Spring 6-16-2022

Analytic Solution of 1D Diffusion-Convection Equation with Varying Boundary Conditions

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by

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An undergraduate honors thesis submitted in partial fulfillment of the

requirements for the degree of

Bachelor of Science

in

University Honors

and

Mathematics

Thesis Adviser

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Portland State University

2022

Abstract

A diffusion-convection equation is a partial differential equation featuring two important physical processes. In this paper, we establish the theory of solving a 1D diffusion-convection equation, subject to homogeneous Dirichlet, Robin, or Neumann boundary conditions and a general initial condition. Firstly, we transform the diffusion-convection equation into a pure diffusion equation. Secondly, using a separation of variables technique, we obtain a general solution formula for each boundary type case, subject to transformed boundary and initial conditions. While eigenvalues in the cases of Dirichlet and Neumann boundary conditions can be constructed easily, the Robin boundary condition necessitates solving a transcendental algebraic equation to determine all the eigenvalues. Thirdly, we use Python to construct and illustrate the solutions for all the cases based on the newly developed solution formulas. Finally, we share all the associated Python code for public access.

1 Introduction

1.1 Importance of Partial Differential Equations

Partial differential equations, PDEs, and their solutions are important mathematical tools that allow scientists to describe the physical world. A PDE provides some information about the behavior of an unknown function by containing its rates of change with respect to more than one variable. Those rates of change are called partial derivatives. Often those variables are location x and time t. A PDE can be thought of as a model, a thorough understanding of what causes the unknown function's change over time. Once it is stated, the PDE needs to be solved, which means finding an explicit form of the previously unknown function of location and time whose rates of change are precisely described by the given equation. [1]

Most of physics and chemistry involves partial differential equations. A powerful example of PDEs is a set of Maxwell's equations that describe the interaction of electric and magnetic fields with each other as well as with electric charges. The knowledge contained in these equations allows us to harness energy in the form of electricity and make it available for day-to-day consumption. Another powerful example is the Schrodinger's equation, whose solutions allow scientists to explain the behavior and properties of the atomic and subatomic structures.

Hence, for those who study physical phenomena, it is imperative to understand the basic techniques of solving PDEs. An example of such an equation is a system involving both diffusion and convection processes which we explain in Section 1.2.

1.2 Diffusion-Convection Equation

In this thesis, we set up the theory of solving a one-dimensional diffusion-convection equation, which is a particular type of PDE. Let us call the unknown function $u = u(x, t)$ which depends on the spatial variable x and time variable t.

1.2.1 Diffusion Process

One of the natural processes occurring in solutions is diffusion, a spontaneous dispersion of particles due to collisions of those particles with each other, and possibly additional molecular-level constituents, present in the solution. Our function u could be a concentration of a compound in question or a temperature of the solution. In either case, it is a quantity that scientists have figured out how to measure precisely. Mathematically, diffusion is represented by:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $\dfrac{\partial u}{\partial t}$ indicates the rate of change of u with respect to time only, c^2 is a constant called a diffusion coefficient, which informs us how likely diffusion is to occur in the particular system. How quickly u will change with time also depends on $\dfrac{\partial^2 u}{\partial x^2}$, which is the second derivative of u with respect to the position and indicates curvature of the function. The derivation of this dependency is based on the conservation of either matter or energy, whether concentration or temperature is studied, and can be found in Farlow's *Partial Differential Equations for Scientists and Engineers.* [1]

A differential equation such as equation (1) describes how the concentration of particles changes with respect of time and location, while the solution to it offers an explicit form for the function $u(x,t)$, which informs us what the concentration of the compound in the solution is at any time and any given location.

1.2.2 Convection Process

Another common process occurring in nature is convection, which arises whenever there is motion induced in a fluid. Convection causes the rate of change of u with respect to time, $\dfrac{\partial u}{\partial t}$, to be proportional to the rate of change of u with respect to position, $\dfrac{\partial u}{\partial x}$,:

$$\frac{\partial u}{\partial t} = -\beta \frac{\partial u}{\partial x}, \tag{2}$$

where β is velocity with which the fluid flows. The steeper the curve is, the more susceptible to convection it will be. A derivation of this dependence is well explained in the case of chemical concentration in solutions in Atkins and De Paula's *Physical Chemistry.* [2]

1.2.3 Simultaneous Diffusion and Convection Processes

When both diffusion and convection processes take place, we simply present them as a sum of the terms from equations (1) and (2):

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial x}. \tag{3}$$

Now, to which extent each process will influence the behavior of our function u will depend on the relative magnitude of the constants c^2 and β in equation (3). This effect will be illustrated in Section 3, where we use the solutions obtained in Section 2 to show the evolution of the function u with time and their dependence on the constants.

The diffusion-convection equation (3), describes how the the change of function u with respect to position influences its change with respect to time. Since a PDE involves partial derivatives, integration techniques are used to solve it. Therefore, the highest degree of a derivative with respect to a variable will determine how much information will be necessary to find a specific solution. In case of a diffusion-convection equation, a second derivative with respect to position, $\frac{\partial^2 u}{\partial x^2}$, is the highest derivative, hence two boundary conditions must be given. Likewise, the only derivative of u with respect to time, $\frac{\partial u}{\partial t}$, necessitates one initial condition.

1.3 Boundary Conditions

Boundary conditions contain information about the function in question, u, on the physical boundary of the system and they describe the interaction between the studied environment and its surroundings. Mathematically, three types of boundary conditions are Dirichlet, Neumann, and Robin conditions. A Dirichlet boundary condition represents a prescribed value of $u(t)$ on the boundary. A Neumann boundary condition is an example of an insulated system in which the value of $u(t)$ at the boundary is allowed to change depending on what happens within the system, but the system and the surrounding environment do not influence one another. A Robin boundary condition arises from Newton's law of cooling in case of temperature, where heat flux on the boundary depends on the difference of temperature between the two interacting parts. In our case, it is the given system and its surroundings. On the other hand, based on Fourier's law, heat always flows from a hotter to a cooler region. Putting the two laws together, for a one-dimensional system of length L where $0 \leq x \leq L$ we get the following expressions which describe the Robin boundary conditions:

$$\begin{cases} -k_0 \dfrac{\partial u}{\partial x}(0,t) = -h\left[u(0,t) - u_{0,\text{surr}}\right] \\ -k_0 \dfrac{\partial u}{\partial x}(L,t) = h\left[u(L,t) - u_{L,\text{surr}}\right], \end{cases} \tag{4}$$

where $k_0 > 0$ is related to the thermal conductivity and is material-specific, while $h > 0$ is the heat transfer coefficient, mostly independent of the temperature difference and materials, but rather it depends on the interface of the system and its surroundings. The temperature on the boundary of the system is given by $u(0,t)$ and $u(L,t)$, while the temperature of the immediate surroundings is given by $u_{0,\text{surr}}$ and $u_{L,\text{surr}}$. [3]

The Dirichlet and Neumann boundary conditions can be viewed as limiting cases of the Robin boundary conditions presented in equation (4). If the relative magnitude of the thermal conductivity is much smaller than the heat transfer coefficient, $k_0 << h$, then we get:

$$\lim_{h\to\infty} \frac{k_0}{h} = 0$$

and the conditions presented in equation (4) can be rewritten as follows:

$$\begin{cases} u(0,t) = u_{0,\text{surr}} \\ u(L,t) = u_{L,\text{surr}}. \end{cases} \tag{5}$$

The above set of equations represent Dirichlet boundary conditions - a prescribed, known value of u at each boundary. If we are given that $u_{0,\text{surr}} = u_{L,\text{surr}} = 0$, then the Dirichlet boundary conditions (5) become homogeneous.

Now, if $h << k_0$, then we get that:

$$\lim_{h \to 0} \frac{h}{k_0} = 0,$$

which forces the Robin boundary conditions presented in equation (4) to be rewritten as:

$$\begin{cases} \dfrac{\partial u}{\partial x}(0,t) = 0 \\ \dfrac{\partial u}{\partial x}(L,t) = 0, \end{cases} \tag{6}$$

which describes homogeneous Neumann boundary conditions, where there is no transfer of u across the boundary.

In this work we will consider each case of homogeneous boundary conditions, where the surrounding's value of u is set to 0: $u_{0,\text{surr}} = u_{L,\text{surr}} = 0$.

1.4 Initial Condition

Finally, an initial condition $u(x,0)$, a set of known values of the function u anywhere within the system at an initial time, is needed to precisely understand how the function u evolves over time. While establishing the theory of solving the diffusion-convection equation with homogeneous boundary conditions we will consider a general form of the initial condition:

$$u(x,0) = f(x).$$

1.5 Motivation

Although the theory of diffusion-convection equation and its analytic solution are well established, few literature sources provide complete details on how to solve the diffusion-convection equation with different kinds of boundary conditions. Farlow [1] and Haberman [3] present several methods to solve some general-type partial differential equations. Arrigo [4] provides specific solutions to the diffusion-convection equation with Dirichlet boundary condition only. Tanveer [5] explains how to handle Robin boundary conditions for a wave equation without considering convection. In this university honors thesis, we will provide a detailed procedure for solving a diffusion-convection equation with all three types of boundary conditions and a general initial condition based on the separation of variables technique. We will then implement Python algorithms to visualize those solutions case by case.

2 Solving a Diffusion-Convection Equation

In this section, we establish the theory of solving the diffusion-convection equation, subject to three different types of homogeneous boundary conditions described in Section 1.3.

2.1 Problem Statement

Consider a one-dimensional system of length L. The following second-degree partial differential equation describes the system in which only diffusion and convection phenomena take place:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} - \beta \frac{\partial u}{\partial x}, \tag{7}$$

where c^2 and β are constants. Equation (7) is valid within the following boundaries:

$$0 < x < L \text{ and } 0 < t < \infty$$

For clarity, we rewrite equation (7) using subscript notation to indicate partial derivatives. Equation (1) then becomes:

$$u_t = c^2 u_{xx} - \beta u_x \tag{8}$$

The model is subject to a general initial condition:

$$u(x, 0) = f(x) \tag{9}$$

and the following three cases of homogeneous boundary conditions:

1. Dirichlet: $u\,(0, t) = u\,(L, t) = 0$,
2. Robin: $u_x\,(0, t) - a_0 u\,(0, t) = u_x\,(L, t) + a_L u\,(L, t) = 0$, where $a_0 > 0$ and $a_L > 0$,
3. Neumann: $u_x\,(0, t) = u_x\,(L, t) = 0$.

2.2 Transforming a Diffusion-Convection Equation into a Diffusion Equation

The first step in solving equation (8) is to transform it into a standard diffusion equation. This process starts with the assumption that the function in question, $u(x, t)$, can be rewritten in the form of:

$$u(x, t) = A(x, t)v(x, t). \tag{10}$$

The objective is to find a function $A(x, t)$ such that the convection term in equation (8), βu_x, disappears. Substituting equation (10) into (8) leads to:

$$A_t v + A v_t = c^2 \left(A_{xx} v + 2A_x v_x + A v_{xx}\right) - \beta \left(A_x v + A v_x\right). \tag{11}$$

After dividing by A we obtain:

$$\frac{A_t}{A} v + v_t = c^2 \left(\frac{A_{xx}}{A} + 2\frac{A_x}{A} v_x + v_{xx}\right) - \beta \left(\frac{A_x}{A} v + v_x\right). \tag{12}$$

Then collecting like-terms will result in:

$$v_t = c^2 v_{xx} + \left(2c^2 \frac{A_x}{A} - \beta\right) v_x + \left(-\beta \frac{A_x}{A} + c^2 \frac{A_{xx}}{A} - \frac{A_t}{A}\right) v. \tag{13}$$

For our transformed equation to have a form of a pure diffusion equation:

$$v_t = c^2 v_{xx}, \tag{14}$$

we ask that:

$$2c^2 \frac{A_x}{A} - \beta = 0 \tag{15}$$

and

$$-\beta \frac{A_x}{A} + c^2 \frac{A_{xx}}{A} - \frac{A_t}{A} = 0. \tag{16}$$

The condition given in equation (15) can be rewritten as:

$$\frac{A_x}{A} = \frac{\beta}{2c^2}. \tag{17}$$

This is an ordinary differential equation whose solution is obtained by first observing that in our case:

$$A_x = \frac{\partial}{\partial x} A = \frac{\mathrm{d}}{\mathrm{d}x} A, \tag{18}$$

which allows us to integrate equation (17) with respect to the variable x:

$$\int \frac{dA}{A} = \int \frac{\beta}{2c^2} \mathrm{d}x. \tag{19}$$

As a result, we obtain the following:

$$\ln(A) = \frac{\beta}{2c^2} x + C(t), \tag{20}$$

or, when we extract our function A, the general solution to equation (17) is:

$$A = C(t) e^{\frac{\beta}{2c^2} x}. \tag{21}$$

Here, $C(t)$ indicates an integration constant with respect to the variable x. Since our function $A(x,t)$ may contain a dependence on variable t, the obtained integration constant may be a function of t, as indicated.
To find an explicit form for the function A, which means finding $C(t)$, we use equation (16), which A must also satisfy. After multiplying by A and rearranging, equation (16) can be rewritten as

$$A_t = c^2 A_{xx} - \beta A_x. \tag{22}$$

This is a partial differential equation of A, so next we must compute the necessary partial derivatives based on equation (21), so with respect to time we get:

$$A_t = C_t e^{\frac{\beta}{2c^2} x} = \frac{C_t}{C} A \tag{23}$$

and with respect to x:

$$A_x = C \frac{\beta}{2c^2} e^{\frac{\beta}{2c^2} x} = \frac{\beta}{2c^2} A, \tag{24}$$

$$A_{xx} = C \frac{\beta^2}{4c^4} e^{\frac{\beta}{2c^2} x} = \frac{\beta^2}{4c^4} A. \tag{25}$$

