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AN ABSTRACT OF THE THESIS OF Gregory Allen Jones for the
Master of Science in Engineering: Civil October 12, 1984.

Title: State Space Approach to Flood Stage Estimation.

APPROVED BY MEMBERS OF THE THESIS COMMITTEE:

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A flood routing and stage prediction model is developed using the techniques of State Space and Kalman filtering. The governing equation is the physically based hydrologic method of flood routing with the output being an optimal estimate of stage given known inputs of streamflow. These equations are developed in state space and the Kalman filter is employed to estimate the flow and river stage.

The model is applied to the Toutle and Cowlitz Rivers in the State of Washington, where the stage is affected by a shifting bed elevation. With the

deterministic inputs at Mayfield Dam on the Cowlitz and Tower Road on the Toutle, the optimal output of stage is predicted at Castle Rock on the Cowlitz River. Addition of the Kalman Filter improves stage prediction based on an application to an observed storm event.

STATE SPACE APPROACH TO FLOOD STAGE ESTIMATION

by

GREGORY ALLEN JONES

A thesis submitted in partial fulfillment of the
requirement for the degree of

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in
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1984

TO THE OFFICE OF GRADUATE STUDIES AND RESEARCH:

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TABLE OF CONTENTS

PAGE

ACKNOWLEDGEMENTS.....	iii
LIST OF TABLES.....	vi
LIST OF FIGURES.....	vii
LIST OF SYMBOLS.....	ix

CHAPTER

I	INTRODUCTION.....	1
II	REVIEW OF LITERATURE.....	5
	Hydrologic Methods.....	6
	Hydraulic Method.....	7
	Correlation.....	10
	Parameter Estimation.....	12
	Alternative Solutions.....	15
	Stage.....	16
	Summary.....	18
III	THEORY AND DEVELOPMENT.....	19
	State Space.....	20
	The Kalman Filter.....	26
	The Cascade of Linear Reservoirs and Stage.	30
	Summary.....	37

CHAPTER	PAGE
IV APPLICATION.....	39
Development.....	41
State Equation and Kalman Filter.....	51
Discussion of Application.....	57
V SUMMARY AND RECOMMENDATIONS.....	61
Summary.....	61
Recommendations.....	64
BIBLIOGRAPHY.....	66
APPENDIX A-Computer Output and Program.....	69
APPENDIX B-Kalman Filter Matrices.....	78

LIST OF TABLES

TABLE	PAGE
I The Celerities for the Toutle and Cowlitz River Found Both Graphically and Analytically.....	42
II The Reach Lengths for the Toutle and Cowlitz Rivers.....	44
III The Reach Lengths, Bedslopes, River Width, and Storage Coefficients for the Toutle and Cowlitz Rivers.....	48
IV The Peak Time, Discharge, and Stage for Castle Rock, Washington.....	57

LIST OF FIGURES

FIGURE	PAGE
2.1 The Stage vs. Discharge Curve for the Cowlitz River at Castle Rock, Washington.....	17
3.1 A Representation of n Cascading Linear Reservoirs for a Typical River Channel....	31
3.2 A Discrete Kalman Filter Time Line.....	38
4.1 Area Map Showing the Impact of the May 18, 1980 Eruption. The Area Flooded can be seen along the Toutle and Cowlitz Rivers.....	40
4.2 The Stage vs. Discharge Curve for the Toutle River at Tower Road, with the Reference Discharge and Tangent Line used for Celerity.....	44
4.3 The Stage vs. Discharge Curve for the Cowlitz River at Castle Rock, with the Reference Discharge and Tangent Line used for Celerity.....	45
4.4 The Stage vs. Discharge Curve for the Cowlitz River at Mayfield Dam, with the Reference Discharge and Tangenet Line used for Celerity.....	46

FIGURE	PAGE
4.5 Map of the Basin with the Rivers Divided into Reaches.....	49
4.6 The Inflows at Mayfield Dam on the Cowlitz River and at Tower Road on the Toutle River vs. the Outflow at Castle Rock, Washington for December 3 and 4, 1982.....	55
4.7 The Comparison of Outflow at Castle Rock, Before and After Lateral Inflow was taken into Account.....	56
4.8 Hydrograph for the Gaging Stations below Mayfield Dam on the Cowlitz River (dash line) and Tower Road on the Toutle River (solid line) during December 3 and 4,1982.....	58
4.9 Observed and Predicted Stage at Castle Rock, with and without the Kalman Filter.....	59

LIST OF SYMBOLS

Symbol	Definition
A	Area of the cross section perpendicular to the flow.
<u>A</u>	A matrix containing the coefficients that relate the state vector to the state vector derivatives.
B	The width of the cross section of a river channel.
<u>B</u>	A matrix containing the coefficients that relate the input to the state variables.
C_0	Routing coefficient that contains Δt , K, and x.
C_1	Routing coefficient that contains Δt , K, and x.
C_2	Routing coefficient that contains Δt , K, and x.
D	The differential operator d/dt .
<u>D</u>	A matrix relating a linear combination of the state variables to the output.
<u>E</u>	A matrix relating the input directly to the output.
<u>G</u>	The coefficient matrix relating the system error to the state variables.
<u>H</u>	A linear matrix relating the measurements to the state variables.
I	The inflow into the system.
<u>I</u>	The identity matrix.
K	Proportionality constant relating storage to inflow and outflow.
<u>K</u>	The Kalman gain matrix.

$\underline{P}$	The covariance matrix of the error and the estimate.
Q	The outflow or discharge of the system.
Q_0	Reference discharge.
$\underline{Q}$	Variance of the system noise.
$\underline{R}$	Variance of the measurement noise.
S	Storage within the system.
S_f	The friction slope.
S_0	The bed slope, equal to the natural gradient of the channel bed.
T	The wave period.
Tr	The time period of rise of the inflow hydrograph.
a	Constant used in the stage discharge equation.
b	Constant used in the stage discharge equation.
c	Constant used in the stage discharge equation.
c	The speed of movement of the flood wave; known as celerity.
$\underline{c}(t)$	The desired output of the output equation.
d^n	The n^{th} derivative.
d_0	The steady uniform flow depth.
g	Acceleration due to gravity.
$g(Q)$	The depth of flow above an arbitrary datum known as stage, as a function of discharge.
k	A parameter used in estimating the maximum reach length with a suggested value of 2.
m	A discrete time step.
q	The lateral inflow into a reach.
t	Continuous time.

$u(t)$	The independent variable (input), as a function of time.
$\underline{u}(m)$	The discrete time independent variable.
v	The velocity of the flow.
$\underline{y}$	A vector of the measurement noise $N(0, \underline{R}(m))$.
x	A weighting term for the space step.
x	A parameter weighting the inflow and outflow.
$\underline{x}$	A vector containing the state variables.
$\dot{\underline{x}}$	A vector containing the first derivatives of the state variables.
$\tilde{\underline{x}}$	The error in the estimate, equal to the difference between the estimated and actual state variables.
$\hat{\underline{x}}$	The estimated state variable.
y	A weighting term for the y step.
y	The depth of flow.
$y(t)$	The dependant variable, as a function of time.
$\underline{z}$	The measurements of the system state.
Δt	A step in time.
Δx	Space step.
Λ	The convolution sum.
τ	A variable of intergration.
$\Phi(t,0)$	The state transistion matrix.
Γ	The convolution sum.
$+,-$	Used to denote just before, and after a measurement is applied.

CHAPTER I

INTRODUCTION

Nature combines the hydrologic cycle with the steep valleys and vast open plains of a river to form one of the most hazardous of all natural disasters, the flood. A flood is defined as any relatively high flow that overtops the natural or artificial banks in the reach of a river.

The processes that produce floods are usually rainfall and snowmelt. When excessive amounts of either are applied to a watershed, runoff is produced, which flows to a river entering as tributary or direct inflow. The flow in rivers is affected by channel storage and resistance to flow, causing flood waves to have a lower magnitude and longer duration at a downstream point in a channel, when compared to a point upstream. This occurrence in a river channel is known as flood wave propagation.

Over the years many physical changes have occurred along the flood plains; areas have been developed and cities have grown. This has increased the possibility of a major disaster. It also has caused millions of dollars to be spent in the prevention and cleanup of floods. This demonstrates the definite need to accurately measure and predict the potential flood.

There are four distinct methods with which to measure the aspects of a flood. The applications of each method varies with the needs of the user. The area inundated can be measured using a contour map and is of interest to the planner who is going to occupy the area in the flood plain adjacent to a river. The peak discharge is measured in terms of flowrate and is usually specified as cubic feet per second. It is of interest to the engineer designing spillways, culverts, bridges, and flood control structures. The volume of flow is of interest to the engineer designing control structures which include storage to be used in irrigation, water supply, and flood control. Volume of flow is commonly measured in units of acre-feet or cubic feet per second per day. The stage of a flood is the elevation of the water surface at a point, above an arbitrary datum. It is of interest to those planning to build or who have already built structures along or near a river. This method is the most widely used and is of the most interest to the general public. Although, stage is useful in developing stage vs. discharge relationship, it can be more useful if it is predicted.

Early flood warning systems can be applied to river systems in an attempt to save lives and limit loss of property. This is illustrated by an event which happened in the town of Putnam, Connecticut on August 19, 1955 when the whole town was buried under 14 feet of water, rock, and silt. Thanks to an early warning system, not one life was

lost (Clark 1982). Another example is the Sprout-Waldron Company in Lycoming County, Pennsylvania. With an early warning system the plant was able to carry out protective measures and save approximately 93.2% of losses compared to an earlier unexpected flood (Flood Loss Reduction Association 1981).

An early flood warning system consists of a model of a river system where the flow at a point upstream is routed through the channel to a point downstream to predict stage at that location. The point downstream is where early warnings would be beneficial. This warning would have to be predictive, so the distance between the upstream and downstream points must be great enough so that the peak of the flow can not travel downstream before the warning is carried out.

This thesis describes in detail a flood wave propagation model for use in the prediction of stage and early flood warnings. The model is conceptual in structure, with the results being an optimal output of stage given a known upstream inflow hydrograph. From the output of stage and time of flow, a topographic map can be consulted. The area to be inundated, can then be estimated and flood warnings issued.

The text of this thesis follows the objectives stated above. Chapter II covers historical background and the formulation of governing equations. Chapter III presents the theory of state space modeling and optimal estimation

(Kalman filtering). Chapter IV presents an application of the model to an actual river system, using the Toutle and Cowlitz Rivers in the State of Washington as examples. Chapter V provides recommendations about future developments and use of the model.

CHAPTER II

REVIEW OF LITERATURE

As a flood wave travels downstream through a channel, the characteristic of the wave changes due to storage in the river channel and resistance to flow caused by friction effects. With no additional inflow, the peak will decrease in magnitude as the wave propagates downstream.

Methods of describing flood wave propagation have developed over the years into two distinct categories: the physically based approaches referred to as "Hydraulic" methods, and the systems or conceptual approach called "Hydrologic" methods (Weinmann and Laurenson 1979). The most established and practiced of the two techniques are the models based on the hydrologic methods (Dooge 1973). This is due to the simplicity of the underlying concepts and operations involved in these routines. However, with the growth of high-technology, new research has been done on the hydraulic methods using modern computers and numerical techniques. A clear understanding of the basic concepts behind these two flood wave propagation methods is essential. This chapter presents an overview of the concepts of each method, including relevant equations and background literature. Additionally, a correlation between

the two techniques, parameter estimation, and alternative means that may be applied to obtain the solution, are discussed.

