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Conventionalizing and Axiomatizing in a Community College Mathematics Bridge Course

Mark Alan Yannotta
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Conventionalizing and Axiomatizing in a
Community College Mathematics Bridge Course

by

Mark Alan Yannotta

A dissertation submitted in partial fulfillment of the
requirements for the degree of

Doctor of Philosophy
in
Mathematics Education

Dissertation Committee:
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Portland State University
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Abstract

This dissertation consists of three related papers. The first paper, *Rethinking mathematics bridge courses—An inquiry model for community colleges*, introduces the activities of conventionalizing and axiomatizing from a practitioner perspective. In the paper, I offer a curricular model that includes both inquiry and traditional instruction for two-year college students interested in mathematics. In particular, I provide both examples and rationales of tasks from the research-based *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum, which anchors the inquiry-oriented version of the mathematics bridge course. The second paper, *The role of past experience in creating a shared representation system for a mathematical operation: A case of conventionalizing*, adds to the existing literature on mathematizing (Freudenthal, 1973) by “zooming in” on the early stages of the classroom enactment of an inquiry-oriented curriculum for reinventing the concept of group (Larsen, 2013). In three case study episodes, I shed light onto “How might conventionalizing unfold in a mathematics classroom?” and offer an initial framework that relates students’ establishment of conventions in light of their past mathematical experiences. The third paper, *Collective axiomatizing as a classroom activity*, is a detailed case study (Yin, 2009) that examines how students collectively engage in axiomatizing. In the paper, I offer a revision to De Villiers’s (1986) model of descriptive axiomatizing. The results of this study emphasize the additions of pre-axiomatic activity and axiomatic creation to the model and elaborate the processes of axiomatic formulation and analysis within the classroom community.

Dedication

For Amy, Kyra, and Evie.

Acknowledgments

First, I am indebted to Sean Larsen, who found me at yet another intersection along a winding academic road and adopted me as his student. Sean, thank you for challenging my thinking, encouraging my potential, and understanding my persistence like no other individual. Second, I need to thank Mike Shaughnessy, who welcomed me along this academic road so many years ago. It was truly an honor to complete five courses with you. I would also like to express my thanks to Karen Marrongelle, who was there almost from the very beginning and saw it through to the end. Your support, along with Sean's, was instrumental in the completion of my dissertation. Thank you to my other committee members, and in particular, John Caughman, who was a continual source of encouragement these past eight years.

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Introduction

The TAAFU Project

The *Teaching Abstract Algebra for Understanding* (TAAFU) project was a collaboration involving Portland State University and Clackamas Community College that centered upon the implementation of an innovative research-based abstract algebra curriculum. The *TAAFU* project had three primary objectives: 1) to develop a set of instructor materials to support the successful implementation of the curriculum, 2) to research the challenges and opportunities that emerge as different faculty implement the curriculum, and 3) to gain insight into how the curriculum materials can enhance students' learning of abstract algebra. Several research strands emerged from the project, and in 2013, the *Journal of Mathematical Behavior* published a special issue (32 (4)) on the *TAAFU* project featuring the work of Sean Larsen and his colleagues (Larsen, Johnson & Bartlo, 2013; Larsen and Lockwood, 2013; Johnson, 2013; Johnson, Caughman, Fredericks, & Gibson, 2013; and Lockwood, Johnson, & Larsen, 2013). In the introduction to the issue, Weber (2013) called attention to two areas in which the *TAAFU* research program has made influential contributions: 1) research on the teaching and learning of abstract algebra and 2) research in scaling up design research to classroom settings.

The TAAFU Research Program

The *TAAFU* research program can be traced back to Larsen's (2004) development of an local instructional theory (LIT) for reinventing group and isomorphism, which made immediate contributions in the areas of teaching and learning abstract algebra. Gravemeijer (2004) used the phrase *local instructional theory* (LIT) "to refer to the

description of and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic” (p. 107). The development of the initial LITs is notable in that they utilized the instructional theory of Realistic Mathematics Education (RME) to support the learning of university-level mathematics. Perhaps most importantly, the continued evolution and implementation of these LITs serve as a model of how design research can evolve into an expansive research program. After several revision cycles, Larsen (2013) refined the LITs for both group and isomorphism and with the addition of an LIT for reinventing quotient group (Larsen & Lockwood, 2013), scaled these instructional units for classroom use. Together, these three LITs anchor the *TAAFU* curriculum, which supports an inquiry-oriented approach to teaching abstract algebra. Following a series of classroom implementations using the *TAAFU* curriculum, other researchers have made significant contributions to the field, which include:

1) investigating the challenges and opportunities that arise when mathematics professors implement inquiry curriculum (Johnson, 2013), 2) scaling up design research for classroom use (Bartlo, Larsen, & Lockwood, 2008), and 3) creating a concept inventory for group theory (Melhuish, 2015).

My Research Agenda

As a co-principal investigator for the *TAAFU* project, my primary research interest was to investigate the challenges and opportunities that arose from implementing a portion of the *TAAFU* curriculum in an experimental mathematics “bridge” (Selden & Selden, 1995) course that was offered at Clackamas Community College. The research reported here differs from the previous *TAAFU* literature in two ways. First, this work provides an existence proof that an exemplary LIT can be tailored for a situation that the

original instructional designer did not intend. Like the university abstract algebra students for which this curriculum was designed, the community college students in this research study successfully reinvented group. In fact, much of their activity was consistent with other implementations of the *TAAFU* curriculum. On the other hand, the pacing and goals of the community college course were different from those of an introductory group theory course taught at a university and afforded unique learning opportunities for these two-year college students. Second, this work elaborates on the mathematical activities of *conventionalizing* (negotiating a set of symbols and representations and establishing procedures for interacting with them) and *descriptive axiomatizing* (formulating and selecting rules in an effort to capture essential properties for a given structure) that supported collective reinvention. In the case of conventionalizing, I explore the role of past experiences and how they can generate ideas and influence decisions regarding how to organize a new mathematical domain. In the case of axiomatizing, I provide a detailed account and discussion of this collective activity and offer a revision to De Villiers's (1986) model of descriptive axiomatizing.

Contents of the Dissertation Papers

The first paper, *Rethinking mathematics bridge courses—An inquiry model for community colleges*, introduces the activities of conventionalizing and axiomatizing from a practitioner perspective. In the paper, I offer a curricular model that includes both inquiry and traditional instruction for two-year college students interested in mathematics. In particular, I provide both examples and rationales of tasks from the research-based *TAAFU* curriculum, which anchors the inquiry-oriented version of the mathematics bridge course.

The second paper, *The role of past experience in creating a shared representation system for a mathematical operation: A case of conventionalizing*, adds to the existing literature on mathematizing (Freudenthal, 1973) by “zooming in” on the early stages of the classroom enactment of the LIT (Larsen, 2013) for reinventing the concept of group. In three case study episodes, I shed light onto “How might conventionalizing unfold in a mathematics classroom?” by reporting on students’ establishment of conventions as they explored a new mathematical domain. I also offer a theoretical argument involving both Tall’s (2004) construct of met-before and Gentner & Markman’s (1997) similarity comparisons that were useful in describing how past experiences influenced the class’s mathematical activity. In addition, I utilized Fishbein’s (1999) *affirmative intuition* to propose a framework that relates students’ establishment of conventions in light of their past mathematical experiences.

The third paper, *Collective axiomatizing as a classroom activity*, is a detailed case study (Yin, 2009) that examines how students collectively engage in axiomatizing. In the paper, I offer a revision to De Villiers’s (1986) model of descriptive axiomatizing. The results of this study emphasize the additions of pre-axiomatic activity and axiomatic creation to the model and elaborate the processes of axiomatic formulation and analysis within the classroom community.

The TAAFU Curriculum

The *TAAFU* curriculum, which was featured in the project, is a research-based, inquiry-oriented abstract algebra curriculum that actively engages students in developing many fundamental concepts of group theory (Johnson, 2013). The *TAAFU* curriculum consists of three units (Groups and Subgroups, Isomorphisms, and Quotient Groups) and

was designed to be implemented in an undergraduate abstract algebra course. Each unit begins with a situational activity that serves as an experientially real starting point for a reinvention phase. It is during this reinvention phase that students build upon their informal understandings and prior knowledge to develop concepts. These concepts are then continually refined to produce formal definitions and/or an axiomatic system accompanied by a set of conjectures. Supporting this reinvention phase is also a deductive phase, in which students engage in proving theorems that often originate as conjectures. In many cases, proving these conjectures from the formal definitions or axioms encourages the students to engage in further reinvention and the cycle continues (Johnson, 2013). While the three units comprise the curriculum, only the first one—Groups and Subgroups—is relevant to my research. In fact, the majority of the data corpus for my dissertation is centered around the *reinvention phase* of this unit, which I will now describe.

Groups and Subgroups. The curriculum begins with Larsen’s (2013) local instructional theory (LIT) for reinventing the group concept, which consists of the following six steps:

Step 1: Identifying, describing, and symbolizing the symmetries of a specific geometric figure (in this case, an equilateral triangle).

Step 2: Combining pairs of symmetries.

Step 3: Developing a calculus for computing combinations of symmetries.

Step 4: Axiomatizing the set of rules for computing combinations of symmetries.

Step 5: Using the axioms as a model-for reasoning about other contexts.

Step 6: Formulating a definition of group.

The first step of the LIT involves identifying, describing, and symbolizing all of the symmetries of an equilateral triangle. During this step, students often flip and rotate plastic triangles, recording these actions with both verbal and symbolic descriptions. After the students have identified six non-equivalent symmetries, they are asked to represent them in terms of a 120° clockwise rotation, R , and a flip across the vertical axis, F . The students must also develop a whole-class convention—additive or multiplicative—for operating on these symmetries. For example, the symmetry that is equivalent to a 240° clockwise rotation, could be represented as $R+R = 2R$ or $RR = R^2$, or even using the symbols $-R$ or R^{-1} .

Once the class agrees on symbols and a convention, the students are then asked to consider all combinations of any two symmetries and determine which symmetry is equivalent to it. Students typically take a variety of approaches when engaging in this task, such as making a series of calculations, an organized list, or an operation table. This task eventually leads back to the discussion of an operation table and to developing shortcuts or properties that assist the students in calculating the combinations of symmetries. As the students complete the operation table, they are encouraged to formulate a list of rules (including the group axioms) that were helpful for calculating all 36 combinations. These rules are then shared with the class and then they are systematically winnowed into an axiomatic system that serves as a *model of* combining the symmetries of an equilateral triangle.

After the students formulate a set of class rules they deem sufficient for completing their operation table, they are then asked to determine and subsequently

eliminate via deduction, any rules that are redundant. Once a minimal list of axioms has been selected, the students then complete the operation table once more using only the minimal list of rules. Rather than having each student recompute 36 calculations, students are typically assigned a few calculations to prove using the minimal list of rules and then collectively, the class verifies that the entire table can indeed be completed using only these rules.

At this point in the enactment of the LIT, the students typically have yet to formulate the closure and inverse axioms. Although it is not listed as a rule, the closure property is often noted in the process of completing the operation table or when proving the *Sudoku property*, which states that each symmetry appears exactly once in every row and every column of the table. While inverses may or may not have been noticed, the students do not need to use this property in axiomatic form for completing the operation table. Therefore, to necessitate the formulation of these remaining axioms, the students are challenged to prove the Sudoku property using only their existing rules. While 36 individual calculations can be made to prove this property holds, a move toward a more general equation-style argument is often made during implementation. By proving that the equation $AX = B$ (or $XA = B$) has a unique solution, it follows that each symmetry appears exactly once in every row (or in every column). When verifying the algebraic steps that are used to solve the equation $AX = B$, students realize that the inverse and closure axioms must be part of their list of rules.

After the students investigate other systems (e.g., rotational symmetries of a square, symmetries of a non-square rectangle, number systems under addition, etc.) to develop similar rules, they are asked to modify and state additional rules for these

systems. For instance, some properties, such as commutativity and infinite order are gained, while other properties, such as the dihedral definition, no longer apply. The students then examine the intersecting properties of these structures and develop a definition of group from the common relations of these systems.

The reinvention phase of the group theory unit culminates with this definition of group and a collection of conjectures, typically including the uniqueness of the identity element and the cancellation law. The second phase of the group theory unit emphasizes deduction and working with the formal definition of group. The students prove a number of results including those that are often conjectured during the previous phase, such as the cancellation law. They also verify whether certain structures satisfy the group axioms and prove characterization theorems, such as those that would establish sufficient rules to show that a structure is a subgroup (Larsen & Zandieh, 2007).

The TAAFU Data

Background and overview. I became involved in the *TAAFU* project when the instructional support materials (ISM) were just being put together for John Caughman, a mathematician who intended to use them in his abstract algebra course. In conjunction with John's first implementation of the materials, feedback about the curriculum was given to Larsen's curriculum design team, which included Larsen, John Caughman, two Ph.D. students, an undergraduate teaching assistant, and myself. The ISM was revised based upon this feedback and then the scaling-up portion of the project commenced.

During the next two years, seven different instructors used some portion of the TAAFU curriculum in a whole-class setting. At the 4-year level, there were three mathematicians who taught at Portland State University (John, Joyce, and Julie), a

Portland State mathematics education doctoral student (Estrella), and a mathematician from a different institution, Lee. All of these instructors used the curriculum to replace and in one case, supplement, the usual curriculum in an undergraduate abstract algebra course. In addition, two community college mathematics instructors (myself and Brian) used the group theory unit in a mathematics bridge course that was primarily designed to expose students to proof and university mathematics in an active and supportive environment.

In the spring of 2009, classroom video data was collected from all regular class sessions in my community college bridge course. Two cameras were used in the room. One was in a fixed location focused on one group of students, while the other was at the back of the room and operated by the undergraduate teaching assistant. In addition, I participated in weekly debriefing and planning sessions at Portland State University that usually began with me and my teaching assistant recounting how the lessons played out in class, identifying aspects of the curriculum or the curriculum support materials that either went really well or that needed to be modified for the bridge course, and discussing what I could expect in the coming lessons. In the spring of 2010, the curriculum was implemented again at Clackamas Community College and my colleague, Brian, taught the class. In addition to myself, who acted as both a mentor teacher and researcher, Estrella supervised data collection and assisted me in conducting the debriefing meetings, which were held on-site.

Relevant whole-class data. Of the massive amount of data that was collected throughout the lifespan of the TAAFU project, I examined a large portion of the whole-class video data taken from the Groups and Subgroups unit that was enacted in the

community college bridge course. The data corpus for these papers was collected during the spring quarter of 2010 at the community college. My colleague, Brian, taught the class for the first time and I acted as both a mentor teacher and researcher throughout the term. Estrella supervised data collection and assisted me in conducting weekly debriefing meetings, which were held on-site. Each one-hour and twenty-minute class session was videotaped using two cameras. One camera was in a fixed location and focused on one group of students, which varied from day to day. The other camera was at the back of the room and was operated by the doctoral student, tracking Brian and whole-class discussions. Seventeen class sessions were filmed and I initially analyzed days six and seven for this study. It is during these class periods that the students were negotiating their representational system and were just beginning to axiomatize, or create rules that described relations within the system they were exploring. Following the initial analysis the study was expanded to data taken from days 6-15.

Structure of the Remainder of the Dissertation

The three papers that comprise the dissertation will each focus on one topic. The first paper is a research-to-practice article divided into three parts. The first part of the paper includes the introduction and describes a mathematics bridge course that runs annually at Clackamas Community College. The second part of the paper introduces an research-based curriculum for teaching abstract algebra (Larsen, 2013) and details two activities (conventionalizing and axiomatizing) that are integral to inquiry-oriented version of the bridge course. The final part of the paper provides some recommendations to community college mathematics departments who may be interested in offering a bridge course for their students.

The second paper presents three case study episodes that describe how conventionalizing might unfold in a mathematics classroom. Each of these episodes includes an introduction, a description, a fine-grain analysis, and a discussion section. The narratives describe how a group of students created a set of shared symbols and negotiated conventions for combining the symmetries of an equilateral triangle and organizing an operation table. In the findings section, the roles of past experiences in conventionalizing are addressed using Tall's (2008) construct of *met-before* and Gentner & Markman's *similarity comparisons* (1997). In the conclusion, I synthesize the findings and offer a framework that connects Fischbein's (1999) work on intuition with these other two constructs to explain the role of prior mathematical experiences in conventionalizing.

The third paper examines the activity of collective axiomatizing as it occurred in a classroom setting. This descriptive case study contains an introduction, a literature review, my research questions, a methods section, which are followed by a case study report on the activity of collective axiomatizing. In the findings section, I elaborate on the process of axiomatization through a discussion of pre-axiomatizing activities, axiomatic creation, and axiomatic analysis. In the conclusion, I offer a revision to DeVilliers's (1986) model of descriptive axiomatizing and coordinate some students' intellectual needs (Harel, 2013) with their axiomatizing. I have also included a 5-page sample of the Axiomatizing Analytic Document (AAD) as an appendix. This document was essential in identifying and initially interpreting the axiomatic changes that took place in the public space, and played a key role in analyzing collective axiomatizing.

Paper 1: Rethinking mathematics bridge courses: An inquiry-oriented model for community colleges

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Abstract

For more than a decade, Clackamas Community College has been offering an elective mathematics bridge course to its STEM students. A key feature of the course is its flexible curriculum, which has led to more than one option for introducing students to abstract mathematics. In addition to offering a traditional survey of advanced mathematics, we have also taught several versions of the course using an inquiry-oriented curriculum for teaching abstract algebra (Larsen, 2013). Like the university abstract algebra students for which this curriculum was designed, our students reinvent the concept of group and are introduced to abstract mathematics through their own activity. One of these activities is conventionalizing (Yannotta, 2016a), which involves negotiating a set of symbols and representations and establishing procedures for interacting with them. A second activity is descriptive axiomatizing (De Villiers, 1986; Yannotta, 2016b), which involves formulating and selecting rules in an effort to capture essential properties for a given structure. When enacting this curriculum, these activities work in tandem as students create and refine a list of axioms that describe the symmetries of an equilateral triangle. At the conclusion of the inquiry course, our STEM students have the experience of building and working with axiomatic systems, which provides a

foundation for more abstract mathematics coursework. Focusing on the broader role that mathematics can play in STEM-transfer, this research-to-practice paper discusses the history of our bridge course and describes the activities of conventionalizing and axiomatizing in the context of mathematical inquiry. I conclude the paper by offering some recommendations to community college mathematics departments that are interested in running a sustainable bridge course at their institution.

Key Words: Mathematics Bridge Course, Axiomatizing, Conventionalizing, Inquiry-Oriented Curriculum, Community College, Abstract Algebra

Introduction

Community colleges continue to play an increasingly significant role in undergraduate Science, Technology, Engineering and Mathematics (STEM) education and represent a largely untapped pool of students for these fields (Barker, Bressoud, Epp, Ganter, Haver, & Pollatsek, 2004). Mathematics is a common thread for all STEM fields, and thus mathematics departments regularly play the role of servicing the needs of other departments throughout the first two years of college. For example, Bressoud, Mesa, and Rasmussen (2015) found that almost 40% of students who take calculus I at a community college major in engineering or computer science. In addition, two-year engineering and computer science programs typically offer their own introductory courses to help orient these students to their respective fields, yet there is no universal analog for the discipline of mathematics. In fact, it is not until math students actually transfer to four-year institutions that the curricular emphasis shifts from a

“procedural/computational understanding of mathematics to a broad understanding encompassing logical reasoning, generalizing, abstraction, and formal proof” (Barker, Bressoud, Epp, Ganter, Haver, & Pollatsek, 2004, pg. 44). Even though the percentage of mathematics majors is quite small compared to those majoring in other STEM disciplines, community college math departments have the responsibility to prepare mathematics majors and minors for more advanced coursework. In that endeavor, they also have an opportunity to support the mathematical development of a larger group of STEM students who may be interested in mathematics beyond the calculus sequence. As two-year colleges consider new directions for STEM education, one of the ways that mathematics departments can contribute is to offer a mathematics bridge course designed for STEM-transfer students.

What is a Bridge Course?

The terms “bridge course”, “transition course”, and “transition-to-proof course” have been used interchangeably in mathematics education literature (Selden & Selden, 2008). Approximately 40% of colleges and universities in the United States offer a dedicated bridge course and more than 80% of those institutions require their mathematics majors to take it (Exner, 2007). Absent from the research on bridge courses are community colleges, but one could argue that other mathematics courses such as linear algebra or differential equations provide two-year colleges opportunities to introduce proof and abstraction to students. On the other hand, a majority of bridge courses are typically designed for a primary purpose: to teach collegiate students how to construct proofs (Selden & Selden, 2008). Smith, Nichols, Yoo, and Oehler (2009) provide a description of what a stereotypical transition-to-proof course might look like: a

university classroom, with a professor at the front of the room teaching various proof techniques to a group of mathematics majors who carefully take notes in order to replicate the presented proofs on a formal assessment. In stark contrast to this stereotype, I describe a very different type of bridge course designed to support a more general STEM population at community colleges and share some research-based mathematical activities that can be used in such a course.

A Brief History of Our Community College Bridge Course

One of the first points that should be made about our bridge course is that it did not originate from an external grant or from an administrative directive. Rather, it started like many initiatives at community colleges—as a faculty response to students’ needs. Toward the end of the Winter quarter in 2005, a group of students approached the chair of our math department and said they wanted to continue studying math, but were unsure of a direction to pursue. A majority of these students were math lab tutors who had taken the entire sequence of math offerings at our college and were transferring at the end of spring term. In response to their request, the chair asked another faculty member and myself if we would each teach a three-week unit that introduced advanced mathematics to these ten students. We agreed and that spring we ran an independent study course entitled Math 299: A Survey of Advanced Mathematics. Divided into three short courses, my colleague taught a unit on proof, the chair taught a unit of point-set topology, and I taught a group theory unit. At the time there was no guarantee that the course would be offered beyond that term, but both the students and the faculty enjoyed the course immensely.

The next year when I was compiling a literature review on students' difficulties with proof, I came across an article by Selden & Selden (1995), who described the function of bridge courses more generally as “designed to ease the transition from lower division, more computational, mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus” (p. 135). Inspired by this broader notion of a bridge course, I began to explore other directions for a sustainable course that could be offered at Clackamas Community College. As community college teachers, we often find ourselves re-teaching mathematical content that students have encountered in previous courses (Sitomer, Ström, Mesa, Duranczyk, Nabb, Smith, & Yannotta, 2012), but at the same time, we have the responsibility to prepare students to be successful in their next math course. Therefore, I wondered if this familiar community college model of developmental preparation could be adapted to better support STEM transfer. Instead of assuming the traditional instructional role of remediation, the focus of the bridge course would be to deliberately teach content that students would encounter in future mathematics courses. In addition, I wanted to provide a low-risk, supportive environment for these students and thus, I began using the term *pre-mediation* to describe this model of instruction.

For pre-mediation to be successful, one must first be familiar with the courses and mathematical content that students will likely experience in the future. Therefore, I began investigating the program requirements for a minor in mathematics at Portland State University—our largest transfer partner. With the exception of four quarters of calculus (16 credits) and linear algebra (4 credits), the only other required course was either advanced calculus or group theory (4 credits). The remaining 9 – 12 credits to

complete the minor could be satisfied with differential equations (4 credits) and any approved electives for the mathematics major. In short, a STEM student could take the entire calculus sequence, linear algebra, and differential equations at our college and then take one more required class and two approved electives to complete a math minor at Portland State. Instead of a survey course or one that focused on proof construction, I thought the students might be better served by explicitly teaching mathematical content covered in more advanced coursework in one of the two required courses. As I had a stronger background in abstract algebra from my graduate studies, I decided to use pre-mediation in the context of group theory.

Using a University Inquiry-Oriented Curriculum for Teaching Abstract Algebra

“A lot of it was exploratory and that I think served a big role to empower me and empower a lot of students around me that we can invent this stuff. We can figure this stuff out. This stuff is here and we can explore it and if we explore it in a sensible way, we will make the same discoveries that everyone else has.”

–Todd, a student who was asked about the activity in the course

After experimenting with different course formats and content for a few terms, I adopted an inquiry-oriented curriculum that was used in many of the group theory classes at Portland State University. The *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum is research-based and divided into three instructional units—groups, isomorphism, and quotient groups, which are all enacted in the 300-level abstract algebra course. Our bridge course at Clackamas Community College was built around only the first unit, which typically spans the first three and a half weeks of the 10-week university-level group theory course. This first unit of the *TAAFU* curriculum supports

the enactment of Larsen’s (2013) local instructional theory (LIT) for reinventing the concept of group. In contrast to the university course, the LIT is enacted at a slower pace (usually 5-6 weeks) in our community college bridge course, which still leaves time for other types of instructional activities (see Figure 1).

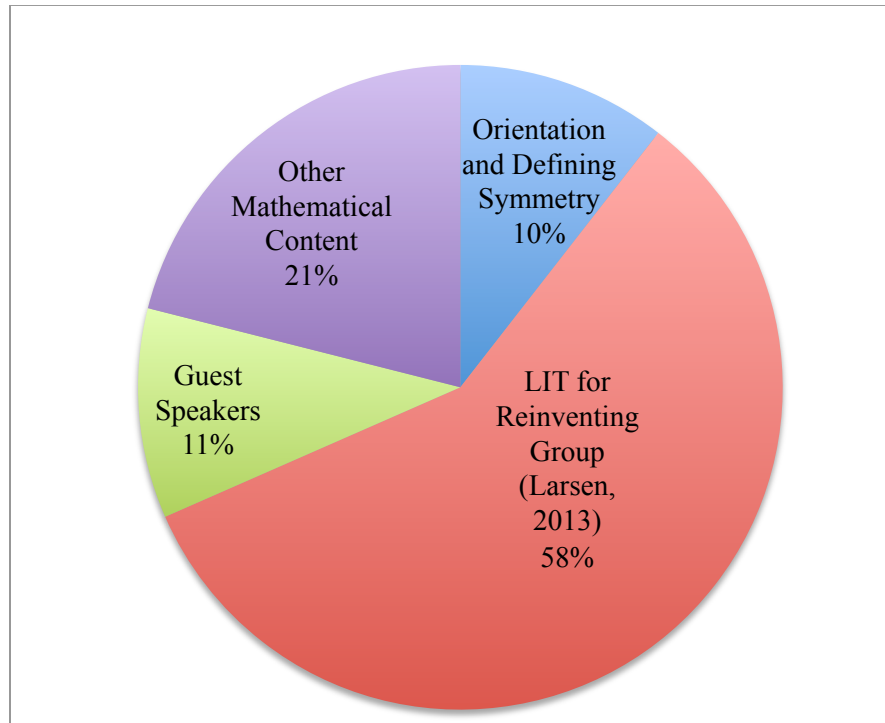


Figure 1: Total instructional time within the bridge course

Larsen’s Local Instructional Theory for Reinventing Group

Gravemeijer (2004) provides a nice description of a local instructional theory (LIT) using a travel metaphor, “... the local instructional theory offers a ‘travel plan’, which the teacher has to transpose into an actual “journey” with his or her students” (p. 107). Larsen’s (2013) LIT describes how the concept of group could be taught in ways that are consistent with some of the design principles of Realistic Mathematics

Education. One of these core principles is guided reinvention, in which one will “invent something that is new to him, but well-known to the guide” (Freudenthal, 1991, p. 48).

The students who take our bridge course engage in many mathematical activities that include symbolizing, conventionalizing, proving and axiomatizing as they enact the steps in the LIT (Table 1) for reinventing the concept of a mathematical group.

Table 1

Steps in the LIT for reinventing the concept of group (Larsen, 2013)

Step number	Activity
Step 1	<i>Identifying, describing, and symbolizing the symmetries of a specific geometric figure</i> (in this case, an equilateral triangle)
Step 2	<i>Combining pairs of symmetries</i>
Step 3	<i>Developing a calculus for computing combinations of symmetries</i>
Step 4	<i>Axiomatizing the set of rules for computing combinations of symmetries</i>
Step 5	<i>Using the axioms as a model-for reasoning about other contexts</i>
Step 6	<i>Formulating a definition of group</i>

In the next section, I elaborate on two of these student activities—conventionalizing and axiomatizing, which were central to the enactment of steps 2, 3 and 4 in the LIT and to my research.

Some Insights From Our Implementation of Larsen’s LIT

If nothing else, the mental discipline of uh, how to organize your thoughts mathematically and just, some different ways of looking at, you know the basics—things that you take for granted, that are the foundation of all of it in arithmetic and in algebra.

– Gene, a student who was asked what he took away from the experience

Conventionalizing

Following an orientation activity that helps establish a community of inquiry, our students began exploring the symmetries of an equilateral triangle. Students often flip and rotate plastic triangles, recording these actions with both verbal and symbolic descriptions. After the students have identified all six non-equivalent symmetries, they are asked to represent them in terms of a 120° clockwise rotation, R , and a flip across the vertical axis, F . The focus then shifts to more collective activity, as the students negotiate a whole-class convention—typically additive or multiplicative—for operating on these symmetries. For example, the symmetry that is equivalent to a 240° clockwise rotation (Figure 2), could be represented as $R + R = 2R$ or $RR = R^2$, or even using the symbols $-R$ or R^{-1} or some other variant. It is during this time, that students engage in conventionalizing (Yannotta, 2016b), which includes developing standard symbols, representations, and procedures.

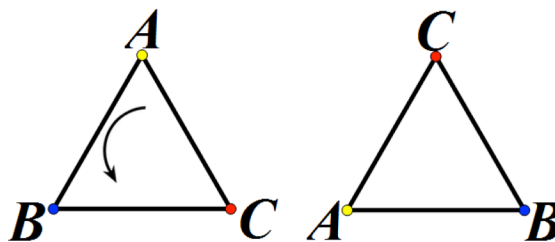


Figure 2: A 240° clockwise rotation of an equilateral triangle

Through discussions of adopting either an additive or multiplicative convention in this new context, students have opportunities to challenge their prior experiences with real numbers and gain some insight into the consequences of the notation itself. For instance, if additive notation is selected to represent the operation of combining

(composing) symmetries, then three rotations would be represented as $R + R + R$ or $3R$. Additive notation may resonate with students because adding the angle of rotation has a high degree of similarity both in its symbolic appearance and relationally to the addition of real numbers. However, as the operation of combining symmetries is non-commutative, comparing the expressions $2F + 2R$ and $2(F + R)$ can present a great learning opportunity for students. For example, students may act on what McGowan and Tall (2010) referred to as a met-before, or “a mental structure that we have *now* as a result of experiences we have met before” (p. 171) and attempt to transform one of these expressions into the other. In this case, a student may try to factor the 2 out of the expression $2F + 2R$ or distribute the factor in $2(F + R)$ to produce what appears to be an equivalent expression. Despite having the knowledge that the variables F and R do not represent real numbers and there is only one non-commutative operation in this new context, some students will conclude that $2F + 2R = 2(F + R)$ and may be validating this equation symbolically using conventions that apply to real numbers.