Now substituting equations (23)-(25) into (22) results in:

$$\frac{C_t}{C} A = c^2 \frac{\beta^2}{4c^4} A - \beta \left(\frac{\beta}{2c^2} A \right). \tag{26}$$

Simplifying and rearranging the above equation with an assumption that A is not trivial gives:

$$\begin{aligned} &\left[\frac{C_t}{C} - \frac{\beta^2}{4c^2} + \frac{\beta^2}{2c^2}\right] A = 0 \\ \Longrightarrow &\frac{C_t}{C} + -\frac{\beta^2}{4c^2} + \frac{\beta^2}{2c^2} = 0 \\ \Longrightarrow &\frac{C_t}{C} = -\frac{\beta^2}{4c^2}. \end{aligned} \tag{27}$$

Equation (27) is again an ordinary differential equation, which we can solve the same way we treated equation (17). The solution is the explicit form of the constant with respect to x, $C(t)$, introduced in equation (20). It was found to be:

$$C\left(t\right) = A_0 e^{-\frac{\beta^2}{4c^2}t}, \tag{28}$$

where A_0 is now a constant not depending on either x, nor on t. Now we can rewrite equation (17), which reveals the function $A\left(x, t\right)$ satisfying conditions (15) and (16):

$$A = A_0 e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x}. \tag{29}$$

Now, returning to the original goal of this section, we substitute (29) into equation (10) to find a form of $u\left(x, t\right)$ such that the diffusion-convection equation is transformed into a diffusion equation form. This happens for:

$$u(x, t) = A_0 e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} v(x, t). \tag{30}$$

Since our transformation of the diffusion-convection equation is possible for any constant A_0, at this point we simplify the form of $u\left(x, t\right)$ by setting A_0 to 1. Hence, our $u\left(x, t\right)$ takes a general form of:

$$u(x, t) = e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} v(x, t). \tag{31}$$

Let us now verify that (31) will allow us to work with a simpler PDE. To do this, we plug in the expression in equation (31) into the diffusion-convection equation (8). We obtain the following:

$$-\frac{\beta^2}{4c^2}Av + Av_t = c^2\left(\frac{\beta^2}{4c^4}Av + 2\frac{\beta}{2c^2}Av_x + Av_{xx}\right) - \beta\left(\frac{\beta}{2c^2}Av + Av_x\right), \tag{32}$$

where $A = e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x}$.

After distributing and simplifying we obtain:

$$v_t = c^2 v_{xx}. \tag{33}$$

From now on we will work with the diffusion equation (33), but this change needs to be reflected by adjusting our initial condition presented in (9). This condition tells us that:

$$\begin{aligned} u\left(x, 0\right) &= A\left(x, 0\right) v\left(x, 0\right) \\ &= e^{\frac{\beta}{2c^2}x} v\left(x, 0\right) \\ &= f\left(x\right). \end{aligned} \tag{34}$$

Hence, we get the initial condition for the transformed diffusion equation:

$$v\left(x, 0\right) = e^{-\frac{\beta}{2c^2}x} f\left(x\right). \tag{35}$$

The next step in the process of seeking solutions to the given equation (33) is to use a separation of variables method. Suppose $v(x,t)$ can be represented as a product of functions, one depending solely on variable x, the other on t:

$$v(x,t) = X(x)\,T(t)\,. \tag{36}$$

Equation (33) can be rewritten as follows:

$$XT' = c^2 X'' T. \tag{37}$$

After rearranging, we obtain that:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T}. \tag{38}$$

We observe now that this equality must hold at any time, t and at any location x, so it must be that

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda. \tag{39}$$

for some constant λ. The assumption noted in equation (36) allows us to rewrite the partial differential equation (33) as a system of two ordinary differential equations:

$$\begin{cases} X'' = \lambda X \\ T' = c^2 \lambda T \end{cases} \tag{40}$$

At this point we need to consider boundary conditions.

2.3 Specific Solution with Dirichlet Boundary Conditions

Our physical system described by equation (8) will first be a subject to the following homogeneous type I boundary conditions:

$$u(0,t) = u(L,t) = 0. \tag{41}$$

Since we have transformed equation (8) into a diffusion equation presented in (33), we should find the corresponding boundary conditions that reflect this change.

At $x = 0$ we have:

$$\begin{aligned} u(0,t) &= A(0,t)\,v(0,t) \\ &= e^{-\frac{\beta^2}{4c^2}t} v(0,t) \\ &= 0 \end{aligned} \tag{42}$$

and at $x = L$:

$$\begin{aligned} u(L,t) &= A(L,t)\,v(L,t) \\ &= e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta L}{2c^2}} v(L,t) \\ &= 0. \end{aligned} \tag{43}$$

We see that for relations in equations (42) and (43) to hold it must be that

$$v(0,t) = v(L,t) = 0 \tag{44}$$

Hence, in our transformed differential equation, (33), we continue working with the Dirichlet boundary conditions.

First we should address the ODE with respect to spacial variable from the system of equations given by (40):

$$X'' = \lambda X \tag{45}$$

Its boundary conditions are:

$$v(0,t) = 0 \implies X(0)T(t) = 0 \implies X(0) = 0 \tag{46}$$

and

$$v(L,t) = 0 \implies X(L)T(t) = 0 \implies X(L) = 0 \tag{47}$$

Next we should show that only negative eigenvalues λ will satisfy the given boundary conditions.

Case 1: $\lambda = 0$
Our equation (45) in this case becomes

$$X'' = 0 \tag{48}$$

which is solved by integrating it twice to obtain a general solution:

$$X(x) = ax + b, \tag{49}$$

where a and b are constants. This solution must satisfy the boundary conditions (46) and (47), hence:

$$X(0) = 0 \implies a \cdot 0 + b = 0 \implies b = 0 \tag{50}$$

and

$$X(L) = 0 \implies a \cdot L = 0 \implies a = 0. \tag{51}$$

Hence, the only solution to equation (45) in case when $\lambda = 0$ is a trivial one.

Case 2: $\lambda = \mathbf{k^2} > \mathbf{0}$
We consider a solution of type:

$$X(x) = e^{rx}, \tag{52}$$

which plugged in to equation (45) will give the following characteristic equation:

$$r^2 - \lambda = 0, \tag{53}$$

from which we obtain valid values for r:

$$r_{\pm} = \pm k. \tag{54}$$

Using this information we can rewrite a general solution as:

$$X(x) = c_1 e^{kx} + c_2 e^{-kx} \tag{55}$$

where c_1 and c_2 are constants which can be found by using known boundary conditions.
Condition (46) will result in:

$$X(0) = 0 \implies c_1 e^0 + c_2 e^0 = 0 \implies c_1 = -c_2 \tag{56}$$

Now, based on condition (47):

$$X(L) = 0 \implies c_1 \left(e^{kL} - e^{-kL}\right) = 0. \tag{57}$$

Since $k = \sqrt{\lambda} > 0$ and $L > 0$ we get that:

$$e^{kL} > e^{-kL}, \tag{58}$$

which in turn means that equation (57) will be satisfied if and only if $c_1 = 0$. This result, by the implication of equation (56), leads to a trivial solution. Therefore, it must be that $\lambda < 0$, which we demonstrate next.

Case 3: $\lambda = -\mathbf{k^2} < \mathbf{0}$
As in the case of positive λ, we expect a general solution presented in equation (52), which leads to the same characteristic equation but in this case the roots are:

$$r_{\pm} = \pm ik. \tag{59}$$

Therefore, the general solution in this case will be:

$$\begin{aligned} X(x) &= c_3 e^{ikx} + c_4 e^{-ikx} \\ &= c_3 \left[\cos(kx) + i\sin(kx)\right] + c_4 \left[\cos(kx) - i\sin(kx)\right] \\ &= (c_3 + c_4)\cos(kx) + i(c_3 - c_4)\sin(kx) \\ &= c_5 \cos(kx) + c_6 \sin(kx), \end{aligned} \tag{60}$$

where c_j for $j = 3, 4, 5, 6$ are constants. As before, to find c_5 and c_6 we should use the known boundary conditions. Condition (46) has to hold, therefore:

$$X(0) = 0 \implies c_5 \cos(0) + c_6 \sin(0) = 0 \implies c_5 = 0 \tag{61}$$

The boundary condition at L will force:

$$X(L) = 0 \implies c_6 \sin(kL) = 0 \tag{62}$$

For the sought-after nontrivial solution we ask that $c_6 \neq 0$, hence it must be that:

$$\sin(kL) = 0, \tag{63}$$

which is true for the following values of k:

$$k = \frac{n\pi}{L} \quad \text{where} \quad n = 1, 2, \ldots \tag{64}$$

Equation (64) leads to the valid eigenvalues:

$$\lambda = -k^2 = -\left(\frac{n\pi}{L}\right)^2 \quad \text{where} \quad n = 1, 2, \ldots \tag{65}$$

and the solution to equation (45) is

$$X_n(x) = \sin\left(\frac{n\pi}{L}x\right) \quad \text{where} \quad n = 1, 2, \ldots \tag{66}$$

Notice, since c_6 was an arbitrary constant, it was set to be equal to 1. The final constants that satisfy the original problem will be determined in subsequent steps.

The next step involves solving the second ODE from the system of equations given in (40). This equation can be solved via a similar method to the one used previously to obtain solutions to equation (15):

$$T' = c^2 \lambda T \implies \ln(T(t)) = c^2 \lambda t + c_7 \implies T(t) = c_7 e^{c^2 \lambda t}, \tag{67}$$

where c_7 is a constant, which for now will be set to 1.
Putting together the solutions obtained for each ODE presented in (66) and (67), we obtain:

$$X_n(x) T_n(t) = \sin\left(\frac{n\pi}{L}x\right) e^{-c^2 \left(\frac{n\pi}{L}\right)^2 t} \quad \text{where} \quad n = 1, 2, \ldots \tag{68}$$

Based on (68), the general solution to equation (33) will be the following linear combination:

$$v\left(x,t\right)=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right)e^{-c^2\left(\frac{n\pi}{L}\right)^2t}, \tag{69}$$

where constants b_n can be extracted from the initial condition in equation (35):

$$\begin{aligned} v\left(x,0\right)&=e^{-\frac{\beta}{2c^2}x}f\left(x\right)\\ &=\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right). \end{aligned} \tag{70}$$

Now we will multiply both sides of equation (70) by $\sin\left(\frac{m\pi}{L}x\right)$ and integrate:

$$\begin{aligned} \int_0^L e^{-\frac{\beta}{2c^2}x}f\left(x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x&=\int_0^L\left(\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right)\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x\\ &=\sum_{n=1}^{\infty}b_n\int_0^L\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}x\right)\mathrm{d}x \end{aligned} \tag{71}$$

By the orthogonality of $\sin\left(n\pi x\right)$, we get the following result:

$$b_n=\frac{2}{L}\int_0^L e^{-\frac{\beta}{2c^2}x}f\left(x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x. \tag{72}$$

Now we are ready to put together all results and obtain the solution to the original partial differential equation given by (8):

$$u_t=c^2u_{xx}-\beta u_x$$

with homogeneous Dirichlet boundary conditions, (41):

$$u(0,t)=u(L,t)=0$$

and general initial conditions given by (9):

$$u(x,0)=f(x).$$

The solution is as follows:

$$\begin{aligned} u\left(x,t\right)&=A(x,t)v(x,t)\\ &=e^{\frac{-\beta^2}{4c^2}t}e^{\frac{\beta}{2c^2}x}\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi}{L}x\right)e^{-c^2\left(\frac{n\pi}{L}\right)^2t}, \end{aligned} \tag{73}$$

where b_n are previously found constants:

$$b_n=\frac{2}{L}\int_0^L e^{-\frac{\beta}{2c^2}x}f\left(x\right)\sin\left(\frac{n\pi}{L}x\right)\mathrm{d}x.$$

2.4 Specific Solution with Robin Boundary Conditions

In this section, we examine the solution to the diffusion-convection equation (8):

$$u_t=c^2u_{xx}-\beta u_x$$

with the general initial condition (9):

$$u(x,0) = f(x),$$

and homogeneous Robin boundary conditions:

$$\begin{aligned} u_x\left(0,t\right) - a_0 u\left(0,t\right) &= 0 \\ u_x\left(L,t\right) + a_L u\left(L,t\right) &= 0, \end{aligned} \tag{74}$$

where $a_0, a_L > 0$ are constants. The constraint on these constants comes from the derivation of Robin boundary conditions based on Newton's law of cooling, and is necessary to ensure unique solutions to the posed problem.

