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TUNNEL ONE GENERALIZED SATELLITE KNOTS

by
JOHN RALPH NEIL

A dissertation submitted in partial fulfillment of the
requirements for the degree of

DOCTOR OF PHILOSOPHY
in
SYSTEMS SCIENCE: MATHEMATICS

Portland State University
1995

DISSERTATION APPROVAL

The abstract and dissertation of John Ralph Neil for the Doctor of Philosophy in Systems Science: Mathematics were presented May 1, 1995, and accepted by the dissertation committee and the doctoral program.

COMMITTEE APPROVALS:

Steven A. Bleiler, Chair

M. Paul Latiolais

Charles B. Balogh

Andrew M. Fraser

James L. Hein
Representative of the Office of Graduate Studies

DOCTORAL PROGRAM APPROVAL:

Beatrice T. Oshika, Director
Systems Science Ph.D. Program

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ABSTRACT

An abstract of the dissertation of John Ralph Neil for the Doctor of Philosophy in Systems Science: Mathematics presented May 1, 1995.

Title: Tunnel One Generalized Satellite Knots

In 1984, T. Kobayashi gave a classification of the genus two 3-manifolds with a non-trivial torus decomposition. The intent of this study is to extend this classification to the genus two, torally bounded 3-manifolds with a separating non-trivial torus decomposition. These 3-manifolds are also known as the tunnel-1 generalized satellite knot exteriors. The main result of the study is a full decomposition of the exterior of a tunnel-1 satellite knot in an arbitrary 3-manifold.

Several corollaries are drawn from this classification. First, Schubert's 1953 results regarding the existence and uniqueness of a core component for satellite knots in the 3-sphere is extended to tunnel-1 satellite knots in arbitrary 3-manifolds. Second, Morimoto and Sakuma's 1991 classification of tunnel-1 satellite knots in the 3-sphere is extended to a classification of the tunnel-1 satellite knots in lens spaces. Finally, for these knot exteriors, a result of Eudave-Muñoz in 1994 regarding the relative position of tunnels and decomposing tori is recovered.

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Introduction

The main result of this study is the classification of tunnel-1 generalized satellite knots. These results are summarized in the main theorem of the paper.

Theorem 12.1 Let M be a 3-manifold with toral boundary such that M has a splitting of genus two and a non-trivial toral decomposition where all decomposing tori are separating. Then, by decomposing M along all such essential tori into simple or Seifert fibered components, then either

1. *M is obtained from $M_1 \in D(2)$ and $M_2 \in V_K$ where the regular fibre of M_1 is glued to a meridian loop of M_2 . Moreover, if M_2 is not simple, then M_2 decomposes into two components, $M'_2 \in A(1)$ and $M''_2 \in M_L$, where the regular fibre of M'_2 is glued to a meridian loop of M''_2 and the other meridian loop is glued to the regular fibre of M_1 .*
2. *M is obtained from $M_1 \in A(1)$ and $M_2 \in L_K$ where the regular fibre of M_1 is glued to the meridian loop of M_2 . Moreover, if M_2 is not simple, then M_2 decomposes into two components, $M'_2 \in D(2)$ and $M''_2 \in M_L$, where the regular fibre of M'_2 is glued to a meridian loop of M''_2 and the other meridian loop of M''_2 is glued to the regular fibre of M_1 .*
3. *M is obtained from $M_1 \in A(2)$ and $M_2 \in M_K$ where the regular fibre of M_1 is glued to a meridian loop of M_2 .*
4. *M is obtained from $M_1 \in \hat{M}\ddot{o}(n)$ ($n = 0, 1$) and $M_2 \in M_K$ where the regular fibre of M_1 is glued to the meridian loop of M_2 .*

This theorem, its proof, and several important corollaries all appear in Chapter 12. The proof of the main theorem draws primarily on material in Chapters 6, 8,

9, 10, and 11. Chapters 1, 2, 3, 5, and 7 recall some basic results in 3-manifold topology. Chapter 4 relates 3-manifold topology to systems science. Chapter 13 contains concluding remarks and several open questions.

There are many which need to be acknowledged with regard to the preparation of this dissertation. First and foremost, I thank my wife and family for the support I have received as I have progressed to this point in my education. Additionally, I thank my dissertation advisor, Dr. Steve Bleiler, for his many long hours devoted to my edification. I thank the members of my dissertation committee for their helpful comments.

Chapter 1

Fundamental issues in topology and algebra

One of the most ancient of mathematical disciplines is geometry. The ancient Greek mathematicians strove to understand the world around them by the properties of the basic shapes they observed. Remarkably, many of the issues which faced mathematicians such as Euclid and Pythagoras are very similar to those faced by modern mathematicians.

In the 18th century, certain geometric properties were discovered which appeared to be true for certain types of figures regardless of how the figure was stretched. Among these was an integer which has come to be called the Euler characteristic. Leonhardt Euler found that if we take a polyhedron and add the number of faces to the number of vertexes, it always equals two more than the number of edges. This turns out to be true regardless of the number of faces required to construct the polyhedron. For example, in the cube there are six faces, eight vertexes, and twelve edges (see Figure 1.1), and in the tetrahedron or *pyramid* there are four faces, four vertexes, and six edges.

Thus was born the field of topology. In the study of topology, we look at the geometry of spaces where shape is flexible. This is in sharp contrast to the field of classical Euclidean geometry where shape is rigid. This flexibility of shape provides both benefit and cost. As a beneficial aspect of this lack of rigidity, if a particular shape doesn't look exactly the way we want it to, we have the freedom to stretch it

Figure 1.1: Two spaces with the same Euler characteristic

until it does look like something we can work with. As a potential problem, topologists make no distinction between circles and squares, for example, since any square can be stretched until it's round. In addition, concepts like volume, area, and perimeter, since we can stretch shapes almost arbitrarily, are not quantities which are usually of interest.

What we look for in topology are methods of distinguishing spaces which are not dependent on a rigid structure. For example, we might trace a loop in the space, as if we had thrown out a fishing line, caught the end and pulled it taut. If the line catches on something, we regard this as significant. We refer to a loop of this kind as an essential loop. This kind of technique allows us to distinguish different types of surfaces. For example, no matter how we throw out a loop around a sphere, we can always pull it back. There isn't anything on the surface of a sphere on which the line will catch. However, if we draw certain loops on a torus, which is what topologists call the surface of a donut, we find that there are certain loops we cannot pull back (see Figure 1.2).

If we start with the original essential loop we found, we can construct multiples of this loop by going around it more than once. If we have more than one essential loop, we can construct a product between them by first going around one of the loops and then around the other. This gives us an algebraic structure in the space based

Figure 1.2: Two distinct topological surfaces

on the essential loops it contains. In the above torus, we see that there are two essential loops which we might consider fundamental. All other loops we might place on the torus will be products of multiples of these two loops. This gives us important information about the group structure of loops in the torus. This algebraic structure is referred to as the fundamental group of the space.

We refer to a connected space where every loop pulls back as being simply connected. The sphere above is simply connected. Every loop we lay down on the sphere can be contracted to a point. This means that the fundamental group of the sphere is trivial. That is, there are no essential loops from which we can construct products.

A closely related concept to the idea of a space being simply connected is that of a space being contractible. We say that one space is contractible as it sits in some larger space if it can be collapsed down to a point in the larger space. Thus, we can refer to inessential loops as being contractible since these are the loops which collapse down to a point without snagging on anything in the space.

A particular type of space which topologists often study is called a *manifold*. A manifold is a space where every point in the space has a small area around it which is topologically equivalent to some standard Euclidean space. We identify the dimension of the manifold by the dimension of these small pieces of the manifold. For example, a one-dimensional manifold looks, when we examine small pieces of it, like a small

line segment. A two-dimensional manifold looks like a disk when we look at small pieces of it. A two-dimensional manifold is, therefore, a surface of some kind.

We are going to be concerned with looking at the fundamental group of certain types of spaces which are called *three-dimensional manifolds* or *3-manifolds*. A 3-manifold is a space which, when we look at small pieces of it, looks like the space around us. There may be something global about a 3-manifold, such as its fundamental group, that might let us distinguish it from other 3-manifolds, but all 3-manifolds look alike locally. An example of a 3-manifold with boundary is a cannonball whose *edge* would be the sphere depicted in Figure 1.2.

A classical result of Whitney shows that all 3-manifolds can be embedded in seven-dimensional space. The 3-manifolds we will study are those which contain all their limit points and, in seven dimensional space, are bounded. We will refer to these types of spaces as closed 3-manifolds. If our 3-manifold has a boundary, we will refer to it as a closed 3-manifold with boundary.

The primary algebraic issue we will attempt to deal with is the minimum number of fundamental loops required to generate all of the group. We refer to this number as the rank of the group. This is a well-known quantity in algebra. We will be looking at what this algebraic quantity tells us about the geometry of the space.

As a geometric question, we will be looking at a generalization of the above idea of essential loops. We regarded a loop as being essential if we couldn't pull it all the way back. In this manner, the essential loop told us something about the nature of the space it lived in. Likewise, when our space is of higher dimension, we can place the loop in a surface, like a torus, and see if it can be pulled back in the larger manifold. If the essential loops in the surface are also essential loops in the manifold it resides in, the surface gives us important information about the nature of the larger manifold it resides in. We regard this surface as essential in the larger manifold.

In the case of a torus, since we have found that there are two fundamental loops which do not pull back, we would regard a torus in a manifold to be essential if an arbitrary product of powers of those two loops in the torus cannot be pulled back through the manifold. We can see in the above picture of a torus that while both of

Figure 1.3: A cannonball with a knotted hole drilled out

the loops indicated give essential information about the torus, if we put the torus in the three dimensional space which exists all around us, both of these loops can be pulled back through space. On the other hand, if we take our cannonball and drill out a knotted hole (as in Figure 1.3), we obtain a 3-manifold whose boundary is a torus. In this case, the boundary of the 3-manifold gives us essential information about the 3-manifold. This is an example of an essential torus.

In all of the above examples, the loops which we placed in the manifold were *tame*. That is, we can take each loop and thicken it as it sits in the manifold until we have a fattened up version of it. In this manner, we can take a loop which is a zero-width mathematical construction and represent it with a rope. This process is referred to as taking a *regular neighborhood* of the loop. We can, in a similar fashion, take the regular neighborhood of any sub-manifold of a manifold. There are examples of loops in 3-manifolds which do not have this property. The following knot is an example

Figure 1.4: A wild knot

of this (see Figure 1.4). It is called a *wild* knot since it does not possess a regular neighborhood in any 3-manifold.

This wild knot would be an example where we would have an algebraic construct without an adequate geometric analogue. In this knot we continue placing smaller and smaller local knots until we reach a limit point. It is at this limit point where the knot cannot be thickened. Thus, the geometry of a manifold does not account for this *wildness*. We will construct our manifolds via handle structures (see Chapter 3). In this case, the fundamental essential loops come with pre-existing regular neighborhoods and, as such, are all *tame*. Thus, the question is just how faithfully the algebra associated with these manifolds is represented in the geometry.

Chapter 2

Dehn filling and 3-manifolds

Before discussing the precise statement of the results of this research, we present some background material and terminology. All spaces under consideration are assumed to be compact, connected, and orientable unless specifically identified otherwise.

We begin our look at 3-manifolds by considering the solid torus V as $D^2 \times S^1$ where D^2 is the unit disc in the complex plane and S^1 is its boundary. This particular manifold has an infinite cyclic fundamental group with generator represented by the simple closed curve $\{0\} \times S^1$. The boundary torus of the solid torus, $\partial V = \partial D^2 \times S^1$, has fundamental group $\mathbb{Z} \oplus \mathbb{Z}$ with generators represented by $\mu = \partial D^2 \times \{1\}$ and $\lambda = \{1\} \times S^1$ (see Figure 2.1). In V , the curve μ bounds a disk and therefore represents the trivial element in the fundamental group. The curve λ is homotopic in V to the simple closed curve we have chosen as a representative for the generator of the fundamental group of V . We will refer to μ as the *meridian* of V and λ as the *longitude* of V .

Stereographic projection shows that the 3-sphere, S^3 , can be obtained by gluing two 3-balls, B_1^3 and B_2^3 , together via a homeomorphism between their boundaries (see Figure 2.2). From this, we identify one of the 3-balls as containing the origin and the other containing the point at infinity. This will be a useful distinction to make as it allows us to distinguish the *inside* of closed surfaces in S^3 from the *outside*. Alexander [1] demonstrated that the 3-sphere is obtained by any homeomorphism of the boundaries in a technique now known as the *Alexander trick*.

Figure 2.1: The solid torus V

Figure 2.2: The decomposition of S^3 into two B^3 's

Let h be the homeomorphism between ∂B_1 and ∂B_2 described above. Then by removing a regular neighborhood of the z -axis in B_1 and attaching it via h to B_2 , we are left with two solid tori, V_1 and V_2 . To reconstruct S^3 from these two solid tori via the homeomorphism of the resulting tori induced by h , we must glue the meridian of V_2 to the longitude of V_1 and the longitude of V_2 to the meridian of V_1 .

As mentioned above, Alexander showed that no matter how two 3-balls are glued together, one obtains S^3 . However, by varying the homeomorphism attaching two solid tori together, we can obtain 3-manifolds which are distinct from S^3 . For example, taking two solid tori and gluing them together via a homeomorphism which attaches the meridian of one solid torus to the meridian of the other as in Figure 2.3 is a 3-manifold, M , with some interesting characteristics. The Alexander trick also shows that any embedding of S^2 in S^3 bounds a 3-ball on at least one side. From this we can conclude that every 2-sphere in the 3-sphere is contractible. In M , the meridian of V_1 bounds a disk in V_1 , and the meridian of V_2 bounds a disk in V_2 , and the gluing homeomorphism glues these two disks along their boundaries forming a 2-sphere. However, in this construction, there is no 3-ball on either side of the 2-sphere. This is easy to see as neither disk separates its respective solid tori. Hence, this 2-sphere does not separate in the 3-manifold. Thus, we have a manifold which is fundamentally different than S^3 , in this case $S^2 \times S^1$.

We need to determine those self-homeomorphisms of the torus which yield the same manifold when used to attach a solid torus. To do this, we first decompose a solid torus V into a 3-ball B^3 and a *hockey puck*, $D^2 \times I$, where $V = B^3 \cup_{D^2 \times \partial I} D^2 \times I$. Let T be the toral boundary of the 3-manifold to which we are gluing the solid torus and consider the attaching of the solid torus as occurring in two steps. First we attach the hockey puck, then the 3-ball. Alexander's trick [1] shows that it does not matter how we attach the ball B^3 , so the information required in the attaching is contained in the attaching of the $D^2 \times I$ to the torally bounded 3-manifold. To track the annulus, $\partial D^2 \times I$, under the attaching, we only need to know the simple closed curve in T to which $\partial D^2 \times \{0\}$ maps. Thus, we need a method of parameterizing simple closed curves on a torus.

Figure 2.3: The attaching of two solid tori to create $S^2 \times S^1$

The torus is the quotient of the real plane, $\mathbb{R}^2$, by the integer lattice points. Each homology class of a simple closed curve on the torus is represented by a line in $\mathbb{R}^2$ which passes through at least two of the integer lattice points. The coordinates through which the line passes, up to deck translation, are a multiple of the homology coordinates of the simple closed curve in terms of the generators represented by μ and λ . Thus, any simple closed curve on a torus can be represented by a rational number. For convenience, we parameterize this by μ/λ where, in homology coordinates, μ is the number of meridians represented by the simple closed curve and λ is the number of longitudes represented by the simple closed curve. Hence, this line has a rational slope and we can say that μ/λ is the *slope* of the simple closed curve.

Our construction of $S^2 \times S^1$ via attaching solid tori together can be generalized in several ways. Our first variation was put forward by Dehn in 1910 [9]. Here we vary the process by the manner in which we remove the dowel from B_1 . Instead of a vertical hole we drill out a hole which has a knot in it. As before, attaching the ends of the dowel to B_2 still results in a solid torus. However, $X_1 = B_1 \setminus \{\text{dowel}\}$ is no longer a solid torus even though its boundary is still a torus (see Figure 2.4). Thus we can form a closed 3-manifold by identifying a homeomorphism $g : \partial V_2 \rightarrow \partial X_1$. Since V_2 is a solid torus, this homeomorphism is determined by the image of a meridian of V_2 .