While initially this may seem problematic, it can create a great learning opportunity for exploring the meaning of iterating a non-commutative operation. One way this can occur is to make comparisons of $2F + 2R = F + F + R + R$ (Figure 3) and $2(F + R) = F + R + F + R$ (Figure 4), both symbolically and using triangles. For example, the symmetry $2F + 2R$ (two flips followed by two rotations) is equivalent to a 240° clockwise-rotation, which is represented as $2R$ in an additive convention. On the other hand, the symmetry $2(F + R)$ (two consecutive sequences of a flip followed by a rotation) is equivalent to the identity symmetry, I .

$$2F + 2R = F + F + R + R = 2R$$

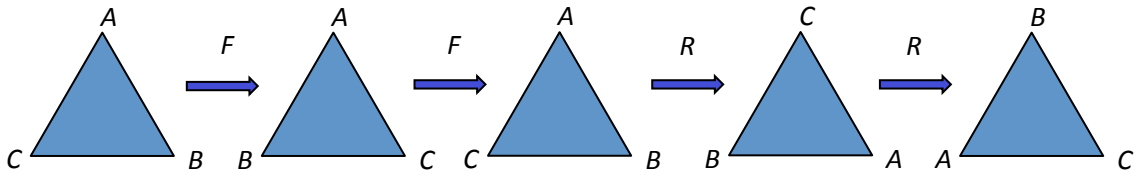


Figure 3: Two flips followed by two rotations

$$2(F + R) = F + R + F + R = I$$

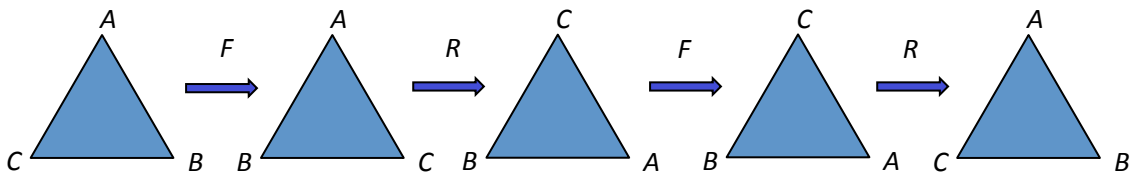


Figure 4: Two consecutive sequences of a flip followed by a rotation

Even if students opt to use multiplicative notation, they still run into the issue that $(FR)^2 \neq F^2R^2$, which provides opportunities to make connections with matrix multiplication and connections within other STEM disciplines. For instance, in our most recent implementation of the curriculum, a chemistry major suggested using multiplicative juxtaposition because it was more efficient and at the same time, the subscripts highlighted that the operation was somehow different from real-number multiplication. Thus, this class represented two consecutive sequences of a flip followed by a rotation as $(FR)_2$ and used this subscript convention until the definition of an abstract group was formulated. It is important to note that, no matter which choices the students make and use throughout much of the curriculum, they are eventually introduced to standard conventions that are recognized within the mathematics community. In

addition, the transition to adopting these normative mathematical conventions is typically smooth for students.

In traditional courses, mathematical notation is often presented authoritatively, and students do not get hands-on experience discerning the advantages and arbitrary features of conventions. In advanced mathematics, additive and multiplicative conventions are regularly used to represent binary operations that do not function in the same ways that they do for real numbers. Thus, something that looks very familiar, like the distributive property, may not be true or may not even apply in a new setting. While it is too impractical to have students reinvent mathematical conventions in every course, students should have this opportunity at least once in their first two years of collegiate mathematics. The previous conventionalizing episode provides an example of how students can explore the concept of iteration and the implications of working with an operation that is non-commutative.

Axiomatizing

I think it's given me a more um, I don't know what word you would use, like a systematic way of looking at things. It's not so much of how to compute something, but of how it works. To kind of sit back and kind of look at the patterns and the relationships that I'm dealing with... maybe more than I would have before. Before I was just been, I'll just compute this, do that. Um, crank the algebra, this is more like, how does it work.

– Kyra, a student who was asked about her experience taking the course

Once the class agrees on a set of symbols and a convention, the students are then asked to consider all combinations of any two symmetries and determine which symmetry is equivalent to it. This task eventually leads back to the production of an

operation table and to developing shortcuts or properties that assist the students in calculating the combinations of symmetries. As the students complete the operation table, they are encouraged to formulate a list of rules (including the group axioms) that were helpful for calculating all 36 combinations. For example, the property that three consecutive 120-degree rotations or two consecutive flips essentially did nothing might be represented by the rules $R^3 = I$ or $F^2 = I$ respectively. These rules are then shared within the classroom in an effort to create a comprehensive list of rules that could be used to complete the entire operation table (Figure 5). This initial phase of axiomatic creation is part of a larger activity of axiomatizing, which is a search for structure. I adapt the term *descriptive axiomatizing* from De Villiers (1986) to describe this collective activity, which includes both the formulation and selection of a subset of essential properties for describing the symmetries of an equilateral triangle.

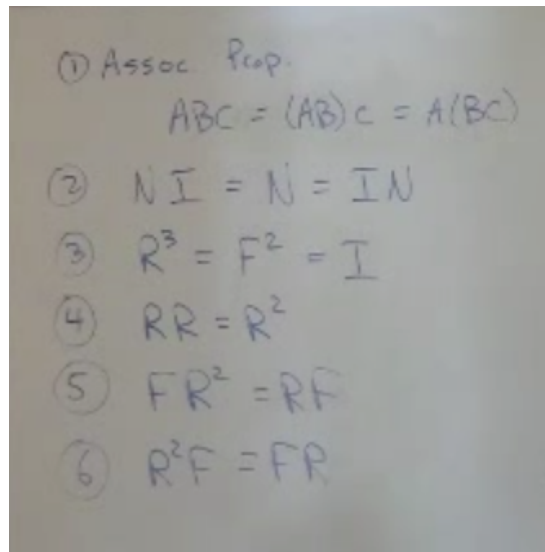


Figure 5: Axioms v. 1.0 (Initial version for the symmetries of an equilateral triangle)

Axiomatizing then progresses to an analysis phase as the students are asked whether all of the rules were necessary or whether some of them could be deduced from

other rules. Students typically put forth different arguments about which rules to keep out of necessity and in the case of equivalent rules, which version to keep, using deduction to create a minimal list. This list of axioms continues to be refined as the students progress through a series of systematization phases (Yannotta, 2016a) over the course of several days. At the conclusion of this instructional step, the students axiomatically describe D_6 —the dihedral group of the equilateral triangle (Figure 6).

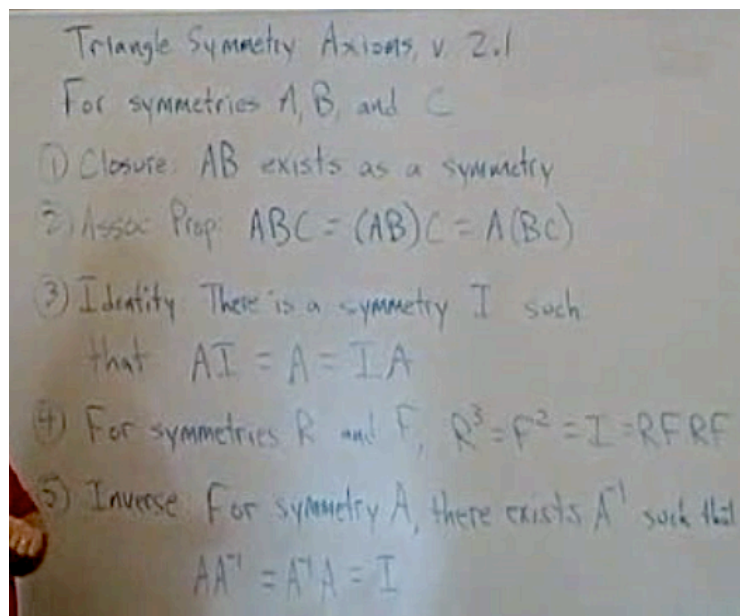


Figure 6: Axioms v. 2.1 (Final version for the symmetries of an equilateral triangle)

One of the most engaging sequences in the *TAAFU* group theory curriculum occurs when the class begins to create a collective list of rules for combining the symmetries of the equilateral triangle. To facilitate axiomatic creation, the instructor will ask students to construct an operation table for combining any two symmetries and to record any short cuts or rules that they used in the process of filling in their tables. As seen in Figure 7, one student has listed several of these rules next to his operation table.

	I	R ²	R	F	FR	R F
I	I	R ²	R	F	FR	R F
R	R	I	R ²	RF	F	FR
R ²	R ²	R	I	FR	RF	F
F	F	RF	FR	I	R	R ²
FR	FR	F	RF	R ²	I	R
RF	RF	FR	F	R	R ²	I

$$FR^2 = RF$$

$$R^2F = FR$$

$$R^3 = I$$

$$F^2 = I$$

$$FRRF = (FR^2)F = RFF = R$$

$$RFFR = RF^2R = RIR = R^2$$

$$RFR^2 = R(FR^2) = RRF = R^2F = RF$$

Figure 7: Kevin's operation table and list of useful rules

To transition from this individual student activity to collective activity, an instructor may then ask a student to share this work, recording a few of these rules on a whiteboard. In this case, the instructor initially selected the first four rules ($FR^2 = RF$, $R^2F = FR$, $R^3 = I$, and $F^2 = I$) as a starting point for collective axiomatizing and then asked other students to share some of their rules. If the class needs to formulate more rules, an instructor could revisit a student's work and examine the rules that were used in a specific calculation. By identifying which rules were used in the calculation, other "stealth" relations such as $IR = R$ or $RI = R$ may emerge and evolve into a generalized identity rule like $NI = N = IN$, as seen in Figure 5. The instructor would then select an initial set of these rules, which typically includes the associative property, an identity rule, a definition for the identity, and multiple versions of the dihedral relation involving F and R , as an initial version of the class's axiomatic system.

Students in traditional proof-based courses are often introduced to formal definitions and axiomatic systems without ever having the opportunity to axiomatize. Furthermore, as Dawkins (in press) points out, "Simply 'telling' students the rules ('use the definition' or 'only using the axioms') does not help them understand why those rules

make sense or are important to doing mathematics” (p. 2). In contrast, one of the most compelling aspects of using the *TAAFU* group theory unit is that it provides multiple opportunities for students to reorganize their own mathematical activity and engage in axiomatizing. Even though I have taught with the *TAAFU* curriculum multiple times, it is always inspiring to guide a new group of students as they abstract properties from their own work with the triangle symmetries and build a mathematical system from the ground up. Although their choices and decisions may vary, each group of students reinvents the concept of group and is introduced to abstract mathematics through their own activity, thus preparing them for more advanced coursework.

Recommendations for Community College Math Departments That Seek to Offer a Mathematics Bridge Course

After more than ten years of experience running the course, I offer a description of a sustainable, research-informed mathematics bridge course model that is not only working for our students, but it is one that could be adopted by other two-year colleges throughout the country. The elective course, which serves mathematics majors, mathematics minors, and STEM students interested in mathematics, should be offered at least one quarter (or semester) per academic year. At colleges that operate on a quarter system, a three-credit course works quite well, meeting twice a week for an hour and twenty minutes. At semester schools, a one or two-credit model provides the flexibility of offering either an 8 or 12-week format or a full-term course. As proof is central to mathematics bridge courses, a minimum pre-requisite of a trigonometry course that includes verification of identities is recommended. To ensure long-term sustainability,

mathematics departments should have a minimum of two faculty committed to teaching the course either annually or every other year.

One of the biggest challenges in running an elective mathematics course is ensuring a critical mass of students to meet the course margin. At Clackamas, we have been fortunate to run this course with smaller numbers, but classes of 10 - 15 students should be targeted enrollment figures. We have found advertising with fliers, making a course announcement in class, and talking to individual students about the class, to be successful strategies for guiding interested students to take the course. Another solution that also helps ensure that the course runs is that we offer two different versions of the bridge course on alternating years. The students can take the same course twice for elective credit and get a completely different and complimentary experience. In one year, the course is designed to introduce abstract algebra through a student-centered approach. The inquiry-oriented *TAAFU* curriculum for reinventing the concept of group (Larsen, 2013) that was described in this paper anchors approximately two-thirds of the course. To establish classroom norms that support inquiry, we typically implement an orientation task that asks students to define symmetry. After the students reinvent group, prove some elementary group properties, and explore some familiar and less-familiar groups, the teacher typically directs the content with the remaining instructional time. In the past, some instructors have continued onto the next unit of the *TAAFU* curriculum (reinventing isomorphism), while others have explored solving equations over abstract groups, introduced other mathematical systems, or presented on topics such as infinity, fractals, or math history. In addition, we usually invite two guest speakers—one from within our mathematics department and one from a four-year institution—to give a math talk

sometime during the term. In the alternating year, the curriculum is broad-based and designed to be a more traditional survey of advanced mathematics. Like many transition-to-proof courses at 4-year institutions, the content typically covers topics such as sets, relations, point-set topology, introductory analysis, logic, and proof, and is taught in an interactive lecture-based format. Students may take both versions of the course in either order for repeated credit and it is intended that together, these two experiences will expose students to the breadth and depth of future mathematics courses.

In either format, the course should serve as a bridge to advanced mathematics, so the official course description should reflect that. The costs for the course should be kept as low as possible. For instance, at Clackamas Community College we do not charge additional math course fees nor do we require a textbook. In contrast to required courses, it is recommended that a high percentage of the course grade be based upon a few homework problem sets and regular class participation. In our course, a standard scale of A, B, and P (pass) is used and there are no in-class examinations. Interested faculty should be encouraged to substitute teach and/or guest lecture in the course. In addition to sharing their expertise and interests with students, their involvement further ensures the sustainability of the course.

Conclusion

It definitely made it [mathematics] a lot more interesting... It's really the main reason I've thought about doing a minor in mathematics because it really, made it interesting. It was a really interesting class. Proofs, the proofs we did, really, it really showed me how to develop rules, axioms, and properties for, for certain, for math, really for math in general. Yeah, it was cool.

–Randall, a student who was asked whether the class had changed his view of mathematics

Community colleges continue to play a larger role in STEM education, providing new opportunities for us to serve this population of students. One way that community college mathematics departments can contribute to this endeavor is to offer an elective mathematics bridge course once a year to STEM students. As an alternative to a traditional transition-to-proof course geared specifically toward mathematics majors, we have found that implementing an inquiry-oriented curriculum for teaching group theory to be an effective way to introduce STEM students to abstraction. The *TAAFU* curriculum that we use is designed to support students' reinvention of the concept of group and has been used successfully with university abstract algebra students, high school teachers, and our STEM students. Through the enactment of this curriculum, students engage in the activity of conventionalizing, as they negotiate a class-wide set of symbols and a mathematical convention for operating with the symmetries of an equilateral triangle. More broadly though, conventionalizing is an organizing activity that addresses the larger question of, "How are we going to do things in this context?", which is applicable to all STEM fields. These students also engage in axiomatizing, which involves formulating and selecting rules in an effort to describe everything that is essential about their system of triangle symmetries. In addition to the experience of

building a mathematical system from scratch, these students are exposed to mathematical content in more advanced coursework and get hands-on experience with abstraction. By providing students opportunities to conventionalize and axiomatize, they learn new ways of thinking and operating, which is useful in all STEM disciplines. While there are many ways to support community college STEM students, running an elective bridge course is a low-cost, sustainable initiative that can be quite effective in serving a diverse group of students who are interested in mathematics beyond the calculus sequence.

Paper 2: The role of past experience in creating a shared representation system for a mathematical operation: A case of conventionalizing

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Abstract

This work examines some of the different roles that past experiences can play when students engage in the activity of conventionalizing. References to past experiences emerged when students acted on a met-before (Tall, 2008) and made similarity comparisons (Gentner & Markman, 1997), as the classroom community negotiated a set of symbols and representations and established procedures for combining (composing) the symmetries of an equilateral triangle. Both the students and the instructor made extensive use of previous knowledge with the additive and multiplicative conventions of the algebra of real numbers to conventionalize this new mathematical domain. The results from this study suggest that confronting a problematic met-before can lead to new conventionalizing knowledge, such as how to iterate a non-commutative operation. In addition, the use of similarity comparisons involving past experiences functioned to: 1) foster precedent thinking and 2) support or reject the adoption of a proposed convention.

Key Words: Conventionalizing, Met-Before, Similarity Comparison, Intuition,
Realistic Mathematics Education

Introduction

As students advance through their mathematics courses, prior experiences provide tools and knowledge that can be drawn upon, serving as referents that help them make sense of new mathematical ideas. In inquiry-oriented instruction, these referents often play out in the public space and therefore, have the further potential to influence the classroom discussion, collective decisions, and even how the curriculum itself unfolds.

One inquiry-oriented curriculum that has been studied by several researchers at the undergraduate level is the *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum (Larsen, 2013). Research associated with the *TAAFU* program has contributed to the areas of: design research (Larsen, 2004; 2013; Larsen & Lockwood, 2013), teaching (Johnson, 2013; Johnson, Caughman, Fredericks, & Gibson, 2013), and scaling up design research for implementation (Bartlo, Larsen, & Lockwood, 2008; Lockwood, Johnson, & Larsen, 2013). This study has a different take and describes a collective activity that is supported by the *TAAFU* curriculum. By “zooming in” on an enactment of this robust curriculum, I report the students’ mathematical activity of creating a shared representation system for a mathematical operation and articulate some ways that prior experiences may influence the collective organization of a new mathematical domain. To describe this activity more generally, I use the term *conventionalizing* to encompass what Gravemeijer, Cobb, Bowers, & Whitenack refer to as “symbolizing and schematizing (developing standard procedures and notations)” (2000, p. 236). This study contributes to the RME literature at the undergraduate level in two ways. First, it elaborates the mathematical activity of conventionalizing, which along with symbolizing (Gravemeijer, 2004) and axiomatizing (De Villiers, 1986;

Yannotta, 2016a), is a mathematical activity supported by the *TAAFU* curriculum.

Second, this paper proposes a theoretical model that links prior mathematical experiences to conventionalizing that may be helpful to design research.

Purpose of the Study

The research reported here sought to better understand how conventionalizing might unfold in a mathematics classroom. Prior to conventionalizing, the students in this study manipulated and explored the symmetries of an equilateral triangle both individually and collectively. Interestingly, the transition from this familiar and well-grounded starting point sometimes led to mathematical decisions that challenged their prior experiences. For example, in a spontaneous exchange, the students were asked to determine whether a mathematical relation that resembled the distributive property also held in this new situation. Another instance involved the class discussion and subsequent vote to use additive or multiplicative notation to represent the non-commutative operation of combining symmetries. Later, the class had to decide how to organize an operation table, which involved setting an order for arranging the rows and columns and establishing a procedural order to denote the entries. As the students mathematized this new content domain, they adapted established mathematical conventions and created novel ones to describe how the elements and the operation interacted in the system. In addition to providing rich descriptions of conventionalizing, the findings of these case study episodes suggest a framework linking past experiences to this mathematical activity that might be helpful for design research.

Theoretical Perspective

The *TAAFU* curriculum is grounded in the instructional design theory of Realistic Mathematics Education (RME), which places a strong emphasis on two key tenets: 1) mathematics is situated as a dynamic human activity and 2) mathematics must be connected to an experientially-real context. One of the instructional design principles of RME that supports these tenets is called *guided reinvention*, in which the individual is intended to “invent something that is new to him, but well-known to the guide” (Freudenthal, 1991, p. 48). Like other inquiry-oriented curricula that embrace guided reinvention, the *TAAFU* curriculum places learners in explorer-like roles, where they are given the opportunity to mathematize, and then guided to create and restructure mathematics. In this study, the central object of inquiry was the mathematical activity of conventionalizing, which involved negotiating standard procedures, notations, and conventions for the non-commutative operation of combining (composing) the symmetries of an equilateral triangle. This activity is featured in the early stages of the enactment of the group theory unit of the *TAAFU* curriculum. When examining the class’s conventionalizing, I situated my analysis within the Emergent Perspective (Cobb and Yackel, 1996), in which individual activity and collective activity are reflexively related. In this case, I sought to make sense of how prior mathematical experiences affected individual student contributions to collective conventionalizing and how they influenced collective decision-making. My analysis of these references to prior mathematical experiences was informed by Tall’s (2004) construct of *met-before*, Gentner and Markman’s (1997) *similarity comparisons*, and Fischbein’s (1999) work on *affirmatory intuition*.

Literature Review

Freudenthal (1991) described guided reinvention as a course of action in which an individual will “invent something that is new to him, but well-known to the guide” (p. 48). As an instructional design principal of Realistic Mathematics Education (RME), a primary goal for guided reinvention is for “learners to come to regard the knowledge they acquire as their own, personal knowledge, knowledge for which they themselves are responsible” (Gravemeijer & Terwel, 2000, p. 786). To achieve this goal, mathematics instruction is designed around activities that build and expand upon the learners’ experiences and reasoning (Johnson, 2013). In courses that use a curriculum that is designed to build on students’ prior understandings through their own activity, past experiences have the potential to greatly influence how the curriculum unfolds because references to these experiences often play out in the public space. In this study, I focus my analysis on prior mathematical experiences, with an emphasis on conventions.

Student contributions that contain referents to past experiences not only inform the discourse, but also shed some light onto how these individuals may have been thinking about the current situation. One construct that can be used to describe the role of past experience as students engage in mathematizing is a *met-before* (Tall, 2004). McGowen and Tall (2010) use the term *met-before* to describe “a mental structure that we have *now* as a result of experiences we have met before” (p. 171). In new contexts, met-befores provide resources that can influence decision-making, and many times this happens in ways that are unproblematic. As noted by Tall (2008), a met-before such as “adding two whole numbers produces another whole number” can be consistent with new situations as it continues to work for arithmetic across the integers, rational, real and

complex numbers. However, there are other times when met-befores are problematic as students engage in new situations. For instance, the met-befores “multiplication makes bigger” and “taking away makes less” apply to whole numbers, but are inconsistent with future situations involving rational and negative numbers (McGowen & Tall, 2010).

While an awareness of students’ met-befores could be useful in any learning environment, it is the special role these met-befores play in inquiry-oriented instruction that is highlighted in this study.

Met-befores are related to a class of intellectual cognitions that are self-evident, which Fischbein (1999) called intuitions. For example, a met-before, such as “multiplication makes bigger”, exemplifies some of the main characteristics of intuitive cognitions, which include both immediacy and intrinsic certainty. The coercive nature of what Fischbein called *affirmatory intuitions*, in which cognitions “appear to the individual to be directly acceptable, self-evident, global, and intrinsically necessary” (Fischbein, 1999, p. 33) often lead individuals to reject an alternate interpretation of a situation or statement. Zandieh, Larsen, and Nunley (2008) reported on the coercive and complex nature of intuitions within the context of proof. The researchers found that on one hand, intuitions provide ideas and a foundation for students to mathematize, but on the other, these same intuitions may inhibit one’s ability to question something that appears obvious.

A second way in which past experience can emerge as students engage in collective mathematizing is through a *similarity comparison*. As noted by Gentner (1989), “much of experiential learning proceeds through spontaneous comparisons—which may be implicit or explicit—between a current situation and prior similar or

analogous situations that the learner has stored in memory (p. 232).” Similarity comparisons involving past experiences can help students orient to a new content domain, but theorists argue that not all similarity comparisons are equivalent. Gentner and Markman (1997) categorize different types of similarity comparisons by identifying two dimensions of similarity—surface features (attributes) and structural features (relations)—arranging them on continuums to form a similarity space (see Figure 8).

At the far end along the horizontal axis are *mere-appearance matches*, in which two representations have a high degree of shared attributes, with few or no shared structural relations. Comparing a planet to a beach ball would be an example of a mere-appearance match in that they are both round objects. While toward the end of the vertical axis, lies *analogy*—the process of understanding a new situation in terms of one that is familiar through shared relations (Gentner & Holyoak, 1997).

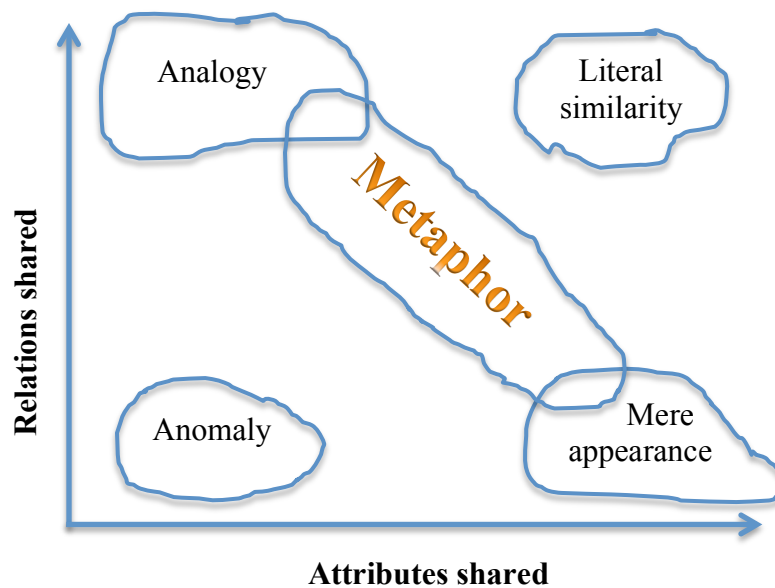


Figure 8: Similarity space showing different kinds of matches in terms of degree of relational versus attribute overlap adapted from Gentner & Markman (1997)

As noted by Gentner and Markman (1997), the dimensions of shared attributes and shared relations are not dichotomous. Thus, if an analogy also has a high degree of shared attributes, it approaches a *literal similarity*, while an *anomaly* would have a low degree of both shared attributes and shared relations. Consistent with the RME design heuristic of guided reinvention (Freudenthal, 1973), when a learner is placed in an explorer-like role, where he or she is given the opportunity to mathematize an unfamiliar domain, similarity comparisons provide foundational resources to create and restructure mathematics.

Methods

Context and Research Questions

The context of this study involves students creating symbols for six non-equivalent symmetries of an equilateral triangle as well as adopting a class-wide convention to represent the operation of combining (composing) symmetries. I describe the mathematical aspects that arise when:

1. students negotiate a shared set of symbols for expressing the elements and the operation.
2. students create rules that describe how the elements and the operation of the system interact.

It is important to note that these two activities are not mutually exclusive, nor are they necessarily sequential. The first aspect above addresses how symbols are used to describe the objects or what actions are done with them. The second addresses utility—specifically, the way the symbols function as a system. Each of these activities supports the other and together they highlight some of the considerations involved in mathematizing a new content domain. My research questions are:

Research Question 1: How might conventionalizing unfold in mathematics classroom?

Research Question 2: How do past experiences influence the conventionalizing of a new mathematical domain?

The first research question is designed to explore the activity of negotiating and adopting mathematical conventions in a classroom setting. The second research question developed from a preliminary analysis of the data, as it became apparent that students looked to their past to inform their decisions about how to conventionalize the new domain of the symmetries of the equilateral triangle.

Research Setting

The data for this research was taken from selected classroom episodes of a mathematics “bridge course” that was taught at a suburban community college in the United States. Selden & Selden (1995) describe a bridge course as being “designed to ease the transition from lower division, more computational, mathematics courses to upper division, more abstract, mathematics courses such as modern algebra and advanced calculus” (p. 135). Inspired by this broad notion of a bridge course and the tenet that students should engage in authentic mathematical activity (Freudenthal, 1973), this bridge course was designed for two-year college students and focused on introducing university-level mathematics through active participation. The content domain that was selected was abstract algebra and the course closely followed Larsen’s (2013) local instructional theory (LIT) for reinventing the group concept. The majority of the students’ conventionalizing activity took place during the enactment of the first portion of the LIT and therefore, only the relevant steps of the LIT are described in this paper.

Introduction to Larsen's (2013) LIT

Gravemeijer (2004) used the phrase *local instructional theory* (LIT) “to refer to the description of and rationale for, the envisioned learning route as it relates to a set of instructional activities for a specific topic” (p. 107). Larsen's (2013) LIT for reinventing group begins with an exploration of the symmetries of an equilateral triangle. During the enactment of this initial step, students often flip and rotate plastic triangles, recording these actions with both verbal and symbolic descriptions. After the students have identified six non-equivalent symmetries, they are asked to represent them in terms of a 120° clockwise rotation, R , and a flip across the vertical axis, F . For example, the symmetry that is equivalent to a 240° clockwise rotation could be represented as $R + R$, $2R$, RR , R^2 , or even using the symbols $-R$ or R^{-1} . As there are many choices available, it becomes necessary for the class to negotiate both a common set of symbols and develop a whole-class convention—typically using additive or multiplicative notation—for operating on these symmetries.

Once the class agrees on common symbols, the students are then asked to consider each combination of any two symmetries and determine which symmetry is equivalent to it. Students typically take a variety of approaches on an individual level when engaging in this task. Some of them manipulate plastic triangles, while other students may use shortcuts to produce pencil-and-paper representations that often manifest as a series of calculations, an organized list, or an operation table. If an operation table is not generated, the teacher can introduce this idea into the whole-class discussion as an effective way to record their mathematical activity. The students are then asked to construct individual operation tables and to keep track of any shortcuts or

rules that assist them in calculating the combinations of symmetries. Although some of these short cuts emerged in students' earlier work through symbolizing, the focus now is on formulating these relations (Yannotta, 2016a). These properties are shared with the class in an effort to form a list of rules (axioms) that is sufficient for completing the operation table without needing to physically manipulate a triangle.

Participants

The participants in this research study included thirteen community college students (three female and ten male) whose ages ranged from 17 to 48 years. In the course, there were three math majors, six engineering majors, one physics major, one psychology major, one biology major and one student who had not yet declared a major. The pre-requisite for this elective course was trigonometry, but the majority of the class had exceeded this minimum requirement. Ten of the students had completed linear algebra, one student had completed calculus I, and the two other students had only completed trigonometry.

Brian, the classroom teacher, was a full-time community college mathematics instructor with more than fifteen years of combined experience at the high school and two-year college level. Although he had never taught group theory, he had taught courses ranging from pre-algebra through differential equations and linear algebra and had tutored proof-based courses prior to becoming a teacher. Brian was familiar with the NCTM (2000) *Standards* and AMATYC (1995) *Crossroads in Mathematics* and was also open to pedagogical approaches that provided alternatives to lecturing. As the researcher, I attended every class session, assisted Brian with lesson planning, and debriefed with

him once a week. I occasionally participated in class discussions with the intent of playing an investigative rather than a pedagogical role in the classroom.