As in the previous section, we will work with the transformed heat equation (14). To do so, we should transform our new boundary conditions to reflect the changes to the governing equation. So, we plug equation (31) into (74), but before we do so, we differentiate (31) with respect to x:

$$u_x(x,t) = \frac{\beta}{2c^2} e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} v\left(x,t\right) + e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} v_x\left(x,t\right). \tag{75}$$

Then, at location $x = 0$, we obtain:

$$\begin{aligned} &\frac{\beta}{2c^2} e^{-\frac{\beta^2}{4c^2}t} v\left(0,t\right) + e^{-\frac{\beta^2}{4c^2}t} v_x\left(0,t\right) - a_0 e^{-\frac{\beta^2}{4c^2}t} v(0,t) = 0 \\ \implies &\left[v_x\left(0,t\right) - \left(a_0 - \frac{\beta}{2c^2}\right) v\left(0,t\right)\right] e^{-\frac{\beta^2}{4c^2}t} = 0. \end{aligned} \tag{76}$$

Equation (76) is satisfied if and only if:

$$v_x\left(0,t\right) - \left(a_0 - \frac{\beta}{2c^2}\right) v\left(0,t\right) = 0. \tag{77}$$

And for the other boundary condition, at $x = L$ we get:

$$\begin{aligned} &\frac{\beta}{2c^2} e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}L} v\left(L,t\right) + e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}L} v_x\left(L,t\right) + a_L e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}L} v(L,t) = 0 \\ \implies &\left[v_x\left(L,t\right) + \left(a_L + \frac{\beta}{2c^2}\right) v(L,t)\right] e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}L} = 0, \end{aligned} \tag{78}$$

which is satisfied when

$$v_x\left(L,t\right) + \left(a_L + \frac{\beta}{2c^2}\right) v\left(L,t\right) = 0. \tag{79}$$

Equations (77) and (79) are the new Robin boundary conditions in our transformed system. Now, our condition for a well-posed problem requires that both of the coefficients of $v(0,t)$ in (77) and $v(L,t)$ in (79) are non negative. Hence, when $\beta \geq 0$, the following condition is always satisfied:

$$a_L + \frac{\beta}{2c^2} \geq 0, \tag{80}$$

since in our problem $a_L > 0$ and $c^2 > 0$. On the other hand, at $x = 0$, we must have that:

$$a_0 - \frac{\beta}{2c^2} \geq 0 \tag{81}$$

So, for our problem to remain well posed we must satisfy:

$$a_0 \geq \frac{\beta}{2c^2} \implies \beta \leq 2c^2 a_0 \tag{82}$$

Relation (82) implies that there must be an upper bound on the value of β, so that it does not dominate the behavior of the system. This condition ensures the solution to the problem is always well-defined.

Likewise, a similar approach is necessary in case when $\beta < 0$. Now condition (81) is always satisfied, while condition (80) requires that:

$$a_L + \frac{\beta}{2c^2} \geq 0 \implies -\beta \leq 2c^2 a_L. \tag{83}$$

In the process of solving our PDE we consider the case when $\beta \geq 0$, since the solution can be easily adapted to accommodate the opposite case.

As before, we now continue with the separation of variables technique, in which our solution, $v\left(x,t\right)$ can be written as a product of two functions, each depending on only one variable, as shown in equation (36) repeated below for readers' convenience:

$$v\left(x,t\right) = X\left(x\right)T\left(t\right).$$

Examining the boundary conditions in (77) allows us to notice that:

$$\begin{aligned} &v_x\left(0,t\right) - \left(a_0 - \frac{\beta}{2c^2}\right)v\left(0,t\right) = 0 \\ \implies &X_x\left(0\right)T\left(t\right) - \left(a_0 - \frac{\beta}{2c^2}\right)X\left(0\right)T\left(t\right) = 0 \\ \implies &X_x\left(0\right) - \left(a_0 - \frac{\beta}{2c^2}\right)X\left(0\right) = 0. \end{aligned} \tag{84}$$

Likewise, the boundary conditions in (79) translate to:

$$\begin{aligned} &v_x\left(L,t\right) + \left(a_L + \frac{\beta}{2c^2}\right)v\left(L,t\right) = 0 \\ \implies &X_x\left(L\right)T\left(t\right) + \left(a_L + \frac{\beta}{2c^2}\right)X\left(L\right)T\left(t\right) = 0 \\ \implies &X_x\left(L\right) + \left(a_L + \frac{\beta}{2c^2}\right)X\left(L\right) = 0. \end{aligned} \tag{85}$$

Given equation (45): $X'' = \lambda X$, we will now examine the eigenvalues that satisfy above boundary conditions.

Case 1: $\lambda = 0$
A previously considered, a general solution to equation (45) is of the form $X(x) = ax + b$ and must satisfy boundary conditions (84) and (85). Hence, at $x = 0$ we get:

$$\begin{aligned} &X_x\left(0\right) - \left(a_0 - \frac{\beta}{2c^2}\right)X\left(0\right) = 0 \\ \implies &a - \left(a_0 - \frac{\beta}{2c^2}\right)b = 0 \\ \implies &a = \left(a_0 - \frac{\beta}{2c^2}\right)b. \end{aligned} \tag{86}$$

Additionally, at $x = L$ we obtain:

$$\begin{aligned} X_x(L) + \left(a_L + \frac{\beta}{2c^2}\right) X(L) = 0 \\ \implies a + \left(a_L + \frac{\beta}{2c^2}\right)(aL + b) = 0. \end{aligned} \tag{87}$$

Then, substituting the result from (86) into (87) leads us to:

$$\begin{aligned} &\left(a_0 - \frac{\beta}{2c^2}\right) b + \left(a_L + \frac{\beta}{2c^2}\right)\left(\left(a_0 - \frac{\beta}{2c^2}\right) bL + b\right) = 0 \\ \implies &\left[\left(a_0 - \frac{\beta}{2c^2}\right) + \left(a_L + \frac{\beta}{2c^2}\right)\left(\left(a_0 - \frac{\beta}{2c^2}\right) L + 1\right)\right] b = 0 \\ \implies &\left[\left(a_0 - \frac{\beta}{2c^2}\right) + \left(a_L + \frac{\beta}{2c^2}\right)\left(a_0 - \frac{\beta}{2c^2}\right) L + \left(a_L + \frac{\beta}{2c^2}\right)\right] b = 0. \end{aligned} \tag{88}$$

Since all the terms inside the square bracket in equation (88) are greater or equal to zero and the last term $a_L + \frac{\beta}{2c^2} > 0$, then

$$\left(a_0 - \frac{\beta}{2c^2}\right) + \left(a_L + \frac{\beta}{2c^2}\right)\left(a_0 - \frac{\beta}{2c^2}\right) L + \left(a_L + \frac{\beta}{2c^2}\right) > 0. \tag{89}$$

Therefore $b = 0$ must hold. This yields a trivial solution based on equation (86). Hence $\lambda \neq 0$.

Case 2: $\lambda = \mathbf{k^2} > \mathbf{0}$
As done previously, we should verify that the general solution in this case, given by (55):

$$X(x) = c_1 e^{kx} + c_2 e^{-kx}$$

will satisfy boundary conditions (84) and (85). At $x = 0$ we obtain the following:

$$\begin{aligned} &X_x(0) - \left(a_0 - \frac{\beta}{2c^2}\right) X(0) = 0 \\ \implies &k(c_1 - c_2) - \left(a_0 - \frac{\beta}{2c^2}\right)(c_1 + c_2) = 0 \\ \implies &\left(k - \left(a_0 - \frac{\beta}{2c^2}\right)\right) c_1 - \left(k + \left(a_0 - \frac{\beta}{2c^2}\right)\right) c_2 = 0. \end{aligned} \tag{90}$$

Similarly, at $x = L$ we get:

$$\begin{aligned} &X_x(L) + \left(a_L + \frac{\beta}{2c^2}\right) X(L) = 0 \\ \implies &k\left(c_1 e^{kL} - c_2 e^{-kL}\right) + \left(a_L + \frac{\beta}{2c^2}\right)\left(c_1 e^{kL} + c_2 e^{-kL}\right) = 0 \\ \implies &\left(k + \left(a_L + \frac{\beta}{2c^2}\right)\right) e^{kL} c_1 - \left(k - \left(a_L + \frac{\beta}{2c^2}\right)\right) e^{-kL} c_2 = 0. \end{aligned} \tag{91}$$

Now we rewrite the results from (90) and (91) in a matrix form:

$$\begin{bmatrix} k - \left(a_0 - \frac{\beta}{2c^2}\right) & -k - \left(a_0 - \frac{\beta}{2c^2}\right) \\ \left(k + \left(a_L + \frac{\beta}{2c^2}\right)\right) e^{kL} & \left(-k + \left(a_L + \frac{\beta}{2c^2}\right)\right) e^{-kL} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0. \end{bmatrix} \tag{92}$$

The system of equations in (92) will have solutions if

$$\begin{vmatrix} k-\left(a_0-\frac{\beta}{2c^2}\right) & -k-\left(a_0-\frac{\beta}{2c^2}\right) \\ \left(k+\left(a_L+\frac{\beta}{2c^2}\right)\right)e^{kL} & \left(-k+\left(a_L+\frac{\beta}{2c^2}\right)\right)e^{-kL} \end{vmatrix} = 0, \tag{93}$$

which means that:

$$\left(k-\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k-\left(a_L+\frac{\beta}{2c^2}\right)\right)e^{-kL} = \left(k+\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k+\left(a_L+\frac{\beta}{2c^2}\right)\right)e^{kL}.$$

The above condition holds only if:

$$\left|\frac{\left(k-\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k-\left(a_L+\frac{\beta}{2c^2}\right)\right)}{\left(k+\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k+\left(a_L+\frac{\beta}{2c^2}\right)\right)}\right| = e^{2kL}. \tag{94}$$

Since $k=\sqrt{\lambda}>0$, we must have:

$$0 \leq \left|\frac{k-\left(a_0-\frac{\beta}{2c^2}\right)}{k+\left(a_0-\frac{\beta}{2c^2}\right)}\right| < 1, \text{ and } 0 \leq \left|\frac{k-\left(a_L+\frac{\beta}{2c^2}\right)}{k+\left(a_L+\frac{\beta}{2c^2}\right)}\right| < 1$$

So, the left-hand-side of equation (94) satisfies:

$$\left|\frac{\left(k-\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k-\left(a_L+\frac{\beta}{2c^2}\right)\right)}{\left(k+\left(a_0-\frac{\beta}{2c^2}\right)\right)\left(k+\left(a_L+\frac{\beta}{2c^2}\right)\right)}\right| < 1,$$

while the right-hand-side gives:

$$e^{2kL} > 1,$$

which is a contradiction. Hence, $\lambda < 0$ is the only case that holds for Robin boundary conditions.

Case 3: $\lambda = -\mathbf{k^2} < \mathbf{0}$
As done previously, we now consider a general solution (60) that satisfies the given ODE (45) and use boundary conditions (84) as well as (85) to find the solution that satisfies them.