Figure 2.4: A 3-ball with a knotted hole drilled out

The other information we need in this case is the identification of the knotted hole we put in B_1 . We do this by identifying the knotted arc which is the core of the dowel we removed and close it with a trivial arc in B_2 . We then have an embedding of S^1 in S^3 appearing as the core of the solid torus V_2 after applying the attaching map g induced by h .

Therefore, if we know the core of the solid torus we remove from S^3 and we know the image of the meridian when we glue it back, we have a well-defined closed 3-manifold called the result of *Dehn filling* on the complement of the knot represented by the core of our solid torus.

Still more generally, instead of removing just one solid torus from S^3 , we can remove several. Then, the information we need is the cores of the various solid tori in S^3 and the images of the meridians of these solid tori when we glue them back in. To quantify these ideas, we need some additional terminology.

Figure 2.5: The 3 basic Reidemeister moves

Recall that a *link* in S^3 is a subset that is homeomorphic to a disjoint union of circles. A knot is thus a one component link. Two links are said to be equivalent if they are *ambiently isotopic* (see [29]). Ambient isotopy on links induces an equivalence relation on link diagrams which is generated by the three Reidemeister moves (see Figure 2.5). That is, two diagrams are equivalent if one can be deformed into the other via a sequence of the three Reidemeister moves.

A *framed link* is a link diagram with a rational number f_i assigned to the i^{th} component of the link. By taking the various components of a framed link as the cores of the solid tori we are removing from S^3 and the framings on each component as the slope of the simple closed curve to which we will glue the meridian curve when gluing solid tori back in, we see that a framed link is a description of a closed 3-manifold. A fundamental theorem of Lickorish and Wallace proves that every closed, orientable 3-manifold can be represented by some framed link (again, see [29]).

Through this construction technique in [27], use of the combinatorial structure of knots in S^3 allowed calculation of certain 3-manifold invariants which arise out of quantum field theory in mathematical physics. In particular, the invariants of 3-manifolds which arise under certain conditions are amenable to calculation (see Chapter 4).

Chapter 3

Handle decompositions and Heegaard splittings

The Dehn filling construction described in the previous chapter is unique to 3-manifolds. While analogous techniques exist in other dimensions, the specific issues involved rarely translate directly. A more general construction technique which can be used in arbitrary dimensions is through the use of a *handle structure*. We will describe this technique in dimension three. The same principles are involved regardless of the dimension of the underlying handle structures.

The basic technique in constructing handle structures is to glue balls of the same dimension together in a manner which can be enumerated. The important information to track is the attaching of the various balls. This is accomplished by looking at the homotopy type of the attaching regions and classifying it by its dimension. We identify a 3-ball as a *k-handle* when it is structured as $B^k \times B^{3-k}$ with attaching region $\partial B^k \times B^{3-k}$. Here is the list of handles in dimension 3 with their corresponding attaching regions (see also Figure 3.1):

Handle	Structure	Attaching Region
0-handles	$B^0 \times B^3$	$\partial B^0 \times B^3 = \emptyset$
1-handles	$B^1 \times B^2$	$\partial B^1 \times B^2 = \{-1, 1\} \times B^2$
2-handles	$B^2 \times B^1$	$\partial B^2 \times B^1 = S^1 \times I$
3-handles	$B^3 \times B^0$	$\partial B^3 \times B^0 = S^2$

We start our construction with 0-handles. We follow by attaching the 1-handles, 2-handles, and 3-handles in sequence.

In the case of 1-handles, the attaching region we use is $\partial B^1 \times B^2$, two disjoint

No attaching region

(a) 0-handle

attaching
regions

$D^2 \times \{-1, 1\}$

(b) 1-handle

$S^1 \times I$

(c) 2-handle

S^2

(d) 3-handle

Figure 3.1: The 3-dimensional handles with their attaching regions

Figure 3.2: The attaching of a 1-handle to a 0-handle

disks. We will attach these disks to the boundaries of the 0-handles with which we began. The manifold with boundary, which is created by attaching 1-handles to 0-handles, will appear to us as a set of 3-balls with *handles* attached (see Figure 3.2). This is where the description of handlebodies comes from. We can always slide the attaching region of any particular 1-handle off any other 1-handles we might have previously attached until the attaching regions of all 1-handles are on the boundaries of the 0-handles with which we began. The union of all 0-handles and 1-handles, if connected, is called a *handlebody*. The important feature of a handlebody is the closed 2-manifold which bounds the set of 0-handles and 1-handles. We will be particularly interested in the genus of this surface.

When we attach 2-handles to handlebodies, the attaching region has more information contained in it. The attaching region of a 2-handle is $\partial B^2 \times B^1 = S^1 \times I$, an annulus. The important information to track in attaching a 2-handle is the core of the annulus, a simple closed curve, on the boundary of the handlebody.

After attaching 2-handles, a closed manifold is formed by filling in the remaining 2-sphere boundary components with 3-handles. As the attaching region of a 3-handle is $\partial B^3 \times B^0 = S^2$, the Alexander trick [1] shows that the resulting space is unchanged regardless of how the 3-handles are attached.

If we turn this picture upside-down and consider the 3-handles to be 0-handles, then the 2-handles attach to 3-handles in the same manner as 1-handles to 0-handles. The set of all 0-handles and 1-handles form one handlebody, and the set of all 2-handles and 3-handles form another handlebody. The boundaries of these two handlebodies will be surfaces with the same genus when the resulting manifold is closed. The closed 3-manifold is the union of these two handlebodies where the attaching map is induced by the 2-handle attaching description.

Moreover, we can also reverse the process. Instead of constructing a closed 3-manifold by gluing handlebodies together, we begin with a closed 3-manifold and try to determine how to split the manifold into the union of two handlebodies. This is called a *Heegaard splitting* of the 3-manifold. In a Heegaard splitting of a 3-manifold, M , we have the triple (H_1, H_2, f) where H_i is a handlebody bounded by a closed 2-manifold of genus g , and f is a homeomorphism attaching ∂H_1 to ∂H_2 such that $M = H_1 \cup_f H_2$. The surface which bounds H_i in M is called a *splitting surface* for M . We define the *Heegaard genus* of a 3-manifold as the minimal genus of all Heegaard splittings of the 3-manifold.

All closed 3-manifolds possess a Heegaard splitting. Here is a sketch of the proof. It is a classical result of 3-manifold theory that all closed compact 3-manifolds are triangulable. Let K be a triangulation of a closed 3-manifold M . Take the second barycentric subdivision of K , K'' . The closed stars in K'' of the vertices of K are our 0-handles. The closed stars in K'' of the barycenters of the 1 simplexes of K attach to our 0-handles in the manner of 1-handles. The boundary of the union of our 0-handles and 1-handles will be the splitting surface for M . For 2-handles we use the closed stars in K'' of the barycenters of the 2 simplexes of K and for 3-handles the closed stars in K'' of the barycenters of the 3 simplexes, giving us the complementary handlebody. This, then, is a Heegaard splitting for the manifold.

Chapter 4

Applications of low-dimensional topology to systems theory

The description of the theory of knots has, to this point, depended on the fact that our knots are in S^3 . Many of the same techniques can be applied regardless of what space knots live in. For example, in physics it is interesting to look at the behavior of knots in spaces of varying dimension and physical characteristics.

As an application of this, in recent years there has been considerable interest in the quantum invariants of 3-manifolds originally proposed by Witten [33]. The invariants arise in several contexts in mathematical physics. This includes the original approach of Witten which involves the computation of certain path integrals in quantum field theory (see [12] and [33]). They also arise via applications of representation theory in the field of quantum groups (see [19] and [28]).

Yet another approach to these invariants via knot theory and the Kauffman bracket was given in [27]. In it, the invariants of certain 3-manifolds are derived from their Dehn filling descriptions. The work was based on the work of Lickorish who discovered the connection between the combinatorial descriptions of these 3-manifolds and the contexts from which they arose via physics (see [22], [23], [24], and [25]).

However, one of the strongest uses of theoretical topology in the fields represented by systems science are the uses of topological techniques to study non-linear systems. Various authors have written extensively on the attempt to classify the behavior of

non-linear systems through knot theory.

Back in 1983, Birman and Williams presented a ground-breaking series of articles on the use of knot theory to answer outstanding questions in the field of dynamical systems ([2] and [3]). In [2], the behavior of the orbits of the Lorenz equations was analyzed using the techniques of knot theory. Several types of equations were classified by the knot type of their orbits via topological techniques. In the second of their two articles, Birman and Williams [3] presented yet another classification scheme whereby knot theory could contribute to the understanding of the orbits of solutions to dynamical systems.

In 1985, a series of articles on utilizing knot theory invariants to classify orbits in suspensions of Smale's horseshoe began. The first of these, by Holmes and Williams [16], dealt primarily with using the correspondence between torus knots and bifurcation sequences to add to the understanding of the orbits of this system.

In particular, Holmes and Williams constructed a suspension of Smale's horseshoe diffeomorphism of D^2 as a flow in a 3-manifold. From this suspension, a knot-holder is constructed so that the periodic orbits are isotopic to those in the 3-manifold. In particular, this arises in suspensions of the Smale horseshoe which occurs in the Duffing equation

$$\dot{x} = y, \dot{y} = x - x^3 - \delta u + \gamma \cos t.$$

In this case the Poincaré map is a diffeomorphism of D^2 . Then, using the techniques of knot theory, various theorems of existence, uniqueness, and nonexistence of various torus knots were obtained. In the proof of their theorems, Holmes and Williams showed how the information about the knots in a dynamical system can deduce results on the bifurcation sequences in the creation of sequences.

The key to what Holmes and Williams contributed was that the knot type of periodic orbits cannot change in these 3-dimensional flows. This, then, is why topological techniques can effectively describe the underlying structure of the attractors in the system.

In 1986, Holmes [13] extended this by increasing the complexity of the knots to both iterated torus knots and cabled knots in general. Holmes showed in the case

of iterated horseshoe knots that approximately half are cabled and that, among the cabled knots, are infinitely many non-isotopic iterated torus knots. From this are obtained certain uniqueness results related to the cascades of period-doubling and period-multiplying bifurcations created by passing through a family of diffeomorphisms.

Holmes [15] concluded this series in 1989 with a look at extended knot families and bifurcation sequences. In this paper he attempted to classify the knot types of periodic orbits. In particular, he notes that in periodically forced nonlinear oscillators, horseshoes naturally arise. Since horseshoes provide a certain structure for the attractors, the manner in which they are created as attractors in the sequences of the bifurcations is important.

In 1987, Holmes [14] gave an extension of this type of analysis of the periodic flows by studying the class of suspensions of diffeomorphisms of the annulus as flows in a particular 3-manifold, $T^2 \times I$. He again finds several existence and uniqueness results with regard to certain torus knots and shows that certain other braids give rise to higher genus. In this case, he deals with the pendulum or Josephson equation

$$\dot{\theta} = r, \dot{r} = -\sin \theta + \nu - \delta r + \beta \cos t$$

in which the Poincaré map is defined on the annulus.

In all of these cases the phase space of the system in question is a 3-manifold and the periodic orbits are embedded closed curves. Thus, the question to be answered in describing the attractors of the dynamical system lies with the structure of knots and links in 3-manifolds.

While this is not intended to be an exhaustive listing of the applications of knot theory to dynamical systems or to systems theory in general, it does give an indication of the depth to which knot theory can be used in describing certain phenomena which occur in other disciplines. One would expect that as our understanding of dynamical systems increases and the systems under study become increasingly more complex, topological techniques to identify certain phenomena might become more prevalent.

Chapter 5

Compression and boundary compression

Given an n -manifold M and a sub-manifold N , a proper embedding of N in M is a one-to-one mapping of N to M such that ∂N is also embedded in ∂M . If N has a non-empty boundary, then ∂M must be non-empty in order for a proper embedding to be possible. If M is closed, then N would have to be closed for a proper embedding to be possible. Since our work concerns itself with 3-manifolds, proper embeddings will be embeddings of surfaces (perhaps with boundary), simple closed curves, and arcs into 3-manifolds (perhaps with boundary).

If N is a properly embedded surface in a 3-manifold M , we can form a *compression* of N in M if there is an embedded disk D in M such that $D \cap N = \partial D \subset N$ where ∂D does *not* bound a disk in N . The disk D is called a *compressing disk* for N . One feature of compressing disks is that they indicate, in the homomorphism mapping $\pi_1(N)$ to $\pi_1(M)$, a non-trivial generator of the kernel of this map. This is because ∂D does not bound a disk in N but does bound a disk (and is therefore trivial in the fundamental group) of M .

As an example of a compressing disk, the properly embedded disk in Figure 5.1 is a compressing disk. A properly embedded surface which has at least one compressing disk in the ambient manifold is a *compressible surface*.

Form the 2-handle $D \times I \subset M$ where $D = D \times \{0\}$ and $\partial D \times I \subset N$. If ∂D is separating in N , we can *compress* N by D into two properly embedded surfaces by

Figure 5.1: A compressing disk

gluing $N_1 = \overline{N \setminus (\partial D \times I)}$ to $D \times \{1\}$ and $D \times \{-1\}$. If ∂D is non-separating in N , we can compress N into a single properly embedded surface by gluing $N_2 = \overline{N \setminus (\partial D \times I)}$ to $D \times \{1\}$ and $D \times \{-1\}$. A result from the classification of surfaces guarantees that the genus of the surface or surface after compression will be no greater than the genus of N .

For a surface N with boundary properly embedded in a 3-manifold M with boundary, there is an additional type of compression involving ∂M . We again find an embedded disk D in M . However, this time, half of ∂D is in ∂M and half of ∂D is in the interior of N . The conditions when D is a *boundary compressing disk* for N in M are slightly different than those for determining whether we have a compressing disk. Let $\alpha = D \cap N$ and $\gamma = D \cap \partial M$ be the two arcs which together comprise ∂D . Then D will be a boundary compressing disk for N in M if there is no arc $\beta \subset \partial N$ such that $\alpha \cup \beta$ bound a disk in N .

As an example of a boundary compressing disk, the embedded disk in Figure 5.2 is a boundary compressing disk bounded by the two arcs α and β . A properly embedded surface which has at least one boundary compressing disk in the ambient manifold is a *boundary compressible surface*.

We can form a boundary compression for N whenever a boundary compressing disk is present in a similar manner to the way in which we formed a compression for N . Form the 2-handle $D \times I \subset M$ where $D = D \times \{0\}$, $\alpha \times I \subset N$, and $\beta \times I \subset \partial M$. If α is separating in N , we can form a pair of properly embedded surfaces by gluing

Figure 5.2: A boundary compressing disk

$N_1 = \overline{N \setminus (\alpha \times I)}$ to $D \times \{1\}$ and $D \times \{-1\}$. If α is non-separating in N , we can form a properly embedded surfaces by gluing $N_2 = \overline{N \setminus (\alpha \times I)}$ to $D \times \{1\}$ and $D \times \{-1\}$. Again, the classification of surfaces guarantees that the genus of the resulting surfaces or surface after boundary compression will be no greater than the genus of N .

If N has no compressing disks in M , then we call N *incompressible*. If N has no boundary compressing disks in M , we call N *∂ -incompressible*. If N is both incompressible and boundary incompressible, we call N *essential*.

The technique of compressing surfaces along disks can be extended to compressing a properly embedded surface N along any embedded surface F where $\partial F \subset N$. Embed $F \times I$ where $\partial F \times I \subset N$ in M and $F = F \times \{0\}$. Then, let $M' = \overline{M \setminus (F \times I)}$ and $N' = \overline{N \setminus (\partial F \times I)}$. This technique is referred to as *surgering* F from M along N . Unless otherwise stated, the surface we will surger along is ∂M .