Data Description

The data corpus for this paper was collected during the spring quarter of 2010 at the community college. My colleague, Brian, taught the class for the first time and I acted as both a mentor teacher and researcher throughout the term. Another doctoral student supervised data collection and assisted me in conducting weekly debriefing meetings, which were held on-site. Each one-hour and twenty-minute class session was videotaped using two cameras. One camera was in a fixed location and focused on one group of students, which varied from day to day. The other camera was at the back of the room and was operated by the doctoral student, tracking Brian and whole-class discussions. Seventeen class sessions were filmed and I analyzed days six and seven for this study. It is during these class periods that the students were negotiating their representational system and were just beginning to axiomatize, or create rules that described relations within the system they were exploring. The data from both of these class periods was transcribed word-for-word by another graduate student and together with the video data, were used in my analysis.

Analytic Methods

The data for this paper was analyzed in ways that were consistent with those outlined in Lesh and Lehrer's (2000) framework for iterative videotape analysis and included five passes through the data. In the first cycle of analysis, I identified all student utterances and how the teacher responded to these utterances in these class periods. If an utterance was mathematical in nature, it was coded as a student contribution. The second

pass through the data involved a second coder, and both he and I separately coded these contributions with the intention of trying to identify the origin or the influences of the student contributions. This second pass was designed to answer the question: What did the researcher perceive as the student's primary source of knowledge for his or her contribution? In five cases, both researchers concurred that the perceived rationale for the contribution came directly from the student in the form of a reason that referenced a specific prior mathematical experience, and those reasons became the focus of the next two passes of analysis.

These five episodes were first analyzed using Tall's (2004) concept of met-before in an effort to explain the role these prior experiences played in the way the students were conventionalizing in this new setting. When coding for a met-before, I looked for evidence of two characteristics. First, I needed to identify the current mental structure for consideration (i.e. What is the met-before?). Second, there needed to be evidence that suggested the met-before was a result a prior experience (i.e. Identify the past referent and make an argument connecting the referent to the met-before). While I was able to find many references to prior experiences, I was only able to find one example (detailed in the first case study episode) of a met-before that met this operational definition.

I then made a fourth pass through the data using Gentner and Markman's (1997) similarity comparison to code the contributions. In addition to the students, I included any similarity comparisons the teacher made, as I considered Brian a member of the class with elevated status. The criteria for identifying similarity comparisons involving past experiences involved finding evidence of a link between a past representation and the current conventionalizing representation on the mathematical agenda. Excerpts of raw

data were used also used to describe the role of prior knowledge in the decision-making process, thus shedding light onto the emergence of mathematical convention(s) within the classroom. At the beginning of this phase, one of the five episodes was removed from the study as it lacked sufficient evidence for identifying a primary knowledge source. Midway through this phase, two of the episodes were combined into a single one because the latter one was judged to be a continuation of the previous episode. For this pass through the data, I was interested in relation-based comparisons involving conventions and sought to adhere to Gentner and Markman's (1997) criteria for identifying analogies. The authors provide three psychological characteristics for analogies: 1) structural consistency, 2) relational focus, and 3) systematicity. With structural consistency, there should be evidence of parallel connectivity and one-to-one correspondence *between* the past and present representation. Second, there needs to be evidence of a *relational focus* between the two representations. Finally, analogies tend to match *systems of relations* within each representation and can be strengthened by interconnectivity.

When coding a similarity comparison, I used the lens of conventionalizing to examine these three characteristics. For instance, when the students were discussing the procedural order of their operation table (the current representation), comparisons were made to both graphing in the Cartesian plane and denoting a position within a matrix. In both of these instances, I inferred there was structural consistency between each pair of representations that was guided by the relation the students were comparing—the operational order of their table and the procedural order of a past convention. In both of these cases, past conventions mapped to how a location could be determined within this operation table. The final characteristic of matching a system of relations between the

representations was the most difficult to address. In the case of the operation table, the students likely had no prior experience with making a table for a non-commutative operation, so comparisons with a conventional multiplication table carried little weight. Even when the students made strong analogies to real number addition and multiplication when debating the operational convention for combining symmetries, the non-commutativity of the operation limited some of the structural relations that could be mapped to their commutative real number counterparts. Therefore, this third characteristic was given considerably less attention and was only used to discuss the strength of analogies.

I then made a fifth pass through the first conventionalizing episode that involved the met-before (Tall, 2004) using Fischbein's (1999) work on intuition to try to better understand how the students transitioned from accepting a relation that at first appeared to be true, to challenging it, and then eventually reinterpreting the relation and rejecting its truth.

Structure of the Paper

To help orient the reader, the body of the paper consists of three case study episodes of conventionalizing, each with four respective sub-sections: an introduction, a description, a fine-grain analysis, and a discussion. Episode 1 and 2 address how a shared set of symbols is created. These episodes also introduce a second aspect related to conventionalizing—how the elements and operation interact, as they involve collectively symbolizing the symmetries of an equilateral triangle and the operation of combining them. Episode 3 explores the negotiation of a convention for an operation table, which involves both arranging the table and establishing a procedural order for identifying a

specific element within the table. The findings section describes three roles that past experiences played across the conventionalizing episodes. The paper concludes with a synthesis of the findings and directions for future research.

Episode 1: Are the Expressions $2F2R$, $2F + 2R$ and $2(F + R)$ Equivalent?

Episode introduction. In this episode, the class first accepts a student contribution in the form of a met-before, but later, its applicability in this new setting is challenged. The ensuing discussion points toward some of the supportive and problematic aspects associated with prior experience. Through a subsequent similarity comparison, the class realizes that combining two kinds of notational systems is problematic, and they also discover that what appears to be the distributive property involving multiplication and addition does not hold in this new system.

Episode description. Following Larsen's (2013) LIT, the students identified six non-equivalent symmetries for an equilateral triangle, and were asked to represent them in terms of a 120° clockwise rotation, R , and a flip across the vertical axis, F . Brian had recorded several variants on the board in the previous class. In the process of exploring the notation $2F2R$, Brian clarified that this combination referred to two flips followed by two rotations and then expressed an alternate form of the expression as $2F + 2R$. Todd immediately responded, "Well, factor the 2 out of that, right there." Kevin supported Todd's suggestion and formulated the expression $2(F + R)$, which Brian also wrote on the board next to the two other forms. After a few minutes had passed, Gene questioned the usage of this notation.

Gene: I need you to help me out (pointing at the board). Those aren't the same.
 $2F2R$ is not the same as 2 times the quantity F plus R .
Jeremy: It's implied.

Kevin: Uh huh.
 Jeremy: You just factor a two out.
 Larry: It's just the notation.
 Todd: It depends on what you mean by $2F2R$.
 Teresa: Well it's still not the same.
 Brian: What, what...
 Gene: So you're saying $2F2R$ is the same thing as $2F$ plus $2R$?
 Todd: Maybe it isn't.
 Teresa: It's...yes it is.
 Gene: $2F$ times ... $2F2R$ is $2F$ times $2R$.
 Brian: Yeah.
 Gene: Not $2F$ plus $2R$.

Gene reiterated that $2F2R$ was not equivalent to " $2F$ plus $2R$ " and further questioned using this notation asking, "...If we're going to use symbols, don't they need to meet some kind of conventions?" Brian affirmed Gene's point by saying "it would be nice if those matched conventions we already have with symbols" and then Gene further clarified his objection to this notation.

Gene: And for those, for that sequence, order matters, right? It matters a lot if you're, if you do the flips in the second one.
 Brian: That's a great question. Does order matter here?
 Students: Yes.
 Gene: What about the idea of a composite, kind of a composite function? You have to do one before you have the other one act upon it. Now that's not addition.

Brian briefly reviewed function composition with the students, then the class was asked to revisit an earlier conjecture about whether $2F + 2R$ was equivalent to $2(F + R)$.

Researcher: I'd like to come back to just one more thing about what Gene said. I'd like everyone to think about the top piece that Brian wrote down. I don't know... the $2F + 2R$ and then the very bottom one (points to $2(F + R)$). Are those two the same? ... How could you tell?
 Todd: It depends on what the two multiplier meant.

Brian then asked whether $2F + 2R$ could also be expressed as $F + F + R + R$ and several members of the class concurred. He then proceeded with a similar line of questioning for the expression $2(F + R)$.

- Larry: Well, what did he officially mean by do *that* twice? Do we do F twice and then R twice? So then, that would be F plus F plus R plus R —if he meant to do F twice and then R twice (Brian writes $F + F + R + R$). That just, that's just what probably comes to mind.
- Brian: Or does it mean F plus R plus F plus R (writing $F + R + F + R$)?
- Todd: They mean different things.
- Teresa: Oh, I see what I was thinking.

Teresa, who was flipping her triangle throughout the previous discussion, later confirmed that $2F + 2R$ “is not” $2(F + R)$ and that she had checked it. Kevin then made a structural observation.

- Kevin: So then you can't factor out the two.
- Jeremy: It all depends on how you read it.
- Brian: Oh man, that stinks.
- Jeremy: You can't factor it, cuz it's not addition.
- Todd: So the distributive law doesn't apply here, maybe, huh.
- Kevin: I would agree with that.
- Brian: Well that stinks.
- Gene: I would agree that distributing does not, does not work here.

The episode concluded with Todd stating that “maybe it's not a good idea to mix the two notations”.

Episode analysis. Although Brian introduced the notation $2F + 2R$ to clarify what he thought a student meant by the concatenated notation $2F2R$, this new interpretation led to a discussion involving the equivalence of other forms of the expression. First, there was $2F2R$, which Kevin had symbolized in the previous class session and Brian had already written on the board. Second, there was $2F + 2R$, which Brian introduced as his clarification of the notation, and finally there was $2(F + R)$,

which Todd suggested, Kevin amended, and Brian subsequently recorded. Todd's spontaneous contribution to "factor out the 2" in the expression $2F + 2R$ was the result of him acting on a met-before (Tall, 2004). In this case, Todd's previous knowledge of factoring and of the distributive property of multiplication over addition produced [what he believed to be] an equivalent form of the expression. The met-before that "factoring produces an equivalent form of the expression" was likely a product of the cumulative experience of working with the operations of addition and multiplication in algebraic settings. More specifically, factoring is a skill that is typically emphasized in a traditional algebra sequence that Todd and many of these students would have taken. Kevin's comment at the end of the episode that "you can't factor out the 2" refuted Todd's initial met-before and points toward the type of similarity comparison that members of the class may have been making earlier in the episode. Although the equation $2F + 2R = 2(F + R)$ looked similar to the distributive property the students had encountered numerous times in their previous experience, they eventually rejected this mere-appearance match, surmising that this relation did not hold in this new setting.

Episode discussion. An analysis of the discussion concerning the representations $2F2R$, $2F + 2R$, and $2(F + R)$ exposed a complicated mathematical issue when trying to iterate the non-commutative operation of combining symmetries. If addition is used to symbolize the operation, then the shorthand $na = a + a + \dots + a$ is a natural consequence of the notational choice. Thus, the notation " $n(a + b)$ " is used to symbolize the process of combining elements a and b and then iterating that element n times. So, while the students eventually concluded that the distributive property did not hold for their structure, this point needs further clarification. The distributive property does not hold in

this situation because D_6 is a group and the distributive property is not applicable to groups. I argue that what the students were actually grappling with was whether the equation $n(a + b) = na + nb$, which is valid for abelian groups, was true for their system.

Notably, the representation $R + R + R$ was already accepted by the students as equivalent to $3R$ and therefore, this experience can be viewed as supportive as multiplication is used to represent repeated addition. Furthermore, when Todd said, “Well, factor the 2 out of that,” it implied that the equation $2F + 2R = 2(F + R)$ was true in this new context. However, I claim this was just a mere-appearance match because his justification pointed toward a past experience involving factoring algebraic expressions. Even though these expressions may have appeared equivalent, the validity of the relation went uncontested likely because it was viewed as the distributive property that members of the class had met before. Not surprising, Todd’s comment about factoring out the 2 went unchallenged for several minutes in the classroom, despite the community’s knowledge that the operation of combining symmetries was non-commutative. In fact, no one—including the teacher—seemed to take issue with equivalence until Gene questioned it several minutes later.

When the class revisited this issue involving the met-before distributive property, it greatly influenced the discussion and the decisions they made in the process of conventionalizing. First, it forced the students to reexamine the notation and question Jeremy’s comment about equivalence being implied by the notation itself. Teresa’s rebuttal helped move the discussion from assuming equivalence based upon their past experience with symbols to considering the two expressions in the present context with the equilateral triangle. Iteration was then formally addressed through a comparison of

the expressions $2F + 2R$ and $2(F + R)$, which led to the expanded forms of $F + F + R + R$ and $F + R + F + R$ that Brian wrote in the public space. Teresa then empirically checked these representations with her triangle and determined them to be non-equivalent. Thus, revisiting the meaning of $2(F + R)$ in this new context moved the discussion away from trying to preserve a conventional relation between the operations of addition and multiplication and toward adopting a notational system to symbolize a single operation.

Episode 2: Should We Use Additive or Multiplicative Notation to Represent a Non-Commutative Operation?

Episode introduction. In this episode, the students must collectively decide the convention they will use to represent the operation of combining symmetries. Brian facilitates this activity as a debate between additive and multiplicative notation and throughout much of this discussion he makes references to prior experiences involving these operations. Brian first uses these referents in a general way to frame this new situation with the triangle in ways that are consistent with the students' past experiences. However, he meets resistance from some of the students, who do not seem to share Brian's viewpoint that multiplicative notation is more appropriate to represent a non-commutative operation. In the end, the class votes to adopt multiplicative notation, but this episode provides some insight into how the students interpret their prior experiences in different ways from the way Brian intended.

Episode description. Brian directed the class's attention to the notational choices they were considering and asked, "Have you ever seen a case where addition is going to be different depending on order?" Following a long pause, several students shook their heads no.

- Brian: In other words I'm saying, you can add numbers, you can add variables, you can add functions, ... you can add a lot of things. When you think of anything you've ever added, if you reverse the order, have you changed it or has it stayed the same?
- Todd: Alternating infinite series, it mattered.
- Brian: Huh (laughing and nodding). Well, I'm just going to deal with plus now (several students laugh). It's kind of a strike against plus.

Brian then asked, "Have you ever seen a case where something is written as multiplication where order matters?" Several students shouted, "Matrices!" and then Brian reviewed how to multiply two matrices and demonstrated that the operation was not commutative. He then asked the group what they thought about adopting multiplicative notation, but Kevin wanted to know why other members of the class did not want to use addition to denote the operation.

- Todd: We said, because the commutative property applies all throughout algebra, the plus implies commutative.
- Kevin: This isn't algebra. In algebra of symmetries it doesn't have to be.
- Jeremy: Well, we don't want to make another definition of a plus sign.
- Brian: What ... what I'm trying to get at is the notation that um ... the symbols we use in mathematics um, already have something of their own personality based on experience. As we develop new areas of mathematics it is not unusual to violate that personality. The first folks who discovered that you could have multiplication that wasn't commutative—it was kind of a shock. (laughs) . . . different symbols and letters kind of carry their own baggage a lot. So, it's nice when you can go with the flow of that.
- Gene: So we can use addition, we just realize that it's not commutative.
- Brian: Or if we use multiplication ... it wouldn't be the first time we've seen multiplication that wasn't commutative.

Although Gene's comment stressed the importance of communicating that the operation was not commutative in this new context, Brian's response about multiplication pointed to a precedent that may have already been established through the students' prior

experience with matrices. After a long pause, Kevin raised his hand and interjected a second time.

Kevin: But then at the same time . . . and feel free to disagree . . . but just seeing $3R$, I think I do R three times (rotating motion with his hand), but to see R^3 ... how do you ... 'cause then you're taking the superscript and changing its definition.

Brian erased the board and reminded the students that the “shorthand notation” for RRR was R^3 and that for $R + R + R$ it was $3R$. He then wrote several versions of different symmetry combinations using both multiplicative and additive notation styles on the board and began to explore them with the class.

Todd: Could I bring a case for an exponential version? For, a case for using the exponents?

Brian: Yeah.

Todd: Well, what we're saying there is F^3 equals F right there. Or three F 's equal F right there. I noticed that negative one cubed is negative one and negative one to the fifth is negative one. So long as you've got an odd number of negative ones, it takes it back home. It takes you back to what you started with.

Brian: So you're saying there are numbers that behave...

Todd: Yeah.

Brian: ...with powers the way that we're talking about... So you're saying, that the numerical operation of taking a negative one and cube it (writes $(-1)^3$ on the board).

Todd: Equals negative one.

Brian: (writes = -1) ...really is the same number as negative one. So there's something satisfying about that. Not that we literally are saying, we're plugging in a number for F , but for you (looks at Todd) that's satisfying that F is behaving the same way.

After Brian privileged Todd's analogy for the class, he then explored the implications for iterating the expressions $F + R$ and FR .

Brian: Keeping in touch with Todd's idea, things that feel consistent with the math that we already do, are, that's, that's something that's appealing. Does this (points to $2(F + R)$) feel consistent with the math that we already do? (pause) Well, not entirely. The math we already do, we'd be strongly inclined to just distribute. Cause everywhere else in math we have ever

worked, if you have an expression like that (points again to $2(F + R)$), whether these are numbers, or functions, or matrices, or integrals, or anything, we can just distribute. But you guys were telling me earlier (writes $\neq 2F + 2R$ next to $2(F + R)$) that's not the same as $2F$ plus $2R$... (pointing to $FRFR$) How would I write this using parentheses if I do this using multiplication (writes a set of parentheses)?

Arthur: FR squared (Brian writes $(FR)^2$).

Brian: (pointing at $FRFR$ and $(FR)^2$) Are these the same?

Teresa: Yes.

Kevin: Mmm. It depends, because if you distribute the squared...

Brian: Yeah, the question is...

Todd: F squared, R squared ain't the same.

Jeremy: We're running into the same problem.

As Jeremy pointed out, one could run into an issue when trying to iterate a flip followed by a rotation using either convention as iteration does not behave the same way as it does for real numbers. Brian then tried to capitalize on this observation by comparing these two representations in light of past experience.

Brian: (writes $(FR)^2 \neq F^2R^2$) And so my question is, are these equally unsatisfying or dissatisfying? Or does one of them feel more problematic than the other?

Gene: A poke in the left eye, versus a poke in the right eye.

Brian: (laughing) I think it's a poke in the left eye and a stick in the right eye, but you know. Um, it's definitely the case that any place you've ever run into addition in mathematics and multiplication by a scalar...um when you're adding things and multiplying them by a number or scalar, the distributive law has always worked. Is the distributive law for powers, has that always been true in everything you've ever worked on? ... It actually doesn't work for matrices. Again, we're in territory where, you know, more than half the class has seen that play out and the other half is just saying, "Okay, well what are matrices again?" but it doesn't always work out. So maybe it's more like a poke in the eye (points to $2(F + R) \neq 2F + 2R$) and a punch in the arm (points to $(FR)^2 \neq F^2R^2$).

As Brian prepared to take a vote on the convention, Larry raised his hand and asked about the importance of another aspect of the notation.

Larry: Is the reason we're jumping through hoops to pick one of these, is so that it becomes very easy to just write a very large number of them down in say, that format?

- Brian: Um, that would be part of the deal. Um, what's gonna happen is, we will be—not in this class—but in later coursework, you'll be working in settings where there are two operations that work. So whichever operation we choose, that will be one of the operations that work in that higher-level stuff, but also, there will be a second operation. And the symbol that we'll have to throw in for that will be whichever symbol we don't use here.
- Larry: Will, will the better one usually be the one that, that's easier to write down? Is that what people generally pick, mathematicians generally?
- Brian: People do generally pick the one that's easier to write down.
- Larry: That...Okay. That makes sense.

Following this exchange about the importance of notational efficiency within the mathematics community and Brian's preview of future experiences involving these conventions, Larry suggested the class vote on the notation. Eight students voted for multiplicative notation, three voted for additive notation, and Gene voted to adopt the symbol " \oplus ", which Larry had introduced to distinguish the operation from conventional addition.

Episode analysis. In contrast to the other two episodes, Brian was less willing to allow the students to conventionalize in a way that differed from that of the mathematics community when choosing the operation symbol. In fact, he tried repeatedly to leverage multiplication over addition by building on students' thinking about their *past* experiences. In this way, I claim that Brian was using a specific type of generative alternative (Rasmussen & Marrongelle, 2006) to move the mathematical agenda forward. The use of generative alternatives supports guided reinvention by allowing the teacher a middle-ground stance between not intervening and assuming the total responsibility of explicating the mathematics within the classroom community. In this case, Brian was steering the class toward adopting multiplicative notation, but he did not want the class to choose multiplication solely because it was more efficient than addition. Instead he

appealed to the students' past experiences with these conventions, highlighting the importance of consistency with those conventions, but he met considerable resistance toward adopting multiplication from the class.

For instance, when Todd mentioned alternating infinite series, it provided an affirmative answer to Brian's question about order mattering with addition. While this contribution was not pursued, it is significant because it was a student-generated counterargument to order not mattering with addition. On the other hand, the class's reference to matrices supported an order matters argument for multiplication, which Brian then extended to emphasize the importance of the connotation that a symbol may already possess within the mathematics community. More specifically, when presented with a choice of two symbols, he argued that a person should choose the one that "does not violate the symbol's personality". Referencing back to matrices, a majority of the class had already used multiplicative notation in a setting where the operation was non-commutative. Therefore, adopting multiplication for this system would likely not be seen as violating the personality of multiplication. Conversely, Brian hedged his central thesis by saying, "As we develop new areas of mathematics, it is not unusual to violate that personality", which may have communicated the message to both Gene and Kevin that it was okay to violate the personality of a symbol in this new domain as they had previously done with matrices. Therefore, Brian's caveat may have inadvertently provided a rationale for using additive notation for the non-commutative operation of combining triangle symmetries. Gene's summative comment supports this conjecture when he confirmed, "So we can use addition, we just realize that it's not commutative."

Kevin also continued to advocate for additive notation, identifying more with the notation “ $3R$ ” rather than “ R^3 ” for representing three consecutive rotations. Kevin justified this preference with a similarity comparison involving the past and present usage of “ $3R$ ” and “ R^3 ”. While both expressions are consistent with their respective operations for iterating R , the additive form may have exhibited a higher degree of similarity to Kevin because it shares both attributes and relations in this new setting. Not only does the equation $3R = R + R + R$ look the same as did in the past, but the relation of iterating rotations is analogous in this new setting because the angle of rotation is being added in the conventional sense. On the other hand, Todd noticed a pattern in which the exponentiation of F behaved similarly to the exponentiation of negative one. Todd’s multiplicative analogy was then privileged by Brian, who acknowledged that F behaved similarly to negative one with regard to exponents, commenting, “... there’s something satisfying about that” before exploring iteration further.

As Brian revisited the expression $2(F + R)$, he reminded the class that until now the [apparent] distributive law had worked consistently throughout their mathematical experience. In this way, I claim that Brian was again attempting to make a similarity comparison for the students by assuming their past experience with the distributive property and comparing it to the current situation with the triangle. Instead of affirming an “order matters” match as he did previously to support the adoption of multiplication, Brian used this similarity comparison to foster the students’ rejection of additive notation because it was inconsistent with their prior experience. Later, when Brian turned his attention to the multiplicative form $(FR)^2$, he made a second similarity comparison to cultivate student support for adopting multiplicative notation. It is worth noting that

these relational similarity comparisons differ from the ones that the students made earlier because they focus on a shared relation that is false, rather than one that is true. In this new situation, the two equations $2F + 2R = 2(F + R)$ and $(FR)^2 = F^2R^2$ were both shown to be false, which was largely inconsistent with the students' past experience.

After reminding the students that the equation was also false for matrices, Brian asked the students to consider which notation felt more dissatisfying. Gene's neutral metaphor, "A poke in the left eye, versus a poke in the right eye" was sharply contrasted by Brian's reply involving a stick in one eye, which implied the additive situation was clearly less desirable than the multiplicative one. Based on Gene's previous comments about the operation behaving more like function composition (see Episode 1), his metaphor that addition and multiplication were equally bad, and his later vote for circle-plus notation, I conjecture that Brian's similarity comparisons further reinforced Gene's position to adopt an alternate notation. Contrary to Brian's intention to elicit a clear choice of multiplication in favor of addition, both of these operational conventions now conflicted with the ways Gene interpreted iterations of a commutative operation. After Larry's clarification about efficiency, a vote was subsequently taken and a majority of the class chose to represent the operation using multiplicative notation—the convention that Brian had been suggesting throughout the episode.

Episode discussion. I argue that there are two main factors that played a role in the students' conventionalizing within this episode. First and foremost, Brian played a central role in the notational debate and in the final decision made by the class. While he encouraged extensive debate and discussion about notation, he also demonstrated a strong preference for adopting multiplicative notation, and was candid at the end of the episode

when he acknowledged this fact. Although he reminded the class on several occasions that it was their decision, Brian both initiated and contributed to arguments in favor of adopting multiplicative notation, making use of assumed past experiences in pedagogical ways to frame this new situation for the students. At the beginning of the episode, Brian reminded the class of a precedent involving matrices and had this example not met resistance, it may have been enough to persuade the class to adopt multiplicative notation. Later, he endorsed Todd's observation about the analogous way that both -1 and F behaved under exponentiation. He then made another similarity comparison demonstrating that the relation $(FR)^2 = F^2R^2$ was also false for matrices. In his closing argument, Brian acknowledged the importance of being efficient and even alluded to the way things would be done in the future, which further supported his steadfast argument for adopting a multiplicative convention.

Second, despite Brian's attempts to persuade the students to adopt multiplication, numerous references to past experiences provided balanced support for both conventions. The resulting point-counterpoint discussion contained the following contributions: (1) general references to past experiences in which the order of the operation mattered (alternating infinite series and matrices), (2) student-generated similarity comparisons associated with iterating a single symmetry (an analogy involving R and an analogy involving F), and (3) Brian's similarity comparisons of the equations produced when iterating a symmetry combination (an analogy involving $F + R$ and an analogy involving FR). Despite the teacher's intention to capitalize on these references to past experiences to support the adoption of multiplication, a class decision regarding the convention was

only reached by a majority vote after Larry questioned the importance of notational efficiency.

Episode 3: How Should This Operation Table Look?

Episode introduction. In the previous class session, the students were asked to determine all possible combinations of any two symmetries of the equilateral triangle. After Brian asked some of the students to share their different calculation methods, he invited the class to discuss and modify an operation table that Todd produced. During the discussion, one of the modifications results in rearranging the original order of both the columns and the rows. However, this action leads to a second alteration that keeps the rows fixed, while switching two of the columns. The discussion then transitions to the computational order of the operation table and whether the class should adopt Todd's original convention of operating column first followed by row or switch to the opposite order. The students make several analogies to support both conventions. Different arguments are made in favor of each procedural order, but ultimately, the class decides to adopt a row-first, column-second convention for the table.

Episode description. As seen in Figure 9, Todd's table was initially set up with the identity element listed first, followed by R (a 120-degree clock-wise rotation). In addition, the operational order was columns first, followed by rows.

		FIRST					
		I	R	R^2	F	FR	RF
SECOND	I	I	R	R^2	F	FR	RF
	R	R	R^2	I	FR	RF	F
	R^2	R^2	I	R	RF	F	FR
	F	F	RF	FR	I	R^2	R
	FR	FR	F	RF	R	I	R^2
	RF	RF	FR	F	R^2	R	I

Figure 9: Todd's operation table

- Brian: Does that remind you of something?
- Jim: A multiplication table.
- Brian: Yeah. That turns out to be a really efficient way to track these operations. It also helps it make sense why we came up with 36. Um, first of all, Todd ordered them I, R, R^2, F, FR, RF and it looks like you did the exact same way along the top. How do we feel about that ordering or do you want to change it to a different ordering? And I'm really not pre-disposed to any particular ordering we've chosen.

Responding to Brian's invitation to discuss the current representation, Todd was the first student who suggested reordering some of the symmetries in the existing table.

- Todd: I might be inclined to put that F over in that second row and second column over there. So it goes, I, F and then the rest of the stuff. Keep I next to F , that's, something about that feels a lot brighter to me.
- Brian: (writing column headers I, F, R, R^2, FR, RF on the board to the left) Like this?
- Todd: Yeah. You get that flip over there next to the identity. You get that modulo 2 thing sitting in the corner.

Todd's suggestion was likely based upon a past experience with a similar representation, but most notably, this initial change opened the door to make further modifications to the table.

Brian: Well, what do we think? (long pause) Would that be cleaner? Yeah, Arthur?
Arthur: Um, also I would suggest switching the columns of R and RF , cuz that way you get identities right along the diagonal.
Brian: Oh, so if we look along this diagonal right now...
Kevin: There you go.
Gene: Yeah.
Brian: We almost have diagonals all the way along here. So, which...
Todd: Could, could that happen?
Jeremy: So, just switch R and R^2 and you would get the identity all the way down the diagonal.
Brian: I thought, Arthur, you were talking about switching R and RF .
Arthur: Uh, I meant R^2 .

If the rows and columns of the table are arranged in the same order, then I 's along the diagonal would indicate the elements that are self-inverses. However, the only way to get *all* I 's down the diagonal for the table was to switch two columns, but keep their rows fixed. Although Arthur's initial suggestion would not accomplish what he intended, he later clarified that he meant to switch the columns of R and R^2 . Despite Todd's skepticism, other members of the class continued to assist Brian in constructing a table that produced identities all along the main diagonal.

Brian: Oh I see. (Brian writes I , R^2 , R , F , FR , and RF for the columns and rows). So if we did this, so the advantage would be, now we have I 's along the diagonal?
Gene: Yeah.
Todd: I wonder. I question that call.
Jeremy: You can't switch the other side though.
Jim: Yeah, you'd have to leave the other side the same.
Jeremy: The other side has to stay the same (Brian erases the rows and writes I , F , R , R^2 , FR , RF). Yeah.
Todd: I think having the same order for the row and the column is important.
Brian: So, we're not so keen on this, even though it makes the table work out?

Because it makes the borders messy.
Todd: It makes the borders less symmetrical.

Brian then redirected the discussion by asking, “Uh, do we feel okay about doing the top operation first, and the uh, side one second? ... Anyone want it to be the other way around?” The first student that responded regarding the procedural order of the table was Jerry, who suggested changing the existing order to “the left side first and then the top second.” Jim supported Jerry’s suggestion, as did Teresa, who said, “you read left to right, so it kind of makes sense to do the left side first and then you go to the top.” Jerry continued by saying, “That way, you can just go across.” However, some students then argued for preserving the existing column-by-row convention.