HYDROLOGIC METHODS

The hydrologic methods represent a systems approach to flood wave propagation. Where an input or inflow is forced at one end of a river reach and an output or outflow is predicted at the other. No information is needed or desired at any other point in the system. When an approach such as this is undertaken the equation for continuity takes a lumped form of the storage equation:

$$I - Q = dS/dt \quad 2.1$$

where, I is the inflow into the system,

Q is the outflow of the system,

dS/dt is the change in storage with respect to time for the system.

In this form of the continuity equation there are two unknowns, storage and outflow; therefore a second equation is required. Another equation that can be used is the Muskingum equation, which relates storage to weighted values of inflow and outflow, as follows:

$$S = K(xI - (1-x)Q) \quad 2.2$$

where, K is a proportionality constant relating storage to inflow and outflow,

x is a parameter weighting the inflow and the outflow.

The classical approach to flood wave propagation is formed when the discrete form of the equation for continuity (Eq. 2.1) is combined with the Muskingum equation (Eq. 2.2) to form a single equation. The form of the equation is:

$$Q(m+1) = C_0 I(m+1) + C_1 I(m) + C_2 Q(m) \quad 2.3$$

where, $Q(m+1)$ and $Q(m)$ are the outflow at time (m) and one time step ahead $(m+1)$,

$I(m+1)$ and $I(m)$ are the inflows at time (m) and one time step ahead $(m+1)$,

C_0 , C_1 , and C_2 being coefficients that contain t , K , and x (Linsley, et al., 1982, p. 275).

This method was developed in the 1930's and has been used in models such as the HEC 1.

A simplified version of the Muskingum equation is used when storage is assumed to be only a function of outflow; that is when x is equal to zero:

$$S = KQ \quad 2.4$$

Equation 2.4 is known as the linear reservoir.

HYDRAULIC METHOD

The hydraulic methods incorporate the use of the Saint Venant equations. The Saint Venant equations describe unsteady open channel flow based on the continuity and momentum (Huggins and Burney, 1982). The continuity equation is:

$$\delta Q / \delta x + \delta A / \delta t = q \quad 2.5$$

where, $\delta Q / \delta x$ is the change in flow with respect to the longitudinal direction x ,

$\delta A/\delta t$ is the change in area of flow with respect to time,

q is the lateral inflow.

Ignoring the effects of lateral inflow, the momentum equation is:

$$S_f = S_0 - \delta y / \delta x - v \delta v / g \delta x - \delta v / g \delta t \quad 2.6$$

where, S_f is the friction slope,

S_0 is the bed slope,

y is the depth of flow,

v is the velocity of the flow,

g is the acceleration due to gravity.

Certain terms in Eq. 2.6 have a special meaning as follows:

$\delta y / \delta x$ represents the effects due to pressure forces,

$v \delta v / g \delta x$ is the convective acceleration resulting from nonuniformity of the flow,

$\delta v / g \delta t$ is the local acceleration due to the unsteadiness.

Different approximate forms of the momentum equation can be used to form different models (Weinmann and Laurenson 1979). The three most popular are the complete dynamic wave, diffusive wave, and the kinematic wave. The complete dynamic wave is formed by using the complete Saint Venant equation. This is a very cumbersome model in the fact that it is computationally difficult, with the needed data out of the realm of practical hydraulic methods. The diffusive wave is based on the observation that the local and convective acceleration terms are of approximately equal magnitude but opposite in sign (Weinmann and Laurenson

1979). When the remaining terms are combined with the continuity equation, a model is formed as follows:

$$\delta Q / \delta t + c \delta Q / \delta x = (Q / 2BS_f) \delta^2 Q / \delta x^2 + q \quad 2.7$$

where, c is the speed of movement of the flood wave; known as celerity,

B is the width of the cross section,

$(Q / 2BS_f)$ is the diffusion coefficient.

This wave exhibits both convection and dampening due to the diffusion coefficient.

Finally the simplest form of the hydraulic methods is the kinematic wave, which produces only convection, so the wave is not attenuated. The kinematic wave assumes the pressure term is negligible to form the equation:

$$\delta Q / \delta t + c \delta Q / \delta x = q \quad 2.8$$

Ponce, et al. (1978) discuss the application and use of the kinematic wave, diffusive wave, and complete dynamic wave. The criteria for use of the different models, kinematic and diffusive wave is based on bed slope and wave period, and is given by:

$$TS_0 (g/d_0)^{1/2} \geq 30 \quad 2.9$$

Where, T is the wave period, and

d_0 is the steady uniform flow depth.

They found that the diffusive wave can be applied to a wide range of bed slopes and wave periods. The inequality in Eq. 2.9 is no longer valid when this limiting criteria is reached. At this point the diffusive wave breaks down and the complete dynamic wave must be used. It was also found

that the kinematic wave gives a good approximation to the diffusive wave when the bed slope reaches a limiting value of ten percent, or when the wave period is long enough.

This phenomenon can be found in overland flow.

Other applications and discussions of the uses of the three formulations of the Saint Venant equation can be found in Weinmann and Laurenson (1979).

CORRELATION

Cunge (1969), demonstrates a correspondence between hydraulic and hydrologic methods by showing that the classical Muskingum routing is analogous to the diffusive wave approximation of the Saint Venant equations.

Cunge (1969) first formulated the kinematic wave (Eq. 2.8) into a numerical solution by a finite difference approximation of the derivatives. This equation is of the same form as Eq. 2.3, the classical Muskingum equation. Both equations showed some wave dampening in the finite difference form, however there should be no dampening of the kinematic wave in the analytical solution. To help explain the phenomenon, the kinematic wave was transformed into a finite difference scheme, using a Taylor series expansion, retaining 2nd order terms. This yielded an equation of the form:

$$\begin{aligned}
& (x(I(n+1)-I(n))/\Delta t) + ((1-x)(Q(n+1)-Q(n))/\Delta t) \\
& + c((y(Q(n+1)-I(n))/\Delta x) + (1-y)(Q(n+1)-I(n+1))/\Delta x) \\
& + c\Delta x((x-.5)+c(.5-y)\Delta t/\Delta x)\delta^2 Q/\delta x^2 = 0
\end{aligned} \tag{2.10}$$

where, y is a weighting term for the time step,

x is a weighting term for the space step,

Δt is the time step,

Δx is a space step.

This new equation is of the same form as the difference equation of the kinematic wave, but with the second order term retained. The new approximation does show attenuation. But, when the second order term is dropped, Eq. 2.10 will still show attenuation. This shows that the wave dampening affects of the finite difference scheme of the kinematic wave is due to numerical error.

Cunge (1969) next took the diffusion coefficient from the diffusive wave (Eq. 2.7) and equated it with the numerical diffusion coefficient of Eq. 2.10. Assuming equal weight for each time step, $y=0.5$, yields:

$$-c\Delta x(x-1/2) = (Q/2BS_f) \tag{2.11}$$

Solving Eq. 2.11 for x , gives it meaning in terms of physical variables:

$$x = 1/2(1 - Q/\Delta x B C S_f) \tag{2.12}$$

Cunge (1969) states that the limiting condition of Eq. 2.12 is that the diffusion coefficient must be between 0.0 and 0.5 for attenuation of the wave. Values of x greater than 0.5 cause the wave to amplify.

The coefficient K , which is normally referred to as the storage coefficient, is shown to represent travel time of the wave through a reach, and can be defined as:

$$K = \Delta x / c \quad 2.13$$

As can be seen from the above equations (Eqs. 2.12 and 2.13), a physical meaning has been given to the conceptually based Muskingum equation.

PARAMETER ESTIMATION

Since Cunge (1969) made his initial criteria for stability based on the parameter x , additional attempts have been made to establish limits on both the coefficients K , and x . Weinmann and Laurenson (1979), Kundzewicz (1980), Kundzewicz and Strupczewski (1980), Ponce and Theurer (1982) and others have either substantiated his suggestions or formulated their own criteria.

The problems that need to be addressed are the possibility of a dip in the hydrograph at the beginning of the Muskingum computation, and assuring the attenuation obtained by the model reproduces the physical attenuation observed in the river flow.

Weinmann and Laurenson (1979) state that, based on Cunge's criteria of the parameter x having limits between 0.0 and 0.5, Δx also has a lower limit, when x is equal to 0.0. This can be seen in Eq. 2.12. But, Ponce and Theurer (1982), and Kundzewicz (1980) state that from numerical

studies and experience there is no practical lower limit on Δx , and therefore no lower limit on x . Both have had experience with using Δx that violates the limit placed on them by Eq. 2.12, without any numerical complications. Kundzewicz and Strupczewski (1980) go as far as to say that x can fall in the range of $(-\infty, 0.5)$.

It has been noted that at times an unrealistic negative outflow at the beginning of the outflow hydrograph has been obtained. Weinmann and Laurenson (1979) show that this can be minimized and short lived if K is kept below a limiting amount:

$$K = \Delta x / c \leq Tr / 2x \quad 2.14$$

where, Tr is the time period of rise of the inflow hydrograph.

This gives an upper limit on Δx when it does not equal zero. Ponce and Theurer (1982) give an upper limit on Δx by:

$$\Delta x \leq (\Delta x_c + \Delta x_D) / k \quad 2.15$$

where, Δx_c is equal to Eq. 2.13 when K is equal to the time step Δt ,

Δx_D is based on Eq. 2.12 with x equal to 0.0,

k is based on experience and a suggested value is 2.

The limits of Δt have been discussed by the authors mentioned above. An upper limit according to Weinmann and Laurenson (1979) comes from the criteria that:

$$\Delta t = \Delta x / c \quad 2.16$$

Larger values of Δt have been used by Weinmann and Laurenson (1979) without stability problems. Kundzewicz and Strupezewski (1980) use the convolution integral in their solution of the Muskingum equation, where the only upper limit on Δt is that the input and output have to correspond. A lower limit also from Weinmann and Laurenson (1979) is that Δt must not be much smaller than $\Delta x/c$, for if it were, the variables at the current section would be evaluated before the disturbances at the previous section has travelled the distance Δx . Ponce and Theurer (1982) on the other hand state that there is no lower limit on Δt . It's up to the modeler to choose, based on computational resources.

The authors that have been quoted use their own criteria on stability to eliminate the dip in the outflow hydrograph. Weinmann and Laurenson (1979) state that the negative outflow can be ignored and suggest the use of a limiting value on the storage constant K , given by Eq.2.14 to keep the negative flows at a minimum. Kundzewicz (1980) as reported by Ponce and Theurer (1982), says that a search for a general condition to suppress negative outflows may prove to be worthless. Ponce and Theurer (1982) on the other hand, state that in order to keep the solution stable, Eq.2.15 must be satisfied.