Therefore, at $x=0$ we get:

$$\begin{aligned} &X_x(0) - \left(a_0-\frac{\beta}{2c^2}\right)X(0) = 0 \\ \Longrightarrow\ & k\left(-c_5\sin(0)+c_6\cos(0)\right) - \left(a_0-\frac{\beta}{2c^2}\right)\left(c_5\cos(0)+c_6\sin(0)\right) = 0 \\ \Longrightarrow\ & -\left(a_0-\frac{\beta}{2c^2}\right)c_5 + kc_6 = 0 \end{aligned} \tag{95}$$

On the other hand, at $x=L$, we get that:

$$\begin{aligned} &X_x(L) + \left(a_L+\frac{\beta}{2c^2}\right)X(L) = 0 \\ \Longrightarrow\ & k\left(-c_5\sin(kL)+c_6\cos(kL)\right) + \left(a_L+\frac{\beta}{2c^2}\right)\left(c_5\cos(kL)+c_6\sin(kL)\right) = 0 \\ \Longrightarrow\ & \left(\left(a_L+\frac{\beta}{2c^2}\right)\cos(kL) - k\sin(kL)\right)c_5 + \left(k\cos(kL) + \left(a_L+\frac{\beta}{2c^2}\right)\sin(kL)\right)c_6 = 0. \end{aligned} \tag{96}$$

Now we rewrite equations (95) and (96) in a matrix form to obtain:

$$\begin{bmatrix} -\left(a_0-\frac{\beta}{2c^2}\right) & k \\ \left(a_L+\frac{\beta}{2c^2}\right)\cos(kL)-k\sin(kL) & k\cos(kL)+\left(a_L+\frac{\beta}{2c^2}\right)\sin(kL) \end{bmatrix} \begin{bmatrix} c_5 \\ c_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{97}$$

Equation (97) has solutions if:

$$\begin{vmatrix} -\left(a_0-\frac{\beta}{2c^2}\right) & k \\ \left(a_L+\frac{\beta}{2c^2}\right)\cos(kL)-k\sin(kL) & k\cos(kL)+\left(a_L+\frac{\beta}{2c^2}\right)\sin(kL) \end{vmatrix} = 0. \tag{98}$$

Hence, the following must be true:

$$-\left(a_0-\frac{\beta}{2c^2}\right)\left(k\cos(kL)+\left(a_L+\frac{\beta}{2c^2}\right)\sin(kL)\right)-k\left(\left(a_L+\frac{\beta}{2c^2}\right)\cos(kL)-k\sin(kL)\right)=0. \tag{99}$$

Multiplying through and collecting like-terms leads to the following:

$$\left(k^2-\left(a_0-\frac{\beta}{2c^2}\right)\left(a_L+\frac{\beta}{2c^2}\right)\right)\sin(kL)-k\left(\left(a_0-\frac{\beta}{2c^2}\right)+\left(a_L+\frac{\beta}{2c^2}\right)\right)\cos(kL)=0. \tag{100}$$

Now, equation (100) can be further represented as follows:

$$\begin{aligned} \tan(kL) &= \frac{k\left(\left(a_0-\frac{\beta}{2c^2}\right)+\left(a_L+\frac{\beta}{2c^2}\right)\right)}{k^2-\left(a_0-\frac{\beta}{2c^2}\right)\left(a_L+\frac{\beta}{2c^2}\right)} \\ &= \frac{k\left(a_0+a_L\right)}{k^2-\left(a_0-\frac{\beta}{2c^2}\right)\left(a_L+\frac{\beta}{2c^2}\right)}. \end{aligned} \tag{101}$$

To find the values of k that satisfy the transcendental equation (101) we should rewrite it in the form:

$$\tan(\alpha) = \frac{L\left(a_0+a_L\right)\alpha}{\alpha^2-L^2\left(a_0-\frac{\beta}{2c^2}\right)\left(a_L+\frac{\beta}{2c^2}\right)}, \text{ where } \alpha = kL. \tag{102}$$

Now we graphically represent each side of (102), as shown in Figure 1. The intersection of the two graphs determines possible values of α_n, from which eigenvalues can be obtained based on the following relationship:

$$\lambda_n = -\frac{\alpha_n^2}{L^2}, \text{ where } n = 1, 2, \ldots \tag{103}$$

The plot illustrating (102) in Figure 1 was obtained for the following values of the constants in (102):

$$a_0 = 1.5,\ a_L = 1,\ \frac{\beta}{2c^2} = 1, \text{ and } L = 6.5$$

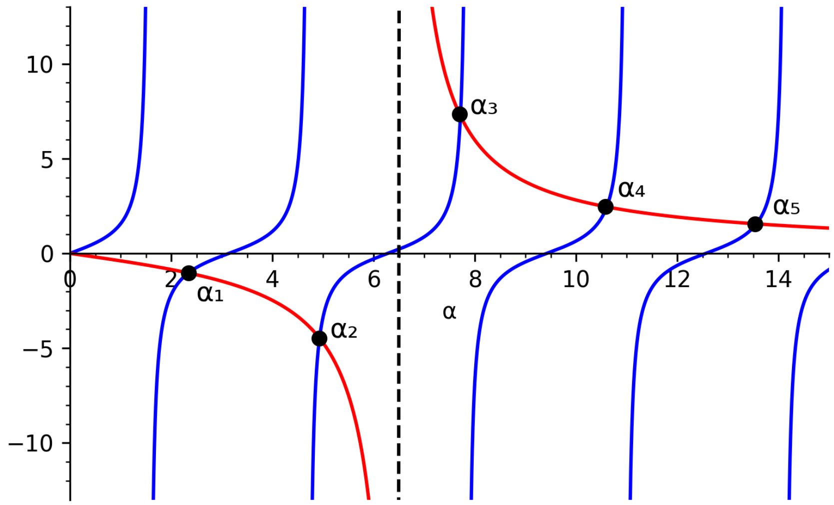

Figure 1: Plot of left- and right-hand side of (102) as a function of α.

Once the eigenvalues are obtained, we can find the corresponding eigen function $X(x)$ by first noticing, based on (95), that:

$$c_6 = \frac{a_0 - \frac{\beta}{2c^2}}{k} c_5, \tag{104}$$

which leads us to rewriting a general solution (60):

$$X(x) = \cos(kx) + \frac{a_0 - \frac{\beta}{2c^2}}{k} \sin(kx), \tag{105}$$

where the arbitrary constant c_5 was set to 1.

Continuing with the separation of variables technique, we find that, as in the case of Dirichlet boundary conditions, the solution to $T(t)$ for (40) is:

$$T(t) = e^{c^2 \lambda t}. \tag{106}$$

Therefore, the solutions to (36) will be:

$$X_n(x) T_n(t) = \left(\cos\left(\frac{\alpha_n}{L} x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\frac{\alpha_n}{L}} \sin\left(\frac{\alpha_n}{L} x\right) \right) e^{-\frac{\alpha_n^2}{L^2} c^2 t}, \text{ where } n = 1, 2, \ldots \tag{107}$$

Based on the above equation, the general solution to equation (33) will be the following linear combination:

$$\begin{aligned} v(x,t) &= \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= \sum_{n=1}^{\infty} A_n \left(\cos\left(\frac{\alpha_n}{L} x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L} x\right) \right) e^{-\frac{\alpha_n^2}{L^2} c^2 t}, \end{aligned} \tag{108}$$

where A_n are constants whose values can be determined from the known initial boundary conditions (35):

$$\begin{aligned} v(x,0) &= e^{-\frac{\beta}{2c^2}x} f(x) \\ &= \sum_{n=1}^{\infty} A_n \left(\cos\left(\frac{\alpha_n}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L}x\right) \right). \end{aligned} \tag{109}$$

Now we multiply both sides of the equation by $X_m(x)$ and integrate over the allowed values of x to obtain on the left-hand side:

$$\int_0^L \left(\cos\left(\frac{\alpha_m}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_m} L \sin\left(\frac{\alpha_m}{L}x\right) \right) e^{-\frac{\beta}{2c^2}x} f(x)\,\mathrm{d}x, \tag{110}$$

and on the right-hand side:

$$\int_0^L \sum_{n=1}^{\infty} A_n \left(\cos\left(\frac{\alpha_n}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L}x\right) \right) \left(\cos\left(\frac{\alpha_m}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_m} L \sin\left(\frac{\alpha_m}{L}x\right) \right) \mathrm{d}x. \tag{111}$$

First, notice that in (111) we are allowed to move the summation outside the integral. Also, since the solutions $X_n(x)$ in the Sturm-Liouville-type problems are orthogonal, asserted by Haberman in Chapter 5 of *Applied Partial Differential Equations with Fourier Series and Boundary Value Problems* we obtain the following result:

$$\begin{aligned} &\int_0^L \left(\cos\left(\frac{\alpha_m}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_m} L \sin\left(\frac{\alpha_m}{L}x\right) \right) e^{-\frac{\beta}{2c^2}x} f(x)\,\mathrm{d}x = \\ &= A_m \int_0^L \left(\cos\left(\frac{\alpha_m}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_m} L \sin\left(\frac{\alpha_m}{L}x\right) \right)^2 \mathrm{d}x. \end{aligned} \tag{112}$$

Solving for the constant A_m and renaming the index back to n leads to:

$$A_n = \frac{\int_0^L \left(\cos\left(\frac{\alpha_n}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L}x\right) \right) e^{-\frac{\beta}{2c^2}x} f(x)\,\mathrm{d}x}{\int_0^L \left(\cos\left(\frac{\alpha_n}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L}x\right) \right)^2 \mathrm{d}x}. \tag{113}$$

With constants A_n in mind given by (113), we return to the original problem and its solution form given by (31) to present the final solution to equation (8) with homogeneous Robin boundary conditions:

$$\begin{aligned} u(x,t) &= A(x,t)\, v(x,t) \\ &= e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} \sum_{n=1}^{\infty} A_n \left(\cos\left(\frac{\alpha_n}{L}x\right) + \frac{a_0 - \frac{\beta}{2c^2}}{\alpha_n} L \sin\left(\frac{\alpha_n}{L}x\right) \right) e^{-\frac{\alpha_n^2}{L^2}c^2 t}. \end{aligned} \tag{114}$$

2.5 Neumann Boundary Conditions

Let us now consider the problem presented in (8) with general initial conditions (9) and homogeneous type II boundary conditions:

$$\begin{aligned} u_x(0,t) &= 0 \\ u_x(L,t) &= 0. \end{aligned} \tag{115}$$

These boundary conditions may be viewed as a limiting case of more general, Robin, boundary conditions (74) considered in Section 2.3, where

$$a_0 = a_L = 0. \tag{116}$$

Again, we have to keep in mind well-posedness of the problem, hence, after transformation of the original equation (8) into a diffusion equation (33), we must satisfy conditions (80) and (82). With our assumption of $a_0 = 0$, condition (80) becomes:

$$a_0 - \frac{\beta}{2c^2} \geq 0 \implies \beta \leq 0. \tag{117}$$

For condition (82) to be satisfied it must be that:

$$a_L + \frac{\beta}{2c^2} \geq 0 \implies \beta \geq 0. \tag{118}$$

Putting the two above conditions together we obtain that the Neumann boundary conditions require that:

$$\beta = 0. \tag{119}$$

From (119) we see that the original problem presented in equation (8) becomes a diffusion equation:

$$u_t = c^2 u_{xx} \tag{120}$$

with a general initial condition (9):

$$u(x,0) = f(x)$$

and boundary conditions (115). Again, the method of separation of variables suggests a solution of type:

$$u(x,t) = X(x)\,T(t)\,, \tag{121}$$

which needs to satisfy (120), hence we obtain:

$$XT' = c^2 X''T \implies \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda, \tag{122}$$

for some constant λ. As before, we obtain a system of two ordinary differential equations:

$$\begin{cases} X'' = \lambda X \\ T' = c^2 \lambda T. \end{cases} \tag{123}$$

The suggested solution (121) must satisfy boundary conditions (115), hence at $x = 0$ we must have:

$$u_x(0,t) = X_x(0)\,T(t) = 0 \implies X_x(0) = 0. \tag{124}$$

Likewise, at $x = L$ we get:

$$u_x(L,t) = X_x(L)\,T(t) = 0 \implies X_x(L) = 0. \tag{125}$$

Now we must check for which values of λ, equation:

$$X'' = \lambda X \tag{126}$$

with boundary conditions (124) and (125) is satisfied.

Case 1: $\lambda = \mathbf{0}$
Our equation (126) in this case becomes:

$$X'' = 0, \tag{127}$$

which is solved by integrating it twice to obtain a general solution:

$$X(x) = ax + b, \tag{128}$$

where a and b are constants. This solution must satisfy the boundary conditions hence:

$$X_x(0) = 0 \implies a = 0 \tag{129}$$

and:

$$X_x(L) = 0 \implies a = 0 \tag{130}$$

as well. With no condition on the constant b we set it for now arbitrarily to 1, knowing that a constant function:

$$X(x) = 1 \tag{131}$$

is a valid and a non-trivial solution to (127).

Now we should examine if positive values of λ are allowed.

Case 2: $\lambda = \mathbf{k^2} > 0$
The characteristic equation of (126) with suggested solution of type $X(x) = e^{rx}$ is:

$$r^2 - k^2 = 0, \tag{132}$$

which leads to the following roots:

$$r_{\pm} = \pm k. \tag{133}$$

A general solution then to (126) is:

$$X(x) = c_7 e^{kx} + c_8 e^{-kx}, \tag{134}$$

where c_7 and c_8 are some constants. To find the solution to (126) we should use the information contained in the boundary conditions. At $x = 0$ condition (129) must hold, which means that:

$$k \cdot c_7 e^{k \cdot 0} - k \cdot c_8 e^{-k \cdot 0} = 0 \implies c_7 - c_8 = 0 \implies c_7 = c_8. \tag{135}$$

On the other hand, at $x = L$ it must be that:

$$k \cdot c_7 e^{kL} - k \cdot c_7 e^{-kL} = 0 \implies k \cdot c_7 \left(e^{kL} - e^{-kL}\right) = 0. \tag{136}$$

Since $k = \sqrt{\lambda} > 0$ and $L > 0$, Similar argument as Case 2 of Dirichlet boundary conditions from Section 2.3 yields $c_7 = 0$. This result, together with equation (135), leads to a trivial solution. Therefore, $\lambda > 0$ will not hold for Neumann boundary condition.