Randall: But you always do it x first before the y .
Jeremy: Right, like on a graph. You do the horizontal first.
Jim: The x ’s.
Jeremy: You have the x ’s and then you do the vertical, or the y .

This brief exchange was then followed by a question from Todd about another mathematical convention that involved matrices.

Todd: How do we denote matrices and location? Isn’t it “I” and “J” for matrix?
Brian: In a matrix, what’s the first number that you give out? Is it the row or the column?
Todd: Isn’t it the column?
Jeremy: The column.
Todd: Or the, it’s the vertical, isn’t it?

As the students were talking, Brian erased part of the white board and began a short, interactive lecture that reviewed the standard row-first convention for reporting a position within a matrix.

Brian: Does that affect what we want to decide in terms of which one’s first?
Cuz we’ve heard some arguments in favor of each reading.
Todd: Well, we were discussing how matrices play a role in the way we multiply

things, so maybe we better carry some notation over from that. That might be wise.

Jeremy: Uh, I like that idea.

Brian: Well, when we fill out a times table do we know which order it is?

Kevin: It doesn't matter.

Brian: (laughing) Yeah. If it's any comfort, I was having this exact conversation with Mark this afternoon. Like, I don't know if there is a standard way to do this and what he reminded me of was you guys get to pick. So, we've heard some arguments for doing the column first—the principal one I recollect being um, when you give coordinates to x first, then y second—you go horizontally first then how far you go up and down. That totally makes sense. Um, what are arguments in terms of going sideways first? I think Teresa, you had given one.

Teresa: Um, just that you read left to right. So it made sense to treat the left first and then the top.

Jeremy: In a lot of books, you find, you go down the list, and then you pick one and then you go over to where it matches. Like with stats, you go down the left side and then you go across to find the value.

Brian: So that would mean we're doing this one (points to the rows on the left side) first and then that one (then moves his hand to the right). And does the matrix notation, um, favor going to the top one first or the side one first?

Several students shouted, "Side." Brian paused and then asked the class, "What do you want to do?"

Jerry: Side first.

Kevin: Vote.

Brian: Let's have a vote. How many um, how many um, so the choices are: I'm so articulate today. How many want this position to be RF versus this position to be FR ? Are we doing the F first, the R second or are we doing the R first and the F second? So, how many want this entry to be RF ? (Larry raises his hand) And how many want it to be FR ? (several students raise their hands) I think we have a majority here.

At the conclusion of this discussion, it was decided (12 to 1) that the tabular convention be switched to make the row operation first, followed by the column.

Brian then began drawing a new table on the white board that had the rows and columns in the same order, incorporating Todd's original contribution that reordered the table headers with $I, F, R, R^2, FR,$ and RF .

Brian: So, just to get back to where we were, one of Todd's comments or one of the things that seemed really appealing about moving F is that we get that really pretty upper left-hand corner (points to the new table).

Todd: It kind of looks like a little identity matrix. What's that about?

Jeremy: Uh, huh. Linear.

Brian did not complete the new table on the board and instead transitioned to the next task, by inviting the students to construct their own operation table.

Brian: Use whatever technique you want at this point. I, I want to uh, not compel you to do any of this shortcut business, but I do want to look at the table and have people make different choices about how to organize the table.

After the students worked on this task for several minutes, Kevin produced a table (see Figure 10) using a row-first, column-second convention that did not have the same order for the rows and columns, but had I 's running all along the main diagonal.

	I	R^2	R	F	FR	R	F
I	I	R^2	R	F	FR	R	F
R	R	I	R^2	RF	F	FR	FR
R^2	R^2	R	I	FR	RF	F	F
F	F	RF	FR	I	R	R^2	R^2
FR	FR	F	RF	R^2	I	R	R
RF	RF	FR	F	R	R^2	I	I

Figure 10: Kevin's operation table

Episode analysis. Responding to Brian's invitation to consider modifications to Todd's operation table, Todd suggested switching the rows and columns of F and R , because having I next to F felt "brighter" to him and then elaborated on this decision to put F next to I by referring to some prior experience with modulo two. By moving F

next to I , the upper corner would form a table of a subgroup of order two, so perhaps that was appealing to him. In addition to this analogy, Todd later commented on the surface features of this corner of the table saying that it looked like “a little identity matrix.” As other students joined in rearranging the table, Todd’s question about whether it was even possible to get identities down the diagonal was noteworthy, as it helps to distinguish how Todd’s conventionalizing differed from other members of the class. Furthermore, when several members of the class assisted in getting I ’s down the main diagonal, Todd’s subsequent comment about the importance of preserving the same order for rows and columns affirmed there was an aspect about the convention the class was proposing that he did not like. One possibility is that perhaps with his initial switch of F and R , he may have realized that “ R ” was not an element of order two and would not produce the desired result. Another conjecture is that Todd was simply unable to accept the aesthetic result inside of the table because he realized it would come at the expense of establishing an order for the rows that differed from the columns. Unlike Todd, Arthur did not reference a specific comparison to explain why he wanted the table to have I ’s down the entire diagonal. Perhaps he merely wanted to extend Todd’s result of getting I ’s in the first two diagonal entries to the rest of the table. Another possibility is that he thought this property might hold some advantage over the current representation. Yet, once the suggestion was made, other students quickly affirmed his recommendation. In particular, Kevin’s comment of “there you go” and Gene’s “yeah”, suggest at least some common understanding that Arthur was proposing something desirable to each of them.

Students’ past experiences continued to play a dominant role in the subsequent discussion of which procedural convention the class would adopt for their operation

table. In their arguments, the students provided rich descriptions of their prior experiences and used similarity comparisons to support the convention they thought the class should implement. For instance, the first argument for supporting Jerry's suggestion of switching to a left-first, top-second convention, came from Teresa, who made a connection with the experience of reading left-to-right in the English language. Randall then made his counterargument for preserving the column-first convention of Todd's original table. Jeremy supported this contribution through a graph reference, where one moves horizontally first and then vertically. In addition to being mathematical in nature, this similarity comparison seemed to capture a salient relation within the table in that each cell could be viewed as a location, specified by the order of two elements. While Randall's analogy attends to a shared relational convention assigned to the x and y -coordinates for naming a location, his comment that "you always do it x first" suggests a second computational order that is not necessarily a consequence of the Cartesian graphing convention.

Todd's question about denoting a location within a matrix is an interesting one because his response to Brian about the column entry being given first was incorrect and further, it did not support the convention he implemented in his original table (see Figure 3). Although this is just a conjecture, this may explain why Todd set up his initial table column-first, row-second and why he was so quick to change the order once he was reminded of the row-first, column-second convention used to specify a location within a matrix. Another possibility for Todd's question about the matrix convention may have originated from the previous similarity comparisons with matrices that resulted in the adoption of multiplicative notation (see Episode 2). In fact, Todd even mentioned that

“we were discussing how matrices play a role in the way we multiply things” to preface why the matrix-entry location convention might be appropriate for deciding the convention for the table. Even though Todd’s analogy of determining a location within a matrix is consistent for describing a cell within the table, this comparison has nothing to do with how the entries within a matrix are computed.

Jeremy’s statistics contribution is important to note because it provided a second mathematical similarity comparison and a third prior experience that supported the row-first, column second convention. Brian not only acknowledged Jeremy’s contribution, but also clarified it for the class by indicating the order before relating it back to matrices. If one assumes Jeremy was referencing how to look up a value in a z -table or t -table, as is a common experience in undergraduate statistics, such a situation could be viewed as similar to what the students were currently deciding. For instance, both of these statistical representations are tables, which is consistent with the current representation and their purpose is to compute a composite statistical value that must be obtained in a specified order. Not only must a location be reached by going row-first, column-second, but the result is a single computational value that is produced from these two parameters.

Episode discussion. As Brian noted, a table is an efficient way to track calculations and helps to justify why there are 36 of them. However, these comments provide little insight into why the class should prefer Todd’s version to Arthur’s, which had a row ordering that differed from that of the columns. In fact, at this stage of the course, there had not been any assigned tasks that provided students the opportunity to leverage these two alternate arrangements against one another. For example, these students already knew the operation was non-commutative, so relying on the table to

determine that fact was superfluous. Furthermore, while questions about self-inverses and the structure of a yet-to-be defined center were discussed later in the course, they were not germane when first conventionalizing the table. Another important feature that is observable in Todd's original table is that it has the Sudoku property, which plays a critical role later in the enactment of the LIT. Yet, this relationship is invariant under Arthur's action of swapping columns as illustrated by Kevin's table (see Figure 3). Later in the LIT, there are tasks that make the utility of the conventional representation more apparent to the students, but Brian's closing comment highlighted what appeared to be the most immediate difference between these two conventions within the classroom community in the moment—it was an aesthetic trade-off. More succinctly, if one ensures that certain patterns hold inside the table, it may come at the expense of the uniformity of the borders. Therefore, if the primary distinction between Todd's representation and the one Arthur suggested was based upon aesthetic differences, it is not surprising that some students preferred the representation that may have been similar to a representation from their prior experience.

Similarity comparisons involving past experiences were used extensively throughout this episode as the students discussed various conventions associated with the operation table. For example, Jim initially commented that Todd's operation table reminded him of a multiplication table, but as Kevin later indicated, the procedural order did not matter in a standard multiplication table. Although it is presumed a multiplication table is a familiar referent for most mathematics students, its conventional representation carries with it some specific characteristics such as: 1) the rows and columns are in the same order, 2) there is a pattern down the main diagonal and symmetry across it, 3) a

given entry appears exactly once in each row and in each column (Latin-square property), and 4) entries are symbolized as simplified computations. While all of these aspects are worthy of discussion, the non-commutativity of the operation adds a level of complexity to the current situation that required the students' immediate attention. While these students had numerous prior experiences with non-commutative operations, they were most likely not represented in tabular form until now. In addition, Brian's unfamiliarity with whether there was an established preference within the mathematics community allowed the students more freedom to decide the convention for procedural order. As a result, the students drew upon similar experiences to help them make sense of the current situation.

With all four of these similarity comparisons playing out in the public space, it is impossible to point to one and conclude it was the most influential in determining the convention the class decided. Yet, the majority of the class voted to change Todd's original convention and operation with a row-first, column-second convention. One possible explanation for this nearly unanimous decision is that there were three similarity comparisons supporting it as opposed to only one that upheld the column-first convention. For example, this convention was consistent with the way one reads in English, the way a matrix location is determined, and the way one would look up a statistical value. Another conjecture that might help explain why the class decided to adopt the row-first convention deals with the perceived sophistication of the mathematical setting involved in the progression of the similarity comparisons. When Teresa initially based her argument on a reading convention, Randall and other students immediately dismissed it in favor of a mathematical one that involved graphing. This

argument stood until it was countered by Todd's suggestion and Brian's exploration of a more mathematically sophisticated experience involving matrices. Jeremy's reference to statistics and Brian's connection back to matrices may have added weight to the general notion that "in higher math, we do it this way", thus further influencing the outcome.

A third factor that may have impacted the decision was the status attributed to the contributions that Brian and Todd made. While Brian played a smaller role in this episode than he did in the previous one involving the notational choice, his reference back to matrices may have revealed his predilection just prior to the vote. Todd, who had taken the most math courses among the students, had some familiarity with abstract algebra and at times throughout the course, had dominated class discussions. In this case, it was Todd who introduced the operation table into the public space and it was his idea to borrow the convention that was consistent with matrix location, which was a departure from the way he initially designed it. Therefore, Todd and Brian's apparent preferences toward a row-first convention may have influenced some of the students' choices.

Findings

There were three findings that emerged following the analysis of these episodes that point toward the different roles that past experiences can play in conventionalizing. First, I discuss some of the functions of similarity comparisons and how student-generated analogies supported conventionalizing within the classroom community. Second, I describe how confronting a student's problematic met-before led to new conventionalizing knowledge. Thirdly, I elaborate upon how students' knowledge of prior conventions was used to validate relations in a new setting. I discuss each of these findings in turn and then conclude with some remarks about conventionalizing.

Table 2

Functions of past referents upon conventionalizing

	Conventionalizing issue	Addition	Multiplication
Foster precedent thinking	Has the operation ever been used in an “order matters” situation?	Yes alternating infinite series	Yes matrices
Support the adoption of a proposed convention	Does iterating one symmetry work like it has in the past?	Yes Analogy $3R = R + R + R$	Yes Analogy $(-1)^3 = -1$ and $F^3 = F$
Reject (and support) the adoption of a proposed convention	How does iterating a combination of two different symmetries work?	$2F + 2R \neq 2(F + R)$ “and until now, this relation was always true”	$(FR)^2 \neq F^2R^2$ “and this was the same way matrices worked”

Similarity comparisons involving past experiences. Similarity comparisons involving past experiences can be used to: 1) foster precedent thinking and 2) support or reject the adoption of a proposed convention. I lay no claim that these functions are exhaustive nor are they mutually exclusive, but they are discussed separately here. To illustrate these different functions, I refer to Episode 2, when the class discussed the adoption of additive or multiplicative (or an alternate) notation for the operation of combining symmetries. This decision is anticipated in the LIT and Brian initially framed the discussion as a collective choice of adopting an additive or multiplicative convention. Although Brian made some of these contributions and privileged others while campaigning for multiplication, Table 2 is meant to illustrate the variety of referents that

were shared within the classroom community and the functions they served during this discussion.

The first set of referents involved an “order matters” relation and served to foster precedent thinking by considering past practices with each of the proposed operations. When Brian asked the students if they had prior experiences with addition or multiplication in which “order matters”, I claim he used a generative alternative to move the mathematical agenda forward. In this case, Brian was fostering a social norm to consider past experience when deciding the mathematical convention. This pedagogical content tool not only established the base domain for a series of similarity comparisons (Genter & Markman, 1997), but it also privileged precedent thinking, as the students were encouraged to look to their past experiences to inform their current choices. Todd’s surprising response of “alternating infinite series” and the anticipated multi-student response of “matrices” both involved similarity comparisons that point to two prior mathematical settings that are typical for undergraduate STEM majors and these past experiences were now becoming part of the decision-making process. With ten of the thirteen students having successfully completed linear algebra, it should not be surprising that ideas connected with matrices influenced all three episodes. Yet there was still resistance to adopting multiplicative notation, even in light of these past experiences.

The second pair of referents came in the form of student-generated similarity comparisons that functioned to support the adoption of a particular convention. Kevin made an analogy involving the iteration of R that supported the adoption of additive notation. His contribution illustrated the way the angles were summed in this new setting and suggested that the equation $3R = R + R + R$ for adding symmetries was consistent

with his prior experiences of addition. On the other hand, Todd made an analogy involving the iteration of F that supported the adoption of multiplicative notation. While “ -1 ” and “ F ” were different entities, they functioned in similar ways with respect to exponentiation. Although Brian acknowledged that Todd’s observation was “satisfying”, the same thing could be said about Kevin’s contribution as each of these similarity comparisons functioned to support one of the two conventions. In fact these contributions did not settle the debate, but evolved into an in-depth discussion of iterating a flip followed by a rotation.

A third set of referents were teacher-initiated and functioned to reject the adoption of a convention, which in turn, would support the adoption of the other convention. I claim that Brian employed a generative alternative this time to explore how “false” relations could be viewed in light of assumed past experience. Until this experience with the triangle, the equation $2(F + R) = 2F + 2R$ had always been true, thus Brian framed this similarity comparison involving the [apparent] distributive property to reject the additive convention. Brian then highlighted the exponential version $(FR)^2 = F^2R^2$ and demonstrated that both of these equations were false in this new context. Gene responded that both forms seemed equally bad, but Brian argued that the additive form was less desirable because it had always been true until now; unlike the exponential equation, which was also not true for matrices. However, for Gene and possibly other students, these comparisons did not privilege multiplication as Brian had intended, but rather they reemphasized the decision as a choice between two equally viable (or problematic) candidates.

Given Brian's strong position regarding multiplicative notation, it may not be surprising that the class adopted a multiplicative convention to symbolize the operation. However, it is noteworthy that the class's decision was not immediate, nor was it unanimous. In fact, the string of similarity comparisons and Brian's comments about the importance of being consistent with a symbol's personality did not seem to have the effect he intended on some students. Although references to past experiences functioned to foster precedent thinking and support or reject a convention, some students still took issue with adopting multiplicative notation. For instance, when the final vote was taken, Kevin voted for addition along with two other students and Gene supported a third alternative, " \oplus ". Ultimately, Larry's contribution about notational efficiency may have been the deciding factor to move forward with multiplicative notation.

Student-generated analogies that supported conventionalizing. In addition to the relational comparisons involving iteration that Kevin and Todd made in Episode 2, the students made a series of analogies involving past experiences in Episode 3 to support either a column-first or row-first convention for the table. As noted by Gentner and Markman (1997), analogy is central in cognitive processing and is a "clever, sophisticated process used in creative discovery" (p. 45). The authors also argue that the more structural relations that are created within an analogy, the stronger it becomes. I claim that such an analogical progression occurred during the class discussion of the operation table. For example, when Randall countered Teresa's analogy with a mathematical one involving graphing, it seemed to gain more traction with the class. One conjecture that might explain this positive reception to Randall's analogy is that it contained a key structural relation that was absent in the one Teresa presented.

Specifically, the graphing analogy specifies the order of any entry through an ordered pair, thus determining a location within the table by giving a column followed by a row. In this analogy, there is a specified ordering—i.e. there is a first-named element and a second-element.

Todd's analogy of determining a location within a matrix is quite similar to Randall's graphing analogy in that it too specifies a location within the table. What makes Todd's analogy different is that he projected additional relations associated with matrices into this comparison. First, when he asked how matrices and location were denoted, he made an analogy involving the shared relation of location to support a convention for denoting a position within the table. Yet later, he acknowledged a previous discussion concerning matrix multiplication to justify why it was wise to "carry over some notation from that." While the operation of multiplying matrices is non-commutative and was a strong similarity match for the operation of combining symmetries, I claim that Todd's reference back to this prior discussion of matrix multiplication involved a different comparison altogether. Matrix multiplication was previously shared as an example in which "order matters", so this fruitful similarity comparison may have added weight to the current one Todd was now making involving a specific location inside a matrix and a table.

Acting on and then confronting a problematic met-before. Like Tall (2013) and McGowan and Tall (2010), this study adds to the body of work that describes how a problematic met-before can serve as an integral component of learning. I claim that the students in this study initially accepted the equivalence of $2F + 2R$ and $2(F + R)$ because the met-before that "factoring produces an equivalent expression" had always worked for

the algebra of real numbers. Furthermore, despite the community's knowledge that the operation was non-commutative and that the variables R and F referred to symmetries as opposed to real numbers, this knowledge was initially overpowered by their prior experiences with factoring or distributing.

Following Gene's initial challenge, the students explored whether the [apparent] distributive property that had always been true in the past now continued to be true in the present. The class confronted this powerful met-before on two fronts—both symbolically and empirically — through: 1) Brian's guided reinterpretation of iteration and 2) Teresa's physical investigation with the triangle. By expanding $2F + 2R$ into $F + F + R + R$ and $2(F + R)$ into $F + R + F + R$, Brian helped expose a syntactical ambiguity associated with Todd's met-before—it hinged upon the operation being commutative. Although the met-before distributive property suggested these expressions were equivalent, Brian's contribution fostered a new way to think about iterating a non-commutative operation using additive notation. Teresa then confirmed the non-equivalence of these expressions by manipulating her triangle, which led Kevin to surmise that one could not factor out a 2 in this situation. By confronting this met-before, the students no longer tried to combine the operations of addition and multiplication as they had in the past and instead redirected their attention toward using a single operation in this new context. Regardless of which convention the class chose to adopt, the symbols would not represent a commutative convention like those associated with the addition and multiplication of real numbers.

Pre-existing conventions as sources of validity. A third role that past references can play in conventionalizing concerns the establishment of truth. As this study shows, the validity of relations in a new setting can be established by accessing referents to a

pre-existing convention. I use the phrase *validating by pre-existing convention* to refer to instances where an individual either implicitly or explicitly attempts to validate a relation in a new setting by referencing a convention from his or her prior experience. An example of attempting to validate by pre-existing convention occurred when Todd told Brian to factor $2F + 2R$, which resulted in an apparently equivalent form, $2(F + R)$. As noted earlier, Todd's contribution was likely the result of him acting on a met-before associated with factoring algebraic expressions and the distributive property of multiplication over addition that was valid for the real numbers. Another example of attempting to validate by pre-existing convention occurred when Gene refuted the equivalence of $2F2R$ and $2(F + R)$ on the grounds that multiplication and addition were not the same operation. Gene's challenge points toward the additive and multiplicative conventions of the algebra of real numbers, where things worked differently. Later, he also noted that $2F2R$ was not the same as $2F + 2R$ using this argument. However, Jeremy still insisted that the notation itself implied that $2(F + R) = 2F + 2R$. Once again, I claim that Jeremy was attempting to validate this equation by pre-existing convention, rather than considering how iteration actually worked in this new context. Jeremy's comment implies that equality could be verified symbolically either by distributing or factoring as Todd had initially suggested. The phenomenon of validating by pre-existing convention may provide some insight into why some students generalize the properties of real numbers to other abstract systems as noted by Selden and Selden (1987). Despite having the knowledge that the variables did not represent real numbers and the operation was non-commutative, the mere-appearance match of this relation to the distributive property was sufficient for members of the class to accept the validity of Todd's contribution for

several minutes. It is important to note that validating by pre-existing convention can be supportive as illustrated by Gene's challenge that $2F2R$ was not equivalent to $2(F + R)$ or $2F + 2R$. However, as Teresa alluded, these previous conventions did not address why $2(F + R) \neq 2F + 2R$ in this new context.

Conclusions

As noted by the instructor in this study, "the symbols we use in mathematics already have something of their own personality based on experience." By the time students reach the transition-to-proof level they have had numerous experiences with symbols and conventions that shape both their understandings of mathematical content and the ways in which the mathematics community has agreed to organize that content. Therefore, when students are given opportunities to reinvent advanced mathematics in a whole-class setting, past experiences serve as valuable resources for mathematical activity in a new context. This study makes a contribution in the area of RME literature at the undergraduate level in two ways. First, it describes the mathematical activity of conventionalizing, which is featured as one of several supporting activities in the *TAAFU* curriculum. Second, this paper proposes a theoretical argument that links prior mathematical experiences to conventionalizing that may be helpful to design research.

Confronting Met-befores

One way past mathematical experiences can emerge in the classroom is through an individual's contribution that points toward a met-before (Tall, 2004). Todd's suggestion to factor the expression $2F + 2R$ was likely the result of him acting on a met-before that applied to the algebra of real numbers. In this new context, addition was now used to represent a non-commutative operation, but this contribution initially went

unchallenged. However, the class later grappled with the notion that $2(F + R)$ and $2F + 2R$ may no longer be equivalent, which continued to challenge students' affirmatory intuitions (Fischbein, 1999). Eventually, the teacher guided the class to reinterpret each expression in ways that were consistent with their new non-commutative operation, which they were able to empirically verify with their triangles. After confronting this problematic met-before, the students no longer validated the truth of $2(F + R) = 2F + 2R$ by pre-existing convention, but instead they now had conventionalized a new interpretation of $2(F + R)$ that was applicable to iterating a non-commutative operation.

Making Analogies

A second way that past experiences emerge in the classroom are through similarity comparisons (Gentner and Markman, 1997), which can influence both the discussion and decisions surrounding conventions. In this study, the instructor's generative alternative (Rasmussen & Marrongelle, 2006) fostered precedent thinking to move the mathematical agenda forward. The resulting discussion evolved into an extensive debate about the convention the class should adopt for the operation of combining (composing) symmetries. As the students discussed this issue, they looked to their past mathematical experiences and made similarity comparisons involving addition and multiplication. In particular, analogies, or relationally-based comparisons, were used by both the instructor and the students to support (or reject) the adoption of a proposed convention. During this debate, the students not only made their own analogies to support one of these conventions, but they continued making new analogies throughout the next episode, when conventionalizing their operation table. As noted by Gentner and

Holyoak (1997), analogy assists in the process of understanding a new situation in terms of one that is familiar through shared relations, and thus these student-initiated analogies were instrumental in negotiating the class's conventions. By considering their past experiences with conventions, the students in this study were able to make and utilize viable analogies when collectively organizing a new mathematical domain.

If one takes a wider view of conventionalizing, it is a human activity that addresses a fundamental question: "How are we going to do things in this context?" In inquiry-oriented instruction, this question is very important and one that should be frequently asked, as student contributions help drive the mathematical agenda. When considering how to do things in a new context, it seems natural to draw upon one's prior experiences with conventions. At the same time, students may be unaware of the implications of adopting a familiar convention in a new setting. Yet, this is precisely the type of situation in which students can learn about mathematical conventions, which includes challenging their affirmatory intuitions about them. For instance, confronting a problematic met-before can create opportunities for students to deepen their understandings of mathematical conventions as they adapt prior conventions to new contexts. Furthermore, analogies can support these adaptations as they involve making relation-based comparisons to past experiences. In turn, analogies can inform decision-making processes by directing attention to how things function, rather than simply how they appear in a new context. As was illustrated by this study, if students are provided opportunities to create and adapt mathematical conventions in new settings, part of this process should include examining the consequences of these conventions in these new contexts and not rely solely on past interpretations of them.

Paper 3: Collective Axiomatizing as a Classroom Activity

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Abstract

This descriptive case study (Yin, 2009) explores the activity of collective axiomatizing as it occurred in a classroom setting. The context for this research occurred in a community college mathematics “bridge course”, which was designed around the implementation of Larsen’s (2013) local instructional theory (LIT) for reinventing the concept of group. The case study report identifies two findings that contribute to the collective body of work of axiomatizing. First, it elaborates upon De Villiers’s (1986) model of descriptive axiomatizing through the inclusion of pre-axiomatic activity and axiomatic creation and a discussion of axiomatic analysis at a classroom level. A second finding coordinates students’ axiomatizing with some intellectual needs (Harel, 2013) associated with these activities.

Key Words: Axiomatizing, Intellectual Need, Realistic Mathematics Education,
Group Theory, Local Instructional Theory

Introduction

In discussing the genesis of the word, mathematizing, Freudenthal (1991) states that “Mathematizing as a term was very likely preceded and suggested by terms such as

axiomatizing, formalizing, schematizing, among which axiomatizing may have been the very first to occur in mathematical contexts” (p. 30). Indeed, axiomatizing is essential to the practicing mathematician, as it serves to create and reorganize knowledge into a fundamental starting point for deductive work in both education and research. Yet, what does it mean for mathematics students to axiomatize? The purpose of this paper is to address that question through the examination of axiomatizing as a classroom mathematical activity. More specifically, if given the opportunity, how do students formulate and select axioms en route to building an axiomatic system?

Purpose of the Study

This paper makes two contributions to the literature. First, it adds to the growing body of research associated with the *TAAFU* curriculum project by describing a collective activity that is supported in the implementation of the curriculum. Previous research involving the curriculum has included the areas of: design research (Larsen, 2004; 2013; Larsen & Lockwood, 2013), teaching (Johnson, 2013; Johnson, Caughman, Fredericks, & Gibson), and scaling up design research for implementation (Bartlo, Larsen, & Lockwood, 2008; Lockwood, Johnson, & Larsen, 2013), but this study has a different take. By “zooming in” on an enactment of this robust curriculum, I explore the students’ mathematical activity of formulating and selecting rules that describe the symmetries of an equilateral triangle. In addition to offering a revision to De Villiers’s (1986) model of descriptive axiomatizing, this paper also makes a contribution in the area of RME in undergraduate mathematics education by elaborating a supportive activity for defining (De Villiers, 1997; Rasmussen and Zandieh, 2010). The duality of being able to consider a group as both the definition of a mathematical object and as a small axiomatic

system is highlighted through the implementation of the *TAAFU* curriculum. In this study, I detail the creation, analysis, and systematization that took place as a classroom community axiomatized a group enroute to creating a formal definition for this mathematical structure.

Axiomatizing Framework

Rationale for the Teaching and Learning of Axiomatizing

The primary goal of this study is to further the notion of axiomatizing by exploring how students collectively engage in this activity. Axiomatizing is an important and authentic activity of mathematicians and has been an instrumental tool for organizing mathematical knowledge for more than two millennia and continues to pervade mathematics today (Freudenthal, 1973). When we teach students to axiomatize, we are introducing them to an important aspect of formal mathematics often deemphasized in traditional instruction—that mathematics is a creative human discipline (Wilder, 1959). When students reinvent their own axiomatic system, opportunities arise to advance their mathematical enculturation through reflection regarding the beauty, purpose, and power in creating such a system.

Perhaps most pragmatically, as students progress through an undergraduate mathematics program, they must transition from computational-based courses to more proof-based courses such as abstract algebra and advanced calculus (Selden and Selden, 1995). Undoubtedly, there are many challenges associated with this transition, but one that cannot be ignored is the cognitive journey that students must undertake en route to formal axiomatic thinking (Tall, 2013). In upper-division courses, there is suddenly an increased importance placed upon proof, and thus the definitions, theorems and axioms

that constitute such proofs now become central objects of study. Yet, as Dawkins (in press) points out, “Simply ‘telling’ students the rules (‘use the definition’ or ‘only using the axioms’) does not help them understand why those rules make sense or are important to doing mathematics” (p. 2). Therefore, if we expect students to understand and reason axiomatically from these objects, the activity of axiomatizing should be considered both teaching matter and a learning matter.

Axiomatization

Mathematicians appreciate the power and elegance of axiomatization as it serves as a principal tool in all branches of the field. As an instrument of abstraction, axiomatization allows researchers to work both creatively and flexibly. Despite the increased emphasis placed upon axiomatic systems in the second half of undergraduate mathematics programs, students are rarely afforded opportunities to create a mathematical system through their own activity. Instead, students in traditional proof-based courses are often introduced to formal definitions and axiomatic systems without ever having the opportunity to participate in axiomatization before moving onto deduction. Freudenthal, a staunch opponent on the New Math movement and this formalist style of teaching mathematics, cautions that students should not be introduced to the axiomatic method using pre-structured axiomatic systems:

Should axiomatics be taught in schools? If it is taught in the form it has been in the majority of projects in the last few years, I say "no". Prefabricated axiomatics is no more a teaching matter in school instruction than is prefabricated mathematics in general. But what is judged to be essential in axiomatics by the adult mathematician, I mean axiomatizing, may be a teaching matter.