The preceding discussion has been made on the assumption that the value of the wave celerity was based on

a representative flow and will remain constant, but it is known this is not true. Wave celerity is defined as follows:

$$c=dQ/dA \quad 2.17$$

Ponce and Yevjevich (1978) demonstrates a method in which the wave celerity is computed as an average sum of the previous known values, which produces a nonlinear effect. The results correspond well with values of a nonlinear function.

ALTERNATIVE SOLUTIONS

One form of the Muskingum equation that is often used is the Linear Reservoir and the Cascade of Linear Reservoirs (Dodge 1973). A physical interpretation of the coefficients can be formed from methods of Cunge (1969). This method is employed in the SSARR model and the Kalinin-Milyukov method of flood routing as stated by Ponce (1980). The linear reservoir (Eq. 2.4) has a physical definition for Δx known as the characteristic reach length (Ponce 1980) and can be found by setting $x=0$ in Eq. 2.12.

In the SSARR model a numerical technique is used to solve the governing equation, but as shown by Kundzewicz and Strupczewski (1981) and the method employed in this thesis, a well known integral operator, the "convolution integral" can also be used.

$$Q(t) = \int_{-\infty}^t \text{Kernal}(t-\tau) I(\tau) d\tau = Q_0(t) + \int_{t_0}^t \text{Kernal}(t-\tau) I(\tau) d\tau \quad 2.18$$

where, $Q(t)$ is the output signal,

$I(\tau)$ is the input signal,

Kernal is the impulse response function,

Q_0 is the term responsible for the impact of the
initial conditions (decay response),

τ is a variable of intergration.

This method has an added advantage in that its stability is not dependent on the time step.

A cascade of linear reservoirs is formed when a series of reservoirs is used; the outflow from one reservoir becomes the inflow to the following reservoir. An n^{th} order differential equation results, corresponding to n cascading linear reservoirs. The method of solution that lends itself nicely to an n^{th} order differential equation is the state space technique as described in Gelb (1974) and Szollosi-Nagy (1976). This technique is convenient, in that it takes an n^{th} order differential equation and reduces it into n , 1^{st} order differential equations. The equations can be formulated in a matrix notation and solved by techniques to be discussed later.

STAGE

Stage is the level of the water surface above an arbitrary point for a given discharge. The relationship between stage and discharge can be shown by a stage discharge rating curve, figure 2.1, and can be defined by the nonlinear relationship (Linsley, et al., 1982):

Figure 2.1. The stage vs. discharge curve for the Cowlitz River at Castle Rock, Washington.

$$g(Q) = a + bQ^c \quad 2.19$$

Where, $g(Q)$ is the stage as a function of discharge,

a , b , and c are the coefficients that relate stage and discharge.

The coefficients a , b , and c can be found as described by Linsley et. al. (1982) or by using linear regression techniques (Walpole and Myers, 1978).

SUMMARY

The review of literature in the field of flood wave propagation has been centered on the hydrologic models. The concept of the physical interpretation of the basis for the hydrologic models is generally accepted, but the exact limits for calculation of the parameters is not completely agreed upon. The solution of the underlying hydrologic equations have been solved mostly using numerical techniques. The state space techniques as applied to hydrologic modeling have not been used extensively.

CHAPTER III

THEORY AND DEVELOPMENT

When a natural system such as a river is modeled, it is probable that error will be present in any prediction. This error or noise, as it is often called, can be due to improper modeling of the system. In the case of a river system, a tributary may be ungaged and therefore unaccounted for, or lateral inflow may pass into a river along its banks. This leads to noise in the system that is difficult to predict. One way to estimate this noise is to take measurements of the system at discrete times to help in updating the model. Unfortunately, measurements are not completely accurate and they too may be corrupted by noise.

What is needed is an estimate of the state of the system, e.g. river flow, that is based on some combination of the output from the model and the measurement corresponding to the current time step. The estimate should be unbiased, have minimum error variance, and be consistent. The Kalman filter as presented by Schweppe (1973) and Gelb (1974) provides for a linear combination of prediction and measurement that meets all of these criteria, as long as some assumptions are made.

The Kalman filter was developed for use with a system that is formulated in the structure of state space; state space being one method to solve a system of n 1st order, ordinary, differential equations. Recall that a cascade of linear reservoirs can be represented this way. The structure of state space, which is matrix-vector is also convenient for multi-input, multi-output systems.

In this chapter, a discussion of the methods of state space and Kalman filtering is presented. A simple, 2nd order model is formulated with these methods using a cascade of linear reservoirs in both continuous and difference forms to demonstrate the approach.

STATE SPACE

State space is an n -dimensional space whose coordinate axes consist of the n state variables, where any state of the system can be described by a point in the space given by the set of state variables. The state variables is defined as the minimum set of n variables that completely describes a system given the initial conditions at time $t=0$, plus information on the input excitation. These variables are sufficient to determine the state of the system at any time $t > t(0)$. A set of n state variables that completely describes the system can be written in vector form and is known as a state vector. A set of state variables is not unique to a system. There may be many different sets of

state variables that would yield the same overall input-output behavior. The choice of a particular set corresponds to choosing a coordinate system; the proper choice may have a considerable impact on the numerical analysis (Gelb 1974).

When the governing equation is an n^{th} order, ordinary differential equation, such as the cascade of linear reservoirs, the equation can be broken down into state space form by a transformation of variables. This transformation results in n first order, linear differential equations. This can be shown by following the examples in Gelb (1974). The governing n^{th} order ordinary differential equation may be written in the form:

$$(D^n + a(n-1)D^{n-1} + \dots + a(1)D + a(0))y(t) = u(t) \quad 3.1$$

where, D is the differential operator d/dt ,

D^n is the n^{th} derivative,

$a(n-1) \dots a(0)$ are time invariant coefficients,

$y(t)$ is the dependent variable,

$u(t)$ is the independent variable (input).

The first step in transformation is to redefine the derivatives in Eq. 3.1 as:

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= dy(t)/dt = \dot{x}_1(t) \end{aligned} \quad 3.2$$

.

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$$x_n(t) = d^{n-1}y(t)/dt^{n-1} = \dot{x}_{n-1}(t)$$

With Eqs. 3.1 and 3.2 the n^{th} order governing differential equation can be broken down to form n , 1^{st} order differential equations:

$$\dot{x}_1(t) = x_2(t) \quad 3.3$$

$$\dot{x}_2(t) = x_3(t)$$

.

.

.

$$\dot{x}_n(t) = -a(0)x_1(t) - a(1)x_2(t) - \dots - a(n-1)x_{n-1}(t) = u(t)$$

The first $n-1$ equations of Eq. 3.3 comes from the definition of the state variables (Eq. 3.2), while the final equation is obtained from Eq. 3.1, the definition of the governing equation.

Eq. 3.3 can be written in matrix notation, as follows:

$$\begin{bmatrix} \dot{x}_1 \\ x_2 \\ . \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ . & . & \dots & . & . \\ -a(0) & -a(1) & \dots & -a(n-2) & -a(n-1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 1 \end{bmatrix} u(t) \quad 3.4$$

or, in a more compact form,

$$\dot{\underline{x}}(t) = \underline{A} \underline{x}(t) + \underline{B} \underline{u}(t)$$

Where, $\dot{\underline{x}}(t)$ is a vector containing the first derivatives of the state variables, with dimensions $n \times 1$,

$\underline{x}(t)$ is a vector containing the state variables with

dimensions $n \times 1$,

A is a matrix containing the coefficients that relate the state vector to the state vector derivatives, with dimensions $n \times n$,

B is a matrix containing the coefficients that relate the input to the state variables, with dimensions $n \times p$, where p is the number of inputs.

If the system error is included, the equation becomes:

$$\dot{\underline{x}}(t) = \underline{A}\underline{x}(t) + \underline{B}\underline{u}(t) + \underline{G}\underline{w}(t) \quad 3.5$$

Where, G is the coefficient matrix relating the system error to the state variables, and

w(t) is the vector containing the system errors.

The homogeneous solution of Eq. 3.5 (when the matrix B and G contain all zeros) can be obtained by several different methods (Wiberg 1971). Among these are Laplace Transforms, Cayley-Hamilton, and the Series method. These methods are used to find the transition matrix which determines the response of the system to the initial conditions. The homogeneous solution of Eq. 3.5, with the initial conditions taken at time $t=0$ is:

$$\underline{x}(t) = \phi(t, 0) \underline{x}(0) \quad 3.6$$

where, x(t) is the the response of the system to the initial conditions,

$\phi(t, 0) = e^{\underline{A}t}$ is the state transition matrix and is equal to

x(0) is the vector of the initial conditions.

The solution of the nonhomogeneous terms of Eq. 3.5, can be solved for the time invariant case, with the convolution integral, Eq. 2.18. This gives the solution of the nonhomogeneous terms as:

$$\int_{t_0}^t \Phi(t-\tau) \underline{B}(\tau) \underline{u}(\tau) d\tau + \int_{t_0}^t \Phi(t-\tau) \underline{G}(\tau) \underline{w}(\tau) d\tau \quad 3.7$$

The complete solution for the nonhomogeneous continuous case Eq. 3.5, can be written as the sum of Eqs. 3.6 and 3.7 as:

$$\underline{x}(t) = \Phi(t, 0) \underline{x}(0) + \int_{t_0}^t \Phi(t-\tau) \underline{B}(\tau) \underline{u}(\tau) d\tau + \int_{t_0}^t \Phi(t-\tau) \underline{G}(\tau) \underline{w}(\tau) d\tau \quad 3.8$$

Real systems are continuous but our knowledge on the inputs and outputs of a system are usually in discrete time, such as with a river system where we take sample measurements at equal time intervals, $t(m)$, $m=1, 2, \dots$. The corresponding discrete time equation is formed directly from Eq. 3.8 as (Timothy and Bona 1968; Gelb 1974):

$$\underline{x}(m+1) = \Phi(m) \underline{x}(m) + \Gamma(m) \underline{w}(m) + \Lambda(m) \underline{u}(m) \quad 3.9$$

where, $\Phi(m)$ is $\Phi(t(m+1), t(m))$,

$$\Lambda(m) \underline{u}(m) \text{ is } \int_{t_0}^t \Phi(t(m+1), \tau) \underline{B}(\tau) \underline{u}(\tau) d\tau,$$

$\Lambda(m)$ is the convolution sum at time m ,

$$\Gamma(m) \underline{w}(m) \text{ is } \int_{t_0}^t \Phi(t(m+1), \tau) \underline{B}(\tau) \underline{w}(\tau) d\tau,$$

$\Gamma(m)$ is the convolution sum at time m .

The difference equation for a cascade of linear reservoirs in state space form is analogous to Eq. 3.4 and Eq. 3.5, except that instead of the state vector being composed of the dependent variable y and the $n-1$ derivatives of y , it is composed of the dependent variable y and the $n-1$ previous values of y . The solution is of the form (Strejc 1981):

$$\underline{x}(m) = \underline{A}^m \underline{x}(0) + \sum_{r=0}^{m-1} \underline{A}^{m-r-1} \underline{B}u(r) + \sum_{r=0}^{m-1} \underline{A}^{m-r-1} \underline{G}w(r) \quad 3.10$$

where, $\underline{x}(m)$ is the state vector at time m , and

$\underline{A}^m$ is the coefficient matrix raised to the power of the time step.