Chapter 6

Essential annuli in handlebodies

A properly embedded arc α in a surface F with boundary where the two components of $\partial\alpha$ lie in different components of ∂F is a *spanning* arc. A spanning arc in an annulus is an *essential* arc. Similarly, a properly embedded annulus A in a 3-manifold M where the two components of ∂A lie in different components of ∂M is a *spanning* annulus. A properly embedded sub-manifold N of a 3-manifold M which is parallel to the boundary is *peripheral*. What follows are primarily new proofs of some results which first appeared in [21].

Lemma 6.1 (Kobayashi [21]) *If A is an incompressible annulus in a solid torus V , then A is peripheral.*

Proof. Let D be a meridian disk for V . There are two annuli A_1 and A_2 in ∂V such that $\partial A = \partial A_1 = \partial A_2$ and $\partial V = A_1 \cup A_2$. Then $A \cup A_i$ ($i = 1, 2$) is a torus in V . Through standard disk-swapping techniques, minimize the intersections of A with D . Let α be an arc in $A \cap D$. Such an arc exists, for otherwise surgering D from V would have A properly embedded in a 3-ball and hence compressible. Let α be outermost in D . The arc α is essential in A , for otherwise we could reduce the number of intersections of A with D . Since α is outermost in D , there is an arc β in ∂D such that $\partial\alpha = \partial\beta$ and $\alpha \cup \beta$ bounds a disk $\Delta \subset D$. The disk Δ is a boundary compressing disk for A . Thus, A separates V into two solid tori, one of which has meridian disk Δ . Without loss of generality, we will assume that $A \cup A_1$ is the solid torus whose meridian is Δ . Then A is parallel to A_1 . $\square$

A properly embedded, non-peripheral disk in a 3-manifold is an *essential* disk. An essential, non-separating disk D in a handlebody V is a *meridian disk*. The set of co-cores of the 1-handles are a *complete system of meridian disks* for V since they are mutually disjoint and surgering them from V results in a 3-ball.

Lemma 6.2 *Let V be a genus two handlebody with a complete system of meridian disks $\{D_1, D_2\}$ and let A be an non-peripheral incompressible annulus in V . Then there is a boundary compression of A such that the disk, D , obtained after the boundary compression, is essential in V . Furthermore, D is disjoint from both D_1 and D_2 .*

Proof. Since A is incompressible in V , the core of A , a simple closed curve which here is denoted by α , represents a nontrivial element of $\pi_1(V)$. As α is isotopic to both boundary components of A , α is isotopic to an essential simple closed curve on the genus two boundary component of V . Therefore, α must have a non-zero geometric intersection number with at least one of ∂D_1 or ∂D_2 . Therefore, A intersects at least one of D_1 or D_2 in an essential arc of A . Passing to an outermost such arc and proceeding as in the proof of Lemma 6.1, it follows that there is an arc β in A and an arc γ in ∂V such that β is essential in A , $\partial\beta = \partial\gamma$, and $\beta \cup \gamma$ bounds a disk Δ in V which boundary compresses A .

After doing the boundary compression on A , we obtain a disk D which is properly embedded in V . If D was peripheral in V , then, after doing the identification indicated by the boundary compression, A would be peripheral. Therefore, D is essential in V .

Let $\Sigma = D \cap (D_1 \cup D_2)$ after minimizing intersections via isotopy in V . Then Σ consists of simple closed curves and arcs. Let α be a simple closed curve in Σ such that α is innermost in D and bounds a disk Δ' in D .

Since D_1 and D_2 are disjoint, without loss of generality we will assume that α is in D_1 . Then α separates D_1 into a disk and an annulus, A_1 . Let $B_1 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$, $\partial\Delta' \times \{1\}$ is one of the boundary components of A_1 , and $B_1 \cap D_1 = \partial\Delta' \times I$. Then $D'_1 = (\Delta' \times \{1\}) \cup A_1$ is a non-separating, essential disk in V . Since α was innermost in D , $D'_1 \cap D_2 = \emptyset$. This implies that D'_1 is a meridian disk for V with a lower geometric intersection number with D .

We proceed to find new disks for a complete system of meridian disks for V until we have a system which contains no simple closed curves of intersection with D .

Let α be an arc in Σ such that α is outermost in D and co-bounds a disk Δ' in D with an arc $\beta \in \partial D$. We will again assume, without loss of generality, that α is in D_1 .

This case is much more difficult since α separates D_1 into two undistinguished disks. We begin by constructing $B_1 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$ and $B_1 \cap D_1 = \alpha \times I$. Then there exist disjoint disks Δ'_1 and Δ'_2 in D_1 such that $\Delta'_1 \cap B_1 = \alpha \times \{1\}$ and $\Delta'_2 \cap B_1 = \alpha \times \{-1\}$ and disjoint, properly embedded disks D'_1 and D''_1 in V such that $D'_1 = \Delta'_1 \cup (\Delta' \times \{1\})$ and $D''_1 = \Delta'_2 \cup (\Delta' \times \{-1\})$. Note that $D_1 = \Delta'_1 \cup (\alpha \times I) \cup \Delta'_2$.

Claim 1. The disks D'_1 and D''_1 cannot both be peripheral.

Proof of Claim 1. Since $\alpha \times I$ is peripheral, if D'_1 and D''_1 were both peripheral then D_1 would be peripheral—a contradiction.

Claim 2. If D'_1 is separating, then D''_1 is not peripheral.

Proof of Claim 2. Since $\alpha \times I$ is peripheral, if D''_1 is peripheral, then D_1 would be separating—a contradiction.

Claim 3. If both D'_1 and D''_1 are separating, they are parallel in V .

Proof of Claim 3. The disk D'_1 separates V into solid tori V' and V'' . Then D''_1 is a properly embedded, separating disk in either V' or V'' . Since $\pi_1(V')$ is torsion free and D''_1 is not a meridian disk, D''_1 is peripheral in V' . If D''_1 is not parallel to D'_1 , then it would be peripheral in V —a contradiction. Likewise, D''_1 would be peripheral in V'' . If D''_1 is not parallel to D'_1 , then it would again be peripheral in V —a contradiction.

Claim 4. At least one of D'_1 and D''_1 , say D'_1 , is non-separating in V .

Proof of Claim 4. If both D'_1 and D''_1 are separating, then by Claim 3, they would be parallel. If Δ'_1 is parallel to Δ'_2 , then D_1 is peripheral—a contradiction. If Δ'_1 is not parallel to Δ'_2 , then D_1 is parallel to D'_1 , a separating disk in V —a contradiction. Therefore at least one of D'_1 and D''_1 must be non-separating in V . Without loss of generality, we will assume this properly embedded, non-separating essential disk is D'_1 . Since D'_1 is disjoint from D_2 it is a meridian disk for V with smaller geometric intersection number with D .

Figure 6.1: A genus two handlebody with a single non-peripheral incompressible annulus

We proceed to find new disks for a complete system of meridian disks for V until $D \cap (D'_1 \cup D'_2) = \emptyset$. $\square$

Lemma 6.3 (Kobayashi [21]) *If A is a non-peripheral incompressible annulus in a genus two handlebody V , then surgering the annulus A from V results in either (see Figure 6.1)*

1. *a solid torus V_1 and a genus two handlebody V_2 and there is a complete system of meridian disks $\{D_1, \Delta\}$ of V_2 such that $D_1 \cap A = \emptyset$ and $\Delta \cap A$ is an essential*

arc of A , or

2. a genus two handlebody V' and there is a complete system of meridian disks $\{D_1, \Delta\}$ of V' such that $\Delta \cap A$ is an essential arc of A .

Proof. From Lemma 6.2 there is a boundary compressing disk Δ for A which compresses A to an essential disk D in V . The disk D is either separating or non-separating. The remainder of the proof is recalled from [21].

If D is separating in V , then surgering it from V results in two solid tori V' and V'' , and there are copies Δ' and Δ'' of Δ on $\partial V'$. Then D_1 is a meridian disk for V' and is disjoint from both Δ' and Δ'' . Since Δ' and Δ'' are identified in V , A separates V into a genus two handlebody V_1 with meridian disks D_1 and Δ and a solid torus V_2 .

If D is non-separating in V , then surgering D from V results in a solid torus V_1 . There are copies Δ' and Δ'' on ∂V_1 . Since Δ' and Δ'' are identified in V , surgering A from V results in a genus two handlebody V' . Since D is non-separating in V , the classification of surfaces tells us that since it is disjoint from both D_1 and D_2 , it must be parallel to one of them. Without loss of generality, we will assume that D is parallel to D_1 . Then D_1 and Δ are meridian disks for V' . $\square$

Lemma 6.4 (Kobayashi [21]) Let V be a genus two handlebody and $\{A_1, A_2\}$ be a system of mutually disjoint annuli in ∂V such that there is a complete system of meridian disks $\{D_1, D_2\}$ of V which satisfies $D_i \cap A_j = \emptyset$ ($i \neq j$) and $D_i \cap A_i$ is an essential arc of A_i ($i = 1, 2$). If A is an incompressible annulus in $(V, \overline{\partial V \setminus (A_1 \cup A_2)})$, then A is parallel to A_1 or A_2 .

Proof. In a similar manner to the proof of Lemma 6.2, we can find a boundary compressing disk Δ for A in V , and doing the boundary compression of A results in a properly embedded disk D . Since A is disjoint from A_i , D is disjoint from A_i ($i = 1, 2$), and so D separates V into two solid tori V_1 and V_2 . We will assume, without loss of generality, that $A_i \subset \partial V_i$ ($i = 1, 2$). By assumption there is a meridian disk D'_i of V_i such that $D'_i \cap A_i$ is an essential arc of A_i . Then, from Lemma 6.3 we know that

A separates V into a genus two handlebody V'_1 and a solid torus V'_2 . We may now assume that $A_2 \subset \partial V'_2$. Since A and A_2 are both in $\partial V'_1$ and disjoint, their respective cores are parallel simple closed curves on $\partial V'_1$. Thus, A is parallel to A_2 . $\square$

Lemma 6.5 (Kobayashi [21]) *Let $\{A_1, A_2\}$ be a system of mutually disjoint, non-parallel, non-peripheral incompressible annuli in the genus two handlebody V . Then surgering $A_1 \cup A_2$ from V results in either (see Figure 6.2)*

1. *a solid torus V_1 and a genus two handlebody V_2 where $A_1 \cup A_2 \subset \partial V_1$, $A_1 \cup A_2 \subset \partial V_2$, and there is a complete system of meridian disks $\{\Delta_1, \Delta_2\}$ of V_2 such that $\Delta_i \cap A_j = \emptyset$ ($i \neq j$) and $\Delta_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i ,*
2. *two solid tori V_1 and V_2 and a genus two handlebody V_3 where $A_1 \subset \partial V_1$, $A_2 \subset \partial V_2$, $A_1 \cup A_2 \subset \partial V_3$, and there is a complete system of meridian disks $\{\Delta_1, \Delta_2\}$ of V_3 such that $\Delta_i \cap A_j = \emptyset$ ($i \neq j$) and $\Delta_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i , or*
3. *a solid torus V_1 and a genus two handlebody V_2 where $A_1 \subset \partial V_1$, $A_2 \cap V_1 = \emptyset$, $A_1 \cup A_2 \subset \partial V_2$, and there is a complete system of meridian disks $\{\Delta_1, \overline{\Delta_2 \setminus \Delta_1}\}$ for V_2 such that $\Delta_1 \cap A_2$ is an essential arc in A_2 and $\overline{\Delta_2 \setminus \Delta_1} \cap A_i$ ($i = 1, 2$) is an essential arc of A_i .*

Proof. From Lemma 6.2 we know that there are boundary compressing disks Δ_1 and Δ_2 for A_1 and A_2 respectively in V that compress A_1 and A_2 to two essential disks D' and D'' . Since A_1 and A_2 are disjoint, D' and D'' are disjoint. Assume that $\{D', D''\}$ is a complete system of meridian disks for V . Then we can move A_2 by a small isotopy into V surgered along $D' \cup D''$ which contradicts that A_2 is incompressible in V . We have the following three cases.

Case 1. The disks D' and D'' are parallel and neither separates in V . This gives us conclusion 1 of the Lemma.

Case 2. The disks D' and D'' are parallel and both separate in V . This gives us conclusion 2 of the Lemma.

(a)

(b)

(c)

Figure 6.2: A genus two handlebody with two non-parallel, non-peripheral incompressible annuli

Figure 6.3: A genus two handlebody with three non-parallel, non-peripheral, incompressible annuli

Case 3. The disks D' and D'' are not parallel. They cannot both be non-separating and non-parallel since they do not form a system of meridian disks for V . Therefore, without loss of generality, we will assume that D' is separating. Surgering D' from V results in two solid tori V' and V'' . The disk D'' is a properly embedded disk in one of them, say V'' . If D'' is peripheral in V'' and is not parallel to D' , then D'' is peripheral in V , a contradiction. Since D'' is, by hypothesis, not parallel to D' , D'' is non-separating in V'' and is therefore non-separating in V . In this case, we have conclusion 3 of the Lemma. $\square$

Lemma 6.6 (Kobayashi [21]) Let $\{A_1, A_2, A_3\}$ be a system of pairwise disjoint, non-parallel, non-peripheral incompressible annuli in a genus two handlebody V . Then surgering A_1 , A_2 , and A_3 from V results in two solid tori V_1 and V_2 , and a genus two handlebody V_3 which satisfy (see Figure 6.3)

1. the annuli appear on the boundaries of the various compression bodies and handlebodies as follows: $A_3 \subset \partial V_1$, $A_2 \cup A_3 \subset \partial V_2$, and $A_1 \cup A_2 \subset \partial V_3$,
2. there is a complete system of meridian disks $\{\Delta_1, \Delta_2\}$ of V_3 such that $\Delta_i \cap A_j = \emptyset$ for $i \neq j$ and $\Delta_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i , and

3. *there is a meridian disk D_3 of V_2 such that $D_3 \cap A_i$ ($i = 2, 3$) is an essential arc of A_i .*

Proof. The annuli $\{A_1, A_2\}$ must satisfy one of the conclusions of Lemma 6.5. If we assume that $\{A_1, A_2\}$ satisfy conclusion 2, then surgering A_1 and A_2 from V results in a solid tori V_1 , and V_2 , and a genus two handlebody V_3 . If A_3 is in ∂V_1 then, by Lemma 6.1, A_3 is parallel to either A_1 or A_2 , a contradiction. If A_3 is in ∂V_2 then, by Lemma 8.1, A_3 is parallel to either A_1 or A_2 , a contradiction. If A_3 is in ∂V_3 , then by 6.4, A_3 is parallel to either A_1 or A_2 , a contradiction. Thus, $\{A_1, A_2\}$ cannot satisfy conclusion 2 of Lemma 6.5.

If we assume that $\{A_1, A_2\}$ satisfy conclusion 1 of Lemma 6.5, then surgering A_1 and A_2 from V results in a collar of a torus V_1 and a genus two handlebody V_2 . By Lemma 6.4, we know that A_3 is not in ∂V_2 . Then $A_3 \subset \partial V_1$ and, by Lemma 8.1, A_3 is parallel to an annulus A' in ∂V_1 . Since A_3 is incompressible and is not parallel to either A_1 or A_2 , $\partial A_1 \cup \partial A_2$ is in A' . Then $\{A_1, A_2, A_3\}$ satisfy the conditions of the lemma. $\square$

If we assume that $\{A_1, A_2\}$ satisfy conclusion 3 of Lemma 6.5, then surgering A_1 and A_2 from V results in a solid torus V_1 and a genus two handlebody V_2 where $A_1 \cup A_2 \subset \partial V_2$ and $A_1 \cap \partial V_1 = \emptyset$. Again, by Lemma 8.1, we find that A_3 must be in ∂V_2 . Since $A_3 \cap (A_1 \cup A_2) = \emptyset$, by Lemma 6.4 we see that A_3 is parallel to an annulus A' in ∂V_2 . Again, since A_3 is incompressible and not parallel to either A_1 or A_2 , $\partial A_1 \cup \partial A_2 \subset A'$. Thus, via a change in subscripts, $\{A_1, A_2, A_3\}$ satisfy the conditions of the lemma. $\square$

Chapter 7

Compression bodies, decompositions by tori and annuli, and generalized satellite knots

Given a surface F (not necessarily closed), we denote the surface obtained from F by removing an open regular neighborhood of a point in the interior of F as $\hat{F}$.