(Freudenthal, 1973, p. 541)

Extending DeVilliers's (1986) notion of *descriptive axiomatization*, the collective activity I describe in this paper involves both the formulation and selection of relations in an effort to capture everything that is essential about a structure. In the sections that follow, I elaborate my argument for the importance of teaching students to axiomatize and then review the literature on this mathematical activity.

Literature Review

Classical Versus Modern Axiomatizing

While Freudenthal (1973) acknowledges that axiomatizing can be traced back to the classical age of Greece, he cautions that the current usage of axioms is quite different from the function they served in Antiquity: "Axiomatics, as we now use this term, is a modern idea, and ascribing it to the ancient Greeks is, in spite of precursors, an anachronism" (p. 30). This distinction between "Classical" and "modern" axiomatizing can be characterized along ontological lines as illustrated by two different descriptions of Euclidean geometry. Harel and Sowder (2007) note the primary difference between Euclid's *Elements* and Hilbert's *Grundlagen der Geometrie*, which both axiomatize Euclidean geometry, is the level of interpretation each work affords. The *Elements* is organized with a single interpretation that was based on a "presumed description of human spatial realization". For the Ancient Greeks, it was essential that this axiomatic system described "reality" and that their truths were self-evident. On the other hand, the *Grundlagen der Geometrie* is open to a wide range of interpretations some of which model a physical reality, but also include a mathematical reality that is defined solely in terms of axioms. Freudenthal (1973) attributes this key distinction of modern axiomatizing to Hilbert who said that it is analogous to chess: "The pieces are not defined

by their shape, but by the rules they have to obey.” Hence, “Classical” axiomatizing relies upon formulating and selecting self-evident or observable truths, while “modern” axiomatizing is born out of convenience or logical necessity and one does not ascribe truth to the axioms that are created.

Constructive Versus Descriptive Axiomatizing

In his paper synthesizing much of the previous literature on axiomatization, De Villiers (1986), explicates two fundamentally different types of axiomatizing—constructive and descriptive, based upon the function these activities (eventually) serve. *Constructive axiomatizing* (a priori) occurs when an existing set of axioms is altered through the omission, generalization, substitution or the addition of axioms resulting in content that can then be organized into a new logical structure. Constructive axiomatizing can be illustrated historically by the systematic discoveries and subsequent inventions of non-Euclidean geometries by varying a negation of Euclid’s Fifth Postulate. Although it was likely unintentional, the primary function of these acts of constructive axiomatizing resulted in the creation of new knowledge and mathematical research domains. This knowledge was created a priori as many of the theorems that were valid for Euclidean geometry still existed and were now either true or false based solely upon the axioms of these new geometries.

On the other hand, *Descriptive axiomatizing* (a posteriori) involves the selection of a subset of essential properties that describes a concept (De Villiers, 1986). Descriptive axiomatizing reorganizes existing knowledge in an a-posteriori way because the concept and its properties are known before the concept is described axiomatically. For instance, Euclid’s *Elements* can be viewed as a massive reorganization of knowledge

into an axiomatic system that describes Euclidean geometry as it may have been understood in Antiquity. It is important to note that many of the theorems presented in the *Elements* were previously known and even later attributed to other mathematicians such as Eudoxes, Pythagorus, and Hippocrates, but it is Euclid who is credited for formulating and selecting the definitions, postulates, and common notions that resulted in the descriptive axiomatic system of plane geometry. When viewed as a process of abstraction, descriptive axiomatizing serves as a method that identifies the essential properties of a particular object or system and then codifies those relations into statements that are accepted without proof. Once the essential axioms have been identified, any relations pertaining to the object or system can then be deduced from the axioms, without reference to their original source (Krygowska, 1971).

Research on Axiomatizing in the Mathematics Classroom

The research on axiomatizing in mathematics classrooms has primarily focused on three issues. One area of research concerns theoretical arguments for teaching students to axiomatize (Dawkins, in press; De Villiers, 1986; Freudenthal, 1973; Krygowska, 1971; Yannotta, 2013), while a second set of findings report on curriculum enactments that involve student axiomatizing across different domains of inquiry: Boolean algebra (De Villiers, 1987), group theory (Larsen, 2009; 2013), ring theory (Cook, 2012), and geometry (Dawkins, in press). The third area of research explores student learning associated with axiomatizing (Dawkins, in press; De Villiers, 1987) and De Villiers's work is elaborated here to situate my study.

De Villiers (1987) reported on how advanced high school students axiomatized within the context of Boolean Algebra. The curriculum he implemented was divided into

two distinct parts by the activities of modeling and reorganization. Part A (Modeling) focused on seven problems whose solutions would provide opportunities to formulate relations that would be consistent with the axioms and theorems of Boolean Algebra. The solutions to these seven problems were explored and revisited over several class periods through analyses of circuit diagrams, truth tables, and symbolic relations. Toward the end of the Modeling unit, the students were asked to make a “systematic summary” of all the properties of switching circuits, which De Villiers reorganized into a “useful reference” for their mathematical model (p. 52, 2010).

In Part B of the curriculum (Reorganization), De Villiers gathered all of the relations that had emerged from the students’ solutions and subsequent reflections and reduced them to a list of eight statements for the class to investigate. Unlike the first part of the course in which the validity of a relation was verified by exhaustion using a truth table or through circuit diagrams, the students were asked to use the set of eight statements to prove two additional “useful statements” and later to explore the logical implications among the statements. De Villiers (1987) noted that the students had little trouble proving the statements, but to assist in axiomatization, he scaffolded the analysis tasks using axiomatic diagrams (see Figure 11).

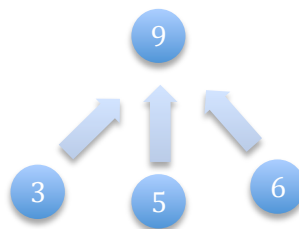


Figure 11: Axiomatic diagram adapted from De Villiers (1987) showing the relations among statements 3, 5, 6 and 9

The diagram is set up hierarchically from bottom to top, demonstrating that axioms 3, 5, and 6 are used to prove statement 9. De Villiers (1987) noted that in the process of constructing these axiomatic diagrams, students also gave attention to completeness and independence of the system they were creating. Over time, the students were not only able to build a suitable diagram that axiomatically described Boolean algebra, they also constructed alternate diagrams based upon simplicity, ease of deduction, and elegance (De Villiers, 1987, p. 532). A more complete illustration of this work is found in De Villiers (1986), which makes a compelling case for teaching axiomatization and provides a general model of descriptive axiomatizing overviewed in Figure 12. During the activity, logical relationships between unrelated or partially related statements are analyzed and then (re)organized and systematized into an axiomatic framework (De Villiers, 1986). Although De Villiers did not elaborate on *how* the analysis or synthesis occurred at the collective level, the model of the activity provides an overview of descriptive axiomatization.

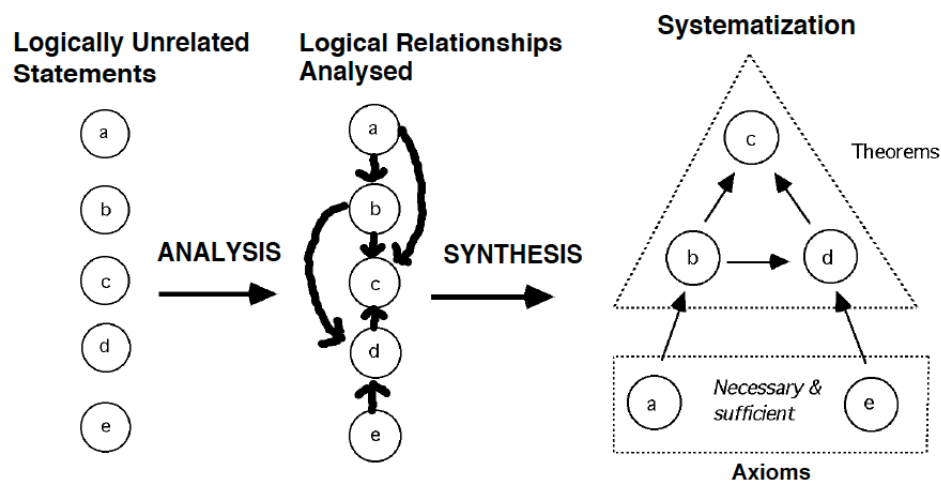


Figure 12: De Villiers's model of descriptive axiomatization

Intellectual Need

Harel's (1998) Necessity Principle states, "Students are most likely to learn when they see a need for what we intend to teach them, where by 'need', is meant intellectual need, as opposed to social or economic need" (p. 501). The Necessity Principle puts forth a conjecture about how students learn (Speer, Smith & Horvath, 2010) and has been used extensively by Harel as a component of a larger conceptual framework called DNR (Duality, Necessity, and Repeated-Reasoning) (Harel, 2001). Harel (1998) initially delineated three forms of intellectual need: *computation*, *formalization*, and *elegance*. The need for computation and the need for formalization have been elaborated upon in the literature, but the need for elegance, which Harel (1998) describes as "what we associate with mathematical beauty, efficiency, and abstraction" (p. 502), has not been given much attention beyond this description. These initial forms of intellectual need have undergone considerable refinement by Harel (2001; 2008; 2013) to form five inextricably-linked categories: *the need for certainty* (to establish that a statement is true), *the need for causality* (to determine why a statement is true), *the need for computation* (to quantify and calculate), *the need for communication* (to persuade others of truth and to agree on conventions), and *the need for structure* (to re-organize knowledge into a logical system).

Intellectual Need as a Tool to Analyze Student Learning

One way of framing Harel's categories of intellectual need is that they offer an explanation and in some cases, more than one explanation, for why individuals engage in certain mathematical activities. For example, proving is an activity that both students and mathematicians may engage in to satisfy a need for certainty, but this is not the only need

that may be involved. While both students and members of the mathematical community view proof as a form of *verification* (De Villiers, 1990; Hanna, 1990; Tinto 1990), mathematicians also use proof as means for: *explaining* (Balacheff, 1988; De Villiers, 1990; Hanna 1990; Hersh, 1993; McCrone and Martin, 2009), *systemization* (De Villiers, 1990; Weber, 2002), *discovery* (De Villiers, 1990; Lakatos, 1976), and *communication* (Balacheff, 1988; De Villiers, 1990). These different functions of proof can be correlated with Harel's categories of intellectual need as explaining serves a way to address a need for causality, systematization as a method to structure, and the proof itself as a way to communicate an idea, a method, a result or all three. Although Harel (2013) notes that many the core activities of mathematizing associated with Realistic Mathematics Education, namely generality, certainty, exactness, and brevity (Gravemeijer, Cobb, Bowers & Whitnack, 2000) correlate with the Necessity Principle, it is not clear how specific intellectual needs influence the mathematical activity of axiomatizing.

Methodology

Research Questions

The purpose of this study is to explore collective axiomatizing as it occurred in a whole-class setting. My primary research question is: How do students engage in collective axiomatizing? By examining this activity in the context of abstract algebra, I seek to build on what we already know about axiomatizing. In an effort to better understand the mechanisms of axiomatizing a mathematical system in a classroom setting, I intend to elaborate upon De Villiers's model of descriptive axiomatization (1986) by analyzing how students collectively engage in this mathematical practice.

Research Setting and Participants

The data for this research was taken from selected classroom episodes of a mathematics “bridge course” that was taught at a suburban community college in the United States. The participants included thirteen community college students (three female and ten male) whose ages ranged from 17 to 48 years and their teacher, Brian. Brian was a full-time community college mathematics instructor with more than fifteen years of combined experience at the high school and two-year college level, but had never taught group theory. The *Teaching Abstract Algebra for Understanding* (TAAFU) curriculum, which was used in project, is a research-based, inquiry-oriented abstract algebra curriculum that actively engages students in developing many fundamental concepts of group theory (Johnson, 2013). The research reported here details the class’s enactment of much of the first unit of the *TAAFU* curriculum, which is based on Larsen’s (2013) local instructional theory (LIT) for reinventing the group concept.

Data Analysis

Following a preliminary analysis, I concluded that De Villiers’s model of descriptive axiomatizing globally *described* what the students were doing in the *TAAFU* classroom episodes, but there were still elements of axiomatization that warranted further analysis. For instance, in addition to examining the remainder of the axiomatizing sequence, I was curious how axiomatic initiation had occurred and how and why the students chose certain rules instead of others. Therefore, I made a few more passes through the data using Lesh and Lehrer’s (2000) framework for iterative videotape analysis. First, I established the bookends of the collective axiomatizing sequence by identifying the initial codification of the relation $FR = RRF$, which Brian recorded on the

whiteboard. This contribution was selected as the starting point for collective axiomatizing because it was the first time an equation had been recorded in the public space. The ending point of the axiomatizing sequence was defined to be the class's axiomatization of an abstract group on Day 15, which was an implementation of the last step in the LIT. After identifying the boundaries of the activity, the class sessions spanning Days 7 – 15 were then fully transcribed. Each class period was approximately 1 hour and 20 minutes in length and the total transcript video spanned eight different instructional days.

In each class session, I identified any student contribution that related to a restructuring of the collective artifact—the class's list of rules. I, along with a second researcher, then coded these student contributions independently to identify those in which a student provided a reason for his or her contribution. For each contribution in which a student's reason could be identified, we each coded it with one or more categories of intellectual need: causality, certainty, communication, computation, (Harel, 2013), connection and [or] structure (Harel, 2008), and elegance (Harel, 1998). During the process of coding, the new category of *generalization* emerged and was operationalized as “to infer from a specific situation”. The second researcher and I then conducted a joint-analysis concerning those student contributions in which one or both of us had identified a reason associated with a student's contribution. Using an agreement/disagreement protocol, we determined that very few of those contributions actually satisfied Harel's (2013) criteria for observing intellectual need, but we were able to identify tasks in the curriculum and some impromptu revisions made by the class that pointed toward intellectual need.

Construction of the Axiomatizing Analytic Document

I then turned my attention back to the contributions that directly affected the reorganization of the axioms. Aided by transcripts and video data, I reviewed each classroom session in an attempt to capture all of the changes to the axioms that were made in the public space over the course of these eight class days. I then logged each of these changes, often including a still video capture of the white board along together with a short description of each change. This pass through the data culminated in the development of the Axiomatizing Analytic Document (AAD), that has served as the descriptive repository of the entire axiomatizing sequence as it occurred in Brian's class (see a sample of the AAD in the Appendix). I then regrouped some of these revision sequences of the public artifact into larger axiomatic episodes. The AAD was further divided into two subsections for each axiomatic episode: a data description section and a data interpretation section. The data description sections contain a brief summary of the axiomatic episode that occurred accompanied by primary data, which could include: a transcript excerpt, a still capture extracted from the video data, or a digital photo that Brian captured in real time on his iPhone. The data interpretation sections vary in length depending on the axiomatic episode and consist of my interpretation of what happened. In each section, I sought to make warranted claims that would help explain what took place in the episode and to situate this event in the larger scheme of axiomatic development. As individuals collaborated to construct a collective artifact, changes to the class's list of rules became the primary object of inquiry. I then went back through the revised AAD and the transcripts, condensing some sequences and expanding others to create this descriptive case study report of collective axiomatizing.

Case Study Report: The Evolution of a Collective Axiomatic System

Like at the beginning, we were given, you know, here are some symmetries of a triangle, and we kind of like created our own ... our own set of mathematical rules to describe that. So instead of learning um, what's already been established, we kind of created our own mathematics. So yeah, that was kind of cool. — Arthur

Pre-Axiomatic Activity and Axiomatic Initiation

Like Zandieh and Rasmussen (2010), who include other types of activity as a part of defining, I consider supportive activities as part of axiomatizing. For instance, there were activities outlined in the LIT that nurture axiomatization prior to the students' formulation of recognizable rules in the public space. First, the students had to conventionalize (Yannotta, 2016b), which involves negotiating a set of symbols and representations and establishing procedures for interacting with them. In this case, the class adopted a multiplicative convention for representing the six non-equivalent symmetries of an equilateral triangle. Conventionalizing was directly related to axiomatizing in that it determined in large part, the symbolic representation of the axioms that were formulated, but it was also a supportive activity throughout the axiomatizing sequence as the students made revisions to the collective artifact. The activity of symbolizing (Gravemeijer, 2004) also played a role as the students chose a representation for each of their six symmetries and like conventionalizing, continued to support axiomatization later in the sequence. As noted in Figure 13, the boxed representations were the ones the class selected by majority vote, but the students formulated alternate versions of RF (FR^2), FR (R^2F), and I (R^3 , F^2 , and 0) in the public space. Although there was no equal sign between these different representations, their juxtaposition foreshadow some relations that later became axioms for combining symmetries.



Figure 13: The seeds of some axioms

Discussion of these equivalence relations progressed further through the activity of combining two symmetries, which corresponds to Step 2 in the LIT. Brian related this salient task to the students in his own words, “For every possible combination of two of these symmetries, I want you to work out which symmetry you will end up with.” The students began this activity in class and then were asked to share their results at the beginning of the next class period.

Attributes of A Record of are Identified

Consistent with the reinvention process described by Larsen (2013), the students exhibited a variety of organizational strategies for calculating the thirty-six symmetry combinations, but one student’s construction of an operation table (Figure 14) served as the primary impetus for axiomatizing when Brian asked the students to make some observations about it.

	FIRST					
	I	R	R^2	F	FR	PF
I	I	R	R^2	F	FR	PF
R	R	R^2	I	FR	PF	F
R^2	R^2	I	R	PF	F	FR
F	F	PF	FR	I	R^2	R
FR	FR	F	PF	R	I	R^2
PF	PF	FR	F	R^2	R	I

Figure 14: Todd's operation table for combining symmetries

- Brian: What are some of the things that you noticed whether I write it in that sort of table or this sort of table, what are some of the patterns that you noticed?
- Alice: The first and last are opposite.
- Brian: I'm not sure what you mean by that.
- Alice: The row is opposite the last one.
- Brian: Oh. So the first line here... and the last line there...
- Teresa: It's the same for F (noting the fourth and second rows)
- Student: (hmm.... cool)
- Brian: I have to say I didn't notice that (laughing).
- Kevin: You don't get the same entry on any one row or any one column.
- Todd: Hmm...
- Brian: Can you explain what you mean by that?
- Kevin: You don't... So if you are looking at just a single column, you won't see...
- Brian: So, say down this column (points to 2nd column)?
- Kevin: Yeah. You will only see one I , one R , one R^2 . You won't see two of any entry... and that happens for rows as well.

Todd's operation table served as his individual *record of symmetry calculations*, which I claim that Brian then used as a transformational record (Rasmussen and Marrongelle, 2006) for axiomatic initiation, as he invited the class to comment on some of the features of this artifact. During this initial examination of Todd's table, the

students seemed to be focused on its aesthetic properties. For instance, Alicia and Teresa made observations about the order patterns of particular rows that were specific to Todd's table. While the mathematical phenomena related to these observations were beyond the scope of the course, Kevin's remark that there are no repeated entries in any row or column—known as the Sudoku or Latin square property—pointed to a property that plays a critical role later in the LIT. Brian redirected the students' attention to comment on some specific computations.

Specific Relations are Formulated Verbally

- Brian: ...I'm not trying to get an exhaustive list right now, just examples of some of the short cuts or properties that people were using. Did you have anything in particular?
- Todd: I was looking at rotates after a flip went apparently backwards. That is to say...
- Brian: So what do you mean? Rotate after a flip...
- Todd: Well, maybe I'm stretching out on a limb here, but after a flip rotate clockwise...it's just like taking before the flip and rotating counterclockwise two and then flipping over.
- Brian: How would I write that?
- Todd: Good question.
- Brian: Let's just walk through this. You told me, "If you flip..." I've got my triangle in my pocket.
- Todd: Flip it and then rotate, it's the same thing as rotating twice
- Brian: If I try this...If I flip
- Todd: ...and then rotate, it's the same thing as rotating twice and then flipping (Brian writes FR and RRF with a space in between).

During the previous exchange, Todd expressed an equivalency in terms of flipping and rotating, but he seemed challenged to formulate a symbolic representation for the relation he was describing. Brian responded to Todd's contribution by first demonstrating the relation he was describing with the triangle and then wrote the expressions FR and RRF next to each other on the white board, leaving a space between

them. Another student joined the discussion and contributed a different relation that he observed.

The Instructor Models Rule Formulation by Writing Equations

- Jeremy: If you have a flip rotate and a flip rotate then you're going to get the identity because they undo each other.
- Brian: Um, just to make it clear that I am talking flip rotate, then flip rotate, I'm going to put parentheses around that (Brian writes $(FR)(FR)$ on the whiteboard). Flip (demonstrating with the triangle) and then rotate and then flip... and then I did it again...flip, and then rotate. That, that's the same as...as which?
- Jeremy: Where it started. It's the identity (Brian writes I next to the expression $(FR)(FR)$). So you do a rotation after a flip, it's reversed.
- Brian: What I'm going to do is I'm actually going to write equals (starts writing equals signs between the expressions FR and RRF and $(FR)(FR)$ and I) when things work out the same. This is the kind of shortcut I was alluding to... I'm just writing down what Todd told me (points to $FR = RRF$) and writing down what Jeremy was telling me (points to $(FR)(FR) = I$).

By noting that a flip reverses the direction of a rotation, Jeremy was able to provide some insight into why FR combined with itself produced the identity, but like Todd, he did not explicitly mention equality with his contribution. A key moment in axiomatization initiation occurred when Brian formulated two equations on the whiteboard by inserting “=” between the two pairs of juxtaposed symbols FR and RRF and $(FR)(FR)$ and I that he had written earlier. Brian clarified that this is what he meant by “shortcut” and justified this symbolizing activity by attributing it to the students’ previous contributions. The students were then asked to make their own operation table—similar to Todd’s, keeping tracking of patterns they noticed as they filled it in. After working for several minutes on this task, Brian invited Gene to present the method he used to compute his symmetry combinations (see Figures 15 and 16).

A Record of Calculation becomes a Potential Tool for Rule Formulation

Gene: Yeah, I kind of let the results drive it. So,...

Brian: So you jotted all of those down.

Gene: I used substitution kind of and then set all of those equal to each other, since they are equal to FR , they are equal to each other...

$$\begin{aligned}(F)(R) &= FR \\ (R)(RF) &= FR \\ (R^2)(F) &= FR \\ (RF)(R^2) &= FR \\ (FR)(I) &= FR \\ (I)(FR) &= FR\end{aligned}$$

Figure 15: A recreation of Gene's rules for FR

Gene: So, going with what I had on that last page, if you go with FF and notice that I , here, $FF = I$, $FRF = I$, and then I just pick all of those out.

Brian: So, I down here and you recognized these are the different ones that end up equaling I .

Gene: Right. So on the next page, that's when I set them equal. That first, uh, horizontal line there is all of the ones that equal I .

Todd: FRF , FRF , the second term in that line, I question that one. I'm a little worried about that.

The image shows a piece of paper with handwritten mathematical expressions. The top line contains a series of equalities: $(F)(F) = (F)(RF) = (R)(R^2) = (R^2)(R) = (RF)(RF) = (FR)(FR) = (I)(I)$. The second line has a horizontal line under the first term, followed by $(F)(R) = (R)(RF) = (R^2)(F) = (RF)(R^2) = (FR)(I) = (I)(FR)$. The third line contains $(F)(R^2) = (R)(F) = (R^2)(FR) = (RF)(I) = (FR)(R) = I(RF)$.

Figure 16: Gene's strings of calculations

Gene's artifacts taken together with his comment, "I kind of let the results drive it", suggest that he started with a symmetry pair and then attempted to determine the other symmetry combinations that produced the same result. He then set each one of these combinations equal to each other to form a string of equivalent expressions for a given symmetry (Figure 7). In the top line of equivalences that produced I , Gene found

six symmetry combinations in addition to a seventh one, $(F)(RF)$, which Todd immediately challenged. Despite this miscalculation, Gene's idea to construct a string of equivalent expressions marks another important advancement in axiomatic initiation. In addition to serving as a *record of* his calculations, these strings of equivalent expressions also foreshadow how substitution could be used as a *tool for* formulating new relations. As Gene noted, because "they are equal to FR , they are equal to each other", any one of these expressions could now be replaced with one equivalent to FR .

Axiomatic Creation

I'd never heard of the word axiom before this class, that's for sure, which seemed to be the entire class. –Randall

A Record of Calculation becomes a Tool for Calculation

Following the discussion of Gene's work, Brian invited another student to share his methods of calculation (see Figure 17) with the class. Kevin explains how he used the relation $FR^2 = RF$ to calculate $(FR)(RF)$.

Kevin: So I figured out that FR^2 is RF , just kind of by computing it out. But, I made a connection, which was that, you know, two rotates after a flip is the same as rotate before a flip and then that kind of let me to think... well I can rearrange how I want to do things and then kind of got, which what Gene was getting was that, things are all equivalent.

Brian: FR^2 is the same as...?

Kevin: RF . So if I move RF to where FR^2 is, that simplifies and then I get two F 's, which F^2 is I and so R times I is R . And that's kind of how I went about it—just substituting things in that we've already figured out.

	I	R ²	R	F	FR	RF
I	I	R ²	R	F	FR	RF
R	R	I	R ²	RF	F	FR
R ²	R ²	R	I	FR	RF	F
F	F	RF	FR	I	R	R ²
FR	FR	F	RF	R ²	I	R
RF	RF	FR	F	R	R ²	I

$$FR^2 = RF$$

$$R^2F = FR$$

$$R^3 = I$$

$$F^2 = I$$

$$FRRF = (FR)F = RFF = R$$

$$RFFR = RF^2R = R^2R = R^2$$

$$RFR^2 = R(FR) = RRF = R^2F = RF$$

Figure 17: Kevin's calculations

Kevin's contribution exhibits more advanced mathematical activity when compared to the previous work that has been shared. Kevin was able to articulate how he computed the entry in row F and column R^2 as well as some of the more complicated entries of the table, such as $(FR)(RF)$, through the use of substitution. I inferred that Kevin misspoke when he said he moved " RF to where FR^2 is" because his work illustrates the exact opposite in that he substituted FR^2 into the calculation $(FR)(RF)$ in place of RF . Nevertheless, both his words and his actions point to a shift in his activity that is important. Kevin adapted some of the initial relations he calculated into a *tool for* computing other combinations.

Collective Axiomatizing

An Individual's Record of/Tool for Calculation becomes a Collective Record of

Following Kevin's explanation, Brian called attention to the first four relations that Kevin had written to the right of his table: $FR^2 = RF$, $R^2F = FR$, $R^3 = I$, and $F^2 = I$.

- Brian: So what I'd like you to do, by now you've been working out those tables, let's figure out what list of things we've got. Um, Kevin has a list of four facts here. Did you use these four to figure out some of these others or did you need...
- Kevin: Yeah.
- Brian: Did you need any other, other, um, entries from the table to jot down to help you figure these out or was, or were those four enough?
- Kevin: I think that's what I was working with, just those four.

Brian: Are there other properties we should list? (Brian copies Kevin's four facts on the board.)

Brian's decision to record Kevin's "facts" $FR^2 = RF$, $R^2F = FR$, $R^3 = I$, and $F^2 = I$ on the whiteboard marks a critical shift in activity in the classroom as he redirected the mathematical agenda toward formulating a collective list of properties. Although axiomatic initiation began earlier with observations about Todd's table, the class is now formally invited to engage in a form of vertical mathematizing (Treffers, 1987), which I call axiomatic creation. As a component of descriptive axiomatizing, this activity consists of formulating logical statements about a known structure. In this case, these "facts" have captured two versions of the dihedral relation as well as two equations involving the identity. Now that Brian has recorded some of these relations, Kevin's statements begin their transition into a collective *record of facts* for computing symmetry combinations. When Brian asked for more of these rules, Todd mentions the calculation $(RF)(FR) = R^2$ in Kevin's work as well as $(R^2)(RF) = F$, which Brian records on the board.

Stealth Properties are Identified

Following Todd's contribution, Brian continues to encourage the activity of axiomatic creation by inviting others to formulate more statements. Brian is particularly interested in identifying any "stealth" rules that the class has been using.

Brian: Have we been using other ones, maybe some stealth properties, that we've been using without noticing or? (long pause) I would like to ask that question, cuz the answer is yes.

Todd: Um, is that like the law of associativity? The fact that we can pick stuff off in order?

Brian: Have we've been using that? I think it was referred to earlier.

Teresa: Yes.

Brian: (writes Associative Property) Um, I can write the words, how do we write it in symbols? Remember that list from way back?
 Kevin: A times B times C is the same as A times B (pause) AB times C is the same as A times BC (Brian writes $ABC = (AB)C = A(BC)$).
 Brian: That's how I interpreted your pauses. Is that what you meant?
 Kevin: Yeah.

Brian guided the students toward considering listing some “stealth” properties that may have been used in the calculations. Multiple students contributed to the creation of the associative property as Todd identified it as a property they had been using, which Kevin then formulated symbolically.

A Record of becomes a Tool for Formulating Stealth Properties

Brian continued to support the collective activity of axiomatic creation by asking the class to consider the properties that were used in Kevin's calculation of $(RF)(FR) = R^2$.

Brian: So, when I'm writing this: $RFRR$, let me actually throw those parentheses in (writes $(RF)(FR)$), we know one of those things that's going on is flip flopping around parentheses. Aside from that, how do I get from here to here (writes $R(F^2)R = RIR$) What rule is being used?
 Alice: The F^2 equals I .
 Brian: From here to here ($RIR = RR$)?
 Kevin: R times I is the same as R and then...
 Brian: Oh, I haven't written that yet (writes $RI = R$).
 Todd: Or we could write $IR = R$. (Brian writes $IR=R$) It really doesn't matter in this case.
 Brian: I just want to make sure I'm writing down the properties we're using—that we're not sneaking properties in.
 Kevin: Then we're using R times R is R^2 (Brian writes $RR = R^2$)

There is a great deal of axiomatic activity happening in the classroom at this time. First, Brian refined the target from identifying rules that the students had been using to complete their tables to identifying the specific rules that were used in each step of one of Kevin's calculations. Following Alice's contribution, Kevin's subsequent formulation

resulted in a new rule $RI = R$ that was quickly followed by Todd's version $IR = R$, which demonstrates the left and right identity properties on R .

Generalizing a Rule

Kevin referred to the last step of the calculation $(RF)(FR) = R^2$ and formulated the rule $RR = R^2$. Jim then turned his attention to formulating rules that were true, but not germane to this calculation.

- Jim: F times I , is then F . IF is F too (Brian writes $FI = F$ and $IF = F$).
 Kevin: Anything times I is what...
 Teresa: is left that way.
 Kevin: ...it is. It's the identity.
 Brian: Okay, so you're saying that kind of these four facts (circles $RI = R$, $IR = R$, $FI = F$ and $IF = F$) could be written in a more shorthand (see Figure 18)?