Case 3: $\lambda = -\mathbf{k^2} < 0$
Here, the characteristic equation of (126) is:

$$r^2 + k^2 = 0, \tag{137}$$

for which the roots are:

$$r_{\pm} = \pm ik. \tag{138}$$

Therefore, as shown previously in Section 2.2 in (60), the following general solution satisfies the given problem:

$$X(x) = c_9 \cos(kx) + c_{10} \sin(kx), \tag{139}$$

where c_9 and c_{10} are constants determined by the considered boundary conditions. We have that at $x = 0$ the following must hold:

$$\begin{aligned} X_x(0) = 0 &\implies k\left(-c_9 \sin(0) + c_{10} \cos(0)\right) = 0 \\ &\implies c_{10} = 0, \end{aligned} \tag{140}$$

which means that:

$$X(x) = c_9 \cos(kx) \tag{141}$$

is a nonzero solution. To find the values of k we should examine the boundary condition at $x = L$, for which:

$$\begin{aligned} X_x(L) = 0 &\implies -kc_9 \sin(kL) = 0 \\ &\implies kL = n\pi, \text{ where } n = 1, 2, \dots \\ &\implies k = \frac{n\pi}{L}, \text{ where } n = 1, 2, \dots \end{aligned} \tag{142}$$

Now, plugging in the value of k obtained in (142) and setting the arbitrary constant $c_9 = 0$ we obtain the following result:

$$X_n(x) = \cos\left(\frac{n\pi}{L} x\right), \text{ where } n = 1, 2, \dots \tag{143}$$

Based on the above, we see that the eigenvalues for (114) will be:

$$\begin{aligned} \lambda_n &= -k^2 \\ &= -\left(\frac{n\pi}{L}\right)^2, \text{ where } n = 1, 2, \dots \end{aligned} \tag{144}$$

Now, similarly to the result (67) from Section 2.2 and the result obtained in (144), we get that the solution to:

$$T' = c^2 \lambda T \tag{145}$$

is:

$$T_n(t) = e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}. \tag{146}$$

Returning to the original problem in this section, (120) and remembering that the available eigenvalues $\lambda \leq 0$, we get that the following linear combination of the above results will be its solution:

$$\begin{aligned} u(x,t) &= 1 \cdot T_0(t) + \sum_{n=1}^{\infty} X_n(x) T_n(t) \\ &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}, \end{aligned} \tag{147}$$

where a_n are constant coefficients whose value we find by examining the initial condition (9):

$$\begin{aligned} u(x,0) &= a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L} x\right) \\ &= f(x). \end{aligned} \tag{148}$$

Now we multiply both sides of (148) by $\cos\left(\frac{m\pi}{L}x\right)$:

$$f(x)\cos\left(\frac{m\pi}{L}x\right) = a_0\cos\left(\frac{0\cdot\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right) \tag{149}$$

and integrate:

$$\begin{aligned}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = a_0\int_0^L \cos\left(\frac{0\cdot\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x+\\ +\int_0^L \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x.\end{aligned} \tag{150}$$

As before, we should use the knowledge of orthogonality of functions $\cos(n\pi x)$. Then for $m = 0$ we obtain:

$$\begin{aligned}&\int_0^L f(x)\cos\left(\frac{0\cdot\pi}{L}x\right)\mathrm{d}x = a_0\int_0^L \cos\left(\frac{0\cdot\pi}{L}x\right)\cos\left(\frac{0\cdot\pi}{L}x\right)\mathrm{d}x\\ \Longrightarrow &\int_0^L f(x)\mathrm{d}x = a_0\int_0^L \mathrm{d}x\\ \Longrightarrow &\, a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x.\end{aligned} \tag{151}$$

In case when $m > 0$ we get:

$$\begin{aligned}&\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = \sum_{n=1}^{\infty} a_n\int_0^L \cos\left(\frac{n\pi}{L}x\right)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x\\ \Longrightarrow &\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x = a_m\int_0^L \left(\cos\left(\frac{m\pi}{L}x\right)\right)^2\mathrm{d}x\\ \Longrightarrow &\, a_m = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{m\pi}{L}x\right)\mathrm{d}x.\end{aligned} \tag{152}$$

Hence, the solution to the problem with Neumann boundary condition is:

$$u(x,t) = a_0 + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi}{L}x\right)e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}, \tag{153}$$

where:

$$a_0 = \frac{1}{L}\int_0^L f(x)\mathrm{d}x \tag{154}$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \text{ for } n = 1, 2, \ldots. \tag{155}$$

3 Analysis and Visualization of the Results for the Diffusion-Convection Equation

In this section, we use the theoretical results obtained in Section 2 to demonstrate the behavior of each analyzed system in order to show how different constants affect the evolution of the initial condition function.

3.1 Homogeneous Dirichlet Boundary Conditions

In Section 2.3 the following problem was investigated:

$$u_t = c^2 u_{xx} - \beta u_x \qquad 0 < x < L \text{ and } 0 < t < \infty,$$

subject to the following boundary and initial conditions:

$$\begin{aligned} u(0,t) &= u(L,t) = 0 \\ u(x,0) &= f(x). \end{aligned}$$

The obtained solution:

$$u\left(x,t\right) = e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t},$$

where the coefficients b_n are given by:

$$b_n = \frac{2}{L}\int_0^L e^{-\frac{\beta}{2c^2}x} f\left(x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x.$$

It is worth noting that the imposed Dirichlet boundary conditions are responsible for the above solution to have a simple equilibrium state. By equilibrium we mean a steady state solution where we examine whether this system will approach a preferable, constant state. This state will be characterized by the following condition:

$$\frac{\partial u}{\partial t} = 0.$$

Notice that the solution contains two exponentially decaying functions of t: $e^{-\frac{\beta^2}{4c^2}t}$ and $e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}$. The first one indicates β's contribution to the overall decay, or rather it is the ratio $\dfrac{\beta^2}{4c^2}$, while the second comes only from the diffusion coefficient, c^2. Calculating the following limit will illustrate the expected decay:

$$\begin{aligned} \lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}\right] \\ &= \lim_{t\to\infty}\left[e^{-\frac{\beta^2}{4c^2}t} e^{\frac{\beta}{2c^2}x} \sum_{n=1}^{\infty} \left(\frac{2}{L}\int_0^L e^{-\frac{\beta}{2c^2}x} f\left(x\right) \sin\left(\frac{n\pi}{L}x\right) \mathrm{d}x\right) \sin\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}\right] \\ &= 0. \end{aligned}$$

Hence, any initial condition will over time decay to zero given Dirichlet boundary conditions.

3.1.1 Effect of Changing the Sign and Magnitude of β on $u(x,t)$

Consider the following constants used in the simulation: $L = 1$, $c = 1$, and let the initial condition be $f(x) = \sin(\pi x)$. Figure 2 presents the solutions for three values of the convection coefficient β : -10, 0, and 5, where the first 70 eigenvalues ($n = 70$) were used. In all three cases the blue curves signify the initial condition, the black curves are the solutions after 0.05 units of time, and the red curves are the solutions at time 0.1.

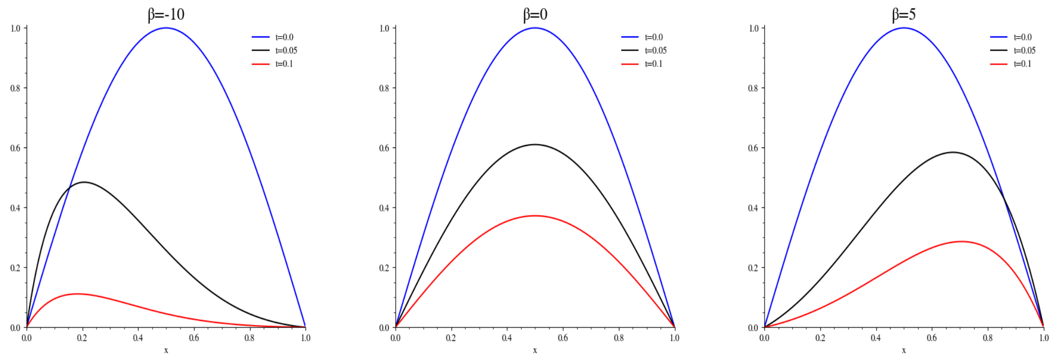

Figure 2: The effect of changing β on the behavior of the system described by the diffusion-convection equation with Dirichlet boundary conditions.

First, homogeneous boundary conditions force the value of the function at $x = 0$ and $x = 1$ to always be 0, which means that over time, with diffusion or convection taking place, one expects that the function will decrease its overall value. This is evident in all three cases presented Figure (2) because the function is forced to have a value of 0 at both boundaries.

Second, changing the sign of β causes the convection to occur in the direction of the sign of the coefficient with respect to the increasing value of assigned position, x. For negative β, convection occurs in the negative x direction, which is shown in the left figure - the curve appears to be "pushed" to the left. One also observes that the function's value is allowed to increase at some locations (e.g., $x = 0.01$). This is because the magnitude of the convection coefficient (10) is much greater than the diffusion coefficient ($c^2 = 1$), so convection in this case dominates the behavior of the system. The figure on the right shows the case where β is positive, so convection occurs in the positive x direction, but its magnitude (5) is half of that of the case with $\beta = -10$, so the function at the corresponding times appears to decrease at a slower rate. At time $t = 0.1$, the red curve in the right figure is much more significant than that in the left figure.

Third, the middle graph in Figure 2 represents the system where no convection occurs, which is described by a pure diffusion equation. Here, we observe an even slower progress of dissipation of the function - but its disappearance is caused by the diffusion, meaning that without it, the function would remain at the blue curve's state at any time t.

3.1.2 Effect of Changing the Magnitude of c^2 on $u(x, t)$

Now, to find how changing c^2 will affect the behavior of the system, we consider the following constants: $L = 1$, $\beta = 5$, and let the initial condition be $f(x) = \sin(\pi x)$. Figure 3 presents the solutions for three values of the convection coefficient c^2 : 1.00, 2.25, and 4.00, where the first 70 eigenvalues ($n = 70$) were used. As before, in all three cases the blue curves signify the initial condition, but now the black curves are the solutions after 0.02 units of time, and the red curves show the solutions at time 0.05.

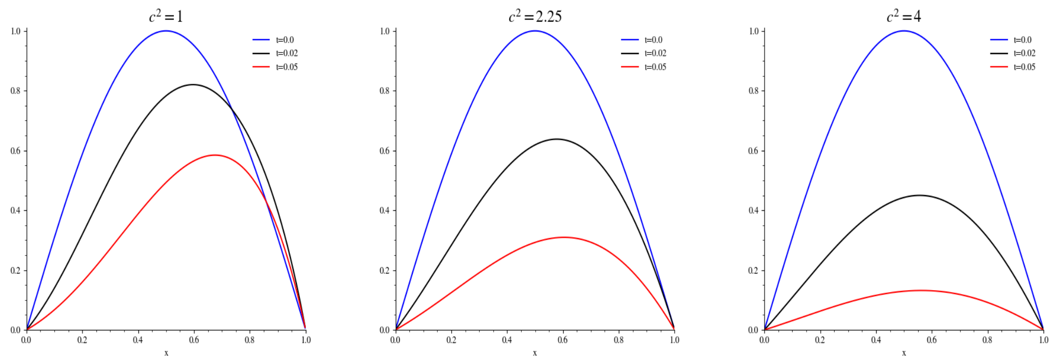

Figure 3: The effect of changing c^2 on the behavior of the system described by the diffusion-convection equation with Dirichlet boundary conditions.

The left-most graph in Figure 3 shows the system with the smallest effect of diffusion. The movement of the curve in the positive x direction as time progresses signifies that convection, or the effect of the β coefficient, is the most prominent. The middle graph in Figure 3 considers an increased value of the diffusion coefficient. With respect to the graph on the left, c^2 increased 2.25 times, which causes the rate at which the function decreases to be larger. The effect of convection, although still visible through the slight tilt of the curve with time progressing, is not as noticeable as in the graph to the left. The graph on the right has c^2 increased four times with respect to the left-most case, which causes the function to dissipate even faster with convection only slightly signified by the tilt of the curve to the right.