A properly embedded, non-peripheral incompressible torus in a 3-manifold is an *essential* torus. Following Jaco and Shalen (see [17] and [18]), a 3-manifold for which every properly embedded incompressible torus is peripheral is *simple*. A *torus decomposition* of an irreducible 3-manifold M is a (possibly empty) collection, $T_1, T_2, \dots, T_n$, of essential disjoint separating tori in M which separate M into pieces which are either simple or Seifert fibered such that n is minimal in the sense that one cannot remove any of the T_i and retain the above property. For closed 3-manifolds and 3-manifolds with incompressible boundary, results of Jaco [17, Chapter VIII], among others, show that such decompositions exist and are unique up to isotopy.

A 3-manifold is said to have a *trivial* torus decomposition if the collection of decomposing tori is empty. A generalized satellite knot is a knot in an arbitrary 3-manifold whose exterior has a nontrivial torus decomposition. For knots in S^3 , this is equivalent to the existence of a non-peripheral incompressible torus in the knot exterior (see Figure 7.1).

A properly embedded, non-peripheral, incompressible, ∂ -incompressible annulus

Figure 7.1: A tunnel-1 satellite knot in S^3 : the Rolfsen-Bailey knot

in a 3-manifold is an *essential* annulus. A 3-manifold is *acylindrical* if all properly embedded, incompressible, ∂ -incompressible annuli are peripheral. For a 3-manifold M with incompressible boundary, an *annular decomposition* is a (possibly empty) collection of essential separating annuli, $A_1, A_2, \dots, A_n$, in M which separate M into pieces which are either acylindrical or Seifert fibered such that n is minimal in the sense that we cannot remove any of the A_i and retain the above property. Results regarding the existence and uniqueness of annular decompositions similar to that for toral decompositions exist (see [17]).

A knot K in a 3-manifold M is *tunnel-1* if there exists a properly embedded arc α in the exterior of K such that removing an open regular neighborhood of α from this exterior results in a genus two handlebody. Dually, the exterior of a tunnel-1 knot is obtained by adding a 2-handle to a genus two handlebody. A *good* 2-handle addition is one where the attaching circle is non-separating. Unless otherwise stated, all 2-handle additions considered here are good. A *compression body* is a collar of a closed, orientable surface with a finite number of 2-handles added to one boundary component and all resulting 2-sphere boundary components capped with 3-handles. We refer to the boundary component of the collar of a surface which contains the attaching circles of the 2-handles as the *inside* boundary component and the other as the *outside*. A compression body obtained by adding n 2-handles to the inside boundary component of the collar of a surface of genus g is said to be of *type* (g, n) .

We denote a compression body of type (g, n) by $C_{g,n}$. In this notation, $C_{g,g}$ is a handlebody of genus g , and $C_{g,0}$ is a collar of a genus g surface. A compression body of type (g, n) is abusively said to be of genus g .

For a 3-manifold M with boundary, a properly embedded, closed, separating surface F in M is a *splitting surface* for M if F separates M into two compression bodies. This splitting generalizes the Heegaard splittings of closed 3-manifolds. The genus of F is the genus of the splitting. The minimal genus over all such splittings is the splitting genus of M .

The geometric intersection number of two simple closed curves in a surface is referred to as the *distance* between those curves.

We now focus our attention on compression bodies of genus at most two. Begin with a collar of a genus two surface and choose representatives μ_1^- and μ_2^- from a pair of distinct isotopy classes of non-separating simple closed curves in one of the boundary components. The collar above μ_1^- (respectively, μ_2^-) is an essential spanning annulus A_{μ_1} (A_{μ_2}) whose other boundary component is a simple closed curve μ_1^+ (μ_2^+) on the other boundary component of the collar of the surface. Corresponding to μ_1^- (μ_2^-) is a simple closed curve λ_1^- (λ_2^-) such that μ_1^- (μ_2^-) and λ_1^- (λ_2^-) meet transversely in a single point and $\mu_1^- \cap \lambda_2^- = \mu_2^- \cap \lambda_1^- = \lambda_1^- \cap \lambda_2^- = \emptyset$. As before, the collar above λ_1^- (λ_2^-) is an essential spanning annulus A_{λ_1} (A_{λ_2}) whose other boundary component is a simple closed curve λ_1^+ (λ_2^+) on the other boundary component of the collar of the surface.

Let C be the compression body of type $(2, 1)$ obtained by adding a 2-handle via μ_1^- . The compression body C has two boundary components: an *outside* component of genus two and an *inside* component of genus one. Since the attaching circle for this 2-handle is μ_1^- , the curves μ_2^- and λ_2^- represent generators of the fundamental group of the inside component of ∂C . The union of A_{μ_1} with the core of the 2-handle added to construct C forms a *meridian disk* D_1 for C . The annulus A_{μ_2} will have a special meaning for us. In several applications, A_{μ_2} will play the role of a meridian disk. To emphasize this meaning, we will denote A_{μ_2} as $\hat{D}_2$. The annulus A_{λ_2} plays a similar role. In particular, λ_2^- will be our choice of longitude for the inner

boundary component of C . To emphasize this role, denote this annulus as A_λ . Refer to $\{D_1, \hat{D}_2, A_\lambda\}$ as a complete disk/annular system for C .

Chapter 8

Essential annuli in compression bodies

Let C be a collar of a torus, that is, a compression body of type $(1, 0)$. The space C has the homotopy type of a torus, so there are two simple closed curves, μ^- and λ^- , at a distance one on one of the boundary components that represent the generators of $\pi_1(C)$. The collars of μ^- and λ^- are essential annuli A_μ and A_λ whose other boundary components are simple closed curves μ^+ and λ^+ on the other boundary component of C .

Lemma 8.1 A non-spanning, incompressible annulus A in a compression body C of type $(1, 0)$ is peripheral.

Proof. Since A is non-spanning, both components of ∂A lie in a single component of ∂C . Note that there are two annuli in this component of ∂C , A_1 and A_2 , such that $\partial A = \partial A_1 = \partial A_2$. Then $A \cup A_i$ ($i = 1, 2$) is a torus in C . Through standard disk-swapping techniques, minimize the intersections of A with A_μ and A_λ . Let α be an arc in $A \cap (A_\mu \cup A_\lambda)$. Such an arc exists, for otherwise both boundary components of A would be inessential simple closed curves in ∂C , and there would be a compressing disk for A in C . Without loss of generality, assume that $\alpha \subset A_\mu$ and is outermost. The curve α is essential in A , for otherwise we could reduce the number of intersections of A with A_μ . Since α is outermost in A_μ , there is an arc β in ∂A_μ such that $\partial\alpha = \partial\beta$ and $\alpha \cup \beta$ bound a disk $\Delta \subset A_\mu$. The disk Δ is a boundary compressing disk for A . Thus, A separates C into a solid torus with meridian disk Δ and a collar of a torus.

Since A and one of A_1 or A_2 bound a solid torus, say A_1 , A is parallel to A_1 . $\square$

A non-spanning, incompressible annulus in a compression body of type $(2, 1)$ where the boundary of the annulus is in the genus two boundary component of the compression body is a *good* annulus.

Lemma 8.2 *Let C be a compression body of type $(2, 1)$ with a complete disk/annular system $\{D_1, \hat{D}_2, A_\lambda\}$, and let A be a good annulus in C . Then there is a boundary compression of A such that the disk, D obtained after the boundary compression is essential in C . Furthermore, D is disjoint from both D_1 and $\hat{D}_2$.*

Proof. Since A is incompressible in C , the core of A , a simple closed curve which here is denoted by α , represents a nontrivial element of $\pi_1(C)$. As α is isotopic to both boundary components of A , α is isotopic to an essential simple closed curve on the genus two boundary component of C . Therefore, α must have a non-zero geometric intersection number with at least one of μ_1^+ , μ_2^+ , or λ_2^+ . Therefore, A intersects at least one of D_1 , $\hat{D}_2$, or A_λ in an essential arc of A . Passing to an outermost such arc and proceeding as in the proof of Lemma 8.1, it follows that there is an arc β in A and an arc γ in ∂C such that β is essential in A , $\partial\beta = \partial\gamma$, and $\beta \cup \gamma$ bounds a disk Δ in C which boundary compresses A .

After doing the boundary compression on A , we obtain a disk D which is properly embedded in C . If D was peripheral in C , then, after doing the identification indicated by the boundary compression, A would be peripheral. Therefore, D is essential in C .

Let $\Sigma = D \cap (D_1 \cup \hat{D}_2)$ after minimizing intersections via isotopy in C . Then Σ consists of simple closed curves and arcs. Let α be a simple closed curve in Σ such that α is innermost in D and bounds a disk Δ' in D . We have two cases: $\alpha \subset D_1$ or $\alpha \subset \hat{D}_2$.

In the case where $\alpha \subset D_1$, then α separates D_1 into a disk and an annulus, A_1 . Let $B_1 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$, $\partial\Delta' \times \{1\}$ is one of the boundary components of A_1 , and $B_1 \cap D_1 = \partial\Delta' \times I$. Then $D'_1 = (\Delta' \times \{1\}) \cup A_1$ is a non-separating, essential disk in C . Since α was innermost in D , $D'_1 \cap \hat{D}_2 = \emptyset$. This implies that D'_1 is a meridian disk for C with a lower geometric intersection number with D .

The second case is where $\alpha \subset \hat{D}_2$. Since α bounds a disk in D and $\hat{D}_2$ is π_1 -injective in C , α cannot be an essential simple closed curve in $\hat{D}_2$. Therefore, α separates $\hat{D}_2$ into a disk and a pair of pants A_2 . As above, we let $B_2 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$, $\partial\Delta' \times \{1\}$ is one of the boundary components of A_2 , and $B_2 \cap \hat{D}_2 = \partial\Delta' \times I$. Then $\hat{D}'_2 = (\Delta' \times I) \cup A_2$ is a non-separating, essential, spanning annulus in C . Since α was innermost in D , $\hat{D}'_2 \cap D_1 = \emptyset$. This implies that $\hat{D}'_2$ is an element of a new complete disk/annular system for C where $\hat{D}'_2$ has a lower geometric intersection number with D .

We proceed to find new meridian disks and annuli for a disk/annular system for C until we have a system which contains no simple closed curves of intersection with D .

Let α be an arc in Σ such that α is outermost in D and co-bounds a disk Δ' in D with an arc $\beta \in \partial D$. We have two cases: $\alpha \subset \hat{D}_2$ or $\alpha \subset D_1$.

In the case where $\alpha \subset \hat{D}_2$, then α separates $\hat{D}_2$ into a disk and an annulus A_2 since α cannot be spanning. Let $B_2 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$, $\partial\Delta' \times \{1\}$ is one of the boundary components of A_2 , and $B_2 \cap \hat{D}_2 = \alpha \times I$. Then $\hat{D}'_2 = (\Delta' \times \{1\}) \cup A_2$ is a non-separating, spanning, essential annulus in C . Since α was outermost in D , $\hat{D}'_2 \cap D_1 = \emptyset$. Thus, $\hat{D}'_2$ is an element of a new complete disk/annular system for C where $\hat{D}'_2$ has a lower geometric intersection number with D .

The case where $\alpha \subset D_1$ is much more difficult since α separates D_1 into two undistinguished disks. We begin by constructing $B_1 = \Delta' \times I$ such that $\Delta' \times \{0\} = \Delta'$ and $B_1 \cap D_1 = \alpha \times I$. Then there exist disjoint disks Δ'_1 and Δ'_2 in D_1 such that $\Delta'_1 \cap B_1 = \alpha \times \{1\}$ and $\Delta'_2 \cap B_1 = \alpha \times \{-1\}$ and disjoint, properly embedded disks D'_1 and D''_1 in C such that $D'_1 = \Delta'_1 \cup (\Delta' \times \{1\})$ and $D''_1 = \Delta'_2 \cup (\Delta' \times \{-1\})$. Note that $D_1 = \Delta'_1 \cup (\alpha \times I) \cup \Delta'_2$.

Claim 1. The disks D'_1 and D''_1 cannot both be peripheral.

Proof of Claim 1. Since $\alpha \times I$ is peripheral, if D'_1 and D''_1 were both peripheral then D_1 would be peripheral—a contradiction.

Claim 2. If D'_1 is separating, then D''_1 is not peripheral.

Proof of Claim 2. Since $\alpha \times I$ is peripheral, if D_1'' is peripheral, then D_1 would be separating—a contradiction.

Claim 3. If both D_1' and D_1'' are separating, they are parallel in C .

Proof of Claim 3. The disk D_1' separates C into a solid torus C' and a compression body C'' of type $(1, 0)$. Then D_1'' is a properly embedded, separating disk in either C' or C'' . Since $\pi_1(C')$ is torsion free and D_1'' is not a meridian disk, D_1'' is peripheral in C' . If D_1'' is not parallel to D_1' , then it would be peripheral in C' —a contradiction. Likewise, since $\pi_1(C'')$ is torsion free and has no meridian disks, D_1'' would be peripheral in C'' . If D_1'' is not parallel to D_1' , then it would again be peripheral in C' —a contradiction.

Claim 4. At least one of D_1' and D_1'' , say D_1' , is non-separating in C .

Proof of Claim 4. If both D_1' and D_1'' are separating, then by Claim 3, they would be parallel. If Δ_1' is parallel to Δ_2' , then D_1 is peripheral—a contradiction. If Δ_1' is not parallel to Δ_2' , then D_1 is parallel to D_1' , a separating disk in C —a contradiction. Therefore at least one of D_1' and D_1'' must be non-separating in C . Without loss of generality, we will assume this properly embedded, non-separating essential disk is D_1' . Since D_1' is disjoint from $\hat{D}_2$, it is a meridian disk for C with smaller geometric intersection number with D .

We proceed to find new meridian disks and annuli for a disk/annular system for C until $D \cap (D_1' \cup \hat{D}_2) = \emptyset$. $\square$

We now classify the non-parallel embeddings of good annuli in compression bodies of type $(2, 1)$.

Lemma 8.3 Let C be a compression body of type $(2, 1)$ with complete disk/annular system $\{D_1, \hat{D}_2, A_\lambda\}$ and let A be a good annulus in C . Then surgering the annulus A from C results in either (see Figure 8.1)

- 1, a solid torus C_1 and a compression body C_2 of type $(2, 1)$, and there is a complete disk/annular system $\{\Delta, \hat{D}_2, A_\lambda^2\}$ for C_2 such that $\hat{D}_2 \cap A = \emptyset$ and $D_1 \cap A$ is an essential arc in A ,
- 2, a collar of a torus C_1 and a genus two handlebody C_2 , and there is a complete system of meridian disks $\{D_1, \Delta\}$ for C_2 such that $D_1 \cap A = \emptyset$ and $D_2 \cap A$ is

(a)

(b)

(c)

Figure 8.1: A compression body of type $(2, 1)$ with a single good annulus

an essential arc in A , or

3. a compression body C_1 of type $(2, 1)$, and there is a complete disk/annular system $\{\Delta, \hat{D}_2^1, A_\lambda^1\}$ for C_1 such that $D_1 \cap A$ is an essential arc in A .

Proof. From Lemma 8.2 there is a boundary compressing disk Δ for A which compresses A to an essential disk D properly embedded in C . The disk D is either separating or non-separating.

We first consider the case where D is separating. Since D is disjoint from both D_1 and $\hat{D}_2$, A is disjoint from at least one of D_1 or $\hat{D}_2$. If $A \cap \hat{D}_2 = A \cap A_\lambda = \emptyset$, then Δ lies on the opposite side of D from $\hat{D}_2$. In this sub-case, D separates C into a collar of a torus C' and a solid torus C'' . Then there are copies Δ', Δ'' of Δ on $\partial C'$. Since Δ' and Δ'' are identified in C separated along A , A separates C into a solid torus C_1 and a compression body C_2 of type $(2, 1)$ where the meridian disk of C_2 is Δ and the essential spanning annulus whose outer boundary component is a meridian of C_2 is $\hat{D}_2$.