$\sum_{r=0}^{m-1} \underline{A}^{m-r-1} \underline{B}u(r)$ is the convolution sum at time m ,

$\sum_{r=0}^{m-1} \underline{A}^{m-r-1} \underline{G}w(r)$ is the convolution sum at time m .

The coefficient matrices, $\underline{A}$ and $\underline{B}$ are of the same form as Eq. 3.4 and Eq. 3.5. The homogeneous solution that is applied to Eq. 3.10 results from a series solution of the coefficient matrix $\underline{A}$, which can also be solved by Z-transforms. The non homogeneous term is the discrete convolution summation.

In certain situations one of the state variables may not be the desired output. The state variables can be either a coefficient matrix times the state variables or a linear combination of the state variables. The equation needed to give the proper output will have the following form and is known as the output equation:

$$\underline{c}(t) = \underline{D}\underline{x}(t) + \underline{E}u(t) \quad 3.11$$

where, $\underline{c}(t)$ is the desired output, having dimension $nx1$,

$\underline{D}$ is a matrix relating a linear combination of the state variables to the output, having dimension nxn ,

$\underline{E}$ is a matrix relating the input directly to the output, having dimensions $nx1$.

At times the governing ordinary differential equation will not only have derivatives of the dependent variable, but will also have derivatives of the independent variable. This will lead to a formulation of the state variables that is different than Eq. 3.2 and Eq. 3.3. The state variables will contain information about the inputs. This is a situation where an output equation (Eq. 3.11) is required.

THE KALMAN FILTER

The Kalman filter can be used to give an estimate of state variables, based on the output of the system model and a measurement of the system. The estimate is found in such a way that it is unbiased, of minimum variance, and consistent. The Kalman filter algorithm is recursive in form, so there is no need to store past measurements for the purpose of computing present estimates. The recursive trait of the filter helps to limit storage requirements and data handling. The derivation of the filter follows from the method used by Gelb (1974). Schweppe (1973) gives a more complete formulation.

The system model used has the form of Eq. 3.9, with the deterministic input, $\underline{u}(m)$ taken as zero. The system noise is assumed to be a zero mean, white sequence of covariance $\underline{Q}(m)$. A white sequence is defined as being uncorrelated in time, meaning that information on its value at one time tells nothing about its value at any other time.

Measurements of the system are linear combinations of the state variables, corrupted by noise; the noise having an assumed mean of zero and being uncorrelated in time. The variance of the measurement noise will be denoted by $\underline{R}(m)$. The measurement equation has the form:

$$\underline{z}(m) = \underline{H}(m) \underline{x}(m) + \underline{v}(m) \quad 3.12$$

where, $\underline{z}(m)$ is the measurement of the system state at time m ,

$\underline{x}(m)$ is the actual state of the system at time m ,

$\underline{H}(m)$ is a linear matrix relating the measurements to the state variables,

$\underline{v}(m)$ is a vector of the measurement noise $N(0, \underline{R}(m))$.

The updated estimate of the state is formed as a linear combination of the prediction and measurement:

$$\underline{x}(m+) = \underline{K}(m) \underline{x}(m-) + \underline{K}'(m) \underline{z}(m) \quad 3.13$$

where, $\underline{x}(m+)$ is the updated estimate at time m ,

$\underline{x}(m-)$ is the prior estimate based on the model at time m ,

$\underline{K}(m)$ and $\underline{K}'(m)$ are time-varying weighting matrices to be specified later,

+ and - are used to denote just before and just after a measurement is made.

The error between the estimate of the state variable and the state variable is defined to be:

$$\tilde{\underline{x}} = \hat{\underline{x}} - \underline{x} \quad 3.14$$

where, $\tilde{\underline{x}}$ is the error in the estimate,

$\hat{\underline{x}}$ is the estimated state,

$\underline{x}$ is the actual state.

For an unbiased estimate the expected value of the error is zero, which will give the covariance or variance of the error as being the expected values of the second moment. In the matrix form the covariance terms are off the diagonal, while the variance terms are on the diagonal. The covariance matrix is designated $\underline{P}$ and defined as:

$$\underline{P} = E(\tilde{\underline{x}}\tilde{\underline{x}}^T) \quad 3.15$$

To derive the Kalman filter $\underline{K}(m)$ and $\underline{K}'(m)$ in Eq. 3.13 must be determined requiring two equations. One equation is based on the requirement of an unbiased estimate of the expected values of the system error. The other requirement is that the variance of the error should be minimized. To accomplish this the error covariance matrix must be minimized (Eq. 3.15). By using the system model (Eq. 3.9) as an unbiased extrapolator, it can be shown that the error covariance matrix can be stepped through a time step without adding any bias. This new error covariance matrix contains values of $\underline{K}(m)$, and therefore to find a value of $\underline{K}(m)$ that minimizes the new $\underline{P}$ matrix the partial derivative is taken with respect to $\underline{K}(m)$ and set equal to zero. This equation is solved for $\underline{K}(m)$. The value of $\underline{K}(m)$ is then substituted back into the error covariance and the updated equation (Eq. 3.15), forming the equations for the Kalman filter:

State Estimate Extrapolation:

$$\underline{x}(m-) = \Phi(m-1)\underline{x}(m-1+) + \Lambda(m-1)\underline{u}(m-1) \quad 3.16$$

Error Covariance Extrapolation:

$$\underline{P}(m-) = \phi(m-1) \underline{P}(m-1+) \phi(m-1)^T + \Gamma \underline{Q}(m-1) \Gamma^T \quad 3.17$$

State Estimate Update:

$$\underline{x}(m+) = \underline{x}(m-) + \underline{K}(m) (\underline{z}(m) - \underline{H}(m) \underline{x}(m-)) \quad 3.18$$

Error Covariance Update:

$$\underline{P}(m+) = (\underline{I} - \underline{K}(m) \underline{H}(m)) \underline{P}(m-) \quad 3.19$$

Kalman Gain Matrix:

$$\underline{K}(m) = \underline{P}(m-) \underline{H}(m)^T (\underline{H}(m) \underline{P}(m-) \underline{H}(m)^T + \underline{R}(m))^{-1} \quad 3.20$$

The details can be found in Gelb (1974).

There are some intuitive concepts related to the idea of the Kalman filter and the equations behind it. The Kalman gain matrix and the error covariance matrix can be looked at in different forms that give insight to the reasoning behind their form. The error covariance update matrix can be expressed in the form:

$$\underline{P}(m+)^{-1} = \underline{P}(m-)^{-1} + \underline{H}(m)^T \underline{R}(m)^{-1} \underline{H}(m) \quad 3.21$$

This form of the equation indicates that large measurement noise will provide only a small decrease in the error covariance when ever a measurement is used, while on the other hand the error covariance will decrease considerably wherever a measurement is used that has small error covariance.

The Kalman gain matrix can be expressed in the form:

$$\underline{K}(m) = \underline{P}(m+) \underline{H}(m)^T \underline{R}(m)^{-1} \quad 3.22$$

Which shows when combined with Eq. 3.18 that if the state estimate errors are large and measurement errors are small,

the measurements should be highly relied upon, and $\underline{K}(m)$ should be large. But on the other hand if the measurement error is large, and the system errors are small, the system output should be relied upon and $\underline{K}(m)$ should be small.

THE CASCADE OF LINEAR RESERVOIRS AND STAGE

The linear reservoir is a form of the Muskingum equation (Eq. 2.2) with the weighting term relating inflow and outflow to storage taken as zero (Eq. 2.4). The linear reservoir was selected because it fits within the state space format, either in the single reach form or the cascade of reaches. As was stated in the literature review, the linear reservoir combined with continuity, is a hydrologic form of routing, with lumped parameters, however it can be related to the diffusive wave. The unknown variable, storage, is eliminated by combining continuity (Eq. 2.1) with the linear reservoir (Eq. 2.4):

$$I - Q = KdQ/dt \quad 3.23$$

The cascade of linear reservoirs is a method in which the outflow from one reach is taken as the inflow to the next and can best be shown by figure 3.1 for n cascading linear reservoirs.

There are two different methods to formulate the governing equation. One method would be to take each reach as being independent of the next and calculating the response of the inflows and initial conditions at each node

Figure 3.1. A representation of n cascading linear reservoirs for a typical river channel.

in the river. A node being defined as the intersection of the end of one reach and the beginning of the next.

The second method to write the governing equation is to tie the reaches together. This is done by formulating the equation such that the outflow from the preceeding reach is the inflow to the next. To demonstrate the method Q_1 will equal I_2 and using Eq. 3.23 to formulate the equation for two cascading linear reservoirs:

$$dQ_2^2/dt^2 + 2dQ_2/Kdt + Q_2/K^2 = I_1/K^2 \quad 3.24$$

This has the form of a transfer function approach, where nothing is known about what happens in the system, only the outputs from known inputs is desired. However, we can still relate the model to the physical system.

The state space approach is very useful in multi-input, multi-output systems; this can be seen in branching river systems, where two or more tributaries come together at a confluence. If the different reaches have equal reservoirs, the governing equation is of the same form as Eq. 3.24. But if the reaches are of an unequal number of reservoirs, then the governing equation will have derivatives of the input associated with them.

The governing equations for the discrete cascade of linear reservoirs is the difference form of Eq. 3.23 which is given as (O'Connor 1976):

$$I - Q = K\Delta Q/\Delta t \quad 3.25$$

Where, ΔQ is the change in discharge due to a change in time Δt .

An operator is convenient for use in this case. The operator is known as the backward difference operator and defined as:

$$\nabla Q(m) = Q(m) - Q(m-1) \quad 3.26$$

When Eq. 3.25 and Eq. 2.4 are combined to form a linear reservoir, and using the backward difference operator (Eq. 3.26) gives an equation of the form:

$$(1 + K\nabla)Q(m) = I(m) \quad 3.27$$

Eq. 3.27 is for a single reach of a river. For a cascade of linear reservoirs the equation is:

$$(1 + K\nabla)^n Q(m) = I(m) \quad 3.28$$

Where, n is the number of cascading linear reservoirs.

The equations for the continuous and difference cascade of linear reservoirs can be set up in state space form, using the relationships defined by Eq. 3.2 and Eq. 3.3, which result in an equation for the continuous cascade of 2 equal linear reservoirs as:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1/K^2 & -2/K \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1/K^2 \end{bmatrix} I \quad 3.29$$

Where, x_1 is the flow Q_2 ,

x_2 is the change in flow with respect to the change in time dQ_2/dt ,

$\dot{x}_1$ and $\dot{x}_2$ are the derivatives of the state variables

x_1 and x_2 .

The solution for the continuous state equation (Eq. 3.24), solved by one of the methods previously discussed is defined by Eq. 3.8 with the noise term zero. The transition and the matrix due to the forcing function coefficients are given as:

$$\Phi = \begin{bmatrix} (1+t/K)e^{-t/K} & te^{-t/K} \\ (-t/K^2)e^{-t/K} & (1-t/K)e^{-t/K} \end{bmatrix} \quad 3.30$$

$$\Lambda = \begin{bmatrix} (-1-t/K)e^{-t/K} \\ (t/K^2)e^{-t/K} \end{bmatrix}$$

For the difference case, the state equation for two cascading linear reservoirs is:

$$\begin{bmatrix} x_1(m+1) \\ x_2(m+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -K^2/C & (2K+2K^2)/C \end{bmatrix} \begin{bmatrix} x_1(m) \\ x_2(m) \end{bmatrix} + \begin{bmatrix} 0 \\ 1/C \end{bmatrix} I(m) \quad 3.31$$

Where, $x_2(m)$ and $x_1(m)$ are the previous outflows one time step back and two time steps back, $Q_2(m-1)$, $Q_2(m-2)$,

$x_1(m+1)$ and $x_2(m+1)$ are the outflows one time step ahead as defined previously,

C is equal to $(1+2K+2K^2)$.