With the convection coefficient kept constant, increased values of the diffusion coefficient should produce a faster disappearance of the function, which is seen going from left to right. This is because the Dirichlet boundary conditions force the value of the function to always be zero at the two boundaries. This is why the diffusion or slow dissipation of the function in both directions over time causes an increase of the value of the function as it approaches the boundaries, but at the same time the imposed boundary conditions force the function to decrease to zero at each end. The same is true of the effect of the convection, which ultimately causes the disappearance of the function, which mathematically is represented by $\lim_{t\to\infty} u(x,t) = 0$.

3.1.3 Effect of Changing the Initial Condition $f(x)$ on $u(x,t)$

In this section, we will investigate how the shape of the initial function $f(x)$ determines the solution of the diffusion-convection equation with Dirichlet boundary conditions. Here we assigned the following values to the constants in the problem: $L = 1$, $c^2 = 1$, $\beta = 5$. The solutions presented were calculated using 70 first eigenvalues ($n = 70$). Figure 4 presents three cases with different initial conditions. The left-most graph presents the solution for $f(x) = \sin^2(2\pi x)\cos^2(3\pi x)$, the middle graph's initial condition was $f(x) = \sin^2(2\pi x)$, and the graph on the right examines $f(x) = \sin^6(\pi x)$. The time points considered is $t = 0.000$, $t = 0.005$, and $t = 0.020$.

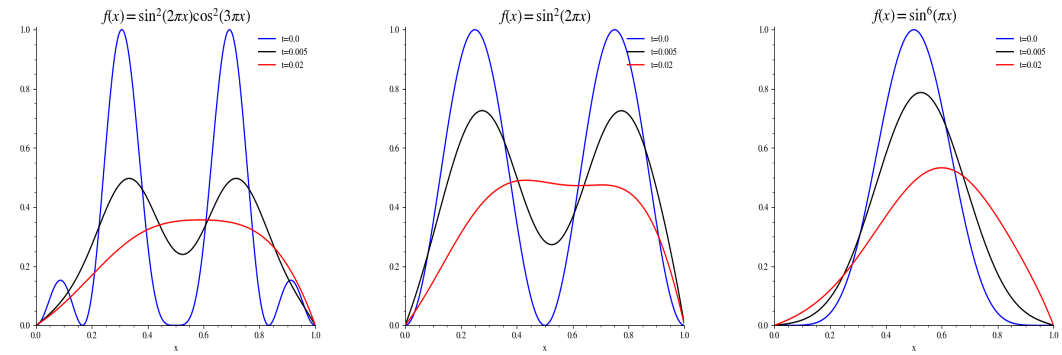

Figure 4: The effect of changing $f(x)$ on the behavior of the system described by the diffusion-convection equation with Dirichlet boundary conditions.

The examples of different initial conditions presented in Figure 4 show how the initial shape of the function determines its evolution over time. The left graph in Figure 4 shows $f(x)$ with four local maxima which quickly merge to become a curve with only two local maxima at $t = 0.005$, followed by the slightly slanted shape with only one maximum at $t = 0.02$. The middle example's initial condition only has two maxima, which start to decrease and shift to the right, but the decrease here is slower than in the left example, which is related to how quickly $f(x)$ is changing with respect to position and its curvature. The first reason relates to the steepness of the curve, which is connected to convection, while the curvature influences the diffusion process. The initial condition in the left graph has a steeper and more curved shape than the starting point of $f(x)$ in the middle graph. To the right, we have an initial condition with only one maximum. Even though the convection shifts the curve to the right, on the lower left hand side we clearly see that a significant curvature there causes the diffusion, occurring in either direction, to visibly increase the value of the function in that corner.

3.2 Homogeneous Robin Boundary Conditions

In section 2.4 the problem investigated was:

$$u_t = c^2 u_{xx} - \beta u_x \qquad 0 < x < L \text{ and } 0 < t < \infty,$$

where c^2 and β are positive constants. This equation was subject to the following boundary conditions:

$$u_x(0,t) - a_0 u(0,t) = 0$$
$$u_x(L,t) + a_L u(L,t) = 0,$$

where a_0 and a_L are positive constants. The following general initial condition was considered:

$$u(x,0) = f(x).$$

The well-posedness of the problem required additional conditions on the constant β:

$$\beta \leq 2c^2 a_0.$$

The solution to this equation was found to be:

$$u\left(x,t\right)=e^{-\frac{\beta^2}{4c^2}t}e^{\frac{\beta}{2c^2}x}\sum_{n=1}^{\infty}A_n\left(\cos\left(\frac{\alpha_n}{L}x\right)+\frac{a_0-\frac{\beta}{2c^2}}{\alpha_n}L\sin\left(\frac{\alpha_n}{L}x\right)\right)e^{-\frac{\alpha_n^2}{L^2}c^2t},$$

where coefficients A_n are given by:

$$A_n=\frac{\int_0^L\left(\cos\left(\frac{\alpha_n}{L}x\right)+\frac{a_0-\frac{\beta}{2c^2}}{\alpha_n}L\sin\left(\frac{\alpha_n}{L}x\right)\right)e^{-\frac{\beta}{2c^2}x}f\left(x\right)\mathrm{d}x}{\int_0^L\left(\cos\left(\frac{\alpha_n}{L}x\right)+\frac{a_0-\frac{\beta}{2c^2}}{\alpha_n}L\sin\left(\frac{\alpha_n}{L}x\right)\right)^2\mathrm{d}x},$$

and constants α_n $(n\in\mathbb{Z}_{\geq 0})$ are solutions to the following transcendental equation:

$$\tan\left(\alpha\right)=\frac{L\left(a_0+a_L\right)\alpha}{\alpha^2-L^2\left(a_0-\frac{\beta}{2c^2}\right)\left(a_L+\frac{\beta}{2c^2}\right)}.$$

As in the case of Dirichlet boundary conditions, we ask about the equilibrium solution:

$$\begin{aligned}\lim_{t\to\infty}u\left(x,t\right)&=\lim_{t\to\infty}\left[e^{-\frac{\beta^2}{4c^2}t}e^{\frac{\beta}{2c^2}x}\sum_{n=1}^{\infty}A_n\left(\cos\left(\frac{\alpha_n}{L}x\right)+\frac{a_0-\frac{\beta}{2c^2}}{\alpha_n}L\sin\left(\frac{\alpha_n}{L}x\right)\right)e^{-\frac{\alpha_n^2}{L^2}c^2t}\right]\\&=0.\end{aligned}$$

The above result signifies that in the system, which interacts with its surroundings according to Newton's law, the solution will also over time diminish to the $u(x)=0$. The following examples will illustrate the major difference between Dirichlet and Robin boundary conditions - while in the Dirichlet case the function at each end of the system was kept at $u(0,t)=u(L,t)=0$, in the Robin case, these values will change in time.

3.2.1 Effect of Changing the Magnitude of β on $u(x,t)$

In this section, we will illustrate how changing the magnitude of the convection coefficient, β, affects the behavior of the system.

Consider a system of length 1 $(0\leq x\leq 1)$, where the following coefficients were kept constant: $c^2=1$, $a_0=10$, and $a_L=0.5$. In each case illustrated in Figure 5, the initial condition was $f(x)=\sin(\pi x)$ and 70 first α_n values were used in obtaining the solutions for times $t=0.00$, 0.03, and 0.20.

Different values of β cause the system to evolve slightly differently. The example on the left in Figure 5 illustrates a solution to the pure diffusion problem ($\beta=0$). The function $f(x)$ is seen to decrease most notably where its curvature is negative, which coincides with a maximum. The most notable difference we observe is in the rate at which the function's value changes on the boundaries. Because the a_0 coefficient was chosen to be much higher than a_L, this means that at $x=0$ the system is allowed to interact with the surroundings to a greater extent than at $x=1$. Over time, this manifests itself by making the curve look tilted - with smaller value achieved at $x=0$ compared to $x=1$. Since the function dissipates through the boundaries, just at different rates at each end, it will continue to decrease until it becomes the zero-function.

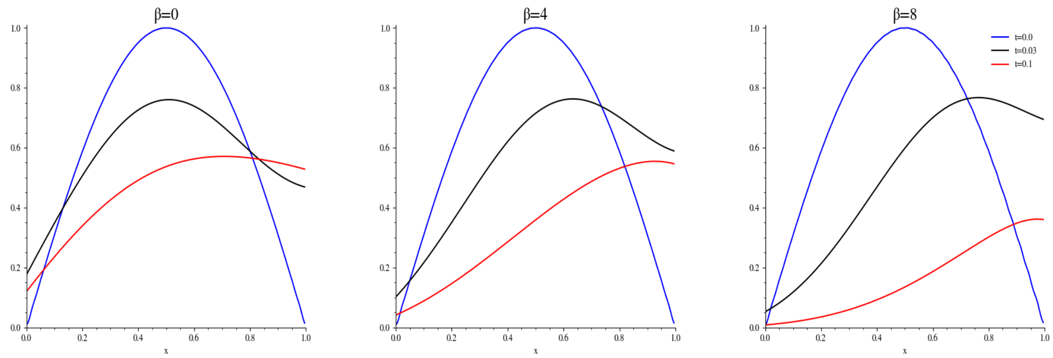

Figure 5: The effect of changing the magnitude of β on the behavior of the system described by the diffusion-convection equation with homogeneous Robin boundary conditions.

The middle graph in Figure 5 shows the same conditions as in the left example, but now convection process is introduced ($\beta = 4$). The convection "pushes" the curve to the right, so the function's value at $x = 1$ at time $t = 0.03$ reaches a higher values than in the left example, but this in turn helps dissipating the function, so at time $t = 0.20$ the curve appears to have decreased faster than the to the left.

In the right example of Figure 5, β was increased by a factor of 2 ($\beta = 8$), which caused the function at the left boundary, $x = 0$, to reach smaller values at different times compared to the left and middle graphs. This is because convection at this end works against the diffusion. On the other hand, at $x = 1$ and at $t = 0.03$, the function takes on the largest value between the three examples, although at $t = 0.20$, it has the smallest of the three values. Even though the exchange with the surroundings at $x = 1$ is inhibited by the small value of $a_L = 0.5$, it still happens, so constant convection in that direction allows for faster progression toward equilibrium.

3.2.2 Effect of Changing the Magnitude of c^2 on $u(x,t)$

We now illustrate how the change of the diffusion coefficient, c^2, affects the solution. Figure 6 presents the system bounded by $0 \le x \le 1$, where the convection coefficient is $\beta = 4$, and the chosen boundary condition coefficients are $a_0 = a_L = 3$. The initial condition is $f(x) = \sin(\pi x)$ and 70 first α_n constants were used in presenting the solutions at times $t = 0.00$, 0.03, and 0.10.

The graph on the left in Figure 6 shows that convection is the dominating process, causing the function to shift to the right as time progresses. This also causes, even though the boundary conditions at each boundary are identical, the function to increase more at $x = 1$ than at $x = 0$.

Increasing the impact of diffusion by a factor of 2 ($c^2 = 2$) in the middle graph shows that the function progresses toward equilibrium much faster and the curves at different times now have smaller curvatures, compared with solutions at corresponding times with the example on the left. Increasing c^2 to 4 in the right example still has visible convection effect, which causes a slight shift of the maximum of $u(x,t)$ to the right. Diffusion now plays a dominant role reducing the relative difference in function values at each boundary, compared with examples from the left and middle graphs. With both convection and higher impact of diffusion, the solution on the right graph at $t = 0.10$ is much closer to equilibrium than the other examples.

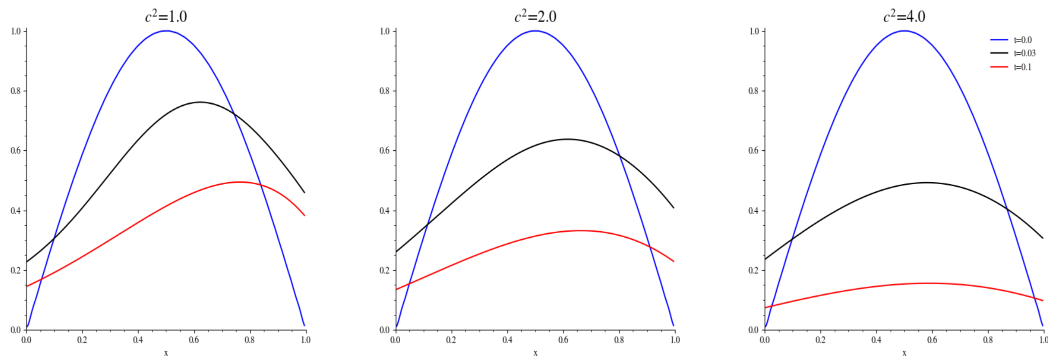

Figure 6: The effect of changing the magnitude of c^2 on the behavior of the system described by the diffusion-convection equation with homogeneous Robin boundary conditions.