If Δ lies on the same side of D as $\hat{D}_2$, then once again D separates C into a solid torus C' and a collar of a torus C'' . Then, as before, there are copies Δ', Δ'' of Δ on $\partial C'$. However, in this case the identification of Δ' and Δ'' determines that A separates C into a collar of a torus C_1 and a genus two handlebody C_2 . In this case, Δ and D_1 act as a complete system of meridian disks for C_2 .

If D is a meridian disk of C , then D separates C into a collar of a torus C' . Since there are copies Δ', Δ'' of Δ on $\partial C'$ and Δ' and Δ'' are identified in C cut along A , surgering A from C results in a compression body C_1 of type $(2, 1)$. Finally, note that Δ is a meridian disk for this compression body. $\square$

Lemma 8.4 Let C be a compression body of type $(2, 1)$ with a complete disk/annular system $\{D_1, \hat{D}_2, A_\lambda\}$ and $\{A_1, A_2\}$ be a system of mutually disjoint annuli in the genus two boundary component of C such that $D_1 \cap A_1$ is an essential arc of A_1 , $D_1 \cap A_2 = \hat{D}_2 \cap A_1 = \emptyset$, and $\hat{D}_2 \cap A_2$ is an essential arc of A_2 . If A is a non-spanning, incompressible annulus in $(C, \overline{\partial C \setminus (A_1 \cup A_2)})$, then A is parallel to A_1 or A_2 .

Proof. We begin by assuming that A is not peripheral. Then, from Lemma 8.2 we know that there is a boundary compressing disk Δ for A giving rise to an essential disk D properly embedded in C . Moreover, by construction and from the nature of the boundary compression, $D \cap (A_1 \cup A_2) = \emptyset$ which means that D separates C into a solid torus C_1 and a collar of a torus C_2 . Without loss of generality we will assume that $A_i \subset \partial C_i$ ($i = 1, 2$). From Lemma 8.3 then, we know that A separates C in one of two ways.

In the first case, A separates C into a compression body C'_1 of type $(2, 1)$ and a solid torus C'_2 . Then, since $A_2 \subset C'_2$, $\text{Im}(i_* : \pi_1(A_2) \rightarrow \pi_1(C'_2)) = \pi_1(C'_2)$ and $A_2 \cap A = \emptyset$. Thus, A is parallel to A_2 .

The second case is when A separates C into a genus two handlebody C'_1 and a collar of a torus C'_2 . Since C'_2 has the homotopy type of a torus and since both A and A_2 are on the same boundary component of C'_2 and disjoint, the cores of both annuli are parallel simple closed curves on $\partial C'_2$. Thus, A is parallel to A_2 . $\square$

Lemma 8.5 Let $\{A_1, A_2\}$ be a system of mutually disjoint, non-parallel good annuli in a compression body C of type $(2, 1)$. Then surgering A_1 and A_2 from C results in either (see Figure 8.2)

1. a collar of a torus C_1 and a genus two handlebody C_2 such that $A_1 \cup A_2 \subset C_i$ ($i = 1, 2$), and there is a complete system of meridian disks $\{\Delta_1, \Delta_2\}$ of C_2 where $D_i \cap A_j = \emptyset$ for $i \neq j$ and $D_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i .
2. a solid torus C_1 , a collar of a torus C_2 , and a genus two handlebody C_3 such that $A_1 \subset \partial C_1$, $A_2 \subset \partial C_2$, and $A_1 \cup A_2 \subset \partial C_3$, and there is a complete system of meridian disks $\{D_1, D_2\}$ of C_3 where $D_i \cap A_j = \emptyset$ for $i \neq j$ and $D_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i , or
3. a collar of a torus C_1 and a genus two handlebody C_2 such that $A_1 \subset \partial C_1$, $A_2 \cap C_1 = \emptyset$, and $A_1 \subset \partial C_2$, and there is a complete system of meridian disks $\{D_1, D_2\}$ of C_2 where $D_1 \cap A_2$ is an essential arc of A_2 and $D_2 \cap A_i$ ($i = 1, 2$) is an essential arc of A_i .

(a)

(b)

(c)

Figure 8.2: A compression body of type $(2, 1)$ with two non-parallel good annuli

Figure 8.3: A compression body of type $(2, 1)$ with three non-parallel good annuli

Proof. From Lemma 8.2 there are boundary compressing disks Δ_1 and Δ_2 of A_1 and A_2 , respectively, and essential disks D' and D'' after doing each boundary compression. Since $A_1 \cap A_2 = \emptyset$, D' and D'' are disjoint. There are three cases: D' and D'' are parallel and non-separating, D' and D'' are parallel and separating, and D' and D'' are non-parallel.

If D' and D'' are parallel and non-separating, we have conclusion 1 of the lemma. If D' and D'' are parallel and separating, we have conclusion 2 of the lemma.

So, we assume that D' and D'' are not parallel. We first show that D' and D'' cannot both be non-separating. If both D' and D'' are non-separating and non-parallel, then $\partial D'$, $\partial D''$, and μ_2^+ are disjoint, non-separating, non-parallel simple closed curves in a genus two surface. However, this is a contradiction of the classification of surfaces. Therefore, at least one of $\{D', D''\}$ must be separating. We will assume that D'' is separating. Since both disks are essential and disjoint, D' must be essential after splitting C along D'' into a solid torus and a collar of a torus. Since there are no essential disks in a collar of a torus, D' must be an essential disk in a solid torus. Thus D' is a meridian disk in this solid torus and is therefore non-separating in C . This gives us conclusion 3 of the lemma. $\square$

Lemma 8.6 Let $\{A_1, A_2, A_3\}$ be a system of pairwise disjoint, non-parallel good annuli in a compression body C of type $(2, 1)$. Then surgering A_1 , A_2 , and A_3 from C results in a collar of a torus C_1 , a solid torus C_2 , and a genus two handlebody C_3 which satisfy (see Figure 8.3)

1. the annuli appear on the boundaries of the various compression bodies and handlebodies as follows: $A_3 \subset \partial C_1$, $A_2 \cup A_3 \subset \partial C_2$, and $A_1 \cup A_2 \subset \partial C_3$,
2. there is a complete system of meridian disks $\{\Delta_1, \Delta_2\}$ of C_3 such that $\Delta_i \cap A_j = \emptyset$ for $i \neq j$ and $\Delta_i \cap A_i$ ($i = 1, 2$) is an essential arc of A_i , and
3. there is a meridian disk D_3 of C_2 such that $D_3 \cap A_i$ ($i = 2, 3$) is an essential arc of A_i .

Proof. The annuli $\{A_1, A_2\}$ must satisfy one of the conclusions of Lemma 8.5. If we assume that $\{A_1, A_2\}$ satisfy conclusion 2, then surgering A_1 and A_2 from C results in a collar of a torus C_1 , a solid torus C_2 , and a genus two handlebody C_3 . If A_3 is in ∂C_1 , then, by Lemma 6.1, A_3 is parallel to either A_1 or A_2 , a contradiction. If A_3 is in ∂C_2 , then, by Lemma 8.1, A_3 is parallel to either A_1 or A_2 , a contradiction. If A_3 is in ∂C_3 , then, by Lemma 6.4, A_3 is parallel to either A_1 or A_2 , a contradiction. Thus, $\{A_1, A_2\}$ cannot satisfy conclusion 2 of Lemma 8.5.

If we assume that $\{A_1, A_2\}$ satisfy conclusion 1 of Lemma 8.5, then surgering A_1 and A_2 from C results in a collar of a torus C_1 and a genus two handlebody C_2 . By Lemma 6.4, we know that A_3 is not in ∂C_2 . Then $A_3 \subset \partial C_1$ and, by Lemma 8.1, A_3 is parallel to an annulus A' in ∂C_1 . Since A_3 is incompressible and is not parallel to either A_1 or A_2 , $\partial A_1 \cup \partial A_2$ is in A' . Then $\{A_1, A_2, A_3\}$ satisfy the conditions of the lemma.

If we assume that $\{A_1, A_2\}$ satisfy conclusion 3 of Lemma 8.5, then surgering A_1 and A_2 from C results in a collar of a torus C_1 and a genus two handlebody C_2 where $A_1 \cup A_2 \subset \partial C_2$ and $A_1 \cap \partial C_1 = \emptyset$. Again, by Lemma 8.1, we find that A_3 must be in ∂C_2 . Since $A_3 \cap (A_1 \cup A_2) = \emptyset$, by [21, Lemma 3.3] we see that A_3 is parallel to an annulus A' in ∂C_2 . Again, since A_3 is incompressible and not parallel to either

A_1 or A_2 , $\partial A_1 \cup \partial A_2 \subset A'$. Thus, via a change in subscripts, $\{A_1, A_2, A_3\}$ satisfy the conditions of the lemma. $\square$

Chapter 9

Generalized bridge number for knots and links

The following definitions are derived from [10]. Note that these definitions have been generalized so that they account for knot exteriors in 3-manifolds with boundary. A properly embedded set of arcs $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ in a 3-manifold M is *trivial* if there is an embedded collection $\{D_1, \dots, D_n\}$ of disks in M such that $\partial D_i \cap \Gamma = \gamma_i$ ($i = 1, \dots, n$) and $\partial D_i \cap \partial M$ is the arc $\alpha_i = \overline{\partial D_i \setminus \gamma_i}$.

A link K in a 3-manifold M is in *n -bridge position* with respect to a splitting surface F if K intersects the closure of each component of $M \setminus F$ in a trivial collection of n arcs. The *genus g bridge number* $b_g(L)$ of a link L in M is the smallest integer n for which L is in n -bridge position with respect to some genus g splitting surface in M . This is the standard form of bridge number when associated with knots and links in S^3 with respect to a 2-sphere.

We begin with a construction which will always result in a knot or link in S^3 in 2-bridge position with respect to a 2-sphere. We will abusively refer to these as 2-bridge knots and links.

Lemma 9.1 Let V_1 and V_2 be 3-balls where α_1 and α_2 are a trivial pair of arcs in V_1 and β_1 and β_2 are a trivial pair of arcs in V_2 . Let h be any attaching map which takes ∂V_1 to ∂V_2 such that $h(\partial \alpha_1 \cup \partial \alpha_2) = \partial \beta_1 \cup \partial \beta_2$. Then $(\alpha_1 \cup \alpha_2) \cup_h (\beta_1 \cup \beta_2)$ is a 2-bridge knot or link in S^3 .

Proof. First note that the Alexander trick [1] guarantees that any attaching map h

chosen will make $V_1 \cup_h V_2$ homeomorphic to S^3 . Then $h(\partial V_1) = \partial V_2$ is an embedded 2-sphere in S^3 and the lemma follows immediately since the resulting knot or link will have at most two components and is, by construction, in 2-bridge position with respect to ∂V_2 . $\square$

There are particular constructions which arise in creating the exteriors of 2-bridge knots and links from the identifications of genus two handlebodies. A version of the following lemma appeared in [21] without proof.

Lemma 9.2 Let V_1 be a genus two handlebody with a complete system of meridian disks $\{D_1^1, D_2^1\}$ and V_2 a genus two handlebody with a complete system of meridian disks $\{D_1^2, D_2^2\}$. Let A_1 and A_2 be incompressible annuli in ∂V_1 such that $A_i \cap D_i^1$ ($i = 1, 2$) is an essential arc in A_i and $A_i \cap D_j^1 = \emptyset$ ($i \neq j$). Let A'_1 and A'_2 be incompressible annuli in ∂V_2 such that $A'_i \cap D_i^2$ ($i = 1, 2$) is an essential arc in A_i and $A_i \cap D_j^2 = \emptyset$ ($i \neq j$). Then, attaching V_1 to V_2 via a map which takes $\overline{V_1 \setminus (A_1 \cup A_2)}$ to $\overline{V_2 \setminus (A'_1 \cup A'_2)}$ yields the exterior of a 2-bridge knot or link in S^3 .

Proof. Let V'_1 be a 3-ball with the open regular neighborhoods of a pair of disjoint trivial arcs α_1 and α_2 removed. Let V'_2 be a 3-ball with the open regular neighborhoods of a pair of disjoint trivial arcs β_1 and β_2 removed. Then, from the Lemma 9.1 we know that any attaching of V'_1 to V'_2 along the specified four punctured sphere yields a 2-bridge knot or link exterior in S^3 . We also know that V'_1 and V'_2 are both genus two handlebodies. Since the α_i 's are trivial arcs, there are meridian disks $D_1^{1'}$ and $D_2^{1'}$ of V'_1 such that $D_i^{1'} \cap \partial\eta(\alpha_i)$ ($i = 1, 2$) is a single essential arc of $\partial\eta(\alpha_i)$ and $D_i^{1'} \cap \eta(\alpha_j) = \emptyset$ ($i \neq j$). Similarly, we know that there are meridian disks $D_1^{2'}$ and $D_2^{2'}$ for V'_2 such that $D_i^{2'} \cap \partial\eta(\beta_i)$ ($i = 1, 2$) is an essential arc of $\partial\eta(\beta_i)$ and $D_i^{2'} \cap \eta(\beta_j) = \emptyset$ ($i \neq j$).

Let h be the identification map which attaches V_1 to V_2 along the four punctured sphere $\overline{\partial V_1 \setminus (A_1 \cup A_2)}$. Then there is a homeomorphism h_1 taking V_1 to V'_1 such that $h_1(D_1^1) = D_1^{1'}$, $h_1(D_2^1) = D_2^{1'}$, and $h_1(A_i) = \partial\eta(\alpha_i)$ ($i = 1, 2$). Similarly, there is a homeomorphism h_2 that takes V_2 to V'_2 such that $h_2(D_1^2) = D_1^{2'}$, $h_2(D_2^2) = D_2^{2'}$, and $h_2(A'_i) = \partial\eta(\beta_i)$ ($i = 1, 2$). From Lemma 9.1 we know that all attachings of V'_1 to

V'_2 along the four punctured sphere $\overline{\partial V'_1 \setminus \partial(\eta(\alpha_1) \cup \eta(\alpha_2))}$ result in a 2-bridge knot or link exterior in S^3 . Thus, since h extends to an attaching of V'_1 to V'_2 along h_1 , $V_1 \cup_h V_2$ is homeomorphic to a 2-bridge knot or link exterior in S^3 . $\square$

Let K be a knot in a lens space L . Then L has a genus one Heegaard splitting and so K will be in n -bridge position with respect to this splitting torus. If K is in 1-bridge position with respect to the splitting torus of L , we will abusively call K a 1-bridge knot in the lens space L .

Lemma 9.3 *Let V_1 and V_2 be solid tori where α is a trivial arc in V_1 and β is a trivial arc in V_2 . Let h be any attaching map which takes ∂V_1 to ∂V_2 such that $h(\partial\alpha) = \partial\beta$. Then $\alpha \cup_h \beta$ is a 1-bridge knot in a lens space L .*

Proof. First note that any attaching map h will create a lens space. Also note that we are including $S^2 \times S^1$ in this list for completeness. While it does not have a finite fundamental group, its fundamental group is cyclic and it does have a genus one Heegaard splitting. Since $h(\partial V_1) = \partial V_2$ is an embedded torus, then $\alpha \cup_h \beta$ is, by construction, a 1-bridge knot in the lens space $V_1 \cup_h V_2$. $\square$

There is a particular construction which always results in a knot exterior homeomorphic to the exterior of a 1-bridge knot in a lens space. A version of the following lemma appeared in [21] without proof.