The difference solution is given by Eq. 3.10 and can be written directly from the state equation.

As was stated in the introduction and formulated in the literature review the desired output from the model and the measurement of the system is stage (Eq. 2.19). The stage-discharge relationship is nonlinear, but the requirements of the Kalman filter are that the $\underline{H}$ matrix, which relates the measurements to the state variables (Eq. 3.12) and the $\underline{D}$ matrix of the output equation (Eq. 3.11) be linear. In order to use the Kalman filter or the output equation, a linear relationship must be developed to relate stage to discharge. This is accomplished by using the Taylor series expansion and ignoring the nonlinear terms:

$$g(Q) = a + bQ_0^c + cbQ_0^{c-1}(Q - Q_0) \quad 3.32$$

Where, Q_0 is the reference discharge, and

Q is the discharge at the desired stage.

The linear stage-discharge in the form of Eq. 3.32 is used for converting the input, output, and measurement from stage to discharge or discharge to stage. It is necessary to manipulate the data in this way since all the measurements are of stage and the formulation of the routing is in terms of discharge.

The Kalman filter is implemented in the following manner. The initial noise conditions can be found from a priori information of the system. The variance of the output is the scatter of the observed stage discharge vs. the theoretical stage discharge given by Eq. 2.19. The variance of the system noise, will be taken in parts, one

for the system noise and the other for the input noise. The error covariance matrix does not need to be found from a priori information, any initial value will be suitable. The only requirement is that the initial values are large enough so that the initial conditions decay at a fast enough rate. A summary of the complete set of equations used in implementing the model and the Kalman filter is as follows (for completeness some of the previous equations will be restated):

System model:

$$\underline{x}(m) = \Phi(m-1) \underline{x}(m-1) + \underline{w}(m-1) + \Lambda(m-1) \underline{u}(m-1) \quad 3.33$$

Where, $\underline{w}(m) \sim N(0, \underline{Q}(m))$

Measurement model:

$$\underline{z}(m) = \underline{H}(m) \underline{x}(m) + \underline{v}(m) \quad 3.34$$

Where, $\underline{v}(m) \sim N(0, \underline{R}(m))$

Initial Conditions:

$$E(\underline{x}(0)) = \hat{\underline{x}}(0) \quad 3.35$$

$$\underline{P}(0) = E(\underline{x}(0) - \hat{\underline{x}}(0))(\underline{x}(0) - \hat{\underline{x}}(0))^T \quad 3.36$$

State Estimate Extrapolation:

$$\hat{\underline{x}}(m-) = \Phi(m-1) \hat{\underline{x}}(m-1+) + \Lambda(m-1) \underline{u}(m-1) \quad 3.37$$

Error Covariance Extrapolation:

$$\underline{P}(m-) = \Phi(m-1) \underline{P}(m-1+) \Phi(m-1)^T + \Gamma \underline{Q}(m-1) \Gamma^T \quad 3.38$$

State Estimate Update:

$$\hat{\underline{x}}(m+) = \hat{\underline{x}}(m-) + \underline{K}(m) (\underline{z}(m) - \underline{H}(m) \hat{\underline{x}}(m-)) \quad 3.39$$

Error Covariance Update:

$$\underline{P}(m+) = (\underline{I} - \underline{K}(m) \underline{H}(m)) \underline{P}(m-) \quad 3.40$$

Kalman Gain Matrix:

$$\underline{K}(m) = \underline{P}(m-) \underline{H}(m)^T (\underline{H}(m) \underline{P}(m-) \underline{H}(m)^T + \underline{R}(m))^{-1} \quad 3.41$$

The order of the implementation of the previous 9 equations, the Kalman filter equation is best seen in a time line such as figure 3.2.

SUMMARY

The application of the Kalman filter and state space is very diversified in its applicability and form in hydrology. In this chapter some different forms of the governing equations, continuous, difference, and discrete have been looked at with the continuous and difference forms placed in state space notation. Some solution techniques have been mentioned, with no preference given for anyone of them.

Figure 3.2. A discrete Kalman filter time line

CHAPTER IV

APPLICATION

The state space approach to flood stage estimation is applied to the Cowlitz and Toutle river systems in the southwest corner of the state of Washington (figure 4.1). This area was chosen due to the high potential of floods and the difficulty of stage estimation at Castle Rock, Washington caused by the eruption of Mount St. Helens on May 18, 1980. The eruption caused massive mudflows to enter the Toutle River in the Mount St. Helens' area. The mudflows were carried downstream and sediment was deposited in the Toutle and Cowlitz rivers as shown in figure 4.1. The high volume of sediment still present around Mount St. Helens results in the aggradation or degradation of the bed of the Toutle and Cowlitz Rivers, the prediction of stage is likely to be a problem for many years to come.

The model that is developed is for stage estimation at Castle Rock, Washington. The inputs that are used are the gaging stations on the Toutle River at Tower Road and the Cowlitz River below Mayfield Dam. These gaging stations are approximately 9.3 and 30 miles from Castle Rock, respectively. The confluence of the Toutle and Cowlitz

Figure 4.1. Area map showing the impact of the May 18, 1980 eruption. The area flooded can be seen along the Toutle and Cowlitz Rivers. Map supplied by U.S. Army Corps of Engineers.

Rivers is approximately 2.6 miles from Castle Rock (see figure 4.1).

DEVELOPMENT

The model is developed from the physical correlation between the hydraulic and hydrologic methods (see LITERATURE REVIEW, CHAPTER II). The governing equation is the simplified Muskingum routing equation known as the linear reservoir (Eq. 2.4), combined with continuity (Eq. 2.1). The decision on the linear reservoir as stated in CHAPTER III, is due to the applicability of forming the governing equation into state space format, which is needed in order to apply the Kalman filter. Using the linear reservoir two parameters are needed, they are the storage coefficients (Eq. 2.13) and reach lengths. To help guide in the final selection of the proper reach lengths, Cunge's criteria is used (Eq. 2.12) with x set equal to zero. This is a function of celerity, friction slope, width of the cross section and reference discharge. In order to find celerity an analytical expression is found for the stage discharge relationships.

First the reference discharge, Q_0 , for the three gaging stations is determined. The reference discharge is chosen by inspection of the hydrographs for the two inflows and the outflow. An average value is used and can be found in Table I. The effect of choosing a constant reference

discharge is that celerity, river width, and reach length will be dependent on the choice of flow; not on the actual flow, as is the case of an actual river system. Width of flow, B , is selected based on United States Geological Survey (USGS) measurements for the river width at the reference discharge.

TABLE I
THE CELERITIES FOR THE TOUTLE AND COWLITZ
RIVERS FOUND BOTH GRAPHICALLY
AND ANALYTICALLY

	Reference Discharge (cfs)	Graph (ft/s)	Analytical (ft/s)
Castle Rock	42900	25.2	28.0
Mayfield Dam	19150	12.0	13.8
Tower Road	19000	15.3	18.75

The stage discharge equations (Eq. 2.19) for the Toutle and Cowlitz Rivers at Mayfield Dam, Tower Road, and Castle Rock are determined by regression analysis (Walpole and Myers 1978). The method is implemented by using the logarithmic transformation of the non linear stage discharge equation to produce a linear relationship. The coefficients are obtained by linear regression using data for the stage discharge relationship supplied by the USGS. The resulting equations are:

Mayfield Dam:

$$g(Q) = g_0 + .498641(Q - Q_0)^{.25} \quad 4.1$$

where, g_0 is 12 feet,

Q_0 is 6950 cfs.

Tower Road:

$$g(Q) = g_0 + .443636 (Q - Q_0)^{.272} \quad 4.2$$

where, g_0 is 20.5 feet,

Q_0 is 3900 cfs.

Castle Rock:

$$g(Q) = g_0 + .00941 (Q - Q_0)^{.6} \quad 4.3$$

where, g_0 is 17.0 feet,

Q_0 is 17500 cfs.

Celerity (Eq. 2.17) is also required for the different reaches of the river and is given by the change in discharge with respect to the change in area. But, if the width is approximately constant; the slope of the stage discharge relationship may be used to determine celerity. The value can be found by either of two methods: graphically from the stage discharge curve, (figure 2.1) or analytically by differentiating the stage discharge equation (Eq. 4.1-4.3). Graphically, celerity is given as the slope of a line that is tangent to the stage discharge curve at a reference discharge. The values for the reference discharge are in Table I and can be found referenced on figures 4.2 through 4.4. The tangent lines used in the graphical calculation of celerity can also be found on these figures. The value of both methods should be approximately equal; for the Toutle and Cowlitz Rivers the values are given in Table I.

Figure 4.2. The stage vs. discharge curve for the Toutle River at Tower Road, with the reference discharge and tangent line used for celerity.

Figure 4.3. The stage vs. discharge curve for the Cowlitz River at Castle Rock, with the reference discharge and tangent line used for celerity.

COWLITZ RIVER BELOW MAYFIELD DAM

Figure 4.4 The stage vs. discharge curve for the Cowlitz River at Mayfield Dam, with the reference discharge and tangent line used for celerity.

Finally, the friction slope must be determined. An assumption often made is that the length of the flow is long enough that normal depth is reached, and in a natural river channel the assumption is approximately valid. Thus, the friction slope is approximated by the bedslope. The bedslope is found from the topographic maps of the USGS that cover the basin.

With the required parameters of equation 2.12 and 2.13 established, the reach lengths and their storage coefficients are obtained. The actual values for the river reaches, as solved by Eq. 2.12, are given in Table II (The analytical solution for celerity is used).

TABLE II
THE REACH LENGTHS FOR THE TOUTLE
AND COWLITZ RIVERS

	Δx (ft)	Number of Reaches	K (1/2 hour)
Upper Cowlitz	9400	15.0	0.378
Lower Cowlitz	4910	2.8	0.097
Toutle River	2260	15.7	0.067

With the cascade of linear reservoirs the number of reaches governs the order of the state equation, n linear reservoirs gives an n^{th} order equation. A large number of reaches and therefore a relatively large transition matrix is required based on the criteria in Eq. 3.4. To lower the number of reaches to a workable size, the reach length is

chosen subjectively, using natural breaks in the river system as a guide. The storage coefficient is increased by the number of reaches given by Eq. 2.12, per chosen reach length, as shown by the following equation:

$$K_{\text{new}} = nK_{\text{old}}$$

Where, n is the number of reaches per reach.

This has the effect of increasing the travel time by the number of reaches.

The reaches for the Toutle and Cowlitz Rivers are as shown on figure 4.5. The values used for K , bedslope, river width, final celerity, and reach lengths are given in Table III.

