (Collopy, 2003).

6. Discussion

This paper was written with two goals in mind. First, we aimed to present the mathematicians' experiences from their own perspective. In doing so, we hoped to provide an opportunity for the research field to gain insight into the factors and considerations that matter to mathematicians. Second, we sought to provide new insights into "the teacher's perceptions and stance towards curriculum materials and the teacher's professional identity" as called for by Remillard (2005, p. 235). In doing so, we hoped to provide needed confirmation of some existing findings and to generate new insights to be explored via careful analyses of classroom practice.

As previously noted, there is little in the research literature about mathematicians as teachers. The empirical research that does exist tends to focus on institutional norms and policies that support or constrain the implementation of student-centered instruction (McDuffie & Graeber, 2003; Walczyk & Ramsey, 2003; Walczyk, Ramsey, & Zha, 2007) or on analyzing mathematicians' teaching practices (Johnson, 2103; Johnson & Larsen, 2012, Speer & Wagner, 2009). In this paper we took a different approach in that we sought to analyze post-implementation reflections in order to identify aspects of the curriculum implementation that were significant to the mathematicians. In doing so, we identified coverage, goals for student learning, and the role of the teacher as significant themes. Interestingly, some direct connections can be made between our findings and

145

some reported research focused on mathematicians' implementation of an inquiry-oriented differential equations course.

Wagner, Speer, and Rossa (2007), discuss Rossa's struggles to, "make decisions about, and monitor the pace and scope of, the course as a whole" (p. 263). For Rossa, a mathematician, this was a challenge because he struggled to see how the individual tasks and lessons fit together to cover the differential equations curriculum. While John also struggled to set the pacing for the TAAFU course, John's struggles were more related to a tension between setting a pace that was fast enough to ensure that the class finished the curriculum, and slow enough to make sure that his students developed a deep understanding of the material. For John, this tension was tied to a belief that, "when you have the material reinvented, in my mind it is going to slow down things". Other researchers and mathematicians have also discussed the difficulty of incorporating non-lecture teaching strategies when a set amount of material must be covered (McDuffe & Graeber, 2003; Wu, 1999). For instance, when discussing the need to cover a set amount of material in a math course for elementary teachers, one mathematician in McDuffe and Graeber's (2003) study stated, "there's an expectation that a certain amount of material be covered… it means you are limited on how much time you can spend to do real constructivist activities where the depth of knowledge is really greater" (p. 336).

Speer and Wagner (2009) presented a study in which they sought to account for the difficulties a mathematician was facing while trying to provide analytic scaffolding during whole class discussions, where analytic scaffolding is used to "support progress towards the mathematical goals of the discussion" (p. 493). Speer and Wagner argue that leading whole class discussions is particularly important for inquiry-oriented curricula

146

because there is a greater emphasis on fostering mathematical communication and valuing students' mathematical reasoning. It is interesting that Julie, a practitioner, described her role as the teacher in a similar vein. For instance, Julie discussed her intention to "figure out how to orchestrate a discussion that will present their ideas in a way that they will be able to make connections and move forward in their thinking". Therefore, we see that Julie's perspective of her role as the teacher mirrors Speer and Wagner's views.

The research reported here raises issues that could be investigated as part of further efforts to understand and support student-centered instruction. Research into the kinds of factors (including concerns about coverage, goals for instruction, and views about the teacher's role) that influence the implementation of such curriculum could support such efforts. For instance, Julie's description of how she views her role as the teacher, and the mathematical work that is necessary to orchestrate whole-class discussions, suggests that more research needs to be carried out to understand how mathematicians draw on their Mathematical Knowledge for Teaching (Ball et al., 2008) in order to implement reform-oriented curricula. Additionally more general research, say into the extent to which the themes described here apply to other mathematicians and other kinds of student-centered curriculum, could contribute to frameworks for understanding the important issues related to mathematicians' implementation of such curricula. Research situated in the context of the TAAFU project will enable the project team to refine the curriculum and instructor support materials or to develop professional development strategies in order to facilitate successful implementation.

Conclusions

Contributions and Future Research

The goal of this dissertation was to build a foundation for investigating teaching and learning in inquiry-oriented classrooms. These three papers are intended to provide lenses for investigating the ways in which teachers' activity and perspectives influence their students' learning. My two papers researching teaching in inquiry-oriented settings, *Implementing Inquiry-Oriented Curriculum: From the Mathematicians' Perspective* and *Teachers' Mathematical Activity in Inquiry-Oriented Instruction* have begun to answer questions about teachers' perspectives and mathematical activity. During the course of researching teachers' activity and perspective, I became aware of a need for documenting student learning in inquiry-oriented contexts. This need motivated the theoretical paper, *Realistic Mathematics Education Design Heuristics and Implications for Documenting Student Learning.* By synthesizing the theoretical research on Realistic Mathematics Education (the instructional design theory used to design the inquiry-oriented curriculum), I was able to explore the implications of RME for documenting student learning. Here I will briefly consider the contributions made by my two empirical studies and consider how my theoretical paper could add to my research efforts.