Lemma 9.4 *Let V_1 be a genus two handlebody with a complete system of meridian disks $\{D_1^1, D_2^1\}$ and V_2 a genus two handlebody with a complete system of meridian disks $\{D_1^2, D_2^2\}$. Let A_1 be an incompressible annulus in ∂V_1 such that $A_1 \cap D_1^1$ is an essential arc of A_1 and $A_1 \cap D_2^1 = \emptyset$. Let A_2 be an incompressible annulus in ∂V_2 such that $A_2 \cap D_1^2$ is an essential arc of A_2 and $A_2 \cap D_2^2 = \emptyset$. Then, attaching V_1 to V_2 via a map which takes $\overline{V_1 \setminus A_1}$ to $\overline{V_2 \setminus A_2}$ yields the exterior of a 1-bridge knot in a lens space.*

Proof. Let V'_1 be a solid torus with an open regular neighborhood of a trivial arc α removed. Let V'_2 be a solid torus with an open regular neighborhood of a trivial arc β removed. Then, from Lemma 9.3, we know that attaching V'_1 to V'_2 along

the specified twice punctured torus yields a 1-bridge knot exterior in a lens space. Since α is a trivial arc, there is a meridian disk $D_1^{1'}$ of V_1' such that $D_1^{1'} \cap \partial\eta(\alpha)$ is a single essential arc of $\partial\eta(\alpha)$. In addition, there is a second meridian disk $D_2^{1'}$ for V_1' such that $D_2^{1'} \cap \partial\eta(\alpha) = \emptyset$. Similarly, there is a meridian disk $D_1^{2'}$ for V_2' such that $D_1^{2'} \cap \partial\eta(\beta)$ is an essential arc of $\partial\eta(\beta)$ and another meridian disk $D_2^{2'}$ of V_2' such that $D_2^{2'} \cap \partial\eta(\beta) = \emptyset$.

Let h be the identification map which attaches V_1 to V_2 along the twice punctured torus $\overline{V_1 \setminus A_1}$. Then, there is a homeomorphism h_1 which takes V_1 to V_1' such that $h_1(D_1^1) = D_1^{1'}$, $h_1(D_2^1) = D_2^{1'}$, and $h_1(A_1) = \partial\eta(\alpha)$. Similarly, there is a homeomorphism h_2 which takes V_2 to V_2' such that $h_1(D_1^2) = D_1^{2'}$, $h_2(D_2^2) = D_2^{2'}$, and $h_1(A_2) = \partial\eta(\beta)$. From Lemma 1:bridge:lensspace:construction we know that all attachings of V_1' to V_2' along the twice punctured torus $\overline{\partial V_1' \setminus \partial\eta(\alpha)}$ result in a 1-bridge knot exterior in a lens space. Thus, since h extends to an attaching of V_1' to V_2' along h_1 , we know that $V_1 \cup_h V_2$ is homeomorphic to a 1-bridge knot in a lens space. $\square$

Let K be a knot in a solid torus V . Let T be an embedded peripheral torus in V such that K meets T transversely in two points. If K is in 1-bridge position with respect to T in V , we will abusively call K a 1-bridge knot in a solid torus.

Lemma 9.5 *Let V_1 be a solid torus and V_2 be a collar of a torus. Then every attaching map which glues ∂V_1 to one of the components of ∂V_2 results in a solid torus.*

Proof. Let D be a meridian disk for V_1 and let h be an identification map taking ∂V_1 to one of the components of ∂V_2 . Then, since $\partial D \subset \partial V_1$, $h(\partial D)$ is a simple closed curve in a single component of ∂V_2 . Collaring $h(\partial D)$ produces a spanning essential annulus A . If we surger D from V_1 , we obtain a 3-ball. If we surger A from V_2 we obtain a solid torus. Let Δ be the disk $D \cup_h A$. Since $\overline{V_1 \setminus \eta(D)}$ and $\overline{V_2 \setminus \eta(A)}$ are attached along an annulus, $\overline{(V_1 \cup_h V_2) \setminus \eta(\Delta)}$ is a 3-ball. Thus Δ is a meridian disk for the solid torus $V_1 \cup_h V_2$. $\square$

From this we can now describe a construction which always results in a knot with a 1-bridge knot in a solid torus.

Lemma 9.6 Let V_1 be a solid torus and α a trivial arc in V_1 . Let V_2 be a collared torus and β a trivial arc in V_2 . Let h be any attaching map which takes ∂V_1 to the component of ∂V_2 containing $\partial\beta$ such that $h(\partial\alpha) = \partial\beta$. Then $\alpha \cup_h \beta$ is a 1-bridge knot in the solid torus $V = V_1 \cup_h V_2$.

Proof. Since α is trivial in V_1 , there is an embedded disk D_1 in V_1 such that α and an embedded arc $\alpha' \subset \partial V_1$ co-bound D_1 . Similarly, there is an embedded disk D_2 in V_2 such that β and an embedded arc $\beta' \subset \partial V_2$ co-bound D_2 . Let $V'_1 = \overline{V_1 \setminus \eta(\alpha)}$ and $V'_2 = \overline{V_2 \setminus \eta(\beta)}$. Then $\Delta_1 = D_1 \cap V_1$ is a compressing disk for V'_1 since $\overline{V_1 \setminus (\eta(\alpha) \cup \eta(D_1))} = \overline{V_1 \setminus \eta(D_1)}$ is a solid torus. Likewise, $\Delta_2 = D_2 \cap V_2$ is a compressing disk for V'_2 and $\overline{V'_2 \setminus \eta(\Delta_2)}$ is a collar of a torus. Since V'_1 has a compression to a solid torus, it is a genus two handlebody. Since V'_2 has a compression to a collar of a torus, it is a compression body of type $(2, 1)$. Since both $\eta(\alpha)$ and $\eta(\beta)$ are 3-balls and all the components of $\partial\eta(\alpha)$ and $\partial\eta(\beta)$ are disks on the boundaries of V_1 and V_2 , $\eta(\alpha) \cup_h \eta(\beta)$ is a solid torus. From Lemma 9.5 we know that $V = V_1 \cup_h V_2$ is a solid torus. Thus, $K = \alpha \cup_h \beta$ is a 1-bridge knot in the solid torus V . $\square$

Under certain circumstances, we will have a genus two handlebody, a compression body of type $(2, 1)$, and an identified essential annulus on the boundary of each. We would like to know under what conditions these two compression bodies can be attached along part of their boundaries to create a knot exterior in a solid torus which has a 1-bridge presentation.

Lemma 9.7 Let V_1 be a genus two handlebody with a complete system of meridian disks $\{D_1^1, D_2^1\}$ and V_2 a compression body of type $(2, 1)$ with a complete disk/annular system $\{D_1^2, \hat{D}_2^2, A_\lambda\}$. Let $A_1 \subset \partial V_1$ be an incompressible annulus such that $A_1 \cap D_1^1$ is an essential arc in A_1 and $A_1 \cap D_2^1 = \emptyset$. Let $A_2 \subset \partial V_2$ be an incompressible annulus such that A_2 is in the genus two component of ∂V_2 , $A_2 \cap D_1^2$ is an essential arc in A_2 , and $A_2 \cap \hat{D}_2^2 = \emptyset$. Then, attaching V_1 to V_2 via a map which takes $\overline{V_1 \setminus A_1}$ to $\overline{V_2 \setminus A_2}$ yields the exterior of a 1-bridge knot in a solid torus.

Proof. Let V'_1 be a solid torus with an open regular neighborhood of a trivial arc α removed. Let V'_2 be a collared torus with an open regular neighborhood of a trivial

arc β removed. Then, from Lemma 9.6 we know that any attaching of V'_1 to V'_2 along the specified twice punctured torus yields a 1-bridge knot exterior in a solid torus. We also know that V'_1 is a genus two handlebody and V'_2 is a compression body of type $(2, 1)$. Since α is a trivial arc, there is a meridian disk D_1 of V'_1 such that $D_1 \cap \partial\eta(\alpha)$ is a single essential arc of $\partial\eta(\alpha)$. Since α is a trivial arc, there is a second meridian disk D_2 for V'_1 such that $D_2 \cap \partial\eta(\alpha) = \emptyset$. Similarly, there is a meridian disk Δ of V'_2 such that $\Delta \cap \partial\eta(\beta)$ is an essential arc of $\partial\eta(\beta)$. In addition, there is a μ -annulus for V'_2 , A_μ , such that $A_\mu \cap \partial\eta(\beta) = \emptyset$.

Let h be the identification map which attaches V_1 to V_2 along the twice punctured torus $\overline{V_1 \setminus A_1}$. Then, there is a homeomorphism h_1 which takes V_1 to V'_1 such that $h_1(D_1^1) = D_1$, $h_1(D_2^1) = D_2$, and $h_1(A_1) = \partial\eta(\alpha)$. Similarly, there is a homeomorphism h_2 which takes V_2 to V'_2 such that $h_2(D_2^2) = \Delta$, $h_2(\hat{D}_2^2) = A_\mu$, and $h_2(A_2) = \partial\eta(\beta)$. From Lemma 9.6 we know that all attachings of V'_1 to V'_2 along the twice punctured torus $\overline{\partial V'_1 \setminus \partial\eta(\alpha)}$ result in a 1-bridge knot exterior in a solid torus. Thus, since h extends to an attaching of V'_1 to V'_2 along h_1 , we know that $V_1 \cup_h V_2$ is homeomorphic to a 1-bridge knot exterior in a solid torus. $\square$

The following lemma is recalled here [32].

Lemma 9.8 *Let M be the exterior of a tunnel-1 link of unknots in S^3 . Then M is a 2-bridge link exterior in S^3 .*

Proof. Let T_1 and T_2 be the two boundary components of M . Let $\mu_i \subset T_i$ ($i = 1, 2$) be the meridians of M . Let τ be a properly embedded, spanning arc in M such that $V' = \overline{M \setminus \eta(\tau)}$ is a genus two handlebody. The arc τ is a tunnel for M . Let μ'_i ($i = 1, 2$) be a simple closed curve in T_i in the same isotopy class as μ_i such that $\mu'_i \cap \tau \subset \partial\tau$. Let V_1 be the closed regular neighborhood of $\mu'_1 \cup \tau \cup \mu'_2$ in M and let $V_2 = \overline{M \setminus V_1}$.

Then V_1 and V_2 are both genus two handlebodies, $V_1 \cup V_2 = M$, and $V_1 \cap V_2$ is a four punctured sphere. Let $A_i = T_i \cap V_1$ ($i = 1, 2$). From the nature of the decomposition of M , there is a complete set of meridian disks $\{D_1^1, D_2^1\}$ for M such that $D_i^1 \cap A_i$ ($i = 1, 2$) is an essential arc in A_i and $D_i^1 \cap A_j = \emptyset$ ($i \neq j$). Moreover, since $\mu'_1 \cup \mu'_2$

are meridians for M , there is a complete system of meridian disks $\{D'_1, D'_2\}$ for V' such that the distance between D'_i and μ'_i ($i = 1, 2$) is one and $D'_i \cap \mu'_j = \emptyset$ ($i \neq j$). Let $A'_i = T_i \cap V_2$ ($i = 1, 2$). Then letting $D_i^2 = D'_i \cap V_2$ ($i = 1, 2$), we form a complete system of meridian disks $\{D_1^2, D_2^2\}$ for V_2 . Then $D_i^2 \cap A'_i$ ($i = 1, 2$) is an essential arc in A'_i and $D_i^2 \cap A_j = \emptyset$ ($i \neq j$). Then, from Lemma 9.2, M is homeomorphic to some 2-bridge link exterior in S^3 . $\square$

From this, since the exterior of a one-bridge knot in a solid torus with a solid torus filling on both components is a tunnel-1 link of unknots in S^3 , we have the following lemma.

Lemma 9.9 Let M be the exterior of a 1-bridge knot in a solid torus with a meridian on each boundary component. Then M is a 2-bridge link exterior in S^3 .

Chapter 10

Lemmata and definitions

Since we are interested in classifying the tunnel-1 generalized satellite knot exteriors, we can take advantage of the properties of the closed 3-manifolds obtainable from them via Dehn filling. The following lemma is the basis for this line of attack.

Lemma 10.1 Let M be the exterior of a tunnel-1 generalized satellite knot. Then every 3-manifold obtained from M via Dehn filling has a genus two Heegaard splitting.

Proof. Since M possesses a genus two splitting, it is the union of a genus two handlebody and a compression body of type $(2, 1)$. Dehn filling, via the Alexander trick [1], amounts to gluing a 2-handle and a 3-handle to the toral boundary component of the above compression body. Therefore, the core of this new 2-handle in the compression body, along with the core of the 2-handle which was already present, induces a system of meridian disks for the resulting handlebody. $\square$

Unless each essential torus along which we decompose M compresses under the filling (see [8]), the resulting closed 3-manifold will be Haken. Then Lemma 10.1 means that, after filling, the closed 3-manifold is subject to the classification of closed, genus two Haken 3-manifolds given in [21].

To construct the exterior of a satellite knot in S^3 , a result of Schubert ([30] and [31]) states that we start with a simple or Seifert fibered torally bounded 3-manifold and attach to it in sequence a finite number of simple or Seifert fibered 3-manifolds with two toral boundary components. The last of these will contain the boundary of our knot exterior. A corollary of our main theorem is that this *nesting* of tori in the

torus decomposition also holds for tunnel one generalized satellite knots. Thus, we must identify the core torally bounded 3-manifold as well as those those simple or Seifert fibered 3-manifolds with two toral boundary components we attach to it.

Definition 1 We define the following classes of torally bounded 3-manifolds:

- $D(n)$ is the class of Seifert fibre spaces over the disk with n exceptional fibres.
- $M\ddot{o}(n)$ is the class of Seifert fibre spaces over the Möbius band with n exceptional fibres.
- M_K is the class of 2-bridge knot exteriors in S^3 .
- L_K is the class of 1-bridge knot exteriors in lens spaces.

Definition 2 We define the following classes of 3-manifolds with two toral boundary components:

- $A(n)$ is the class of Seifert fibre spaces over the annulus with n exceptional fibres.
- $\hat{M}\ddot{o}(n)$ is the class of Seifert fibre spaces over a once punctured Möbius band with n exceptional fibres.
- M_L is the class of 2-bridge link exteriors in the 3-sphere.
- V_K is the class of 1-bridge knot exteriors in solid tori.

In the torus decomposition of a generalized satellite knot exterior, we denote the complementary component containing the boundary of the knot exterior as the *outside* component.

The following theorem appears in [17, Chapter VI] as the object of that chapter. We state it here as a lemma. It appears in [17] as Theorem VI.34.

Lemma 10.2 (Jaco) *Let M be a compact, orientable Seifert fibered manifold. If F is a two-sided, incompressible surface in M then one of the following alternatives holds:*

1. F is a disk or an annulus and F is parallel into ∂M .
2. F does not separate M and F is a fibre in a fibration of M as a surface bundle over S^1 .
3. F does separate M and $M = M_1 \cup M_2$ where $M_1 \cap M_2 = \partial M_1 = \partial M_2 = F$ and M_i ($i = 1, 2$) is a twisted I -bundle over a compact surface (possibly with boundary).
4. F is an annulus or a torus and F is saturated in some Seifert fibration of M .

Chapter 11

Essential Tori in 3-manifolds with boundary

The following is recalled from [17]. Let F and S be properly embedded surfaces in a 3-manifold M such that F and S intersect in essential simple closed curves and arcs and F is separating in M . Suppose further that there is an embedded disk Δ in M such that $\alpha = \Delta \cap S$ is an arc in $\partial\Delta$, $\beta = \Delta \cap F$ is an arc in $\partial\Delta$, $\partial\alpha = \partial\beta$, and $\alpha \cup \beta = \partial\Delta$. Then an *isotopy of type A* at α is performed by sliding α across Δ and past β . This isotopy moves a part of S which was on one side of F in M to the other side of F in M . The result of this isotopy on the part of S which was on the original side of F is equivalent to cutting that surface along α . Refer to Figure 11.1.