TABLE III

THE REACH LENGTHS, BEDSLOPE, RIVER WIDTH, FINAL CELERITIES, AND STORAGE COEFFICIENTS FOR THE TOUTLE AND COWLITZ RIVERS

	K (1/2 hour)	Δx (ft)	B (ft)	S_0	c (ft/s)
Reach 1-3	1.496	37171.2	290	5.09×10^{-4}	13.8
Reach 4	1.052	29351	290	5.09×10^{-4}	15.5
Reach 5	1.052	35508	230	1.95×10^{-3}	18.75
Reach 6-7	0.236	6864	357	8.74×10^{-4}	16.2

After an estimate of the proper number of reaches the governing equation is written using the difference form of the cascade of linear reservoirs (Eq. 3.25). As stated in Chapter III, p. 25; when the number of reach lengths are unequal or the storage coefficient K is unequal for the

Figure 4.5. Map of the basin with the rivers divided into reaches.

branches of the river, the inflows will be a function of the inflow at time m plus the inflow back in time corresponding to the difference in the number of reach lengths or the difference in the number of K values for the reaches. This led to difficulty in formulation of the state equation. To overcome this difficulty the river system was broken up into two governing equations. One equation with equal branches and the other without branches.

The first governing equation was written for the lower section of the Cowlitz River consisting of reservoirs 4, 6, and 7, and reservoir 5 of the Toutle River. As stated previously, in order to overcome the problem with K having to be the same value for the different branches, reach 4 and 5, K was found for reach 5, then while holding celerity constant for reach 4 the reach length was chosen. The values in Table III take this into consideration. This gave an equation in the following form:

$$(1+K\Delta)^2 Q_7(m) = I_4(m) / (1+K\Delta) + I_5(m) / (1+K\Delta) \quad 4.4$$

where, Q_7 is the outflow of reservoir 7,

I_4 is the inflow into reservoir 4 which is equal to outflow from reservoir 3,

I_5 is the inflow at Tower Road on the Toutle River.

The remaining 3 reservoirs along the Cowlitz River were changed to compensate for the change in length of reach 4. The second governing equation was then written for the remaining reaches (1, 2, and 3):

$$(1+K\Delta)^3 Q_3(m) = I_1(m) \quad 4.5$$

where, Q_3 is the outflow of reservoir 3,

I_1 is the inflow at Mayfield Dam.

STATE EQUATION AND KALMAN FILTER

The state equation for the system, developed according to equations 3.2 to 3.4, with the coefficients solved from the proper value of K and the governing equations is:

$$\begin{bmatrix} x_1(m) \\ x_2(m) \\ x_3(m) \\ y_1(m) \\ y_2(m) \\ y_3(m) \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ .0502 & -.4584 & 1.2161 & 0 & 0 & .1922 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & .2154 & -1.0779 & 1.7983 \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \\ y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix}$$

plus,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & .1922 \\ 0 & 0 \\ 0 & 0 \\ .0643 & 0 \end{bmatrix} \begin{matrix} I_1(0) \\ I_5(0) \end{matrix}$$

where, $x_1(0)$ is the outflow from reservoir 7 at time m-3

$x_2(0)$ is the outflow from reservoir 7 at time m-2

$x_3(0)$ is the outflow from reservoir 7 at time m-1

$y_1(0)$ is the outflow from reservoir 3 at time m-3

$y_2(0)$ is the outflow from reservoir 3 at time m-2

$y_3(0)$ is the outflow from reservoir 3 at time $m-1$.

The first step in implementing the Kalman filter is determining the parameters including the variance of the measurement, the variance of the model error, and the terms of the measurement equation. The terms of the measurement equation (Eq. 3.13) are the coefficients in the $\underline{H}$ matrix that relate the measurements to the state variables. The only state variables that are being measured are the outflows at Castle Rock, therefore the only terms in the $\underline{H}$ matrix are the ones that relate the stage measurement to the discharge at Castle Rock. The terms in the $\underline{H}$ matrix are found from the Taylor series expansion of the stage discharge equation (Eq. 3.32). The constant term in the expansion will be subtracted from the stage when the measurement is used. The Taylor series expansion for the stage discharge equation at Castle rock is:

$$g(Q) = 16.81 + 9.76804 \times 10^{-5} Q \quad 4.6$$

where, Q_0 is equal to 42900 c.f.s. at a stage of 21.0 ft.

The $\underline{H}$ matrix takes the form:

$$\underline{H} = \begin{bmatrix} 9.76804 \times 10^{-5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 9.76804 \times 10^{-5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.76804 \times 10^{-5} & 0 & 0 & 0 \end{bmatrix}$$

The variance matrices are required for both the system error and the measurement error. They can be determined from an ensemble of a priori data of the system and model. The variance of the measurement error will be the mean

square of the error. The error being the difference between the value of the stage at Castle Rock; given by the Taylor series expansion and the actual stage from the stage discharge curve, for the given discharges. The values are:

$$\underline{R} = \begin{bmatrix} .399 & 0 & 0 \\ 0 & .399 & 0 \\ 0 & 0 & .399 \end{bmatrix}$$

The variance for the system error is found by taking the outflow hydrograph for a given storm at Castle Rock and comparing it to the outflow hydrograph from the model, given the inflow hydrographs for the same storm. The mean square values of the system error are:

$$\underline{Q} = \begin{bmatrix} 10717364 & 0 & 0 \\ 0 & 10717364 & 0 \\ 0 & 0 & 10717364 \end{bmatrix}$$

As stated earlier the values of the error covariance matrix need not be found (Eq. 3.37), all that is required is to start with a high enough value, such that the matrix converges to a constant after a few time steps.

Further refinement is required in the selection of celerity. The values given in Table I are for specific locations on the rivers, they do not represent the entire system. As a starting value, an average was selected between known values. Then further refinement is made through calibration of the model, matching the time to peak of the model, to the observed time to peak at Castle Rock.

The values are given in Table III (K in Table III is the final value).

As shown in figure 4.6, the inflow hydrographs vs. the outflow hydrograph for a storm between Dec. 3rd and 4th, 1982 from data supplied by the USGS, there is a problem with mass conservation. Not enough inflow is present to account for the outflow due to ungaged streams and lateral inflow between the inflow and outflow gaging stations. In order to compensate for the ungaged portion of the watershed, it was assumed that the precipitation that produced the inflow and outflow was uniformly distributed, both spatially and temporally. Then an analysis was carried out, comparing the ungaged to the gaged area. It was found that the ungaged area was approximately one-half as large as the gaged area of the Toutle River, and was mostly on the Cowlitz River between Mayfield Dam and Castle Rock. Therefore a third inflow was added between reaches 3 and 4 that was one-half the Toutle inflow. As can be seen from figure 4.7 this helped compensate for loss in mass.

Since the model output is discharge and the desired output is stage, an equation known as the output equation (Eq. 3.11) is used. The output equation is of the same form as the previously described measurement equation, except that there is no measurement error term. The D matrix of the output equation is the same as the H matrix of the

Figure 4.6. The inflows at Mayfield Dam on the Cowlitz river and at Tower Road on the Toutle river vs. the outflow at Castle Rock, Washington for December 3 & 4, 1982.

Figure 4.7. The comparison of outflow at Castle Rock, before and after lateral inflow was accounted for.

measurement equation, but now the constant is added to the stage.

DISCUSSION OF APPLICATION

The inflow from Mayfield Dam and Tower road (figure 4.8) were applied to the model, with the Kalman filter and without. The resulting hydrograph of stage versus time (figure 4.9) shows a good correspondence between the actual stage and the stage predicted by the model with the Kalman filter attached, but a poor correspondence between the actual stage and the stage predicted by the model without the Kalman filter. A comparison of the numerical values found on figure 4.9 are given in Table IV.

TABLE IV
THE PEAK TIME, DISCHARGE, AND STAGE
FOR CASTLE ROCK, WASHINGTON

	Time (hours)	Discharge (cfs)	Stage (ft)
Observed	25	67300	23.2
Predicted w/o Kalman Filter	23.5	66320	23.3
Predicted with Kalman Filter	25	64393	23.1

The Kalman filter as applied to the model reaches a steady state value, this is due to the time invariant form of the governing equation and the constant values for the variance matrices of the measurement and system error. An

Figure 4.8. Hydrograph for the gaging stations below Mayfield Dam on the Cowlitz River (dash line) and Tower Road on the Toutle River (solid line) during December 3 and 4, 1982.

Figure 4.9. Observed and predicted stage at Castle Rock, with and without the Kalman Filter.

optimal update of the model is not given until steady state condition of the estimation variance matrix, $\underline{P}$, is reached. As can be seen from figure 4.9 this is approximately the 11th hour. The filter over estimates the stage on the increasing flow and under estimates on the decreasing flow, due to memory contained in the model. During the peak stage or when the increase or the decrease in stage is linear the filter respnds well and gives a consistently good answer. As shown in Table IV the model peaks and observed peaks are within one-tenth of a foot. The main difference between the model with and without the Kalman filter is the time to peak, the model peak without the Kalman filter arrives 1.5 hours before the actual, while with the Kalman filter arrival times are coexistent.

CHAPTER V

SUMMARY AND RECOMMENDATIONS

SUMMARY

As stated in the introduction the purpose of this thesis is the development of a model to optimally predict the stage of a river at a downstream point, given the appropriate upstream inflows. The model is written to predict stage on the Cowlitz River at Castle Rock, Washington, given known inflows at Mayfield Dam on the Cowlitz River and Tower Road on the Toutle River. The flood routing model is developed from the physical correlation between the hydraulic and hydrologic methods of flood wave propagation. The equation used in the routing is the simplified Muskingum routing equation known as the linear reservoir, combined with a lumped form of the continuity equation. Cunge's physical correlation between the hydraulic and hydrologic methods is used to guide in the estimation of the parameters. The selection of reach length, Δx , was carried out subjectively, with reach length decided upon by natural breaks in the river system. Other parameters, such as channel width and wave celerity are estimated for a selected reference discharge and are

considered constant. The optimal update of the model is given by the Kalman filter, which is a weighted value between the model output and a measurement of the system.

The linear reservoir equation is formulated in terms of discharge, but the desired output of the model and measurement of the system is stage, therefore a stage discharge equation is used to relate stage to discharge. To implement the stage discharge equation for use in the measurement and output equation a linear approximation is used. The approximation of the actual nonlinear stage discharge relationship by a linear stage discharge equation is found using the Taylor series expansion of the nonlinear equation around the reference discharge. The linear transformation allowed use of the output equation and Kalman filter.

The Kalman filter parameters for the variances are found by an ensemble of a priori information. The variance of the measurement is found by using the actual values of stage given by the stage discharge curve and the value given by the Taylor series expansion of the stage discharge equation. The variance of the model error is found by taking the outflow hydrograph for a given storm at Castle Rock and comparing it to the outflow hydrograph from the model, given the inflow hydrographs for the same storm. The required matrices to complete the application of the Kalman filter are found in appendix B. The ungaged inflow was

adjusted by an area analysis of the ungaged area of inflow to the gaged area inflow. Adjusting the ungaged inflow as the ratio of ungaged to gaged

The results of an application of the model to a storm event between December 3rd and 4th, 1982 showed a good prediction with the Kalman filter attached. The model with the filter optimally updated the output when a measurement of the system was applied. The model without the Kalman filter showed the same general response to the inflows, but the correlation between the actual stage and estimated stage was not as good.