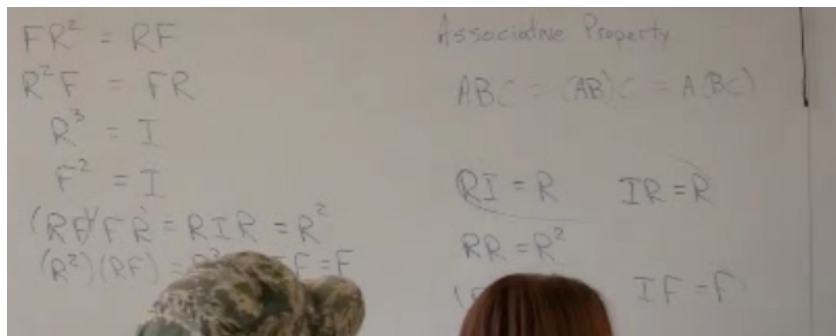


Figure 18: A list of twelve statements

- Kevin: I hope so.
 Jerry: $NI = N$.
 Brian: (starts writing) NI ...
 Jerry: Yeah. Equals N .
 Todd: Equals IN .
 Jerry: Yeah.
 Todd: So, order doesn't matter in this instance (Brian puts a box around $NI = N = IN$).

Jim's contribution is key in moving toward a general identity rule because it shifts the process from rules that were used in a calculation on the board to other properties that are true about the system. Jim's observation that $IF = F$ begins a generalization process

that is quickly picked up by other members of the class. The criteria for axiomatizing the identity rule is not deductive, but inductive, as the class attended to the economy of formulating a rule that would apply to all symmetries. When Brian capitalized on Kevin's contribution, Jerry and Todd then worked together to formulate an algebraic rule $NI = N = IN$, which Brian recorded on the board and then placed a box around.

A List of Seven Axioms is Selected

Brian's decision to place a box around the identity rule $NI = N = IN$ signifies an important shift in axiomatizing as the students' previous activity is selected for further mathematization. Following this initial act, Brian enclosed six other statements each within a box (Figure 19), and then erased the four specific identity statements ($IR = R$, $RI = R$, $IF = F$, and $FI = F$).

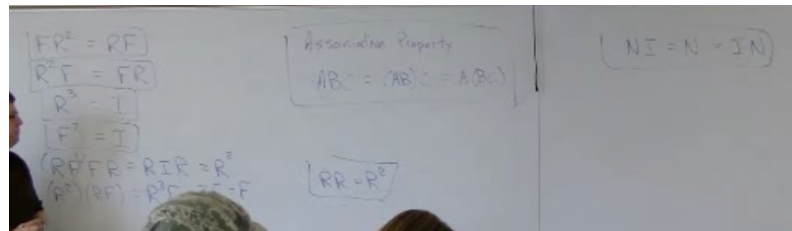


Figure 19: A list of seven axioms

While the students have yet to use the term axiom, Brian's selection and deletion of these particular statements signifies that some rules are more important than others, thus setting the stage for further development of the class's axiomatic system. Brian's decision to leave the statements $(RF)(FR) = R^2$ and $(R^2)(RF) = F$ on the whiteboard, but unboxed is a bit of a mystery. One conjecture for not boxing $(RF)(FR) = R^2$ is that Brian may have viewed this statement as a theorem because Kyle had already demonstrated the derivation of this rule. The second statement, which was not reflected in Kyle's work,

was more trivial and perhaps Brian left it off because it did not add to the system of rules he had promoted. Nevertheless, Brian did not select either one, nor did he explain why he made this decision.

Is the List of Rules Sufficient for Completing the Operation Table?

Another significant transition point in axiomatization occurs when Brian asked the class to consider whether this list of seven rules that he selected is long enough to complete the operation table. This is the first in a sequence of eight *systematization tasks* that tested whether the system of rules accomplishes the collective work that needs to be done. Kevin's solution to ensuring the class could complete the table was to add another rule to their list.

- Brian: What I'm curious about first, before I try and make this list shorter, is I want to know, is it long enough? Could I fill out this table without ever looking at a triangle, but using these rules?
- Gene: If we have all of the rules, you should be able to....
- Brian: Where are you worried? Where on this table are you concerned that maybe we don't have the machinery?
- Todd: In the lower right-hand corner, right there.
- Brian: (laughing) Yeah. The bottom right-hand corner is trouble and everyone knows because you've all filled this in a couple of times. So uh, let's take a look at some of the bottom right-hand corner.
- Kevin: We do have this one in green over there that you didn't circle and we could fill the lower right-hand corner with that. On your right, Brian.
- Brian: Oh, this one. Oh, so maybe we should include that one too. FR, FR (Brian writes $(FR)(FR) = I$ near the other boxed rules)

Rather than immediately exploring whether their set of seven rules was sufficient for completing the table, Kevin suggested adding a statement to their list for filling in some of the more complicated entries in the table. This activity is different from axiomatic formulation because the rule Kevin mentioned was already on the whiteboard. Brian had written the rule $(FR)(FR) = I$ earlier in the class period, when he modeled

axiomatic formulation with an “=” sign. Instead, Kevin’s suggestion to add another rule signals a possible axiomatic revision to the initial system. Brian seems amenable to this suggestion and recopied it near the other rules. However, Brian did not place a box around it like the other seven statements.

One Student’s Record of becomes a Collective Tool for Axiomatic Revision

Todd continues to make suggestions on how to revise their list of rules, citing Gene’s previous string of equalities as a way to create a lengthier rule involving combinations of symmetries that produce the identity.

- Todd: Could we just take and combine three of those together in one piece? How about $R^3 = F^2 = FRFR = I$ —make it all on one line.
- Kevin: And then $RFRF$ too.
- Brian: I’m sorry, I was all into the technology, could you say that again, Todd?
- Todd: Kind of like what Gene did, just pile up all of those I ’s up into one big line.
- Kevin: And then, $RFRF$.
- Teresa: $RFRF = I$
- Todd: Yes, let’s pile all the identities on one line.
- Brian: R cubed equals F squared equals? (writes $R^3 = F^2 =$)
- Students: $FRFR$
- Kevin: Equals $RFRF$ (Brian finishes writing Todd’s suggestion $R^3 = F^2 = (FR)(FR) = I$ and then stops.)

Todd’s reference to Gene’s contribution seems to provide encouragement for other members of the class to help him create a single rule involving combinations that produce the identity.

Should We Add this Useful Fact as An Axiom?

Although Brian initially adds $(FR)(FR)$ to the revised rule involving I , he stops abruptly and challenges the students to deduce $(FR)(FR) = I$ from their existing set of rules.

Todd: There's one more that goes with that set I think.
 Kevin: Yes, $RFRF$.
 Brian: (circles the rule $(FR)(FR) = I$) Do we need this one?
 Kevin: (shrugs his shoulders) I like it.
 Brian: What I want to challenge you guys to do right now is see if you can calculate that using one of these rules, using some collection of these rules we've already got...Everybody work on your own. Private think time for a moment.

Despite the students' collaborative efforts, Brian changed his mind about revising the rule $R^3 = F^2 = I$ to include $(FR)(FR) = I$, which he assisted in formulating earlier. Brian instead asked the students to deduce this statement from the existing seven rules, but it is unclear why Brian charged the students with this task. Perhaps he was anticipating the minimization task later in the curriculum or maybe he thought the class was trying to add too many rules to their list; but in any case, this unexpected teacher move marks a pivotal moment in the axiomatizing sequence. With this challenge, Brian raises the question that some rules may be redundant and encourages the students to begin using deductive criteria for selecting axioms. Furthermore, in contrast to the previous systematization task that addressed axiomatic completeness, this impromptu systematization task addresses axiomatic independence.

An Axiomatic System Emerges

After combining $R^3 = I$ and $F^2 = I$ into a single rule, Brian organized the six rules into a numbered list (Figure 20). The ordering of the list is another structural element of the system that supports further axiomatizing as members of the class can now refer to an axiom by number instead of having to restate the relation.

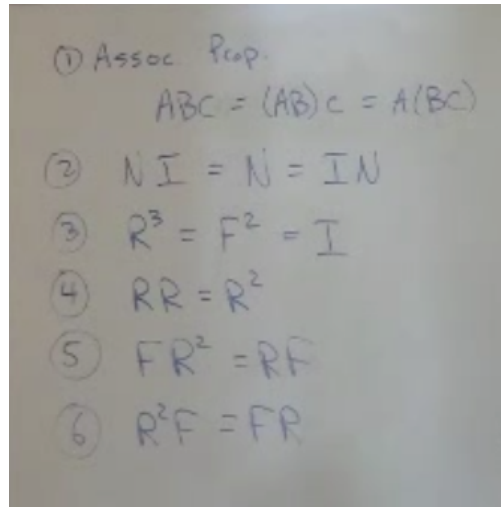


Figure 20: Axioms v. 1.0 (an ordered list of rules)

During this time, the students worked quietly for about five minutes until Brian invited one student to share his proof of $(FR)(FR) = I$ at the board (Figure 21).

Arthur: Okay so I can use this FR (points to Rule 6), and er, yeah, and replace the first one with R^2F and just leave the second FR alone. And then I see that, um, F , from that one, (points to Rule 1: Associative Property) I can see that if I just move these parentheses make this F and F and that will become an I . And then, R here. And then we know that I times anything on any side is just going to leave us with what's left, so that leaves us with, um R^3 , and that leaves us with I .

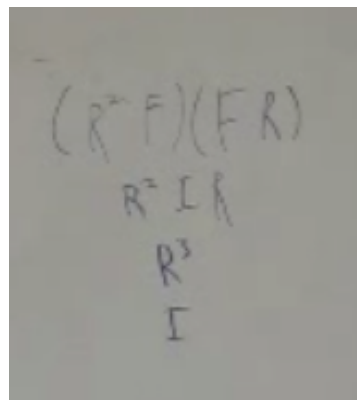


Figure 21: Arthur's proof that $(FR)(FR) = I$

Following Arthur's explanation, Brian asked the class to comment on each rule that Arthur used in his proof.

Teresa: You have to use six.
 Brian: Yeah, but we never actually called it by name. He definitely used property 6 as his first step and then...
 Kevin: And then he combined one and three.
 Brian: And then?
 Gene: Rule of exponents then.
 Brian: Yeah, now we wrote $RR = R^2$ as a rule, should we write like $R^2R = R^3$ as a rule?
 Jim: Yeah, maybe we should write that all as one rule.

Brian's construction of the organized list of rules was consistent with the statement selection process described by De Villiers (2010), who took his students' results and statements and put them into a numbered list of properties for future deductive work. Later, when validating Arthur's proof, it was discovered that he used the "stealth rule" of $R^2R = R^3$, which prompted Brian to ask the class whether they should add it to their list. In terms of axiomatizing, the identification of this rule fostered the subsequent revision of the rule $RR = R^2$. Of particular interest is the ensuing discussion about this rule, which includes three very different responses to Jim's suggestion of adding it to their list.

Generalize the Base so the Rule of Exponents Applies to Any Symmetry

Brian: (writes $R^2R = R^3$) Any other variations we should have?
 Gene: Just a suggestion—take it to the general case and you can get rid of all that.
 Brian: (scratches out the two rules) Okay, so what would the general case be?
 Kevin: N times N equals N squared or N times N squared equals N to the third.
 Brian: I'm not sure that would be much different from writing them with R 's.
 Kevin: Well, then it would apply to F as well.
 Brian: Yeah.
 Jim: Yeah. It kind of makes sense with N .
 Kevin: You wouldn't have to write F . (Brian writes $NN = N^2$, $N^2N = N^3$)

Kevin's suggestion seems to be focused on the types of symmetries to which the rule applies. By suggesting that the R 's be replaced by N 's, he was attending to both combinations of R and F . Another student then contributed an idea of her own.

The Rule of Exponents is “Obvious” and Should Not be on Our List of Rules

- Jenny: Why do you have to write that answer anyway? Isn't it so self-explanatory?
Brian: That's a good question.
Jeremy: But then you have N times N times N equal to N cubed.
Jenny: Yeah, but you can, do you really have to have that as a property—it's so obvious.

Despite Brian privileging Jenny's concern that the rule is “self-explanatory”, her objection was not explored at this time. It could be that she saw this rule as a consequence of their notational choice or that she viewed this rule as a true statement that was so self-evident, that it did not warrant being listed as part of the class artifact. Nevertheless, she was in the vocal minority, quite possibly because this rule was already accepted as one of their rules and was on the mathematical agenda to be generalized.

Generalize the Exponents so the Rule of Exponents Applies to Any Number of R 's

- Gene: How about R to the m plus...or, er, um, R to the m not times, but, you know operate on it, R to the n , equals R to the m plus n (Brian writes: $R^m R^n = R^{m+n}$).
Kevin: There you go.
Teresa: I like that, it's better.
Brian: Do we like that more?
Teresa: Yeah. (Brian crosses out Kevin's rules involving N)
Todd: That covers both of those cases and the other case. Yeah.

Gene's suggestion seems more compatible with what Brian may have originally anticipated as a generalized rule for adding the exponents of rotations, which like rule 4, only deals with R . However, Brian put it back to the class to decide whether this rule should only apply to R .

Generalize Both the Base and the Exponents

- Brian: Is that for just R 's or is that for R 's and F 's?
 Gene: Yeah, it's just for R ...
 Brian: (crosses through $R^m R^n = R^{m+n}$) So maybe change this to...
 Todd: General case for N 's.
 Brian: So what should I use?
 Todd: Huh. (laughing) How about n, p, q ?
 Brian: (looks confused) Well, if we want to be with both R 's and F 's, we don't want to use R , but we could use...
 Jim: C .
 Gene: C .
 Jim: C for constant (Brian writes C)
 Student: How about A ?
 Todd: No, but I'm pretty sure A is reserved. I was going to say N to the p , N to the q ,...
 Jeremy: Yeah, that would work (Brian erases C and writes: $N^p N^q = N^{p+q}$).

As the students continued to refine this rule, Brian suggested that they reconsider their variable choices. While C and A were both offered as bases instead of N , Todd opposed these suggestions and instead formulated the general rule of $N^p N^q = N^{p+q}$, incorporating both Kevin and Gene's suggestions. Brian recorded this chain of revisions until rule 4 was axiomatized into the generalized law of exponents $N^p N^q = N^{p+q}$ (Figure 22).

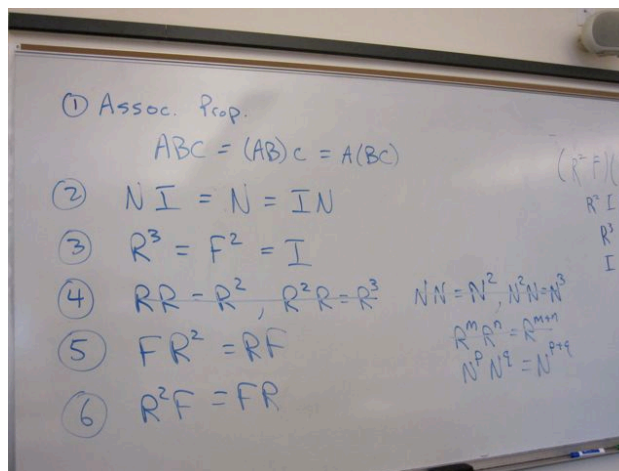


Figure 22: Axioms v. 1.1 (generalizing a rule)

The generalization of the rule of exponents was a collective effort and demonstrated the influence of both deductive and inductive processes when axiomatizing. This activity began deductively when the relation $R^2R = R^3$ was identified as a “stealth rule” in Arthur’s proof. In order to make the axiomatic system complete for proving $(FR)(FR) = I$, the equation $R^2R = R^3$ was appended to the existing rule $RR = R^2$. While Jenny did not see the need for having a rule of exponents in their list, Gene’s suggestion to “take it to the general case”, moved the mathematical agenda forward. Perhaps what is most interesting about this revision is that two different students—Kevin and Gene, produced different, yet viable ways of generalizing this axiom, which Todd utilized in his final version of the rule.

Minimizing the List of Rules

At the beginning of the next class Brian presented the revised list of rules and asked the students to discuss the ideas they had to make this list shorter.

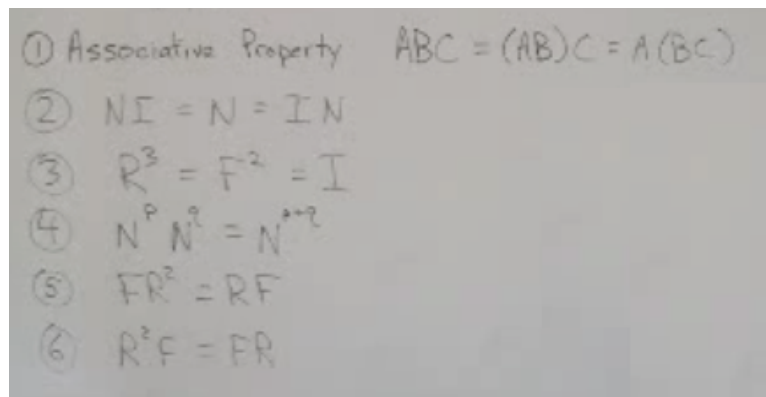


Figure 23: Axioms v. 1.2 (includes a generalized rule of exponents)

As Brian readdressed the class, he noted two observations among the students: Rules 5 and 6 are saying something similar and one could be eliminated and that Rules 3 and 4 could be modified.

Brian: Let's start on a concrete level. It's actually true that you can eliminate Rule 6, but what I want to get a handle on is what does that mean? So here's, here's the task I want to give everyone: In order to eliminate Rule 6, what has to be the case is what we have written now as Rule 6, you could derive using Rules 1 through 5. What I'd like everyone to do right now, um, some of you have done this, but most of you have not, is see if you can prove that $R^2F = FR$, using only rules 1 through 5.

After the students worked individually, Brian invited four students to the board to prove Rule 6 using Rules 1 through 5. Although all four proofs were discussed, the one Jeremy produced (Figure 24) seemed to get the most attention as it "adds" an identity symmetry to the right-hand side of the expression R^2F . Through a series of substitutions that progress from the right to the left, the identity eventually appears again on the left side of an expression as IFR , which is then transformed into the desired expression FR .

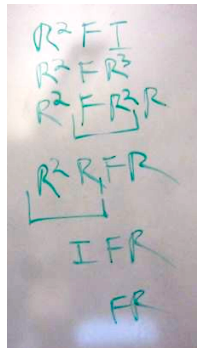


Figure 24: Jeremy's proof of $R^2F = FR$

An Alternate Rule is Proposed

Following the discussion of eliminating Rule 6 from their list of axioms, Brian now moves on to Todd's suggestion regarding restructuring. Todd revisits the idea he had in the previous class period of including several combinations of symmetries that produce the identity into one rule.

Todd: I make the claim that um, if we rewrite number 3 right there, the identity piece right there, by adding one more identity into that set right there, we can eliminate 5 and 6.

Brian: So, Todd's got an assertion that we can always do away with Rule 5 if we adjust rule 3.

Todd: Uh huh. And I said add on to rule, Rule 3 right there...

Brian: Yeah.

Todd: I say add another identity, which would be an identity amongst the functions.

Brian: So?

Todd: Such as, I know that rotate, fold, rotate, er, rotate, flip, rotate, flip always gets me the identity.

Brian: So Todd's suggesting a Rule 3 ALT...

Todd: Well, actually I'd add it onto that. Go ahead and say equals,

Brian: By three ALT, I'm saying, R^3 equals F^2 equals...

Todd: I equals $RFRF$ (Brian writes the Rule 3 ALT as $R^3 = F^2 = I = RFRF$). And I say, with that fact, I can produce Rule 5.

Brian: Mmm hmm. So Todd's asserting that, that by adding on this (circles $I = RFRF$), basically by having that, we can do away with 5...

Todd: I'm not sure that's shortening our list. It's just changing Rule 5 into that statement I've circled.

Brian: It would be one less rule, but one rule happens to be longer.

Todd's suggestion to modify Rule 3 to include the relation $RFRF = I$ is quite interesting. He claims that appending this relation to the existing Rule 3 will make Rule 6 unnecessary. Not surprisingly, Brian did not view this as making the list shorter, but Todd argued that it even though it is a longer rule, it is one less rule in the list. Although Brian's comment stresses the importance of axiomatic efficiency, Todd's remark about having fewer rules suggests a different type economy. While both Rule 5 and Rule 6 express the dihedral relation, neither involves the identity symmetry. By combining a version of the dihedral relation into a rule that already includes each generator and the identity, Rule 3 ALT resembles the conventional presentation that can be used to describe a dihedral group. In addition, this alternate version of Rule 3 not only makes the list of axioms superficially shorter, but perhaps axiomatically more powerful as well. For example, unlike the relations $R^2F = FR$ and $FR^2 = RF$, the angle of rotation

can be redefined so that the $I = RFRF$ version of the dihedral relation is applicable to any regular polygon.

An Alternate Axiomatic System is Proposed

Brian: And just, and just to clarify, which would you rather? Would you rather have this (points to current version Rule 3) or would you rather have this (points to Rule 3 ALT)?

The students discussed the different versions of Rule 3 at their tables. Much of the discussion seemed to be focused on which version is easier to substitute when making calculations and then Brian addressed the class.

Brian: Keep mulling this over. There's, there's kind of a discovery phase going on. Here's something that may help you decide which you prefer because for a lot of folks, you might not have a preference one way or another. Um, I know we have three or four voices that do have a preference right now... What if you wanted to demonstrate that this is true (writes $R^2F = FR$ —i.e. Rule 6) using $I = RFRF$ instead of $FR^2 = RF$ (Rule 5)?

Jeremy: Oh, you're switching them.

Todd: Mmm hmm

Brian: Uh, this was not written on my list of things we're doing today at all, but we're going to go with it... it's sort of like um, this is your loaner vehicle (points to $I = RFRF$), see if you like it as much as the car you had been driving. You might like it more. It's like those Ford commercials. So just to be clear, the original one through four that we had here is available completely, but under this task, we're not going to use $FR^2 = RF$. We're gonna try to do it instead using $I = RFRF$. And after you've had a chance to drive it around for a little while, we'll see what the group prefers.

This episode marks a significant and an unanticipated change in axiomatic development in the classroom. Instead of having the class prove Rule 5 with the proposed changes, Brian had them consider using an alternate axiomatic system with Rule 3 ALT and no Rule 5, to check to see if they can deduce Rule 6 as they did with the previous system. This suggestion does not prove that Rule 5 can be deduced from Rule 3 ALT along with the other rules (as Todd claimed), but what it does show is that up to this

point, Todd's alternative axiomatic system accomplishes the same work that the previous system was shown to do.

An Alternate Axiomatic System is Adopted

After Jim and Teresa share their proofs that $R^2F = FR$ (Rule 6) could be derived using Todd's Rule 3 ALT along with Rules 1, 2 and 4, Brian puts it back to the class to decide the next axiomatic revision. The debate centers upon whether to keep the original system of Rules 1 through 5 or to adopt Todd's Rule 3 ALT system that eliminates Rule 5.

- Brian: Are we more fond of this (Rule 5) or this (Rule 3 ALT)? Because they appear to be interchangeable... so, let's vote. How many people want to use $I = RFRF$? (counting hands) Four, five, six, seven. And how many people want to use $FR^2 = RF$? One, two, three...
- Randall: I would like use both. Can they not both be there?
- Brian: Well, what we've discovered is that you don't need both.
- Randall: Yes.
- Brian: Because they do the same work.
- Alice: He wants it.
- Brian: I know, there's a certain utility to having them, but we're trying to make a short list.
- Randall: Okay, I'd take the last one.
- Brian: So we had a vote of seven to four, with two abstentions, so this one wins (circles $I = RFRF$).

Like the previous axiomatic revision that eliminated Rule 6, deduction initially serves as the tool to determine if this new system is axiomatically equivalent (up to this point) as the students' previous system. As the two systems were shown that they apparently accomplish the same axiomatic work, the decision of which system to keep is decided by majority vote. Yet Randall, who values the utility of both Rule 5 and Rule 3 ALT makes a plea to keep them both. Although Brian seems to convince Randall that one of these rules is redundant, Randall's desire to preserve both of them adds another

dimension to the axiomatic selectivity process. Randall's objection suggests that perhaps minimization through deduction should not be the ultimate criteria for determining whether a rule should be included or excluded from the system. Rather a theorem, such as Rule 5, still has a place in the axiomatic system even if it can be deduced from the existing axioms.

An Alternate (and Minimized) Version of the Axioms

After a class vote in favor of adopting Rule 3 ALT, Brian began writing the rules for the revised axiomatic system on the whiteboard. In the middle of the process, Jeremy suggested a change of variables for Rule 4 ($N^p N^q = N^{p+q}$) leading to further revision.

- Jeremy: Can we change all like N 's in there to A 's, uh, because N can be used as something else that we do? ...
- Brian: Did you want to use N ? Did you have something else in mind for N ?
- Jeremy: Yeah. Like, instead of R^3 , if we put R^n then any number...
- Brian: Oh, so that we get to use lower case n as a...
- Jeremy: Yeah, cuz if we, instead of R^3 , it could be brought into even a square, if you rotate it four times, it's going to be an identity.
- Brian: Any objections to turning the capital N into capital A ? (students shake their heads)
- Todd: It's consistent with matrix notation.
- Jeremy: That's why, that's my reason for doing it.

This new version of the axiomatic system (Figure 25) contains two attributes that are worth mentioning.

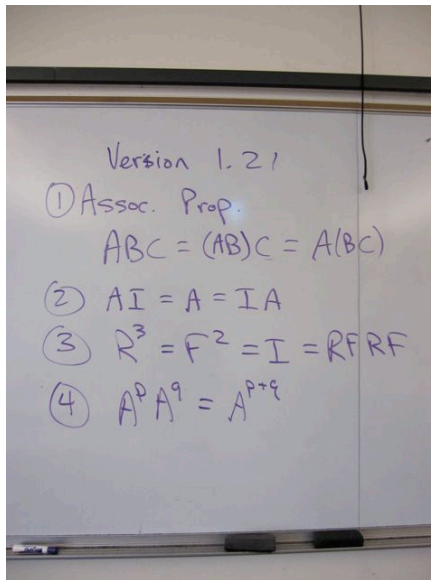


Figure 25: Axioms v 1.21 (Using A for an arbitrary symmetry)

As stated earlier, Rule 3 ALT includes a generalized version of the dihedral relation that would also apply to any regular polygon. Jeremy's suggestion to replace the N 's with A 's also reduces the number of variables used in the axiomatic system by one, with the letter A being used in three axioms to signify a generic symmetry. Brian recorded these changes into "version 1.21" of the artifact, thus creating a minimized list that the students would now refer to for future mathematization.

Throughout the next two class periods, the students were assigned specific calculations within the operation table to prove using the minimal list of rules (Figure 18). Many of these derivations were shared within the classroom community, which allowed for opportunities to discuss both the nature of proof and proof construction. For example, in Kevin's proof that $(R^2)(FR) = RF$, he used the theorem $RFRF = I$, which had been proven earlier (Figure 26). Some students questioned this technique and this challenge led to a fruitful discussion of axioms and theorems, and even introduced the term lemma.

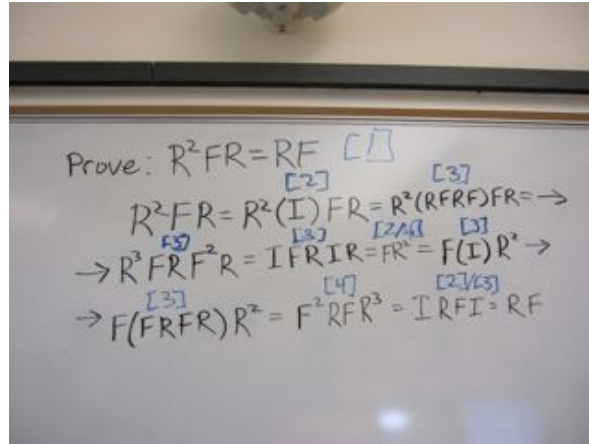


Figure 26: Kevin’s proof using the theorem $RFRF = I$

The students had spent nearly two days verifying all 36 calculations and proving. As anticipated in the LIT, one of the products of these proof-sessions was that this minimized list of rules is sufficient for completing the entire operation table.

Another Stealth Rule is Formulated

Continuing the discussion of the table, Brian then presented the linear equation $XA = B$ informally as “a symmetry in column X combined with a symmetry in row A produces symmetry B ”, thus providing an algebraic representation of an entry in the table. The students were then asked to solve a specific equation of this form referencing only the minimal list of rules. During an analysis of Alice’s solution to $X(FR) = R^2$ (Figure 27), Brian questioned the technique of “adding a symmetry to both sides” that she used.

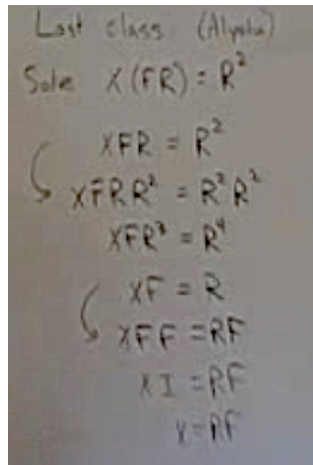


Figure 27: Alice's solution to $X(FR) = R^2$

- Brian: There's something happening from this line to this line (adding arrows from XFR to $XFRR^2$, that's also happening from this line to this line (adding arrows from XF to XFF) What is it?
- Jim: Multiplying on a certain side.
- Brian: We're multiplying on the right. Now, I think people were pretty comfortable that you would need to multiply on the right on both sides, right? Because order matters. Do we have an axiom that let's us do that? Or could it be an axiom of equality that let's us do that? What is it that let's us do that? It seems fair. Nobody had any big qualms about it.
- Gene: Isn't it a multiplication property of equality?
- Brian: That would be great, uh. That's actually the name for it—the multiplicative property of equality, when you're multiplying. But these are rigid motions of a figure onto itself.
- Gene: The rigid motion property of equality.
- Brian: How would I write the general version of what she's doing here? She has something equals something and then multiplied by the same thing on both sides.
- Alice: AC equals BC
- Brian: (writes $AC = BC$) And so the line that's before that you would have had?
- Alice: A equals B .
- Brian: (writes $A = B$ a line above and draws an arrow from it to $AC = BC$)

Brian called attention to Alice's use of "multiplying" by a symmetry on the right-hand side of each expression in two different parts of her proof. Following Gene's recognition of the multiplicative property of equality, Brian challenged the class to write a rule for it. However, instead of focusing on the fact that if $A = B$, the products AC and

BC are still equal, Brian directed the conversation toward the existence of the products AC and BC . The students worked for a few minutes individually and then Brian asked one of the students to share his rule.