Let M be a tunnel-1 generalized satellite knot exterior. Let T be an essential torus in the decomposition of M . Let V_1 and V_2 be the genus two handlebody and compression body of type (2, 1) in the splitting of M . Much of the following is from [20]. Let $T_i = T \cap V_i$ ($i = 1, 2$). Via isotopies of type A (see [17]), we may assume

Figure 11.1: An isotopy of type A

Figure 11.2: Various types of arcs in T_2

that each element of T_1 is a disk, and that the number of components of $T \cap V_1$ is minimal among all tori isotopic to T in M . As in [17] we have the hierarchy $(T_2^{(0)}, \alpha_0), \dots, (T_2^{(m)}, \alpha_m)$ for T_2 which gives rise to a sequence of isotopies of type T in M where the first isotopy is of type A at $\alpha_0, \dots$, and the $(m+1)^{\text{st}}$ isotopy is of type A at α_m . Since T was embedded in M , $\alpha_i \cap \alpha_j = \emptyset$ ($i \neq j$). Each α_i is an arc on T_2 .

If α_i joins distinct components of ∂T_2 , then we refer to α_i as being of *type 1*. If α_i joins a single component S of ∂T_2 and there is an arc $\beta \subset S$ such that $\partial \alpha_i = \partial \beta$ and $\alpha_i \cup \beta$ bound a disk in T , then we refer to α_i as being of *type 2*. If α_i joins a single component S of ∂T_2 and there is an $\beta \subset S$ such that $\partial \alpha_i = \partial \beta$ and surgering $\alpha_i \cup \beta$ from T results in an annulus, we refer to α_i as being of *type 3*. If α_i is of type 1 and there is a component S of ∂T_2 such that α_i is the only arc incident to S , we refer to α_i as a *d-arc*. Refer to Figure 11.2.

Lemma 11.1 (Kobayashi [20]) If any α_i is a *d-arc*, then there is an ambient isotopy h of T in M such that each component of $h(T) \cap V_1$ is a disk and the number of components of $h(T) \cap V_1$ is less than the number of components of $T \cap V_1$.

Proof. If we let h be the converse of an isotopy of type A at α_i , then $h(T) \cap V_1$ will consist entirely of disks and will have one fewer components than $T \cap V_1$. $\square$

Lemma 11.2 (Kobayashi [20]) If α_0 is of type 1 or type 2, then there is an ambient isotopy h of M such that each component of $h(T) \cap V_1$ is a disk and the number of components in $h(T) \cap V_1$ is less than the number of components of $T \cap V_1$.

Proof. The following is recalled from [20]. If α_0 is of type 1, let h be the converse of an isotopy of type A at α_0 and the conclusion follows. Assume that α_0 is of type 2. Then there is an arc $\beta \subset \partial T_2$ such that $\partial\alpha_0 = \partial\beta$ and $\alpha_0 \cup \beta$ bounds a planar surface P in T_2 . Since α_0 is an essential arc in T_2 , some α_i in P is a d -arc. The conclusion now follows from Lemma 11.1. $\square$

Lemma 11.3 (Kobayashi [20]) Suppose that α_0 is of type 3 and one of the following conditions is satisfied:

1. α_1 is of type 1,
2. α_1 is of type 2, or
3. α_1 is of type 3 and α_1 intersects the same component of ∂T_2 that α_0 intersects.

Then there is an ambient isotopy h of M such that each component of $h(T) \cap V_1$ is a disk and the number of components in $h(T) \cap V_1$ is less than the number of components in $T \cap V_1$. $\square$

Proof. The following is recalled from [20]. If condition 1 holds, then the lemma follows by letting h be the isotopy of type A at α_1 . If condition 2 holds, then the lemma follows from the proof of Lemma 11.2. If condition 3 holds, then $\alpha_0 \cup \alpha_1$ separate T_2 into one or two planar surfaces and, as in the proof of Lemma 11.2, there exists a d -arc and the lemma follows from Lemma 11.1. $\square$

Lemma 11.4 Let M be a tunnel-1 generalized satellite exterior in which the torus decomposition is a single torus T which separates M into two components M_1 and M_2 . Furthermore, in the genus two splitting of M into a genus two handlebody V_1 and a compression body V_2 of type $(2, 1)$, $T_1 = T \cap V_1$ consists entirely of disks and

the number of components in T_1 is minimal among all tori isotopic to T in M . Then T_1 consists of at most two components. $\square$

Proof. Much of the following is recalled from [21]. Assume that $T \cap V_1$ consists of $n \geq 3$ components, $D_1, \dots, D_n$. Then, from Lemmas 11.2 and 11.3 we know that α_0 and α_1 are of type 3 and that $T^{(1)} \cap V_1 = A_1 \cup D_2 \cup \dots \cup D_n$ and $T^{(2)} \cap V_1 = A_1 \cup A_2 \cup D_3 \cup \dots \cup D_n$ where A_i ($i = 1, 2$) is a non-peripheral incompressible annulus in V_1 . If D_1 and D_2 are separating in V_1 and A_1 and A_2 are parallel in V_1 , then there are two annuli A' and A'' in ∂V_1 such that $A' \cap (A_1 \cup A_2) = A' \cap A_1 = \partial A' = \partial A_1$, $A'' \cap (A_1 \cup A_2) = \partial A''$, and $A' \cap A''$ is a component of ∂A_1 . We may assume that A' and A'' are in different components of M , say M_1 and M_2 . Therefore, from Lemma 10.2, we know that $M_1 \cup_T M_2$ admits a Seifert fibration with $A' \cap A''$ as the regular fibre. This contradicts the minimality inherent in that T was an element of a torus decomposition.

If D_1 and D_2 are separating in V_1 and A_1 is not parallel to A_2 in V_1 , then $D_1, \dots, D_n$ are parallel in V_1 . Since Lemma 11.1 tells us that none of the α_i are d -arcs, α_2 is of type 3 and we can assume that $T^{(3)} = A_1 \cup A_2 \cup A_3 \cup \dots \cup D_n$ where A_3 is a non-peripheral incompressible annulus in V_1 . Then from Lemma 6.4 we know that A_3 is parallel to either A_1 or A_2 , and we arrive at the same contradiction we had above.

If D_1 is separating and D_2 is non-separating in V_1 , then there exist annuli A' and A'' as above and we again have a contradiction. If D_1 and D_2 are non-separating in V_1 , then $D_1, \dots, D_n$ are mutually parallel in V_1 . Again, we look to $T^{(3)}$ and find that A_3 will be parallel to either A_1 or A_2 and we have again arrived at a contradiction.

Therefore, $T \cap V_1$ consists of at most two components. $\square$

Chapter 12

The main theorem and some corollaries

We now state the main theorem of the paper. We follow this up with some important corollaries.

Theorem 12.1 Let M be a 3-manifold with toral boundary such that M has a splitting of genus two and a non-trivial toral decomposition where all decomposing tori are separating. Then, by decomposing M along all such essential tori into simple or Seifert fibered components, then either

- 1. M is obtained from $M_1 \in D(2)$ and $M_2 \in V_K$ where the regular fibre of M_1 is glued to a meridian loop of M_2 . Moreover, if M_2 is not simple, then M_2 decomposes into two components $M'_2 \in A(1)$ and $M''_2 \in M_L$ where the regular fibre of M'_2 is glued to a meridian loop of M''_2 and the other meridian loop is glued to the regular fibre of M_1 .*
- 2. M is obtained from $M_1 \in A(1)$ and $M_2 \in L_K$ where the regular fibre of M_1 is glued to the meridian loop of M_2 . Moreover, if M_2 is not simple, then M_2 decomposes into two components $M'_2 \in D(2)$ and $M''_2 \in M_L$ where the regular fibre of M'_2 is glued to a meridian loop of M''_2 and the other meridian loop of M''_2 is glued to the regular fibre of M_1 .*
- 3. M is obtained from $M_1 \in A(2)$ and $M_2 \in M_K$ where the regular fibre of M_1 is glued to a meridian loop of M_2 .*

4. M is obtained from $M_1 \in \hat{M}\ddot{o}(n)$ ($n = 0, 1$) and $M_2 \in M_K$ where the regular fibre of M_1 is glued to the meridian loop of M_2 .

Proof. We divide the proof into several cases. Since all decomposing tori are separating, the first case is that M is divided into two simple or Seifert fibered components, M_1 and M_2 , by a single essential torus. The second case is when there are two non-isotopic essential tori separating M into simple or Seifert fibered components, M_1 , M_2 , and M_3 . Finally, we show that there cannot be more than two non-isotopic essential tori and thus no more than three simple or Seifert fibered components.

Case 1. M is decomposed into two components, M_1 and M_2 . Let T be the torus which separates M into M_1 and M_2 . Let $(V_1, V_2; F)$ be the genus two splitting of M . Since $\partial M \cap F = \emptyset$, we will assume, without loss of generality, that V_1 is a genus two handlebody and V_2 is a compression body of type $(2, 1)$. Via isotopies of type A (see [17]), we may assume that all the components of $T \cap V_1$ are disks and that the number of components of $T \cap V_1$ are minimal among all tori isotopic to T in M such that all of the components of intersection of those tori with V_1 are disks. Let $T_2 = T \cap V_2$. As in [17], we have a hierarchy $(T_2^{(0)}, a_0), \dots, (T_2^{(m)}, a_m)$ of T_2 and a sequence of isotopies of type A which realizes this hierarchy. Let $T^{(1)}$ be the image of T after an isotopy of type A at a_0 and $T^{(k+1)}$ ($k \geq 1$) be the image of T after an isotopy of type A at a_k .

We know from Lemma 11.4 that $T \cap V_1$ consists of at most two components. We therefore have two sub-cases.

Case 1.1. The intersection $T \cap V_1$ consists of a single disk D_1 . Since T separates in M , D_1 separates in V_1 . Thus, D_1 separates V_1 into two solid tori. Let $A_1 = T^{(1)} \cap V_1$ and $A_2 = T^{(1)} \cap V_2$. Then, by Lemma 6.3, A_1 separates V_1 into a solid torus V_1^1 and a genus two handlebody V_1^2 . Since T is separating in M , then A_2 is a separating, essential, non-spanning, properly embedded annulus in V_2 . Then, by Lemma 8.3, either A_2 separates V_2 into a solid torus and a compression body of type $(2, 1)$, or it separates V_2 into a collar of a torus and a genus two handlebody. We treat these two cases separately. |

Case 1.1.1. The annulus A_2 separates V_2 into a solid torus V_2^1 and a compression body V_2^2 of type $(2, 1)$. By attaching V_1^1 and V_2^1 along $\overline{\partial V_i^1} \setminus A_i$ ($i = 1, 2$), we have

$M_1 \in D(2)$. By attaching V_1^2 and V_2^2 along $\overline{\partial V_i^2 \setminus A_i}$ ($i = 1, 2$), we have, from Lemma 9.7, $M_2 \in V_K$. Then we have conclusion 1 of the theorem.

Case 1.1.2. The annulus A_2 separates V_2 into a collar of a torus V_2^1 and a genus two handlebody V_2^2 . Again, by attaching V_1^1 and V_2^1 along $\overline{\partial V_i^1 \setminus A_i}$ ($i = 1, 2$), we have $M_1 \in \mathcal{A}(1)$. By attaching V_1^2 and V_2^2 along $\overline{\partial V_i^2 \setminus A_i}$ ($i = 1, 2$), we have $M_2 \in L_K$ (see Lemma 9.2). Then we have conclusion 2 of the theorem.

Case 1.2. The intersection $T \cap V_1$ consists of two disks, D_1 and D_2 . In this case $T^{(2)} \cap V_1$ consists of two essential annuli $\{A_1, A_2\}$, and $T^{(2)} \cap V_2$ consists of two essential annuli $\{A'_1, A'_2\}$.

From [21], we know that if A_1 and A_2 are parallel, then they satisfy conclusion 1 of Lemma 6.5. Thus, since $T^{(2)}$ is separating in M , $A_1 \cup A_2$ satisfy conclusion 1 or conclusion 2 of Lemma 6.5.

Next, we claim that if A'_1 and A'_2 are parallel, then they satisfy the conditions of conclusion 1 of Lemma 8.5. As in [21], we first demonstrate that A'_1 is non-separating in V_2 . If A'_1 is separating in V_2 , then there are annuli A' and A'' in ∂V_2 such that $A' \cap (A'_1 \cup A'_2) = A_1 \cap A'_1 = \partial A' = \partial A'_1$, $A'' \cap (A'_1 \cup A'_2) = \partial A''$, and $A' \cap A''$ is a component of $\partial A'$. Then $A' \subset M_1$ and $A'' \subset M_2$. By the minimality of T , we may assume that A' and A'' are essential annuli in their respective components. Hence, by Lemmas 9.4 and 10.2, M_1 and M_2 admit such Seifert fibrations that $A' \cap A''$ is a regular fibre of both manifolds. Hence, M admits a Seifert fibration. This contradicts that M had a non-trivial torus decomposition. Therefore, A'_1 is non-separating in V_2 and, by Lemma 8.3, A'_1 separates V_2 into a compression body C' of type $(2, 1)$. Let $A_1^{1'}$ and $A_1^{2'}$ be the copies of A_1' in $\partial C'$. Then, there is a complete disk/annular system $\{D_1, \hat{D}_2, A_\lambda\}$ such that $A_1^{1'} \cap D_1$ is an essential arc of $A_1^{1'}$ and $A_1^{2'} \cap \hat{D}_2$ is an essential arc of $A_1^{2'}$. Then, since A'_2 is also parallel to $A_1^{2'}$, A_1' and A_2' satisfy conclusion 1 of Lemma 8.5.

By the above claim and Lemma 8.5, we know that $A_1' \cup A_2'$ must satisfy one of the conclusions of Lemma 8.5. However, just as above, $A_1' \cup A_2'$ cannot satisfy condition 3 of Lemma 8.5 since $T^{(2)}$ is separating in M . Therefore, $A_1' \cup A_2'$ satisfy conclusion 1 or conclusion 2 of Lemma 8.5.

Just as in [21], if A_1 and A_2 satisfy conclusion 2 of Lemma 6.5 and A_1' and A_2' satisfy condition 2 of Lemma 8.5, then we have, after the identification induced by the genus two splitting, $T^{(2)}$ being the disjoint union of two tori. Since $T^{(2)}$ is connected, this is a contradiction.

We therefore have three sub-cases.

Case 1.2.1. The annuli $A_1 \cup A_2$ satisfy conclusion 1 of Lemma 6.5, and $A_1' \cup A_2'$ satisfy conclusion 1 of Lemma 8.5. Then, $A_1 \cup A_2$ separate V_1 into a solid torus V_1^1 and a genus two handlebody V_1^2 , and $A_1' \cup A_2'$ separate V_2 into a collar of a torus V_2^1 and a genus two handlebody V_2^2 . Then, $A_1 \cup A_2 \subset \partial V_1^1$, $A_1 \cup A_2 \subset \partial V_1^2$, $A_1' \cup A_2' \subset \partial V_2^1$, and $A_1' \cup A_2' \subset \partial V_2^2$. By attaching V_1^1 and V_2^1 along $\overline{\partial V_1^1 \setminus (A_1 \cup A_2)}$, we find that $M_1 \in \hat{M}\ddot{o}(n)$ (where $n = 0, 1$). By attaching V_1^2 and V_2^2 along $\overline{V_1^2 \setminus (A_1 \cup A_2)}$, we find that $M_2 \in M_K$ by Lemma 9.2. Thus we have conclusion 4 of the theorem.

Case 1.2.2. The annuli $A_1 \cup A_2$ satisfy conclusion 1 of Lemma 6.5, and A_1' and A_2' satisfy conclusion 2 of Lemma 8.5. Then $A_1 \cup A_2$ separate V_1 into a solid torus V_1^1 and a genus two handlebody V_1^2 , and $A_1' \cup A_2'$ separate V_2 into a solid torus V_2^1 , a collar of a torus V_2^2 , and a genus two handlebody V_2^3 . Then $A_1 \cup A_2 \subset \partial V_1^1$, $A_1 \cup A_2 \subset \partial V_1^2$, $A_1' \subset \partial V_2^1$, $A_2' \subset \partial V_2^2$, and $A_1' \cup A_2' \subset \partial V_2^3$. By attaching V_1^1 , V_2^1 , and V_2^2 along $\overline{\partial V_1^1 \setminus (A_1 \cup A_2)}$, we find that $M_1 \in A(n)$ ($n = 0, 1, 2$). By attaching V_1^2 and V_2^3 along $\overline{\partial V_1^2 \setminus (A_1 \cup A_2)}$, we find that $M_2 \in M_K$. Thus we have conclusion 3 of the theorem.