Linearization of the stage discharge equation and the choice of reference discharge cause limitations in the model. The analytical stage discharge equation is found by linear regression, which gives more weight to the higher flows and stage, therefore giving a better fit at the higher end of the stage discharge curve. When a reference discharge is chosen, and a linear approximation is made of the nonlinear stage discharge equation about the reference discharge, bias is introduced. The model over predicts the stage both above and below the reference discharge, because the linear approximation is above the nonlinear stage discharge curve. This gives a limit as to the range in flow the model will accurately predict stage. The flow model, at the flows used does not introduce bias, as can be seen in

figure 4.7. The bias in the model is introduced in the output equation, where flow is transformed into stage.

RECOMMENDATION

The model can be improved by using either of two methods, the nonlinear time variant capabilities of state space and Kalman filtering, or using the nonlinear stage discharge equation for the output equation.

State space has the capability of having the state equation updated with time, therefore the reference discharge becomes the current discharge. The coefficients that change with the reference discharge are the storage coefficients, K , river width, B , and celerity, c . The computation proceeds as follows; with each new reference discharge, a new river width is chosen, then a new celerity is obtained. With the new celerity a new storage coefficient is found, which is used to solve for the coefficients in the state equation. The H matrix and output equation would then need to be updated by a Taylor series expansion of the new equation for celerity around the new reference discharge. By doing this the model would give a nonlinear approximation to a nonlinear system in finite steps.

The other method is to use the linear system, with the initial reference discharge and assume the flow model does not require updating. But, with the linear model the use of

a nonlinear output equation is employed. This method would remove the bias as shown in figure 4.9, but there would still be questions about whether the flow model would function properly under extreme conditions.

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APPENDIX A

COMPUTER OUTPUT AND PROGRAM

This Appendix contains the tabular computer output associated with the given inputs. Also included is the computer program that was written for and used on an Apple IIe computer.

C A KALMAN FILTERING-CASCADE OF LINEAR RESERVOIRS MODEL

C

C

C THE REQUIRED MATRICES;

C

C PK0:INITIAL ERROR COVARIANCE, TRANS:TRANAITION,

C TRANST:TRANPOSE OF TRANSITION, K:KALMAN GAIN,

C H:MEASURMENT, HT:TRANPOSE OF MEASURMENT,

C RK:MEASURMENT NOISE, COVARIANCE, Z:MEASURMENT,

C X0:INITIAL CONDITIONS, X1:UPDATED, CONDITIONS,

C PK1:UPDATED ERROR COVARIANCE, ID:IDENDITY,

C DEL:CONVOLUTION, Q:SYSTEM ERROR IN:INPUT

C

C ALL THE REST ARE DUMMY MATRICES TO RUN THE PROGRAM

C

REAL PK0(6,6), PK1(6,6), TRANS(6,6), TRANST(6,6)

REAL K(6,3), H(3,6), HT(6,3), RK(3,3), Z(3,1),

REAL X1(6,1), IN(2,1), DEL(6,2), Q(3,3), ID(6,6)

REAL GAMMA(6,3), GAMMAT(3,6), Z1(97), X0(6,1)

REAL DUB(3,6), DUMM(6,6), DUMB(6,6), DUMMY(3,3),

REAL DDUMM(6,1), DDUMB(6,1), INV(6,6), DUME(6,3)

REAL DUMME(3,6), DDUM(3,1), DDUME(3,1), DUM(6,6)

INTEGER R,A

REAL N2,IN1,IN11,IN2,IN22

C

C READ IN THE TRANSITION MATRIX AND THE TRANSPOSE OF

C THE TRANSITION MATRIX; THE H AND TRANSPOSE OF H;

C THE INPUT

C

DATA DEL/0.,0.,0.,0.,0.,0.,0643,0.,0.,0.,1614,0.,0.,0./

DATA TRANS/.0,0.,0842,0.,0.,0.,1.,0.,-5914,0.,0.

X,0.,0.,1.,1.3458,0.,0.,0.,0.,0.,0.,0.,0.,2154,0.,

X0.,0.,1.,0.,-1.0779,0.,0.,1614,0.,1.,1.7983/

DATA TRANST/0.,1.,0.,0.,0.,0.,0.,0.,1.,0.,0.,0.,

X.0842,-5914,1.3458,0.,0.,1614,0.,0.,0.,0.,1.,0.,

X0.,0.,0.,0.,0.,1.,0.,0.,0.,2154,-1.0779,1.7983/

DATA H/.0000976804,0.,0.,0.,0000976804,

X0.,0.,0.,0000976804,0.,0.,0.,0.,0.,0.,0.,0./

DATA HT/.0000976804,0.,0.,0.,0.,0.,0.,0000976804,

X0.,0.,0.,0.,0.,0.,0000976804,0.,0.,0./

DATA ID/1.,0.,0.,0.,0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,1.,

X0.,0.,0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,1.,0.,0.,0.,0

X.,0.,0.,1./

DATA PK0/15168000.0,2507410.,-495149.,0.,0.,0.,

X2507410.,14404200.,6510810.,0.,0.,0.,-495151.,6510800.

X,17478100.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,

X0.,0.,0.,0.,0.,0.,0./

DATA Q /10717364.,0.,0.,0.,10717364.,0.,0.,0

X.,10717364./

```

DATA RK /.3807,0.,0.,0.,.3807,0.,0.,0.,.3807/
DATA Z/.9,.9,.9/
DATA GAMMA/1.,0.,0.,0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,0.,
X1.,0.,0.,0./
DATA GAMMAT/1.,0.,0.,0.,1.,0.,0.,0.,1.,0.,0.,0.,0.,0.,
X0.,0.,0.,0./
DATA Z1/17.4,17.4,17.5,17.5,17.6,17.6,17.7,17.7,17.8,
X17.8,17.9,18.0,18.1,18.1,18.2,18.2,18.2,18.3,18.3,18.2
X18.2,18.3,18.4,18.6,18.6,18.7,18.8,19.0,19.1,19.2,19.4
X19.6,19.8,20.0,20.3,20.6,20.9,21.2,21.6,21.8,21.8,21.9
X22.1,22.4,22.5,22.5,22.8,22.9,22.9,23.1,23.2,23.1,23.0
X23.0,23.1,23.1,23.1,23.1,23.1,23.1,23.2,23.1,23.1,23.0
X23.0,22.9,22.7,22.4,22.2,22.1,22.1,22.0,21.9,21.9,21.8
X21.7,21.6,21.6,21.5,21.4,21.4,21.3,21.2,21.2,21.1,21.0
X20.9,20.9,20.8,20.8,20.7,20.7,20.7,20.7,20.7,20.7,20.7
OPEN(6,FILE='PRINTER:')
OPEN(4,FILE='INPUT',ACCESS='DIRECT',RECL=45)
N=6
A=1
M=3
R=2
K1=49
M1=6
N2=-.5
3   FORMAT(4F10.1)
5   FORMAT(F20.2)
4   FORMAT(3F10.2)
C
C   THE INITIAL STATE VARIABLES
C
WRITE(*,'(A)') 'INITIAL STATE VARIABLES'
DO 21 I=1,N
21  READ(*,5) X0(I,1)
WRITE(6,'(A)') '
WRITE(6,'(A)') 'TIME STAGE      STAGE MAYFIELD TOWER ROAD
X   UNGAGED'
C
C   EXTRAPOLATION ACROSS A TIME STEP
C
C   ERROR COVARIANCE EXTRAPOLATION
DO 50 I=1,K1
N2=N2+.5
WRITE(6,'(F4.1$)') N2
CALL MATPLY(DUM,TRANS,PK0,N,N,N)
CALL MATPLY(PK1,DUM,TRANST,N,N,N)
CALL MATPLY(DUME,GAMMA,Q,N,M,M)
CALL MATPLY(DUMM,DUME,GAMMAT,N,M,N)
CALL MATADD(PK1,PK1,DUMM,N,N)
1   FORMAT(A)
C
C   STATE ESTIMATE EXTRAPOLATION
C   ENTER THE INPUT

```

```

      IF (I.LT.50) THEN
      READ(4,3,REC=I) IN1,IN11,IN2,IN22
      ENDIF
      IF (I.GT.49) THEN
      L=I-49
      READ(4,3,REC=L) IN1,IN11,IN2,IN22
      ENDIF
      IF (I.GT.49) IN(1,1)=IN11
      IF (I.LT.50) IN(1,1)=IN1
      IF (I.GT.49) IN(2,1)=IN22
      IF (I.LT.50) IN(2,1)=IN2
      IN(2,1)=1.5*IN(2,1)
      CALL MATPLY(DDUMM,TRANS,X0,N,N,A)
      CALL MATPLY(DDUMB,DEL,IN,N,R,A)
      CALL MATADD(X1,DDUMM,DDUMB,N,A)
C
C   KALMAN GAIN MATRIX
C
      CALL MATPLY(DUME,PK1,HT,N,N,M)
      CALL MATPLY(DUMME,H,PK1,M,N,N)
      CALL MATPLY(DUMMY,DUMME,HT,M,N,M)
      CALL MATADD(DUMB,DUMMY,RK,M,M)
      CALL MATINV(DUMB,INV,DUB,M,M1)
      CALL MATPLY(K,DUME,INV,N,M,M)
C
C   STATE ESTIMATE UPDATE
C
      Z(3,1)=Z1(I)
      Z(3,1)=Z(3,1)-16.81
      CALL MATPLY(DDUM,H,X1,M,N,A)
      WRITE(6,'(F8.1$)') Z(3,1)+16.81
      CALL MATSUB(DDUME,Z,DDUM,M,A)
      CALL MATPLY(DDUMB,K,DDUME,N,M,A)
      CALL MATADD(X1,X1,DDUMB,M,A)
      WRITE(6,'(F8.1$)') X1(3,1)*.0000976804+16.81
      WRITE(6,'(F10.1$)') IN(1,1)
      WRITE(6,'(F10.1$)') IN(2,1)/1.5
      WRITE(6,'(F10.1$)') IN(2,1)/1.5*.5
C   COVARIANCE MATRIX UPDATE
C
      CALL MATPLY(DUMM,K,H,N,M,N)
      CALL MATSUB(DUMB,ID,DUMM,N,N)
      CALL MATPLY(PK0,DUMB,PK1,N,N,N)
      DO 22 I3=1,N
22      X0(I3,1)=X1(I3,1)
      DO 23 I4=2,M
      I3=I4-1
23      Z(I3,1)=Z(I4,1)
50      CONTINUE
C   THE END SO FAR!!!
      END