- Brian: Larry, yours was very simple, what did you say?
Larry: Uh, AB equals defined.
Brian: (writes $AB =$ defined) Like I said, I thought this captures the spirit of it perfectly and this would be a good starting point for coming up with something that's, uh, a little more polished. Um, cuz I, you know that this has the idea, but um, uh we could probably dress this up. One thing that I think we should put an end to is this business of referring to things like A , B , and C , without ever saying what they are. Way up here (pointing to another part of the board), we specified that A , B , and C represent symmetries, so one thing we could do with this is um... Uh, so what does it mean for it to be defined?
Jim: There exists...
Todd: Exists (laughing).
Brian: Jim, you had, in your version what did you write?
Jim: I didn't write that, but that's what Noah had.
Brian: Oh, oh, Noah, in your version what did you write?
Noah: Well, I just said for all symmetries B and A , there exists an A times B , which equals C .
Brian: (writes "there exists a symmetry) You said, there exists a symmetry (writes C , such that $AB = C$). How does that look? ...

Although this discussion began as an exploration of the multiplicative property of equality, Brian turned it into a search for a rule that would ensure that when any two symmetries were combined, the result was another symmetry. When combining symmetries A and B , Noah's response guaranteed the existence of the product AB , which he represented as a single symmetry, C .

Systematizing (Define the set of objects at the beginning)

In response to Brian's call for identifying that A , B , and C are symmetries in the axiom the class was discussing, Athur suggested a way to make this qualification more efficient.

Arthur: Well, I would say that if we're going to have a long list of axioms, we should have at the top, like, um, "For symmetries A , B , and C ...", and then list all of our axioms.

Brian: Okay.

Arthur: Cuz otherwise we'd have to write "For symmetries A and B " for each one, so that we know what we're doing.

After Brian recorded Arthur's suggestion, he then redirected the class back to the current version of the axiom, "There exists a symmetry C , such that $AB = C$ ".

Brian: I think we're wandering from the simplicity of this. I, I want to come back to Larry's. You said, " AB equals defined." Although it probably needs a little polishing, Larry did it without ever bringing C into this picture.
(Brian writes AB)

Todd: Exists.

Brian: Exists as what?

Todd: As a symmetry.

Brian: (writes "exists as a symmetry") Is that too terse?

Gene: Nah.

Brian later asked if this axiom they created had a name. Todd responded, "I think they call it closure. It's kind of like when we add two rational numbers together, you only get rational numbers." Brian wrote the word "Closure" to this axiom and then addressed Rule 4 ($A^p A^q = A^{p+q}$).

Brian: (Brian circles $A^p A^q = A^{p+q}$) We've had some differences of opinion about this. Some folks have felt that we need it, to avoid ambiguity and other folks have said, yeah, but it's kind of, that's what you do. I think there's a case to made for including this, but we have discussed this, and I'm going to make the umpire's call (crosses out Rule 4), that we're going to cut it.

Kevin: If you do that, then why don't we cut number 2 (the identity property)?

Brian: This one (points to $AI = A = IA$)?

Kevin: Because that's just what we do and that's your basis for cutting number 4.

Kevin's reaction to eliminating the rule of exponents seemed to surprise Brian, who clarified that the law of exponents was "a result of the notational choice and doesn't really merit extra inclusion." Brian also added, "Not everyone's in agreement with me on that", but Brian went on to say, "The moment we chose to use the multiplication notation,

as our shorthand, that implied that all of our exponent rules came along and didn't require special mention."

An Inverse Axiom is Anticipated

Brian's comments about the laws of exponents being a consequence of the multiplicative convention the class had adopted opened the discussion to other exponent rules.

- Jeremy: So then, if we do like, negative one or a 120-rotation counter-clockwise, how would we write that? ... Is it going to be like an A inverse or...?"
- Brian: How would you want to write it—a counter-clockwise 120-degree rotation? If we're choosing to use the exponent notation?
- Jeremy: It would be a negative exponent probably (Brian writes R^{-1}).
- Ted: Would that hold true for rest of the laws of exponents? Or the rest of the system?
- Brian: Would that work if R inverse represents a counter-clockwise 120, R to the minus one represents that? Would that combine according to the rules of exponents, the way, the way I said, we didn't need that (pointing to the adding exponents rule) because the rules of exponents work? Like if I did that followed by R cubed (writes R^3 after R^{-1}), the rules of exponents says that that should be...
- Students: R squared (Brian writes R^2).
- Brian: Is that true? Does this really work?
- Students: Yeah.

It is important to note that Brian could have supported the formulation of an inverse axiom earlier by exploring *why* Alice chose the symmetries she multiplied in her proof. Both times, Alice multiplied each side by the symmetry that would produce the identity when combined with the given symmetry, which is its inverse. Instead, he used Jeremy's question how negative exponents would behave in this system to introduce inverses. By demonstrating that $R^{-1}R^3 = R^2$, Brian reminded the class of other consequences of their multiplicative convention and that inverses could be incorporated into their list of rules. Brian then wrote a new version of these rules, with the closure

axiom listed first, and named it “Axioms v. 2.0” (Figure 28). This version of the axioms included the now-named closure property, a generalized introduction of A , B , and C as arbitrary symmetries, and an anticipated identity axiom (Figure 28). Brian closed the lesson by challenging the students to come up with an axiomatic way to express inverses as part of their system, saying “We want to make sure you’re not being, talking about just for R , you should be able to talk about how an inverse works for any of our elements.

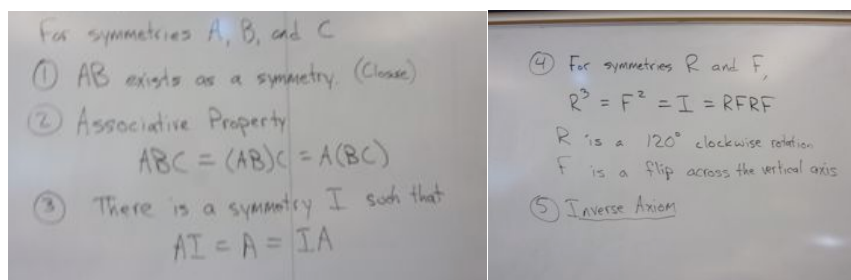


Figure 28: Axioms v. 2.0 (Anticipating an inverse axiom)

An Inverse Axiom is Formulated

At the beginning of the next class period, the students were asked to share their versions of an inverse axiom that had been anticipated in their collective list of rules.

- Brian: What did you have?
 Arthur: Oh, I just listed a bunch of different ones. Like um, R squared equals R inverse and um... [Arthur writes a list of equivalences showing the inverse of each symmetry]
 Brian: I know I heard several people...who else had inverse statements like that? I know Teresa did and a lot, a lot of folks did.
 Todd: I kind of generalized mine over too.
 Brian: So, what'd you have, Todd?
 Todd: There exists for every symmetry, maybe not start with that wording..., but the guts of it is A , A inverse equals—yeah, just the algebra behind it. A , A inverse equals A inverse A equals identity. That's the general workings of it. I didn't know what English leads up to it, but...

Although Arthur's list demonstrated the inverse for each of the six symmetries, Todd's contribution seemed more consistent with the form of their other axioms. Brian

recorded Todd's contribution and then included it as part of the revised set of "Triangle Symmetry Axioms v. 2.1" (Figure 29) on the whiteboard.

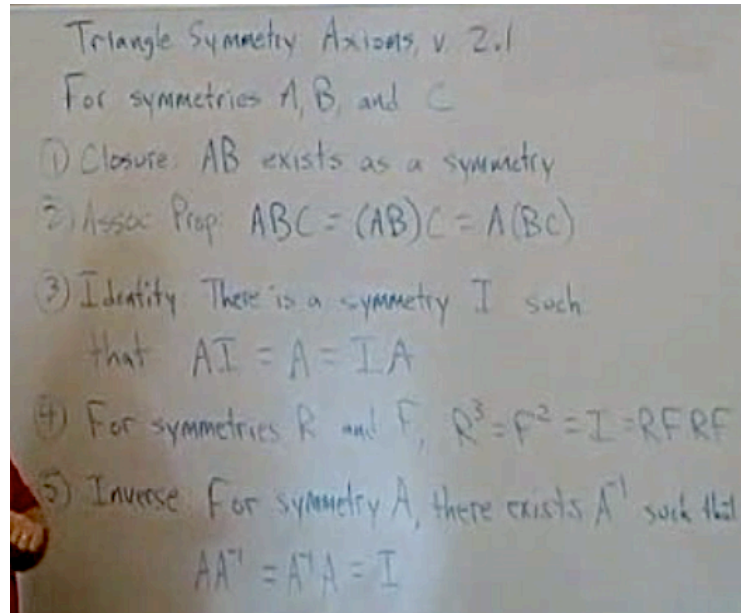


Figure 29: Axioms v. 2.1 (Final version for the symmetries of a triangle)

The Proof of the Sudoku Property

As noted in Figure 29, this list of rules captures the essential relations for describing the symmetry group of an equilateral triangle. Brian then challenged the students to determine whether this list of rules is sufficient for proving the Sudoku property, breaking this task into two parts (uniqueness and existence). First, the students showed that a given symmetry appears no more than once in any row by proving that if $AX = AY$, then $X = Y$. Second they showed that a given symmetry appears at least once in any row by solving the general equation $AX = B$, justifying each step with their existing axioms. As the inverse axiom had already been formulated, Brian concluded the discussion of the Sudoku property by emphasizing the power of axioms the class had created.

Axiomatic Concept Expansion

The class was then split into two groups with one half of the class assigned to create a list of rules for the rotational symmetries of a square while the other half was charged with formulating a list of rules for the symmetries of a non-square rectangle. The students worked on these tasks for several minutes while Brian made an axiomatic comparison chart (Figure 30) that included the symmetries of the equilateral triangle, the rotational symmetries of a square, and the symmetries of a non-square rectangle.

Triangle Symmetry Axioms, v. 2.0 For symmetries A, B, and C	Triangle	Rotations of a Square	Symmetries of a non-square rectangle
1) Closure: AB exists as a symmetry	Y		
2) Assoc: Prop $ABC = (AB)C = A(BC)$	Y		
3) Identity: There is a symmetry I such that $AI = A = IA$	Y		
4) For symmetries R and F, $R^3 = F^2 = I = RFRF$	Y		
5) Inverse: For symmetry A, there exists A^{-1} such that $AA^{-1} = A^{-1}A = I$	Y		
6) Other	N		

Figure 30: Axiomatic comparison chart

The axiomatic comparison chart is an implementation of Step 5 in the LIT and called attention to the shared rules among the symmetries of the different figures. Brian’s inclusion of a sixth “other” axiom also fostered an opportunity for the students to axiomatize some relations that may not have applied to their triangle symmetries. Starting with the rotations of a square, Brian went through each axiom asking the students whether the new figures satisfied the triangle axioms, writing “Y” or “N” next to each. In addition to noting that Axiom 4 ($R^3 = F^2 = I = RFRF$) did not hold, Todd added

that, “That order doesn’t quite matter so much with the squares.” Although this is not true for all the symmetries of the square, commutativity holds when combining the rotational symmetries. Brian then had them consider the symmetries of the non-square rectangle, which they denoted as I (identity), R (a 180° clockwise rotation), H (a flip across the horizontal axis) and V (a flip across the vertical axis), which were also commutative.

Brian: Is the non-square commutative?
 Kevin: I think so.
 Todd: Does HV equal VH ?
 Kevin: It’s R , yeah.
 Brian: VR gives me H . RV gives me H . Did I fill this [table] in right? So is that one commutative as well?
 Todd: Yeah, it sure looks like it. Sure does! Wow!

Although Todd wanted to check the pairs HV and VH , Brian ensures that both combinations of a vertical flip and a 180° rotation produce an equivalent symmetry. In addition to adding that the rotational symmetries of the square and the symmetries of the non-square rectangle satisfied the commutative property, the students then formulated a sixth axiom expressing the identity symmetry in terms of a 90° rotation as $R^4 = I$ for the rotations of a square and rule $H^2 = R^2 = V^2 = I = HVHV$ for the rectangle, which Kevin immediately revised.

Kevin: Just that, that last identity on the, the rectangle $HVHV$? I meant to say HVR is the identity (Brian makes the correction on the chart).
 Todd: Oh, that’s cool. Mixing all three of ‘em.
 Brian: That certainly has a real appeal that it’s got all three showing up.
 Todd: Yeah that’s the connection between all the parts amongst the identity.

Kevin’s original contribution of $H^2 = R^2 = V^2 = I = HVHV$ incorporates multiple symmetry combinations that produce the identity like the original Rule 3 ALT

$(R^3 = F^2 = I = FRFR)$, but the notational convention is different. The symmetry group of the non-square rectangle is isomorphic to the Klein-V group and is generated by two elements instead of three, thus making one of the flips (either vertical or horizontal) redundant in the rule. After completing the chart, Brian called particular attention to the four rules that the symmetries of the equilateral triangle, the symmetries of the non-square rectangle and the rotational symmetries of the square all have in common (Figure 31) and then closed the lesson.

Triangle	Rotations of a Square	Symmetries of a non-square rectangle	
Y	Y	Y	Clos.
Y	Y	Y	Assoc.
Y	Y	Y	Id.
Y	N	N	
Y	Y	Y	Inv.
N			

$A^4 = I$ (Cyclic) $H^2 = R^2 = V^2 = I = HVR$ (commutative set)

Figure 31: Brian identifies the common axioms and introduces the term group

Brian: So not only do the closure, associative, identity and inverse seem to hold all the time, but it seems that sometimes you have something that IS commutative. And sometimes even though we think of the equilateral triangle as a simpler figure, and it is a simpler figure, it does NOT have commutativity, commutativity. Um, sometimes you get it and sometimes you don't. Well, here's the big thing that we've been leading up to for the past, five-and-a-half weeks? Anything that follows THESE four properties: closure, associative, identity and inverse is called a group. The name of this Math 344 that I keep talking about by number is "Group Theory". In a traditional curriculum they tell you the definition of a group and then you work through a bunch of examples. In this curriculum, we develop for ourselves the definition of a group the same way the mathematicians who came up with this did.

Definition of Group

Brian began the next class period saying, “Last class we ended with the definition of a group. Uh, I want to uh, tighten up that definition a little bit.” He then spent several minutes leading the students in modifying the list of four symmetry axioms (closure, associativity, identity, and inverse) to create a formal definition of group. In addition to generalizing to a set and operation, Brian introduced standard notational conventions that included the use of lower-case letters and the symbol “ e ” for identity when defining group (Figure 32).

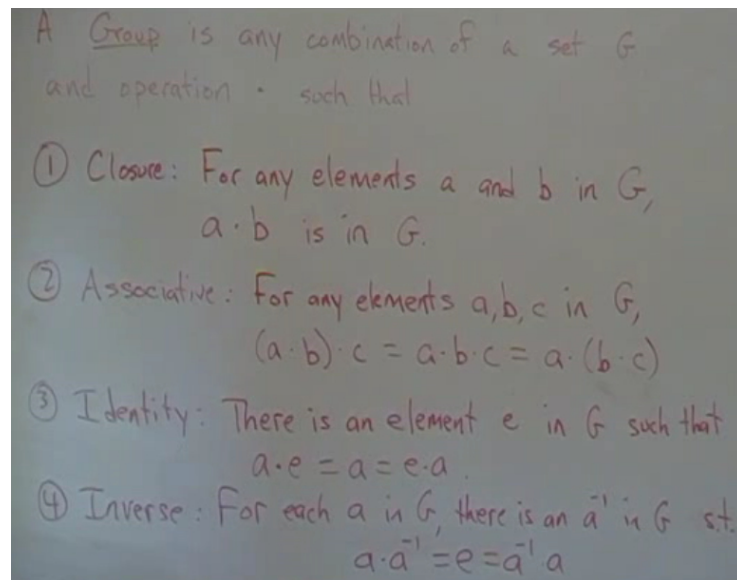


Figure 32: Formal definition of a group

Findings

Supporting Pre-Axiomatic Activities

One of the findings that emerged from this study points to how pre-axiomatizing activities supported the students' axiomatization. For example, conventionalizing (Yannotta, 2016b), influenced both the form and content of the axioms the class created and refined throughout the axiomatizing sequence. Thus, the multiplicative convention the class adopted allowed the students to formulate rules that were both efficient and elegant by not requiring a symbol to represent the operation of combining symmetries. The students even axiomatized a rule that was a consequence of this multiplicative convention. Beginning with the notation R^2 that Teresa explained was "shorthand notation for RR ", this relation was later formulated into the axiom $RR = R^2$ to justify a step in a symmetry calculation. Attending to both the base and the exponent, this rule was then collectively generalized into a rule for adding exponents to $N^p N^q = N^{p+q}$. This exponent rule underwent further revision, until Brian addressed it as a consequence of their notation and removed it from the public artifact.

Another example of a supporting pre-axiomatizing activity was symbolizing (Gravemeijer, 2004). For example, early in the enactment of the curriculum, the students chose I to represent the identity symmetry as opposed to R^3 or F^2 , which were shown to be equivalent. At that time, there were no equations on the whiteboard relating these equivalent expressions, but these different representations formed the seeds of an important rule for their system. Todd later combined these equivalences into Rule 3 ALT as $R^3 = F^2 = I = RFRF$, including a version of the dihedral relation that produced the identity. The activities of conventionalizing and symbolizing, which preceded the

formulation of symbolic equivalence relations, had a lasting impact on the students' axiomatization and they continued to refine their list of rules.

Axiomatic Creation

In addition to elaborating supportive pre-axiomatizing activities, I was able to describe how the students gradually transitioned from examining an operation table of thirty-six symmetry combinations to selecting seven statements that would serve as an initial set of axioms. I claim there were three key phases in the classroom activity that took place during axiomatic creation, which concluded with the formulation and selection of an initial set of statements for axiomatization. These three phases included: orientation, formulation, and selection.

The orientation phase began when Brian asked the students to make comments about “patterns” they noticed in Todd’s operation table. The patterns that students identified were initially specific to Todd’s representation (i.e. the order of the elements in a given row). However, Kevin’s observation that there were no repeated entries in any row or column was invariant of the table’s arrangement and would later be revisited when proving the Sudoku property. Brian then initiated a shift in orientation to identifying any “short cuts” or “properties” that the students noticed when making some of the lengthier calculations. The students’ responses now came in the form of verbal formulations of equivalent expressions, which Brian began recording in the public space. It is important to note that Todd did not mention symbols when he responded with “Flip it and then rotate...it’s the same thing as rotating twice and then flipping”, so Brian symbolized the expressions as FR and RRF and left a space in between them. Brian then repeated this process, recording the expressions $FRFR$ and I when Jeremy was explaining that when

you “have a flip rotate and a flip rotate then you're going to get the identity,” again leaving a blank space between them. Brian capitalized on Todd and Jeremy’s contributions by adding an equals sign between each pair of expressions that were on the whiteboard. Brian then emphasized these relations as the type of “short cut” to keep track of when he assigned the students the task of making a new operation table. While Brian’s comments directed attention to finding more relations, his act of adding the “=” sign also provided an example of how to formulate them into statements.

Axiomatic formulation was the second and most critical phase in axiomatic creation, and thus is outlined more thoroughly. Although the students had been using and tracking short cuts in their individual work, it was the emphasis on capturing these relations and recording them in the public space that marks the shift to this phase. It began when Brian asked Kevin to share both his operation table and the short cuts that he used to complete it. Kevin’s comment about “just substituting things in that we’ve already figured out” suggests that he not only formulated some key relations, but that he also used them to formulate new ones to complete his table. In fact, Kevin acknowledged Gene’s method of substitution and explained how he used a subset of the relations $FR^2 = RF$, $R^2F = FR$, $R^3 = I$, and $F^2 = I$ to calculate other relations like $(RF)(FR) = R^2$. Consistent with the notion of a transformational record (Rasmussen and Marrongelle, 2006), Brian then privileged Kevin’s short cuts as “four facts”, recording them on the whiteboard and thus initiating a collective process of formulating statements. By recording these statements on the whiteboard, Brian brought the activity back to the public space, as Kevin’s individual rules became the genesis for the collective artifact that the class would construct.

Axiomatic formulation continued as Todd contributed the relation $(RF)(FR) = R^2$, which Kevin had listed underneath his four facts, as well as the relation $(R^2)(RF) = F$. Brian recorded these two versions of the dihedral relation on the board with the others and then asked the students to consider “stealth rules” that they may have been using. This search for stealth rules led to Todd’s identification of the associative property, which Kevin formulated as $ABC = (AB)C = A(BC)$. The formulation of the associative property is significant because this “stealth rule” is different than the previous kinds of rules that have been shared. Until now, the rules on the board involved either R ’s or F ’s or both, but the associative property does not mention R or F at all.

Brian continued the class’s search for stealth rules by analyzing the rules that Kevin had used to show that $(RF)(FR) = R^2$. Deduction now became a larger factor in the formulation of new axioms as the class verified each step in this calculation using their emerging list of rules. While the associative property and the relation $F^2 = I$ were rules that were already on the collective list, the next step in the calculation referenced one that was not. Kevin said he used the rule $IR = R$ in the calculation $RIR = R^2$, but Todd replied that they could have also used $RI = R$ in the same step. After Brian recorded both of these statements, Kevin formulated $RR = R^2$, which corresponded to the final step in the calculation. Jim then introduced two other statements $FI = F$ and $IF = F$, which immediately followed the similar observations that were made regarding I and R and then Kevin generalized this property as “Anything times I is what it is. It’s the identity.” Jerry and Todd then formulated this generalization as $NI = N = IN$, thus bringing the total number of student-generated statements on the board to twelve.

The final phase in axiomatic creation began immediately following the formulation of the identity statement, when Brian placed a box around the rule $NI = N = IN$. He continued to box six other statements that the students had formulated and then erased the four specific identity statements that preceded the students' generalization. I claim that Brian's act of placing a box around these seven statements was comparable to the statement selection process described in De Villiers (2010). Like students in the previous study, these students had formulated several relationships through their own activity, but it was the teacher who called attention to these and formalized the selection process. In this case, the selection process sorted the students' statements into three types of rules: the seven that were boxed, the ones that remained unboxed on the board, and those that were erased.

Axiomatic Analysis

Once the list of statements became the central object of inquiry, the emerging axiomatic system was subjected to several systematization tasks that took place over the course of seven instructional days. I was able to identify seven such tasks that corresponded to the axiomatic system of the symmetries of the equilateral triangle and one systematization task that extended to other structures. Many of these systematization tasks are built into the LIT, and thus what is reported here focuses on Brian's implementation of these tasks and their impact on the students' axiomatizing activity. Globally, these tasks posed the question whether the current list of axioms accomplished the work it was continually being asked to do and many led to the revision of the collective list of rules. Table 3 presents my interpretation of the first seven of these systematization tasks listed chronologically (the eighth task is discussed separately). For

each task the teacher implemented, I included what I interpreted to be the class's axiomatic response to this question along with any changes to the artifact that resulted.

Table 3

Systematization tasks, axiomatic responses, and artifact revisions

Systematization task	Axiomatic response	Change in the artifact
1. Do we have enough rules to complete the entire table?	Kevin: Add $FRFR = I$ to the list Todd: Create one long rule	$F^2 = I$ and $R^3 = I$ were combined into a single rule
2. Do we need to add this rule to our list?	Arthur proved that $FRFR = I$	$FRFR = I$ was not added to the list
3. Can we remove an existing rule on our list?	Four students proved that $R^2F = FR$	$R^2F = FR$ was removed from the list
4. Can we prove the same rule using an alternative axiomatic system?	The class showed that both systems can be used to prove $R^2F = FR$	$R^3 = F^2 = I = RFRF$ replaced an axiom and $FR^2 = RF$ was removed from the list
5. Can we still complete the operation table with a minimized list of rules?	After proving all 36 calculations, the class agreed the list was sufficient	No change
6. Will our rules allow us to solve a specific linear equation?	Alice's technique of multiplying a symmetry to both sides of the equation $X(FR) = R^2$ evolved into a discussion of closure	Closure property was formulated Inverse axiom was anticipated and later formulated
7. Can our list of rules prove the Sudoku property?	The students proved $AY = BY$ implies $A = B$ and showed the existence of a solution to $XA = B$	No change

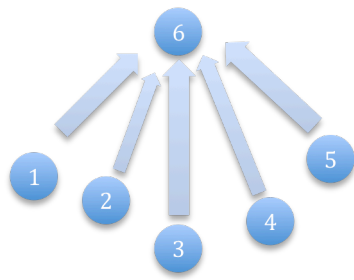
The first task was to determine if the selected list of statements contained enough rules to complete the operation table, and thus served as the starting point for axiomatic analysis. Kevin's tactic was to add another rule to their list, which he argued would assist them in computing some of the more challenging symmetry combinations. Todd supported Kevin's contribution and further suggested that they also bring together several versions of the identity symmetry into a single rule. Brian combined $R^3 = I$ and $F^2 = I$ into one rule, thus reducing their total number of rules by one, and then refocused the students on the main question by challenging the necessity of the rule $FRFR = I$.

The second and third systematization tasks address axiomatic independence in that they show that some rules can be eliminated through deduction. These separate-but-related tasks functioned differently as one prevented a rule from being added to their list, while the other eliminated an existing rule. Rather than adding the rule $FRFR = I$ to their list, Brian implemented the second task by asking the students to show that $FRFR = I$ could be obtained using their existing list of six rules. After the students worked on this impromptu task, Brian asked Arthur to share his proof of this calculation, which showed that indeed this rule was redundant and did not need to be added to the list of axioms. Although Brian may have introduced this minimization task pre-maturely, the discussion of Arthur's proof provided the students with an example of how to prove a rule was unnecessary, which is essential for the next anticipated minimization task.

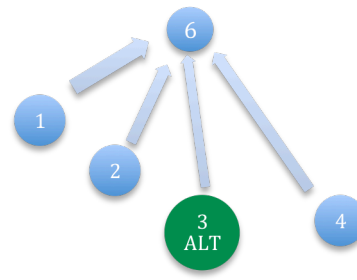
The third systematization task was one that is critical to the LIT and addresses axiomatic efficiency (Larsen, 2013) by deductively eliminating redundant rules from the class's existing list. This activity is consistent with what De Villiers (1986) described as the analysis of logical relationships between statements, which is part of the process of

descriptive axiomatizing. In addition to addressing axiomatic independence through the elimination of Rule 6 ($R^2F = FR$), the implementation of this task also provided opportunities for the class to discuss proof and opened the door for more revisions to the list of rules.

One of these revisions was a “radical” suggestion that Todd had for changing a rule and thus eliminating two other existing rules. Thus, Brian implemented a fourth systematization task as a spontaneous response to Todd’s second attempt to create this longer rule involving the identity that would eliminate both Rule 5 and Rule 6. However, Brian did not ask the students to verify Todd’s first claim, or even his second claim that with this change he could deduce Rule 5 (which would be sufficient for proving his first claim that it could replace both rules). Instead, Brian challenged the class to prove Rule 6 again by deriving it from the new system that Todd was proposing (Figure 33). This does not show that the two systems are axiomatically equivalent, but rather it proves a weak form of equivalence, in that it shows that both systems can be used to deduce Rule 6. It is unknown why Brian chose to address Todd’s contribution in this way, but his analogy comparing the driving experiences of your own car versus a loaner vehicle suggests that he was encouraging the students to consider whether the Rule 3 ALT system could do the same job their current system was just shown to do.



Original system



Rule 3 ALT system

Figure 33: A comparison of systems using axiomatic diagrams (De Villiers, 1987)

Consistent with Larsen’s (2013) LIT, the systematization function of proof is introduced as a supporting activity for axiomatizing as students check to see if this minimized list of axioms is sufficient for completing the operation table. This fifth systematization task was implemented over the course of two class periods as the students shared their proofs of various calculations using the minimal list of rules. All 36 of these calculations were verified using the minimal list of rules and each student had an opportunity to derive at least one entry in the operation table to share with the class. Engagement in this task was quite fruitful for students as they negotiated a variety of proof techniques and even discussed the role of axioms and theorems in their work.

The sixth task asked the students to consider solving the equation $X(FR) = R^2$ and was designed to scaffold the seventh task—the proof of the Sudoku property. Alice’s solution, which included “multiplying both sides by a symmetry” first by R^2 and then by F , was examined. Gene initially identified this rule as the multiplicative property of equality, which was framed as the rule that allowed one to say that if $A = B$, then

$AC = BC$. Although this property deals with the multiplicative property of equality, Brian did not pursue Gene's contribution, nor did he examine the specific symmetries that Alice chose, which might have led to a discussion of inverses. Instead, Brian went in a completely different direction and fostered a discussion about ensuring that the result of multiplying on both sides of the equation was still a symmetry. During this discussion, Brian asked if they could formulate an axiom that would capture Jon's observation that there were "no blank spots in the table." Eventually Larry and Nick collaborated to axiomatize " AB exists as symmetry", which Todd named as the closure property following the formulation of this rule.

After this addendum of Rule 1 (Closure), Brian cut Rule 4 ($A^p A^q = A^{p+q}$), arguing that it was just one of many exponent rules that were a consequence of the class's multiplicative convention. Jeremy then asked about how to represent a 120° counter-clockwise rotation, which led to the class's symbolizing of R^{-1} as the inverse of R . While he did not connect this notation back to Alice's proof, Brian seized this opportunity to challenge the students to create an inverse axiom for the next class, recording Rule 5 generically as "Inverse Axiom". In the next class period, a few students shared a list of the six symmetries and their respective inverses, but Brian pressed them for a general rule. Todd formulated this anticipated axiom as: "For symmetry A , there exists A^{-1} , such that $AA^{-1} = A^{-1}A = I$." and Rule 5 was then added to their list.

The seventh task was to prove the Sudoku property, which is intended in the LIT to elicit either the cancellation law or an inverse axiom through the systematization function of proof (De Villiers, 1990; Larsen, 2013). However, in this implementation of the curriculum, the students had already axiomatized an inverse rule, as it had emerged in

the discussion after the previous scaffolding task. Hence, these students' list of rules was already sufficient for proving the more general property that each symmetry appeared exactly once in a given row (or column) in the table. As Brian led the class through the proof of the Sudoku property, the students verified that their list of rules was sufficient for proving that if $AX = AY$, then $X = Y$ and for solving the equation $AX = B$. By proving the first statement, the class showed that there are no repeated symmetries in a given row; and the existence of the solution $X = A^{-1}B$ demonstrated that each symmetry appeared at least once in a given row. By showing that each symmetry appeared no more than once and at least once in a given row, the class demonstrated that their list of rules showed that each symmetry appeared exactly once in a given row. Brian closed the discussion of this systematization task by having the students prove the column case, if $XA = YA$, then $X = Y$, and then argued that the solution to $XA = B$ would follow "similarly to the preceding row proof."