Case 1.2.3. The annuli $A_1 \cup A_2$ satisfy conclusion 2 of Lemma 6.5, and A_1' and A_2' satisfy conclusion 1 of Lemma 8.5. Then $A_1 \cup A_2$ separate V_1 into two solid tori V_1^1 and V_1^2 and a genus two handlebody V_1^3 , and $A_1' \cup A_2'$ separate V_2 into a collar of a torus V_2^1 and a genus two handlebody V_2^2 . Then $A_1 \subset \partial V_1^1$, $A_2 \subset \partial V_1^2$, $A_1 \cup A_2 \subset \partial V_1^3$, $A_1' \cup A_2' \subset \partial V_2^1$, and $A_1' \cup A_2' \subset \partial V_2^2$. By attaching V_1^1 , V_1^2 , and V_2^1 along $\overline{(\partial V_1^1 \cup \partial V_1^2) \setminus (A_1 \cup A_2)}$, we find that $M_1 \in A(n)$ ($n = 0, 1, 2$). By attaching V_1^3 and V_2^2 along $\overline{V_1^3 \setminus (A_1 \cup A_2)}$, we find that $M_2 \in M_K$. Thus, we again have conclusion 3 of the theorem.

Case 2. M is decomposed into three components M_1 , M_2 , and M_3 by the torus decomposition along two tori. Let T_1 and T_2 be the tori giving rise to this decompo-

sition and let $T = T_1 \cup T_2$. We will again assume that $(V_1, V_2; F)$ is the genus two splitting of M , that V_1 is a genus two handlebody, and that V_2 is a compression body of type $(2, 1)$. We may assume that the components of $T \cap V_1$ are all disks obtained via isotopies of type A. Then, from [21] we know that $T \cap V_1$ consists of at most two components. Since $T_i \cap F$ is non-empty by the essentiality of T_i , $T \cap V_1$ is exactly two disks. Let $T' = T \cap V_2$. Then we have a hierarchy $(T'^{(0)}, a_0), \dots, (T'^{(m)}, a_m)$ of T' and a sequence of isotopies of type A which realizes this hierarchy. Let T^1 be the image of T after an isotopy of type A at a_0 , and T^2 be the image of T^1 after an isotopy of type A at a_1 . Then $T^2 \cap V_i$ will be a pair of essential annuli. Let A_1, A_2 be the two annuli in V_1 and A'_1, A'_2 be the two annuli in V_2 . From [21] we know that $\{A_1, A_2\}$ must satisfy one of the conclusions of Lemma 6.5. From the argument in case 1.2 above, we know that $\{A'_1, A'_2\}$ must satisfy one of the conclusions of Lemma 8.5.

Since each T_i is separating in M , each A_i is separating in V_1 and each A'_i is separating in V_2 . Thus, $\{A_1, A_2\}$ satisfies conclusion 2 of Lemma 6.5 and $\{A'_1, A'_2\}$ satisfies conclusion 2 of Lemma 8.5. Therefore, $A_1 \cup A_2$ separate V_1 into two solid tori V_1^1 and V_1^2 , and a genus two handlebody V_1^3 and $A'_1 \cup A'_2$ separates V_2 into a solid torus V_2^1 , a collar of a torus V_2^2 , and a genus two handlebody V_2^3 . Then $A_1 \subset \partial V_1^1$, $A_2 \subset \partial V_1^2$, $A_1 \cup A_2 \subset \partial V_1^3$, $A'_1 \subset \partial V_2^1$, $A'_2 \subset \partial V_2^2$, and $A'_1 \cup A'_2 \subset \partial V_2^3$. By attaching V_1^1 and V_2^1 along $\overline{\partial V_1^1 \setminus A_1}$, we find that $M_1 \in D(2)$. By attaching V_1^2 and V_2^2 along $\overline{\partial V_1^2 \setminus A_2}$, we find that $M_2 \in \mathcal{A}(1)$. By attaching V_1^3 and V_2^3 along $\overline{\partial V_1^3 \setminus (A_1 \cup A_2)}$, we find that $M_3 \in M_L$. This is the second half of conclusion 1 or conclusion 2 of the theorem.

Case 3. Finally, we show that M cannot be decomposed into more than 3 components. Assume that $M = M_1 \cup M_2 \cup \dots \cup M_k$ by a torus decomposition along the tori $T_1, T_2, \dots, T_{k-1}$. Then, via isotopies of type A, we may assume that $T \cap V_1$ consists entirely of disks. Since each T_i is essential in M , T_i essentially intersects F . However, from [21], we know that $T \cap V_1$ consists of at most two components. Therefore, $k - 1 \leq 2$ and M can be decomposed into at most 3 components. $\square$

This theorem has an immediate corollary which generalizes the results of Schubert ([30] and [31]) regarding the uniqueness of the core manifold in the toral decomposi-

tion of a satellite knot in S^3 to tunnel-1 satellite knots in arbitrary 3-manifolds.

Corollary 12.2 *Let M be a tunnel-1 satellite knot exterior in an arbitrary 3-manifold. Then, in the torus decomposition of M , there is a unique component with a single toral boundary component.*

Proof. The proof of this corollary lies in recognizing that in each of the cases listed in the theorem, there is a unique component in the torus decomposition which possesses a single toral boundary component. $\square$

It is important to note that the secondary statements in conclusions 1 and 2 are the same decomposition. These decompositions yield information regarding the classifications of tunnel-1 satellite knots with either cyclic or reducible fillings. This, then, extends the work of Morimoto and Sakuma [26] and Eudave-Muñoz [11] which classified the tunnel-1 satellite knots in S^3 .

Corollary 12.3 *Let M be a tunnel-1 satellite knot exterior in a lens space. Then M possesses a torus decomposition into either*

1. *two components M_1 and M_2 where $M_1 \in D(2)$ and $M_2 \in M_L$ such that the regular fibre of M_1 is glued to a meridian of M_2 ,*
2. *two components M_1 and M_2 where $M_1 \in A(1)$ and M_2 is a 1-bridge knot exterior in the lens space S^3 , or*
3. *three components M_1 , M_2 , and M_3 where $M_1 \in D(2)$ is a torus knot exterior, $M_2 \in M_L$, and $M_3 \in A(1)$ such that the regular fibre of M_1 is glued to one of the meridians of M_2 and the regular fibre of M_3 is glued to the other.*

Proof. Let M be the exterior of a tunnel-1 generalized satellite knot. Since M is tunnel-1 and non-simple, it must be one of the 3-manifolds described in Theorem 12.1. Since M is meant to be a knot exterior in a lens space, it must have a Dehn filling to a lens space. We will consider each of the cases in Theorem 12.1 separately.

Case 1. M decomposes according to conclusion 1 of Theorem 12.1. We first consider the case where M_2 is simple. Since M_1 is attached to M_2 along the component

of M_2 we knew had a solid torus filling, if there were a curve on the other boundary component that filled M_2 to a solid torus, then Lemma 9.9 implies that M_2 is an element of M_L . In this case, we fill along this curve producing a solid torus whose meridian is glued to the regular fibre of M_1 thus producing a lens space. Note that the tunnel-1 satellite knots in S^3 fall into this category if we restrict M_1 to being a torus knot exterior and M_2 to being a 2-bridge link exterior.

If M_2 is not simple, then M_2 further decomposes into M'_2 , an element of M_L , and M''_2 , an element of $A(1)$. If we fill the free component of M_2 along the regular fibre of M'_2 , we produce the connected sum of a lens space and a solid torus whose meridian is the other meridional component of M'_2 and is attached to the regular fibre of M_1 . This produces the connected sum of two lens spaces. The only way this could be a lens space is if one of the summands is S^3 . The first lens space summand created cannot, by construction, be S^3 . Thus, M_1 must have an S^3 meridian and is therefore a torus knot exterior. If we fill M''_2 along a curve whose distance from the meridian is greater than one, we will have produced an element of $D(2)$, and none of the resulting tori will compress. If we fill at a distance one to the regular fibre, we produce a solid torus whose meridian is induced by the type of the exceptional fibre of M''_2 and the slope of the simple closed curve on which we filled. Since this does compress one of the tori, it has the potential to create a 3-manifold with cyclic fundamental group.

Case 2. M decomposes according to conclusion 2 of Theorem 12.1. We only consider the case where M_2 is simple since the non-simple case was handled above. If we fill M_1 along the regular fibre, we produce a lens space summand and a solid torus whose meridian is attached to the meridian of M_2 producing the connected sum of two lens spaces. This will be a lens space only if the meridian of M_2 is an S^3 meridian. This means that M_2 is the exterior of a 1-bridge knot in the lens space S^3 where bridge number is taken from a toral decomposition of S^3 rather than a spherical decomposition. If we fill at a distance greater than one to the regular fibre of M_1 , we get an element of $D(2)$ and the resulting manifold will remain Haken. Filling M at a distance one to the regular fibre of M_1 produces a solid torus whose meridian is dependent on the type of the exceptional fibre of M_1 and the slope of the

simple closed curve on which we filled M . This compresses the essential torus and will produce a lens space if it is the S^3 meridian of M_2 .

Case 3. M decomposes according to conclusion 3 of Theorem 12.1. If we fill M_1 along the regular fibre, we will produce a lens space connect sum an element of $D(2)$. Since the boundary of the element of $D(2)$ will have an incompressible boundary, the resulting closed 3-manifold will be Haken. If we fill M_1 at a distance greater than one to the meridian, we will produce an element of $D(3)$ which also has an incompressible boundary and will have constructed a closed Haken 3-manifold. If we fill M_1 at a distance one to the regular fibre, we produce an element of $D(2)$. The resulting manifold will remain Haken.

Case 4. M decomposes according to conclusion 4 of Theorem 12.1. If we fill M_1 along the regular fibre, we will have produced the connected sum of a lens space with an element of $M\ddot{o}(n)$ ($n = 0, 1$) whose boundary is incompressible. This means that the resulting 3-manifold will remain Haken. If we fill M_1 at a distance greater than one to the regular fibre, we will produce an element of $M\ddot{o}(n)$ ($n = 1, 2$), and the resulting 3-manifold will remain Haken. If we fill M_1 at a distance one to the regular fibre, we will produce an element of $M\ddot{o}(n)$ ($n = 0, 1$) which has an incompressible boundary. Thus, the resulting manifold will remain Haken. $\square$

In 1994, M. Eudave-Muñoz [11] gave some characteristics of the non-simple 3-manifolds which arise by adding 2-handles to simple 3-manifolds. While a difference between the uses of the term *simple* between this work and that used by Eudave-Muñoz in [11] exists, we can recover his main result for tunnel-1 generalized satellite knots in the following lemma.

We consider two tunnels τ_1 and τ_2 to be *tunnel-equivalent* if there are a continuous series of homeomorphisms taking (M, τ_1) to (M, τ_2) where at each stage, if the boundary of τ' is two points, they are in ∂M , and if the boundary of τ' is one point, then τ' has a non-manifold point. The tunnels in Figure 12.1 are tunnel-equivalent and exhibit sliding the tunnel across itself to a tunnel with a single boundary point.

Figure 12.1: Two equivalent tunnels

Corollary 12.4 Let M be the exterior of a tunnel-1 generalized satellite knot with tunnel τ and toral decomposition T . Then τ is tunnel-equivalent to a tunnel τ' such that $T \cap \tau'$ is at most two points.

Proof. Let V_1 and V_2 be the genus two handlebody and compression body of type $(2, 1)$ and F the genus two splitting surface in the splitting of M . Let T' be the collection of decomposing tori with minimal intersections with F such that the components of $T' \cap V_i$ ($i = 1, 2$) are all incompressible annuli. From the proof of Theorem 12.1 it follows that $T' \cap V_2$ is at most two components. We begin by letting τ be the co-core of the 2-handle added to create V_2 . Removing an open regular neighborhood of τ from M results in a genus two handlebody. If $T' \cap V_2$ is a single annulus, then it must be listed in 8.3. In any one of these cases, the tunnels shown in Figure 12.2 meet these annuli, and hence T' , at most twice. If $T' \cap V_2$ is two annuli, then they must conform to one of the pairs enumerated in Lemma 8.5. In any one of these cases, the tunnels shown in Figure 12.3 meet these annuli, and hence T' , twice. Since $T' \cap V_2$ cannot consist of more than two components, this proves the corollary. $\square$

Figure 12.2: The tunnel in minimal position with respect to a single annulus

(a)

(b)

(c)

Figure 12.3: The tunnel in minimal position with respect to two annuli

Chapter 13

Questions and remarks

There is clearly work remaining to be done. The main theorem stands on its own and the corollaries to it provide generalizations of important work by others. Notably, the corollaries provide a generalization of Schubert's ([30] and [31]) nesting of companion tori in satellite knots in S^3 to nesting tori in tunnel-1 generalized satellite knots in Corollary 12.2. In addition, the work of Morimoto and Sakuma [26] and Eudave-Muñoz [11] which classified the tunnel-1 satellite with S^3 meridians was generalized to the tunnel-1 generalized satellite knots with cyclic fillings in Corollary 12.3. Finally, the work of Eudave-Muñoz [11] is recovered for the special case of tunnel-1 generalized satellite knots in Corollary 12.4.

The following problems all arose during the course of this investigation and remain open. After each is a discussion of the issues known and the potential of each.

Problem 1 Find a complete list of the tunnel-1 generalized satellite knots with reducible Dehn fillings and classify the slopes of the filling curves.

It is left to show that if an element of V_K is not Seifert fibered, there are no reducible fillings. This result would also demonstrate the generalized cabling conjecture for tunnel-1 generalized satellite knots.

Problem 2 Find a complete list of the tunnel-1 generalized satellite knots with finite fillings and classify the slopes of the filling curves.

The fillings yielding lens spaces are enumerated in Corollary 12.3. What remains here is to enumerate those tunnel-1 generalized satellite knots which have fillings to

the other known Seifert fibered 3-manifolds with finite fundamental groups: the prism manifolds and the special Seifert fibre spaces over S^2 with three exceptional fibres.

Problem 3 Find a complete list of the tunnel-1 generalized satellite knots with Seifert fibered fillings and classify the slopes of the filling curves.

Problem 4 Enumerate the slopes yielding lens spaces for tunnel-1 generalized satellite knots and, based on a parameterization of those slopes, indicate the lens spaces which arise.

This problem is a refinement of Corollary 12.3. Rather than just list the structure of those tunnel-1 generalized satellite knots which have cyclic fillings, enumerate the slopes and determine which lens spaces arise.

Problem 5 Eliminate the duplications in the list of tunnel-1 generalized satellite knots in Theorem 12.1.

In the constructions indicated in the proof of Theorem 12.1, most of the components have well-defined constructions, but a certain arbitrariness regarding the parameters of the resulting space. In particular, the trefoil knot exterior is simultaneously an element of $D(2)$, M_K , and L_K . In fact, the set M_K is a subset of the set L_K .

Problem 6 Develop a proof that link complements with two meridional generators are two bridge that does not depend on the orbifold theorem.

This generalizes Lemma 9.9 to 2-generator links.

Problem 7 Show that, for 2-generator link exteriors, the rank of the fundamental group is identical to the minimal splitting genus. Note that this is false for closed 3-manifolds [7].

This is the question of how faithfully the algebra associated with a 3-manifold is reflected in the geometry. In the case of tunnel-1 generalized satellite knots, this amounts to showing that the list of 2-generator generalized satellite knots is the same

as the list given in Theorem 12.1. The recent work of Bleiler and Jones ([5] and [4]) gives work towards this result.

Problem 8 Classify the genus two splittings of tunnel-1 generalized satellite knots up to isotopy.

There are numerous examples in the literature of closed 3-manifolds which have several non-isomorphic Heegaard splittings (for example, see [6]).

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