3.2.3 Effect of Changing the Boundary Constants a_0 and a_L on $u(x,t)$

We move on to demonstrate how changing the character of the boundary condition at each end will impact its solution. Increasing a_0 or a_L constants will cause the boundary to interact with the surroundings more, which having homogeneous conditions means more "Dirichlet" character, while decreasing them will cause the boundary to become more insulated, or "Neumann" in character.

Consider again a bounded system $0 \leq x \leq 1$, where $c^2 = 1$, $\beta = 5$. The initial condition was set to $f(x) = \sin(\pi x)$ and 70 values of α_n coefficients were used in producing the graphs in Figure 7. The solutions are shown for three different time $t = 0.00$, 0.05, and 0.15.

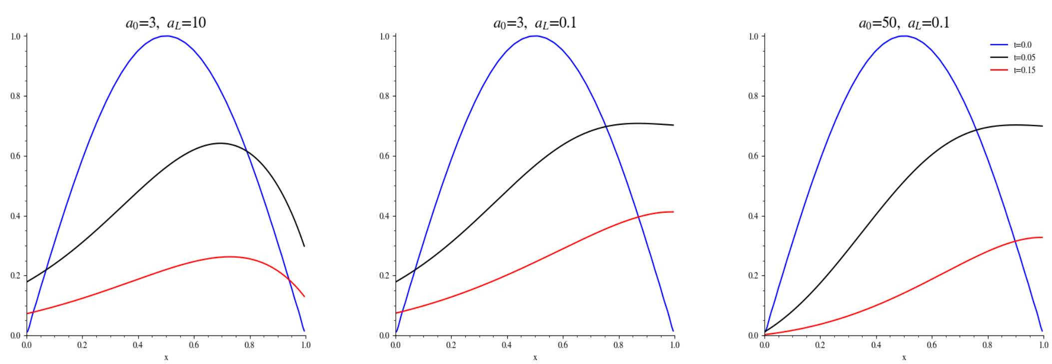

Figure 7: The effect of changing the relative values of a_0 and a_L on the behavior of the system described by the diffusion-convection equation with homogeneous Robin boundary conditions.

Comparing the left and middle graphs in Figure 7 we see that decreasing the magnitude of a_L from 10 to 0.1 made the right boundary more insulated - the function increased in value at $t = 0.05$ and 0.15, compared to the corresponding curves in the left graph. Keeping $a_L = 0.1$ and changing the values of a_0 from 3 to 50 as illustrated in the right graph of Figure 7 shows how close the left boundary appears to the one presented in Section 3.1.1 in Figure 2. This change did affect how the function behaved at the right boundary. While at time $t = 0.05$ there is little if any difference between the value of the function between the middle and right graph, but at $t = 0.15$, one can observe a noticeable decrease in the function's values at that point in space.

3.2.4 Effect of Changing the Initial Condition $f(x)$ on $u(x, t)$

In this section, we show three examples of initial functions to illustrate that, given the same conditions, the systems' behavior, while similar, will differ. Figure 8 presents the solutions with the following constants: $c^2 = 1$, $\beta = 10$, $a_0 = 10$, and $a_L = 0.5$. In all graphs, the blue curves represent $t = 0.000$, the black curves $t = 0.005$, and the red curves correspond to $t = 0.020$. The explicit formulas for each $f(x)$ is presented as a title for each of the graphs in Figure 8.

In all the graphs in Figure 8 we see clear impact of convection, where all curves slowly shift to the right with time passing. The difference between the magnitudes of a_0 and a_L demonstrate themselves by increased value of the function at $x = 1$ as compared to the value of the function at $x = 0$. It is interesting to see how low the $u(0, t)$ remains on the right graph in comparison to the other examples. The reasons for that are the relative magnitude of β and c^2, as well as how close to zero $f(x)$ is for $x \in (0, 2)$.

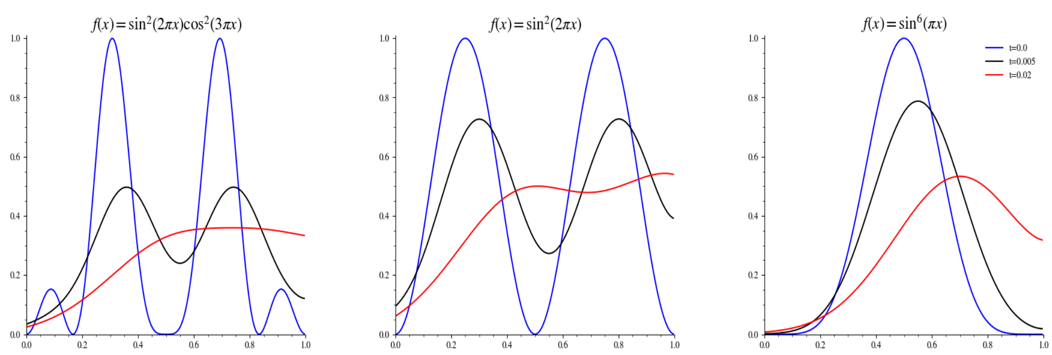

Figure 8: The effect of changing $f(x)$ on the behavior of the system described by the diffusion-convection equation with Robin boundary conditions.

3.3 Homogeneous Neumann Boundary Conditions

In Section 2.5 we investigated an insulated system which necessitated the following value of the convection coefficient: $\beta = 0$. Hence, the problem in the case of Neumann boundary condition turned out to be a pure diffusion equation:

$$u_t = c^2 u_{xx} \qquad 0 < x < L \text{ and } 0 < t < \infty,$$

subject to the following boundary and initial conditions:

$$\begin{aligned} u_x(0, t) &= u_x(L, t) = 0 \\ u(x, 0) &= f(x). \end{aligned}$$

The obtained solution was as follows:

$$u\left(x,t\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t},$$

where:

$$\begin{aligned}
a_0 &= \frac{1}{L}\int_0^L f(x)\mathrm{d}x \\
a_n &= \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x, \text{ for } n = 1, 2, \ldots
\end{aligned}$$

As in the Dirichlet and Robin boundary conditions, we should ask if an equilibrium state is expected, and if so, is it different from the previous problems? Again, to do this we are interested in computing the following limit:

$$\begin{aligned}
\lim_{t\to\infty} u(x,t) &= \lim_{t\to\infty}\left[\frac{1}{L}\int_0^L f(x)\mathrm{d}x + \sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right)\cos\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}\right] \\
&= \lim_{t\to\infty}\left(\frac{1}{L}\int_0^L f(x)\mathrm{d}x\right) + \lim_{t\to\infty}\left[\sum_{n=1}^{\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right)\cos\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}\right] \\
&= \frac{1}{L}\int_0^L f(x)\mathrm{d}x + \sum_{n=1}^{\infty}\left[\lim_{t\to\infty}\left(\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{L}x\right)\mathrm{d}x\right)\cos\left(\frac{n\pi}{L}x\right) e^{-c^2\left(\frac{n\pi}{L}\right)^2 t}\right] \\
&= \frac{1}{L}\int_0^L f(x)\mathrm{d}x
\end{aligned}$$

The above result shows that indeed, the system which is subject to the Neumann boundary conditions will approach the equilibrium state when at each position the value of the function $u(x)$ will be constant and an average of the given $f(x)$.

3.3.1 Effect of Changing the Magnitude of c^2 on $u(x,t)$

We will now illustrate how changing the value of the diffusion coefficient changes the behavior of the system.

Consider a system of length $L = 1$: $0 \le x \le 1$ and the initial condition given by: $f(x) = \sin(\pi x)$, where 70 first eigenvalues were used to approximate the solutions. Figure 9 presents three cases where the diffusion coefficient c^2 was chosen to be: 1.00 (left), 2.25 (middle), and 4.00 (right). The insulated system manifests itself by changing the value of the function at $x = 0$ and at $x = L$. Due to the diffusion process, the function increases where its average value is below the aforementioned average of $f(x)$, and it decreases where its value was above the starting average. This process continues, slightly slowing down with time, most visible on the right graph in Figure 9 at $x = 0.5$, as the curvature of the function decreases.

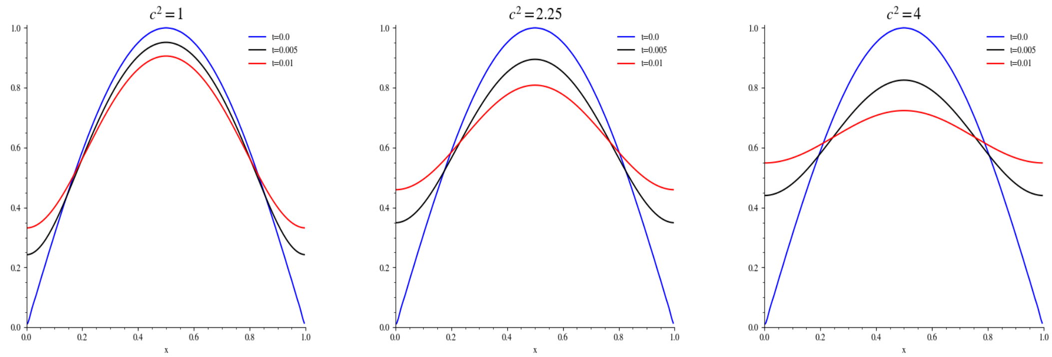

Figure 9: The effect of changing c^2 on the behavior of the system described by the diffusion-convection equation with Neumann boundary conditions.

Figure 9 shows how, with increasing diffusion coefficient, the system changes at a faster rate, which supports the claim that the second derivative with respect to position term in the differential equation is a mathematically represented physical process of diffusion. Notice, now that $\beta = 0$, no convection occurs and the symmetry of the system as given by the initial condition $f(x)$ is preserved.

3.3.2 Effect of Changing the Initial Condition $f(x)$ on $u(x,t)$

We now illustrate in Figure 10 how the change of the system with Neumann boundary conditions depends on the shape of the initial condition $f(x)$. In Figure 10, the graph on the left has an initial condition given by: $f(x) = \sin^2(2\pi x)\cos^2(3\pi x)$, while the one in the middle has $f(x) = \sin^2(2\pi x)$, and the initial condition in the right graph is $f(x) = sin^6(\pi x)$. In each case the spatial variable x is bounded by $x = 0$ on the left, and $L = 1$ on the right. The chosen value of the diffusion coefficient was $c^2 = 1$. The times considered in each case were $t = 0.000$, 0.005, and 0.020.

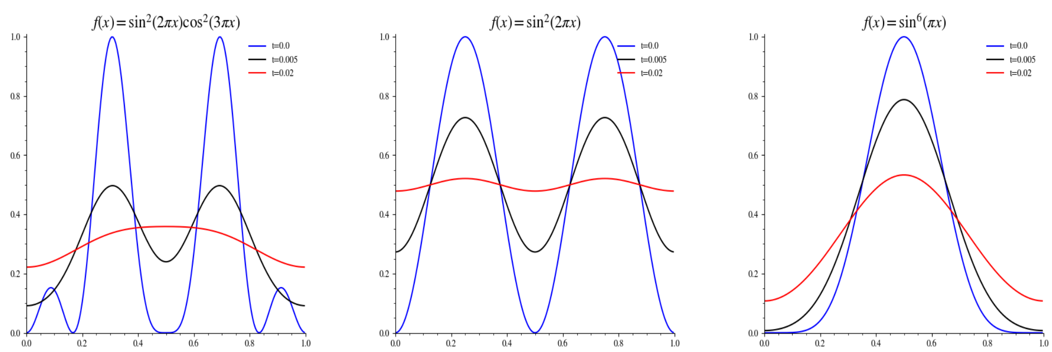

Figure 10: The effect of changing $f(x)$ on the behavior of the system described by the diffusion equation with Neumann boundary conditions.

With time passing, in each case, the function decreases where its curvature is negative and increases where it is positive. This is visible in the left example, where at time $t = 0.005$ each local maximum exhibits some decrease. The decrease is most predominant in the two middle maxima. The two peripheral maxima, both well below the average of $f(x)$ also show slight decrease, until, very quickly their curvatures decrease at which time they no longer are local maxima. Once this happens, those points start increasing, as they appear to be lower than the average of $f(x)$. The graph in the middle shows two local maxima in $f(x)$. The curvature at those points is smaller than in the left example, which explains why their decrease after time $t = 0.005$ was much smaller as compared with the case on the left. The example on the right in Figure 10 demonstrates how much slower the process is when there is one dominant curvature at the maximum with two areas with smaller curvature present on each side of the curve. In comparison, the middle example has two local maxima with about equal curvature each (as compared to the right example) with an additional local minimum having large positive curvature. The number and the magnitude of the points with dominant curvatures causes the system in the middle to evolve quicker than the one on the right.