The eighth systematization task was quite different from the first seven because it directed attention *outside* the current system to explore the rules that would apply to the rotational symmetries of a square and the symmetries of a non-square rectangle. These figures allowed the students to axiomatize new rules such as the commutative property and new relations like $R^4 = I$ for the rotations of a square and $H^2 = R^2 = V^2 = I = HVR$, which did not apply to their triangle symmetries. Brian then led the class through an examination of both figures asking which rules each one satisfied. After completing the axiomatic comparison chart, Brian identified the common relations among all three figures, calling a structure that satisfied those four properties a group. The newly formed relations as well as the rule $R^3 = F^2 = I = RFRF$ were then left off the next version of the

axioms because they did not apply to all three figures. It is important to note that following this comparison task and the identification of the four common rules, the students were no longer describing the symmetries of an equilateral triangle, but a more general symmetry group. At the beginning of the next class period, Brian then led the class through a final axiomatic revision, which culminated in the definition of an abstract group.

The Group Axioms

The four group axioms (closure, associativity, identity and inverse) emerged over the course of several days prior to the formalization of the definition of an abstract group (Figure 25). Therefore, this condensed overview details the emergence of the four group axioms and their formalization, providing more insight into the development of the formal definition of group within the classroom community.

Closure. One could argue that the seeds of the closure axiom might be traced back to early counting arguments students made to show there were only six non-equivalent symmetries of an equilateral triangle, but the axiom came into the public space when Brian asked the class to justify the rules that were used in Alice's solution of $X(FR) = R^2$. In the first step, Alice did not use the inverse property for FR , but rather, she "multiplied" both sides of the equation on the right by R^2 in an effort to eliminate R . When initially asked why she was allowed to do this, Gene responded, "Isn't that just the multiplicative property of equality?" Brian continued to challenge the students after acknowledging Gene's contribution and then asked, "Is this multiplication? What are we multiplying?" and then Todd recalled, "Doesn't something weird happen if we're doing matrices those though?" It was unclear exactly what Todd was referring to, but Brian

then introduced an example of dividing by zero, which addressed whether the operation of multiplying was always defined, which was not the issue in Alice's work. Brian commented that they did not have a property in their list that allowed them to say the result of multiplying two symmetries is always a symmetry to which Jim responded, "But when we filled out our table, there were no spots that were undefined." Brian then asked the class to come up with an axiom that captured what Jim said. It was during this exchange that Gene mentioned that they were working in "a *closed system*", which seemed to help move the discussion to creating a rule to describe closure. After Larry's contribution of " AB equals defined", Jim added the quantifier, "There exists", which was followed by Noah's contribution, "For all symmetries B and A , there exists." The working version of the closure axiom then reverted simply to " AB exists as a symmetry", with the quantification, "For symmetries A , B , and C ", listed before the numbered axioms of the system. Toward the end of the axiomatizing sequence, Brian reincorporated the universal quantifier back into the axiom when recording the formal definition as "For any elements a and b in G , ab is in G ."

Associativity. The axiomatization of associativity was rather uneventful. Following the listing of Kevin's "four facts", Brian asked, "Have we been using other ones, maybe some stealth properties, that we've been using without noticing or?" Todd immediately responded, "Um, is that like the law of associativity? The fact that we can pick stuff off in order?". When asked how to write this axiom, Kevin provided the version $ABC = (AB)C = A(BC)$, which remained unchanged until Brian introduced the quantifier "For any elements a, b, c in G ", in the formalized version. One interesting note about Todd's comment about "picking stuff off in order" is that it is consistent with

the operational ordering that has been reported in the literature. Larsen (2009) found that in the initial stages of the curriculum, it was not uncommon that students would view algebraic expressions as a sequence of actions, rather than as a binary operation acting on two elements. While these students did not seem to be resistant to axiomatizing the associative property, there also was little attention given to this axiom after it became one of their rules.

Identity. The identity axiom also had some early roots when the students were choosing a class-wide set of symbols for their symmetries. The juxtaposition of F^2 , R^3 , and I on the whiteboard, not only foreshadowed Rule 3 Alt, but also the existence of a symmetry that “didn’t do anything”. Later, Kevin identified the relation $RI = R$, in a deductive step to show $(RF)(FR) = R^2$, which Todd followed with the version $IR = R$ that could also have been used in the deduction. Jim then noted that these rules applied to F as well, and axiomatized $FI = IF = F$. After Kevin recognized that I times any symmetry is equivalent to that symmetry, Jerry and Todd combined these four rules together, then formulated this axiom as $NI = N = IN$. Jeremy later revised this rule, when he suggested changing all the N ’s in the axioms to A ’s. The identity axiom remained $AI = A = IA$ until the formulation of the closure axiom. During this revision cycle, Brian asked, “For number 3, I mean, I’m trying to write this (circles $AI = A = IA$), but just dress it up a little bit so that I isn’t coming out of thin air. I want to say what I is.” Todd then responded, “How about, there exists an I such that, that statement is true.” Brian then recorded this rule and it was only changed in the formal definition to “There is an element e in G such that $ae = a = ea$.”

Inverse. Early in the instructional sequence, inverses were discussed in terms of R^2 being the same as “ R rotating backwards”. Like the closure axiom, little attention was given to the inverse axiom until the discussion of the rules Alice used to solve the equation $X(FR) = R^2$. Instead of focusing on the reason Alice picked R^2 to multiply on both sides of the equation, the discussion migrated to the multiplicative property of equality. Later, Brian followed up on a conjecture made by Teresa that the equations $AX = I$ and $XA = I$ always had the same solution. This led to a brief discussion of ensuring a well-defined operation and revising Teresa’s conjecture, by replacing the word “same” with “equivalent”. The students then replaced each A with one of their six symmetries and brute-force computed the solutions to these twelve equations. Todd then made an observation, “This obviously has been creeping up for quite a while. This kind of points at an inverse function for every function.” The students worked individually to formulate an inverse axiom, and then Arthur showed his proof that if $XA = BA$, $X = B$, which is the right-hand cancellation law. In his proof, he multiplied each side on the right by A^{-1} , but Brian immediately challenged how he knew A^{-1} existed for every symmetry. Although Arthur acknowledged it was not efficient, he had computed the inverse for each symmetry. Brian then pushed back on the class for an axiom, which Todd formulated, as “There exists for every symmetry, maybe not start with that wording..., but the guts of it is AA^{-1} equals—yeah, just the algebra behind it— AA^{-1} equals $A^{-1}A$ equals identity.” This axiom was referenced in the proof of the Sudoku property, but remained intact until the formalization of “For each a in G , there is an a^{-1} in G , such that $aa^{-1} = e = a^{-1}a$.”

Intellectual Need and Axiomatizing

A final finding elaborates how students' intellectual need (Harel, 2013) influenced collective axiomatization. I describe two instances in which students' intellectual need for generalization and intellectual need for elegance (Harel, 1998) influenced the class's axiomatic re-organization. The first instance was observed following the discovery of the stealth rule $RR^2 = R^3$ that was added to the existing rule $RR = R^2$. I claim that Gene's comment of "take it to the general case and then you can get rid of all of that" identified a solution to a problem with the current form of the rule and thus, he was attending to an intellectual need for generalization. However, when Brian asked for this generalization, it was Kevin who first offered a suggestion to resolve this problem. Kevin's comment about modifying the rule to $NN = N^2$ and $N^2N = N^3$, pointed toward a generalization of the symmetry base as it "would apply to F as well". On the other hand, Gene followed Kevin's suggestion with $R^mR^n = R^{m+n}$, generalizing the exponent rule to any number of rotational symmetries. Jerry and Todd then collaborated to complete the generalization as $N^pN^q = N^{p+q}$, which later Jeremy revised as $A^pA^q = A^{p+q}$. Gene's intellectual need to generalize two relations involving iterations of R led to a series of revisions of this collective axiom. While these generalizations were competing with one another at first, Jerry and Todd were able to combine both of these contributions into a single axiom that generalized iterative combinations of any symmetry of the equilateral triangle.

I found evidence of an intellectual need for elegance during the implementation of the minimization task from the LIT when Todd suggested changing rule 3 ($R^3 = F^2 = I$) in order to eliminate rule 5 ($FR^2 = RF$) and rule 6 ($R^2F = FR$). Although it is completely

possible that Todd was just responding to the teacher’s directive to make the list shorter, this structural revision had been voiced earlier following Kevin’s suggestion to add the rule $FRFR = I$ to their list. In fact, Todd and Kevin had collaborated to formulate something quite similar in the previous class session before this task had ever been implemented. At that time, Todd suggested that they “pile all the identities on one line” as Gene had done previously, and incorporate the $FRFR = I$ -version of the dihedral relation into the existing rule. I claim that Todd’s contribution of revising an existing rule to include multiple combinations of symmetries that produced the identity, attended to an intellectual need for elegance (Harel, 1998). When Brian challenged that this did not shorten the list of rules, Todd justified his suggestion, saying “It would be one less rule, but one rule happens to be longer.”

The formulation of Rule 3 ALT incorporated Todd’s previous idea to create a more inclusive rule involving the identity and addressed the systematization task of making the list shorter. At the system level, the inclusion of the dihedral relation $FRFR = I$ made other rules containing the dihedral relation redundant, but at the same time extended the length of the original rule. Although Todd did not elaborate any further on the structural implications of this revision, one should consider Todd’s formulation of $R^3 = F^2 = I = RFRF$. Most notably, Rule 3 ALT closely resembles a presentation of the group of symmetries for the equilateral triangle: $\langle R, F \mid R^3 = F^2 = (RF)^2 = I \rangle$, which is an efficient way to describe the structure of this group. On the other hand, axiomatically, Brian was correct that Rule 3 ALT did not make their list of rules any shorter because the revision was now just replacing Rule 5 ($FR^2 = RF$) in an existing rule. Following an

analysis of both the original system and the 3 ALT system that included two deductions of Rule 6 ($R^2F = FR$), the class selected Rule 3 ALT as their axiom of choice.

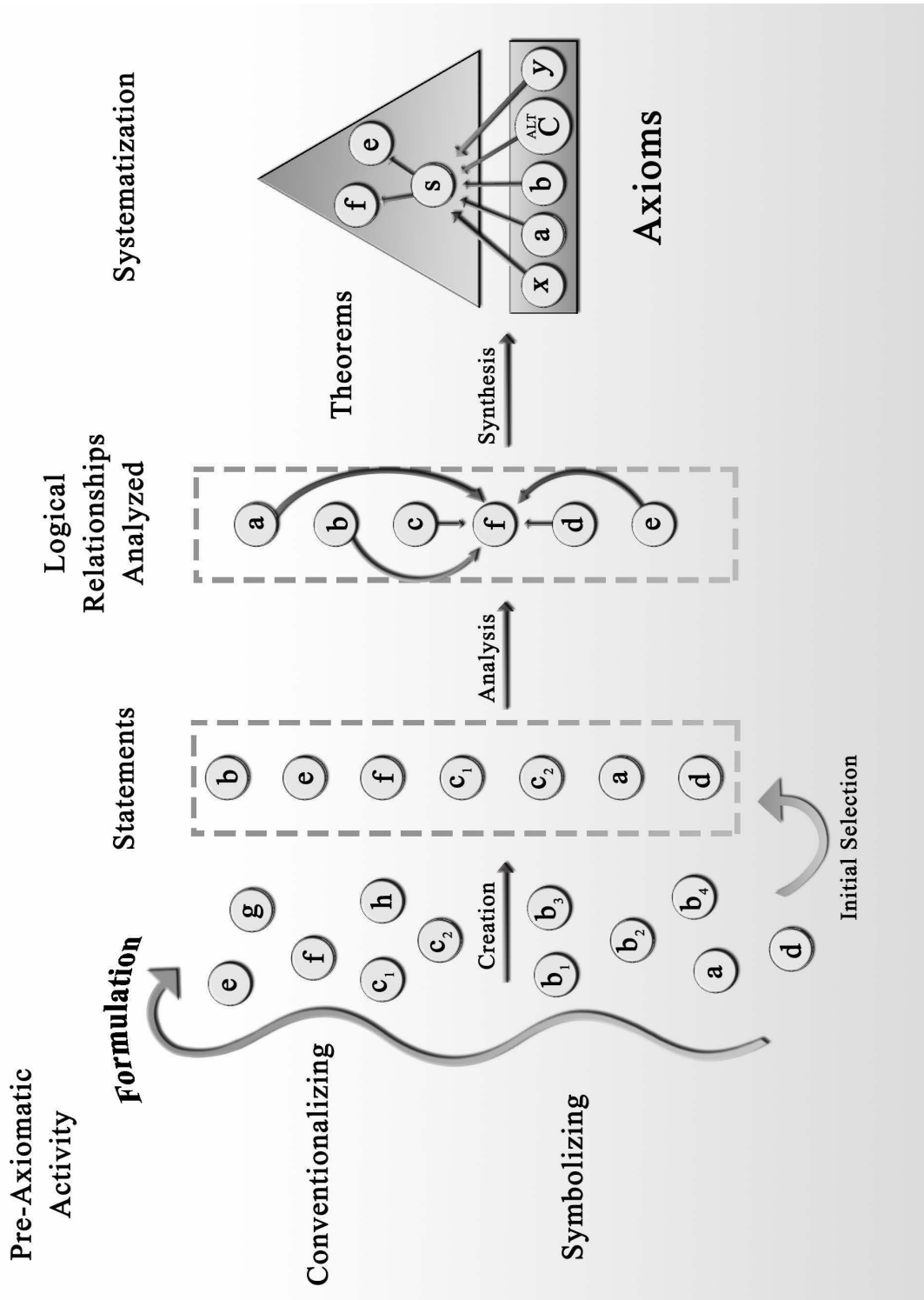
Conclusion

Two findings emerged in the development of this case study description of collective axiomatizing, but the most significant of these offer a revision to De Villiers's (1986) model of descriptive axiomatizing. The results of this study emphasize the additions of pre-axiomatic activity and axiomatic creation to the model and elaborate the processes of axiomatic formulation and analysis within the classroom community. The second finding addresses the role of intellectual need (Harel, 2013) in collective axiomatizing through a discussion of the addition and eventual removal of two axioms that the class had reinvented.

A Revision of De Villiers's (1986) Model of Descriptive Axiomatizing

As noted by De Villiers (1986), descriptive axiomatizing is a complicated process and thus, my revision of his work offers a refinement that attends to two main features concerning the original model of this activity. The revised model provides some insight into how axiomatic creation and analysis can occur at the collective level when students engage in descriptive axiomatizing. In discussing axiomatic creation, I also describe how some pre-axiomatizing mathematical activities support the formulation and selection of an initial set of statements, which De Villiers (1987) had included in an earlier part of his Boolean algebra curriculum. The process of descriptive axiomatizing is overviewed in Figure 34 and the revisions to De Villiers's (1986) model of this activity are then summarized.

Figure 34: A revision to De Villiers's (1986) model of descriptive axiomatizing



Axiomatic Creation

The process of descriptive axiomatizing begins with pre-axiomatic activities that include both conventionalizing (Yannotta, 2016b) and symbolizing (Gravemeijer, 2004). Consistent with De Villiers (1987), who reported these activities occurring during the Modeling portion of his Boolean algebra curriculum, their addition to the process diagram recognizes the role these initial activities play throughout descriptive axiomatizing. For instance, the adoption of a multiplicative convention fostered exponential forms of iterations and the development of a rule for adding exponents. As students transition to axiomatizing, there is a global shift toward identifying essential relations that describe the structure or concept (Krygowska, 1971). This orientation may begin informally by identifying patterns that describe a particular representation associated with the structure, such as an operation table or a switching circuit (De Villiers, 2010). A critical shift occurs in orientation when students begin to identify short cuts for their calculations and make use of substitution. Students may not yet describe these relations using equations, so a teacher can assist in both symbolizing and formulating exemplar statements, before asking students to track their usage.

As the activity shifts to collectively formulating statements, the teacher's role is pivotal. An approach that is consistent with guided reinvention (Freudenthal, 1991) is to use a combination of student contributions and transformational records (Rasmussen and Marrongelle, 2006) to elicit statements from students. By selecting a few key statements from a student's work, the teacher can initiate collective axiomatization by listing these rules in the public space. The act of privileging what was once an individual's list of rules for not only models statement formulation for students, it also signals the transition

to constructing a public artifact that will serve as a collective record of the class's subsequent activity.

Once a few rules have been recorded in the public space, a teacher can ask other students to contribute statements they found useful for calculation. Later, the teacher may modify the formulation criteria by asking the students to find relations that serve different purposes. For instance, one could vary the criteria to searching for properties that students may have used implicitly. In this study, the associative property was identified as one of these “stealth rules” that applied to multiple calculations, but others emerged when examining the deductive steps within a specific calculation. In fact, multiple rules were identified when the students verified each step in the calculation $(RF)(FR) = R^2$, which included both a left and right identity rule for R and the rule $RR = R^2$, which was actually a consequence of the students' notational choice.

During the process of identifying rules, students may start revising statements they have already formulated. For instance, after axiomatizing four specific rules involving the identity, the students in this study worked together to generalize the statement $NI = N = IN$, which would apply to any of their symmetries. Axiomatic creation begins to shift to axiomatic analysis when a subset of these statements is initially selected for further deductive work. As illustrated in both De Villiers (2010) and in this study, the teacher's selection of an initial set of statements shifts attention toward future deductive work as the class begins analyzing their collective artifact.

Axiomatic Analysis

Central to axiomatizing is the analysis and reorganization of the logical statements that were previously formulated and selected. This work includes attending to

the completeness and independence of axioms (De Villiers, 1987) and can be framed in terms of asking whether the current system of rules accomplishes the axiomatic work one asks of it. For instance, systematization tasks that ask questions such as “Do we have enough rules to complete an operation table?” or “Do our current rules allow us to prove the Latin square property?” address completeness. While asking “Can certain rules can be deduced from others?” attends to independence. As students engage in systematization tasks, their responses can serve a variety of functions that foster changes in a collective list of axioms. For instance, students’ proofs can assist in deciding whether to promote a statement or to relegate an axiom, thus reinforcing the use of deductive criteria for adding or removing a statement from the list of axioms. Student responses to systematization tasks may impact the axiomatic system through the creation of a more descriptive axiom that makes the total list of rules shorter or the formulation of “stealth” rules that may have previously been overlooked. Finally, some systematization tasks may not lead to any immediate artifact revisions, but rather, they may indicate that the current list of rules is sufficient for doing the current work it was assigned.

Intellectual Need in Axiomatizing

In addition to the intellectually necessitated tasks that are integrated into a well-designed curriculum, the results from this study suggest that students’ intellectual needs can also drive axiomatization in unanticipated ways. For instance, Gene’s suggestion of taking the rule $RR = R^2$, $RR^2 = R^3$ “to the general case” led to a flurry of collective axiomatizing. Both Kevin and Gene had different ideas about how to generalize this rule, which Todd was able to incorporate into a rule that generalized both the base and the exponents. This rule underwent a series of revisions to eventually become a variant of

the familiar product rule for exponents: $A^p A^q = A^{p+q}$, which Kevin opposed cutting from their list even after Brian's explanation that it was a result of their notational convention. The task of eliminating redundant rules is a component of the LIT (Larsen, 2013), but the way this played out was atypical compared to other implementations of the curriculum. Instead of simply eliminating rules through deduction, Todd's intellectual need for elegance seemed to inspire him to make what Brian referred to as a "radical change" to the system. The formulation of Rule 3 ALT ($R^3 = F^2 = I = RFRF$) made the list of rules shorter and perhaps more elegant to Todd, particularly when this rule was shown to do the same work as the existing rule. The subsequent adoption of this axiom played a pivotal role in the class's reorganization of the public artifact, as it contained both a version of the dihedral relation and information about the generators for the structure they were axiomatizing.

As posited by Harel (2013), some of the specific goals of mathematizing in Realistic Mathematics Education (RME) can be correlated with intellectual need. I claim the intellectual needs associated with the axiomatization of two rules in this study align with two of RME's goals for mathematizing (Gravemeijer, 1994), namely generalization and brevity. Consistent with other implementations of Larsen's LIT, the students in this study reinvented group and en route to their definition, they axiomatized several statements that are recognizable within the mathematics community. In addition to the closure, associative, identity, and inverse axioms that define a group, these students also attended to their intellectual needs by axiomatizing two enduring statements that were not part of the formal definition of group. Specifically, Gene's intellectual need for generalization evolved into a sequence of collective generalizations of the product rule

for exponents. Further, Todd attended to brevity by incorporating an alternate version of the dihedral relation into an axiom that at least superficially, made the list of rules shorter.

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Appendix: Example from the Axiomatizing Analytic Document

Pre-axiomatizing activity as part of a whole-class discussion (Day 6)

The reinvention of the group concept begins with an investigation of the symmetries of an equilateral triangle (Larsen, 2013). After developing their own description and notation for each symmetry, the students are then asked to express these symmetries in terms of R (a 120-degree clockwise rotation) and F (a flip across the vertical axis). After a discussion about whether to adopt additive or multiplicative notation, the class decides by a majority vote to represent their six symmetries as follows:

Symmetry	Notation
A flip across the vertical axis	F
A 120° clockwise rotation	R
A 240° clockwise rotation	R^2
A 120° clockwise rotation followed by a flip across the vertical axis	RF
A flip across the vertical axis followed by a 120° clockwise rotation	FR
A 360° clockwise rotation or two flips across the vertical axis	I

Table A.1: The class's notational choices for the symmetries of an equilateral triangle

Data interpretation

As noted in Figure A.1 below, the class had discussed different notations before agreeing on the convention in Table A.1. In particular, alternate versions of RF (FR^2), FR (R^2F), and I (R^3 and F^2) were formulated by the students. Although they were not selected as the preferred expression for a given symmetry, these equivalencies form the seeds of some of the relations that will later become axioms for combining symmetries.



Figure A.1: The seeds of some axioms

Formulating relations individually

At this point in the enacted curriculum, it has been taken-as-shared that if any of the two symmetries are combined, it will result in one of the existing six symmetries. Brian then relates the task to the class in his own words, “For every possible combination of two of these symmetries, I want you to work out which symmetry you will end up with.”

The students work on this task in groups for the remainder of the class. The camera focuses on one student, Kyle, and his partial list of calculations. In the upper-right hand corner, one can see the equations $RR = R^2$ and $RRF = FR$.

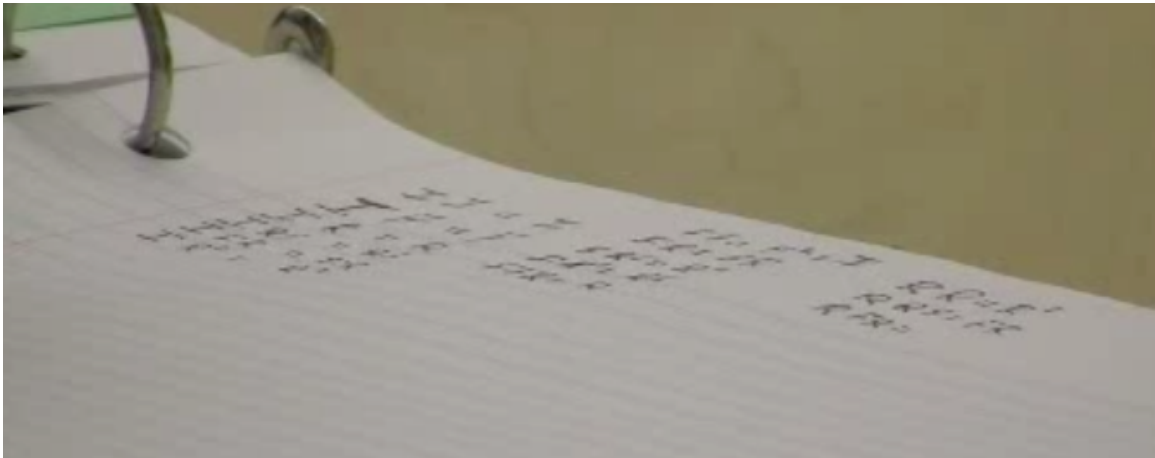


Figure A.2: Kyle’s calculations of symmetry combinations

Data interpretation

This is first time we have evidence of an equivalence relation between two symmetries being expressed in a conventional way. Kyle has not only computed the symmetry combinations, but by using an equals sign, he has constructed a *record* of the relations. Two of these relations, $RR = R^2$ and $RRF = FR$, will later become part of an initial set of axioms in the public space.

Formulating and sharing relations in small groups (Day 7)

In the previous class, the students were asked to find all possible combinations of any two symmetries, so Brian begins this episode by asking how many they came up with. One student, Tessa, responds, “30”, while several others say they got 36. In an effort to clarify this discrepancy, Brian has the students discuss their calculations in their groups as illustrated in this excerpt:

Leland: I don’t know why some of these result in the identity.

Tessa: Hmm?

George: Yeah, a lot of them do. I have one, two, ...
Leland: Why, why is RF and RF and FR and FR become the identities?
George: Which one?
Leland: Uh, FR combined with another FR —with itself basically, and then RF and RF .
George: $FR FR$, yeah it does, give the identity.
Leland: But why?
George: Um, ...
Leland: It's not one that's blatantly obvious.
George: Yeah, but it does. I had one, two, three, four, five, six... Did you have seven of them come up with the identity?
Leland: RF and R becomes F , right? But then FR and F becomes R squared, so there's no relation. But why? Oh, how many?
George: Did you? I got seven of them listed for the identity.
Tessa: I'm finding six.

Data interpretation

Leland has noticed some symmetry combinations that surprised him. In particular, he is curious why the combinations of $(FR)(FR)$ and $(RF)(RF)$ both produce the identity symmetry, while George seems content with simply knowing that they do. Leland also observes that $(RF)(R)$ results in F , while $(FR)(F)$ produces R^2 , noting that there does not seem to be a connecting between the two calculations. Other students also reference these latter two relations, but they do not appear to play a significant role in the axiomatizing sequence for the class. However, the relations $(FR)(FR) = I$ and $(RF)(RF) = I$ each will play a pivotal role in whole-class axiomatizing.

Formulating and sharing relations as a class

Brian now asks some students to share their calculations and their method of organizing these calculations on the document camera. The first student to share, Jesse, has made a list of relations similar to those made by Kyle.

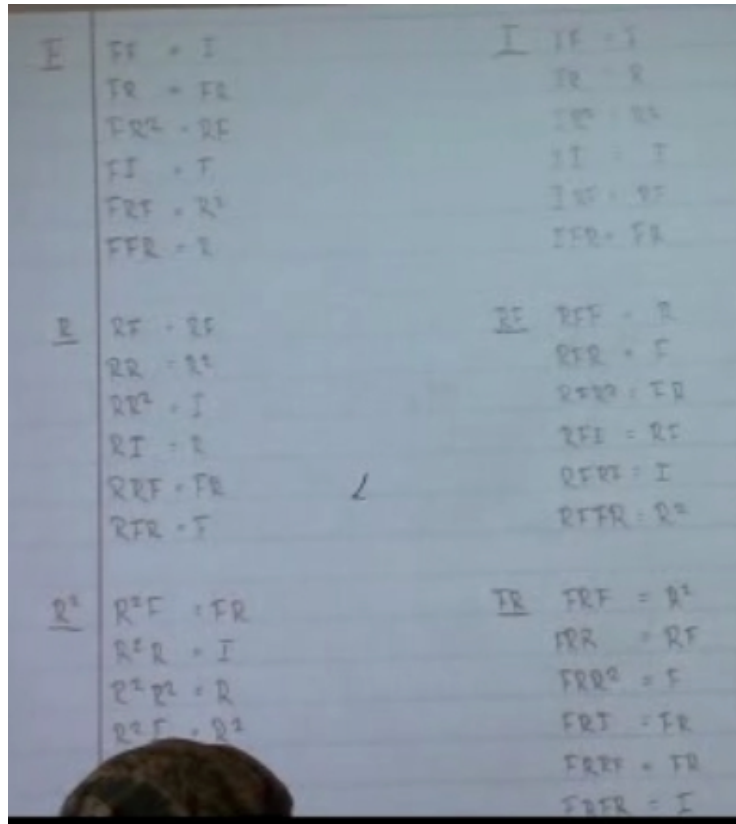


Figure A.3: Jesse's organization of symmetry combinations

Data interpretation

Although Jesse's organization is quite systematic, in that he combined each symmetry in the order F , R , R^2 , I , RF , and FR six different times and has consistently used equality relations, his calculation of $FRRF = FR$ is incorrect. These thirty-six calculations demonstrate the closure property of the system, but this one miscalculation, which goes unchallenged, suggests that some of the students have not yet realized that in each of his six groupings, a given symmetry can only appear one time.

Following a brief discussion of Jesse's list, Brian then asks another student, Ted, to share his calculations. Unlike Kyle and Jesse, who made lists of relations, Ted has used an operation table to record his calculations.

	I	R	R ²	F	FR	RF
I	I	R	R ²	F	FR	RF
R	R	R ²	I	FR	RF	F
R ²	R ²	I	R	RF	F	FR
F	F	RF	FR	I	R ²	R
FR	FR	F	RF	R	I	R ²
RF	RF	FR	F	R ²	R	I

Figure A.4: Ted's Table of Calculations

The Sudoku property is identified

As the students look through Ted's table, they are asked to comment on any patterns they notice.

Brian: What are some of the things that you noticed whether I write it in that sort of table or this sort of table, what are some of the patterns that you noticed?

Alysha: The first and last are opposite.

Brian: I'm not sure what you mean by that.

Alysha: The row is opposite the last one.

Brian: Oh. So the first line here... and the last line there...

Tessa: It's the same for F (noting the fourth and second rows)

Student: (hmm.... cool)

Brian: I have to say I didn't notice that (laughing).

Kyle: You don't get the same entry on any one row or any one column.

Ted: Hmm...

Brian: Can you explain what you mean by that?

Kyle: You don't... So if you are looking at just a single column, you won't see...

Brian: So, say down this column (points to 2nd column)?

Kyle: Yeah. You will only see one I , one R , one R -squared. You won't see two of any entry... and that happens for rows as well.

Brian: Everywhere? (long pause)

Students: Uh huh.

Brian: That's an interesting observation. It's going to turn out to be a really productive one too.