Implementing Inquiry-Oriented Curriculum: From the Mathematicians' Perspective presents an alternative avenue for investigating inquiry-oriented instruction. In this paper we investigated and documented the teaching experiences of three mathematicians, as opposed to analyzing the mathematicians' teaching practices. In doing so, we hoped to gain insight into the factors and considerations that matter to mathematicians; in particular, mathematicians that are willing and excited to implement student-centered

148

curricula. Because of the unique source of these insights (teachers' reflections rather than analyses of practice) our findings provide both a useful confirmation of some existing findings and new ideas to be explored via careful analyses of classroom practice. For instance, research could be carried out to investigate the impact of the teachers' perspectives (including concerns about coverage, goals for instruction, and views about the teacher's role) on their teaching practices. Additionally more general research, say into the extent to which the themes described here apply to other mathematicians and other kinds of student-centered curriculum, could contribute to frameworks for understanding the important issues related to mathematicians' implementation of such curricula.

In *Teachers' Mathematical Activity in Inquiry-Oriented Instruction,* I analyzed classroom video data in order to identify instances in which teachers engaged in observable mathematical activity in response to the mathematical activity of their students. In addition to documenting the mathematical work of teachers, my analysis considered the interactions between teachers' mathematical activity and the mathematical activity of their students. This analysis suggests that teachers' mathematical activity can be a significant component in supporting students' mathematical development, both in terms of subsequent students' mathematical activity and in terms of the mathematical discourse of the classroom community. This work makes two significant contributions to the mathematics education research literature. First, it provides a lens, Teachers' Mathematical Activity, for investigating teachers' in the moment mathematical activity. While examples of teachers' responsive mathematical work had been reported and studied throughout the mathematics education literature (ranging from K12 teachers, to

pre-service elementary teachers, to mathematicians), the establishment and definition of this construct allows for communication between researchers and focused lens for analysis. Second, this work begins to investigate why such mathematical work done on the part of the teacher is valuable in terms of the impact it has on students.

With the addition of my theoretical work for documenting student learning, I now have a comprehensive approach for analyzing classroom interactions – one that accounts for both teacher activity and student learning. This offers several affordances for my future research efforts. For instance, my characterizations of student learning provide a lens for identifying key aspects of students' mathematical activity in a classroom episode. An analysis focused on aspects of student activity that provide evidence of student learning could be coordinated with an analysis of the teacher's mathematical activity in order to generate more precise conjectures relating student learning and teachers' mathematical activity. Additionally, these two lenses could support a comprehensive analysis of classroom activity in which the impact of each instance of teacher mathematical activity is documented in regards to student learning. Such a study would allow me to provide a complete characterization of the relationship between teachers' mathematical activity and student learning in that classroom.

Some Additional Theoretical Considerations

In its current form, *Realistic Mathematics Education Design Heuristics and Implications for Documenting Student Learning,* is written to make sense of the complex ways in which RME is discussed in the research literature and begin to consider how RME can inform the analysis of student learning. While I addressed a number of

150

theoretical issues related to documenting student learning in the context of inquiry-oriented instruction, there are a number of questions that could be raised in light of my discussion. Here I will briefly discuss three of these questions without taking a firm stance on any of them, as it is my intention that my theoretical contributions could be of use to individuals with a variety of perspectives.

The first question that I would like to address is the question of whether one (when attempting to document student learning in the context of inquiry oriented instruction) should consider learning in terms of participation or in terms of acquisition. In my paper, I developed characterizations tied to RME design heuristics from each perspective and it is my intention that one could rely on either one or both of these characterizations when attempting to document student learning. Researchers may choose to maintain a tight theoretical focus and consider learning strictly from one of these perspectives, or they may choose to attempt to coordinate these two perspectives.

The second question I would like to address deals with the relationship between my work and established learning theories. I feel that the theoretical ideas that I have developed are consistent with any number of learning theories, but that there are some clear connections that could be made with some specific theories. In particular, the notion of reification is an idea that is consistent with constructivism (Ernest, 1996). For example, the APOS theory (Dubinsky & McDonald, 2002) is based on constructivism and the reification of mental objects is an important part of that theory. From my point of view, a researcher who thinks about reification in terms of APOS theory would find my characterization of student learning in terms of reification to be useful analytically.

However, my intention is that a researcher who, for example, thinks about reification in terms of procepts (Tall, 1999), would find my characterization just as useful.

Finally, I would like to address the question of whether the ideas I have developed are related to learning at the individual or collective level. From my perspective, these ideas are appropriate to consider learning at either or both of these levels. For example, I conjecture that one could use the characterizations I have developed in conjunction with the emergent perspective (Cobb & Yackel, 1996), which is designed to support analyses of learning at the individual and collective levels.

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