3.4 Comparing the Solutions with Three Different Boundary Conditions for $\beta = 0$

As a final example we present the solutions to the pure diffusion equation and illustrate the effect the three different, homogeneous boundary conditions have on the system. For the graphs presented in Figure 11, the following constants were used: $c^2 = 2$, $\beta = 0$, necessarily, $a_0 = 10$, and $a_L = 0.1$. In each case, 70 eigenvalues were used to compute the solution for $f(x) = \sin^4(\pi x)$. The times at which the solutions are graphed are: $t = 0.000$, 0.002, 0.010, and 0.050.

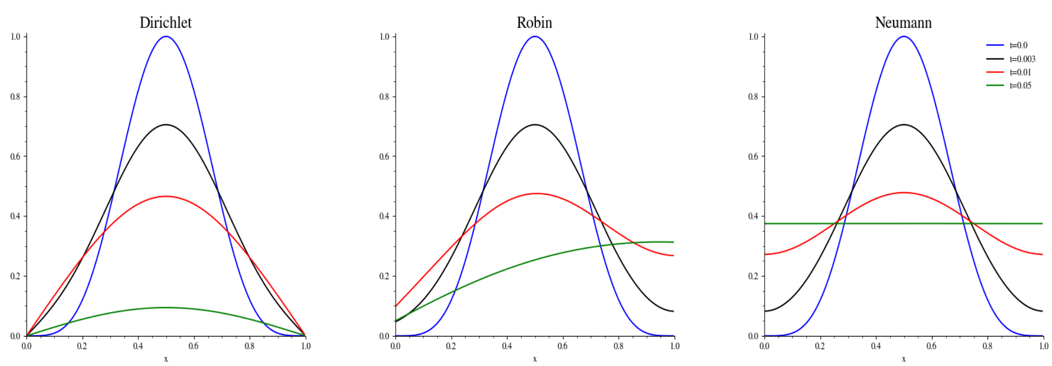

Figure 11: The effect of changing the type of boundary conditions on the behavior of the system described by the diffusion equation with $f(x) = \sin^4(\pi x)$.

While both, the Dirichlet and Robin functions will over time decay to zero, we can clearly see the differences between them. In the Robin case, the interaction with the surroundings is inhibited, hence it will decrease at a lower rate than the Dirichlet system. Somewhat surprisingly, the system with insulated boundary conditions, Neumann on the right, reaches its equilibrium much faster than the Dirichlet case on the left. This may be because in the Neumann case the curvatures close to the boundaries, while decreasing with time, remain and continue driving the diffusion process along with the curvature present in the middle. In the Dirichlet case, the originally present curvatures close to the boundaries quickly diminish leaving only one dominant one, where the function has its maximum, to be the source of diffusion.

4 Python Code

In this section, we share the Python code used to visualize the theoretical results obtained in the previous sections. Visualizing the evolution of the system given different boundary conditions was essential to understanding the physical processes of convection and diffusion, and while derivation of their mathematical forms were referenced from other sources, the rate of change and the curvature of a given function clearly relate to those processes, respectively.

4.1 Dirichlet Boundary Conditions

```
import numpy as np
import scipy.integrate
from numpy import exp
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from matplotlib.animation import PillowWriter

dt = 0.001 #time increment
tmin = 0.0 #initial time
tmax = 0.2 #simulate until

nx = 500 #number of position points (spatial resolution)
xmin = 0.0 #left bound
xmax = 1.0 #right bound
x = np.arange(xmin, xmax, (xmax - xmin)/nx)

#set initial condition:
def f(x):
   return np.sin(np.pi*x/xmax)

#define constants:
c = 1 #diffusion coefficient
beta = 5 #velocity

#number of eigenvalues:
n = 70
N = np.arange(1,n+1,1)

#compute eigenvalues:
a = np.empty(n)
for i in N:
   a[i-1]=(-(i*np.pi/xmax)**2)

#solution:
def u(x,t,beta):
   sum = 0
   for i in N:
      sum= sum+2/xmax*scipy.integrate.quad(lambda x:exp(-beta/(2*c**2)*x)*f(x)*
         np.sin(i*np.pi*x/xmax),0,xmax)[0]*np.sin(i*np.pi*x/xmax)*
         exp(-c**2*(i*np.pi/xmax)**2*t)
```

```
   return sum*exp(-beta**2/(4*c**2)*t)*exp(beta/(2*c**2)*x)

#producing animation
fig, ax = plt.subplots()
ax.set_xlabel('x')
plotLine, = ax.plot(x, np.zeros(len(x))*np.NaN, 'r-')
plotTitle = ax.set_title("t=0")
ax.set_ylim(0,1.2)
ax.set_xlim(xmin,xmax)

def solution(t):
   p = u(x,t,beta)
   return p

def animate(t):
   pp = solution(t)
   plotLine.set_ydata(pp)
   plotTitle.set_text('t = ' + str(round(t,3)))
   return [plotLine,plotTitle]

ani = animation.FuncAnimation(fig, func=animate, frames=np.arange(tmin, tmax, dt),
        blit=False)
plt.show()

#saving animation
ani.save("Dirichlet.gif",writer=PillowWriter(fps=24))
```

4.2 Robin Boundary Conditions

```
import numpy as np
import scipy.integrate
from numpy import exp
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from matplotlib.animation import PillowWriter

dt = 0.001 #time increment
tmin = 0.0 #initial time
tmax = 0.4 #simulate until

nx = 200 #number of position points (spatial resolution)
xmin = 0.0 #left bound
xmax = 1.0 #right bound
x = np.arange(xmin, xmax, nx)

#define constants:
c = 1 #diffusion coefficient
beta = 5 #velocity
a0=5 #x=0 BC constant
aL=1 #x=L BC constant
pertub=1e-4 #perturbation value to determine bisect interval
```

```
#number of alpha constants:
n = 70
N = np.arange(1,n+1,1)

#set initial condition
def f(x):
   return np.sin(np.pi*x/xmax)**4

#compute the vertical asymptote
asympt = xmax*np.sqrt((a0-beta/(2*c**2))*(aL+beta/(2*c**2)))
if asympt > 0:
   print('The vertical asymptote happens at x = ', asympt)
else:
   print('beta is out of range for Robin boundary condition to ensure
                                                        solution existence')

#function to be used in the bisection method
def func(y):
   return np.tan(y)-xmax*(a0+aL)*y/(y**2-xmax**2*((a0-beta/(2*c**2))*
                                                        (aL+beta/(2*c**2))))

#find the interval containing the vertical asymptote
for i in N:
   if np.pi*(i-1) <= asympt and asympt < np.pi*i:
      sep = i-1
      break
print('The interval containing the vertical asymptote is ', sep)

#compute alpha constants:
a = np.empty(n)

for i in N:
   if i <= sep:  #interval before the vertical asymptote
      if func((i-1)*np.pi+np.pi/2*(1+pertub))*func(i*np.pi) < 0:
         print("found alpha in interval ", i)
         a[i-1]=optimize.bisect(func, (i-1)*np.pi+np.pi/2*(1+pertub), i*np.pi)
   elif i > sep+1:  #interval after the vertical asymptote
      if func((i-1)*np.pi)*func((i-1)*np.pi+0.5*np.pi*(1-pertub)) < 0:
         print("found alpha in interval ", i)
         a[i-1]=optimize.bisect(func, (i-1)*np.pi, (i-1)*np.pi+0.5*np.pi*(1-pertub))
   else:  #interval containing the vertical asymptote
      if asympt < (i-1)*np.pi+0.5*np.pi:
         if func(asympt*(1+pertub))*func((i-1)*np.pi+0.5*np.pi*(1-pertub)) < 0:
            print("found alpha in interval ", i)
            a[i-1] = optimize.bisect(func, asympt*(1+pertub),
                                              (i-1)*np.pi+0.5*np.pi*(1-pertub))
      elif asympt > (i-1)*np.pi+0.5*np.pi:
         if func((i-1)*np.pi+0.5*np.pi*(1+pertub))*func(asympt*(1-pertub)) < 0:
            print("found alpha in interval ", i)
            a[i-1] = optimize.bisect(func, (i-1)*np.pi+0.5*np.pi*(1+pertub),
                                                             asympt*(1-pertub))
      else:
```

```
        a[i-1] = asympt
print(a)

#solution:
def u(x,t):
   sum = 0
   for i in N:
      An = (scipy.integrate.quad(lambda x:exp(-beta/(2*c**2)*x)*f(x)*
         (np.cos(a[i-1]*x/xmax)+(a0-beta/(2*c**2))*xmax/a[i-1]*np.sin(a[i-1]*x/xmax)),
            0,xmax)[0])/(scipy.integrate.quad(lambda x:(np.cos(a[i-1]*x/xmax)+
            (a0-beta/(2*c**2))*xmax/(a[i-1])*np.sin(a[i-1]*x/xmax))**2,0,xmax)[0])
      sum = sum+An*exp(-(a[i-1]**2*c**2*t/(xmax**2)))*(np.cos(a[i-1]*x/xmax)+
            (a0-beta/(2*c**2))*xmax/(a[i-1])*np.sin(a[i-1]*x/xmax))
   return sum*exp(-beta**2/(4*c**2)*t)*exp(beta/(2*c**2)*x)

#producing animation
fig, ax = plt.subplots()
ax.set_xlabel('x')
plotLine, = ax.plot(x, np.zeros(len(x))*np.NaN, 'r-')
plotTitle = ax.set_title("t=0")
ax.set_ylim(-np.max(f(x))*1.1,np.max(f(x))*1.1)
ax.set_xlim(xmin,xmax)

def solution(t):
   p = u(x,t)
   return p

def animate(t):
   pp = solution(t)
   plotLine.set_ydata(pp)
   plotTitle.set_text('t = ' + str(round(t,3)))
   return [plotLine,plotTitle]

ani = animation.FuncAnimation(fig, func=animate, frames=np.arange(tmin+2*dt,
                                                    tmax, dt), blit=False)
plt.show()

#saving animation
ani.save("Robin.gif",writer=PillowWriter(fps=24))
```

4.3 Neumann Boundary Conditions

```
import numpy as np
import scipy.integrate
from numpy import exp
import matplotlib.pyplot as plt
import matplotlib.animation as animation
from matplotlib.animation import PillowWriter

dt = 0.003 #time increment
tmin = 0.0 #initial time
```

```
tmax = 0.4 #simulate until

nx = 200 #number of position points (spatial resolution)
xmin = 0.0 #left bound
xmax = 1.0 #right bound
x = np.arange(xmin, xmax, (xmax - xmin)/nx)

#set initial condition:
def f(x):
   return np.sin(np.pi*x/xmax)

#define constants:
c = 1/2 #diffusion coefficient
beta = 0 #no velocity allowed

#number of eigenvalues:
n = 70
N = np.arange(1,n+1,1)

#compute eigenvalues:
a = np.empty(n)
for i in N:
   a[i-1]=(-(i*np.pi/xmax)**2)

#solution:
def u(x,c,t):
   sum = 0
   for i in N:
      sum= sum+2/xmax*scipy.integrate.quad(lambda x:f(x)*
         np.cos(i*np.pi*x/xmax),0,xmax)[0]*np.cos(i*np.pi*x/xmax)*
         exp(-c**2*(i*np.pi/xmax)**2*t)
   return 1/xmax*scipy.integrate.quad(lambda x:f(x),0,xmax)[0]+sum

#producing animation
fig, ax = plt.subplots()
ax.set_xlabel('x')
plotLine, = ax.plot(x, np.zeros(len(x))*np.NaN, 'r-')
plotTitle = ax.set_title("t=0")
ax.set_ylim(0,1.2)
ax.set_xlim(xmin,xmax)

def solution(t):
   p = u(x,c,t)
   return p

def animate(t):
   pp = solution(t)
   plotLine.set_ydata(pp)
   plotTitle.set_text('t = ' + str(round(t,3)))
   return [plotLine,plotTitle]

ani = animation.FuncAnimation(fig, func=animate, frames=np.arange(tmin, tmax, dt),
        blit=False)
```

```
plt.show()

#saving animation
ani.save("Neumann.gif",writer=PillowWriter(fps=24))
```

5 Summary and Conclusions

In this paper, we follow Arrigo [4] and Tanveer [5] to establish a complete procedure of solving 1D diffusion-convection equation with all three kinds boundary conditions (Dirichlet, Robin, and Neumann). The Dirichlet boundary condition does not enforce any restrictions on the convection coefficient. However, the Robin boundary condition indeed enforces an upper bound on the convection coefficient in order to ensure solution existence and uniqueness for the problem. The Neumann boundary condition, as a special case of Robin boundary condition, requires that the convection coefficient remains zero. Therefore, a diffusion system cannot be insulated from its surroundings once convection takes effect. Finally, we use Python code to construct and visualize the solution of the diffusion-convection equation with varying boundary conditions.

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