```

```

C  A SUBROUTINE FOR MATRIX INVERSION
C
      SUBROUTINE MATINV (A, AINV,B, N, N1)
      REAL A(N,N), AINV(N,N),B(N,N1)
C  MOVE A TO THE N BY 2*N MATRIX B, WHICH IS USED FOR
C  WORKING STORAGE. THIS METHOD DELIBERATELY WASTES
C  STORAGE IN THE INTEREST OF UNDERSTANDIBILITY. THE
C  METHOD IS EASILY MODIFIED TO USE LESS SPACE.
      DO 1 I=1,N
      DO 1 J=1,N
1     B(I,J)=A(I,J)
C  LOAD RIGHT HALF OF B WITH UNIT MATRIX
      J1=N+1
      J2=2*N
      DO 2 I=1,N
      DO 2 J=J1,J2
2     B(I,J)=0.0
      DO 3 I=1,N
      J=I+N
3     B(I,J)=1.0
C  START THE PIVOTAL CONDENSATION.
C  K NAMES THE PIVOTAL ROW
      DO 610 K=1,N
      KP1=K+1
C  FIND TERM IN COLUMN K, ON OR BELOW MAIN DIAGONAL,
C  THAT IS LARGEST IN ABSOLUTE VALUE. AFTER THE SEARCH,
C  L IS THE ROW NUMBER OF THE ELEMENT. THERE IS NO
C  SEARCH FOR THE LARGEST ELEMENT WHEN K=N,
C  BUT THE ELIMINATION OF ELEMENTS ABOVE THE DIAGONAL
C  IS STILL NECESSARY
      IF(K.EQ.N) GO TO 500
      L=K
      DO 400 I=KP1,N
400    IF (ABS(B(I,K)) .GT. ABS(B(L,K))) L=I
C  CHECK WHETHER L=K, WHICH MEANS THAT THE LARGEST ELEMENT
C  IN COLUMN K WAS ALREADY THE DIAGONAL TERM, MAKING ROW
C  INTERCHANGE UNNECESSARY
      IF (L.EQ.K) GO TO 500
C  INTERCHANGE ROWS L AND K, FROM DIAGONAL TO RIGHT,
C  INCLUDING IDENTITY
      DO 410 J=K,J2
      TEMP=B(K,J)
      B(K,J)=B(L,J)
410    B(L,J)=TEMP
C  DIVIDE THE ELEMENTS IN ROW K BY A(K,K)
C  HOWEVER, DO NOT DIVIDE DIAGONAL TERM BY ITSELF,
C  SINCE SUBSEQUENT DIVISIONS WOULD BE INCORRECT,
C  AND DIAGONAL TERM IS NEVER USED
C  AFTER THIS ANYWAY
500    DO 501 J=KP1, J2
501    B(K,J)=B(K,J)/B(K,K)

```

```

C  ELIMINATE ALL ELEMENTS IN COLUMN K ABOVE THE DIAGONAL
C  THE ELEMENTS SPECIFICALLY IN COLUMN K ARE NOT, IN FACT,
C  MODIFIED AT ALL, SINCE THEIR VALUES ARE NEEDED LATER.
C  WHAT IS DONE IS TO PERFORM THE ELIMINATION OPERATIONS
C  ON ALL COLUMNS TO RIGHT OF COLUMN K, WHICH ARE BASICALLY
C  ALL WE CARE ABOUT IN OBTAINING THE INVERSE. SEE TEXT.
C  MUST SKIP THIS OPERATION FOR FIRST ROW, OF COURSE
      IF (K.EQ.1) GO TO 600
      KM1=K-1
      DO 510 I=1,KM1
      DO 510 J=KP1,J2
510   B(I,J)=B(I,J)-B(I,K)*B(K,J)
C  ELIMINATE ALL ELEMENTS IN COLUMN K BELOW MAIN DIAGONAL
C  SKIP THIS OPERATION WHEN K=N
      IF (K.EQ.N) GO TO 700
600   DO 610 I=KP1,N
      DO 610 J=KP1,J2
610   B(I,J)=B(I,J)-B(I,K)*B(K,J)
C  MOVE THE INVERSE TO THE OUTPUT MATRIX
700   DO 701 I=1,N
      DO 701 J=1,N
      K=J+N
701   AINV(I,J)=B(I,K)
      RETURN
      END
C  A SUBROUTINE FOR MATRIX MULTIPLICATION
C
      SUBROUTINE MATPLY(C,A,B,M,N,R)
      INTEGER R
      REAL A(M,N) , B(N,R) , C(M,R)
      DO 1 I=1,M
      DO 1 J=1,R
      C(I,J)=0.0
      DO 1 K=1,N
1     C(I,J)=C(I,J)+A(I,K)*B(K,J)
      RETURN
      END
C  A SUBROUTINE FOR ADDITION OF A MATRIX
C
      SUBROUTINE MATADD(C,A,B,M,N)
      REAL A(M,N) , B(M,N) , C(M,N)
      DO 1 I=1,M
      DO 1 J=1,N
1     C(I,J)=A(I,J)+B(I,J)
      RETURN
      END

```



```
C  A SUBROUTINE FOR MATRIX SUBTRACTION
C
      SUBROUTINE MATSUB (C,A,B,M,N)
      REAL A(M,N) , B(M,N) , C(M,N)
      DO 1 I=1,M
      DO 1 J=1,N
1     C(I,J)=A(I,J)-B(I,J)
      RETURN
      END
```

TIME	STAGE	UPDATED	MAYFIELD	INFLOW	UNGAGED
		STAGE		TOWER ROAD	
.0	17.4	18.0	10200.0	4640.0	2320.0
.5	17.4	17.7	10250.0	4895.0	2447.5
1.0	17.5	17.7	10250.0	5150.0	2575.0
1.5	17.5	17.7	10200.0	5420.0	2710.0
2.0	17.6	17.8	10200.0	5690.0	2845.0
2.5	17.6	17.8	10200.0	5850.0	2925.0
3.0	17.7	17.8	10200.0	6010.0	3005.0
3.5	17.7	17.9	10200.0	6930.0	3465.0
4.0	17.8	18.0	10200.0	7500.0	3750.0
4.5	17.8	18.0	10200.0	9810.0	4905.0
5.0	17.9	18.1	10200.0	9660.0	4830.0
5.5	18.0	18.2	10200.0	9600.0	4800.0
6.0	18.1	18.3	10700.0	9800.0	4900.0
6.5	18.1	18.3	10700.0	9810.0	4905.0
7.0	18.2	18.4	10800.0	9900.0	4950.0
7.5	18.2	18.4	10800.0	10700.0	5350.0
8.0	18.2	18.4	10800.0	11000.0	5500.0
8.5	18.3	18.5	10800.0	11250.0	5625.0
9.0	18.3	18.5	10800.0	11500.0	5750.0
9.5	18.2	18.5	10750.0	11733.0	5866.5
10.0	18.2	18.5	10700.0	11967.0	5983.5
10.5	18.3	18.5	10700.0	12200.0	6100.0
11.0	18.4	18.6	10700.0	11700.0	5850.0
11.5	18.6	18.7	10700.0	12050.0	6025.0
12.0	18.6	18.8	10700.0	12400.0	6200.0
12.5	18.7	18.8	10700.0	12300.0	6150.0
13.0	18.8	18.9	10700.0	12300.0	6150.0
13.5	19.0	19.0	10700.0	12000.0	6000.0
14.0	19.1	19.2	10700.0	12000.0	6000.0
14.5	19.2	19.3	10725.0	13900.0	6950.0
15.0	19.4	19.4	10750.0	15000.0	7500.0
15.5	19.6	19.6	10775.0	17800.0	8900.0
16.0	19.8	19.9	10800.0	19200.0	9600.0
16.5	20.0	20.0	11625.0	19200.0	9600.0
17.0	20.3	20.3	12450.0	21500.0	10750.0
17.5	20.6	20.6	13200.0	22000.0	11000.0
18.0	20.9	20.9	13950.0	24600.0	12300.0
18.5	21.2	21.2	14975.0	25800.0	12900.0
19.0	21.6	21.5	16000.0	25900.0	12950.0
19.5	21.8	21.7	17050.0	24100.0	12050.0
20.0	21.8	21.8	18100.0	26500.0	13250.0
20.5	21.9	21.9	18800.0	30900.0	15450.0
21.0	22.1	22.1	19500.0	31600.0	15800.0
21.5	22.4	22.4	20625.0	31400.0	15700.0
22.0	22.5	22.6	21750.0	33800.0	16900.0
22.5	22.5	22.7	21650.0	35400.0	17700.0
23.0	22.8	22.8	21550.0	33900.0	16950.0
23.5	22.9	23.0	21525.0	31000.0	15500.0
24.0	22.9	22.9	21500.0	28200.0	14100.0

TIME	STAGE	UPDATED	INFLOW		UNGAGED
		STAGE	MAYFIELD	TOWER ROAD	
24.5	23.1	23.0	23525.0	28500.0	14250.0
25.0	23.2	23.1	25550.0	26700.0	13350.0
25.5	23.1	23.1	25550.0	26700.0	13350.0
26.0	23.0	23.1	25550.0	26400.0	13200.0
26.5	23.0	23.0	25550.0	25500.0	12750.0
27.0	23.1	23.0	25550.0	26100.0	13050.0
27.5	23.1	23.1	25550.0	26100.0	13050.0
28.0	23.1	23.1	25550.0	25700.0	12850.0
28.5	23.1	23.1	25550.0	25400.0	12700.0
29.0	23.1	23.1	25550.0	23900.0	11950.0
29.5	23.1	23.0	25550.0	23500.0	11750.0
30.0	23.2	23.1	25550.0	22500.0	11250.0
30.5	23.1	23.0	25550.0	22800.0	11400.0
31.0	23.1	23.0	25550.0	21800.0	10900.0
31.5	23.0	22.9	23500.0	20200.0	10100.0
32.0	23.0	22.9	21450.0	19600.0	9800.0
32.5	22.9	22.8	21450.0	18400.0	9200.0
33.0	22.7	22.6	21450.0	17800.0	8900.0
33.5	22.4	22.4	21400.0	17000.0	8500.0
34.0	22.2	22.2	21350.0	17200.0	8600.0
34.5	22.1	22.0	21350.0	16800.0	8400.0
35.0	22.1	22.0	21350.0	16000.0	8000.0
35.5	22.0	21.9	21350.0	16200.0	8100.0
36.0	21.9	21.8	21350.0	15600.0	7800.0
36.5	21.9	21.8	21350.0	15800.0	7900.0
37.0	21.8	21.7	21350.0	15000.0	7500.0
37.5	21.7	21.6	21350.0	14400.0	7200.0
38.0	21.6	21.5	21350.0	13600.0	6800.0
38.5	21.6	21.5	21350.0	14000.0	7000.0
39.0	21.5	21.4	21350.0	13300.0	6650.0
39.5	21.4	21.3	21350.0	13100.0	6550.0
40.0	21.4	21.3	21350.0	13300.0	6650.0
40.5	21.3	21.2	21175.0	13100.0	6550.0
41.0	21.2	21.2	21000.0	12700.0	6350.0
41.5	21.2	21.1	21000.0	12300.0	6150.0
42.0	21.1	21.1	21000.0	12200.0	6100.0
42.5	21.0	21.0	21000.0	12400.0	6200.0
43.0	20.9	20.9	21000.0	11900.0	5950.0
43.5	20.9	20.8	20600.0	11500.0	5750.0
44.0	20.8	20.8	20200.0	11400.0	5700.0
44.5	20.8	20.7	20200.0	11200.0	5600.0
45.0	20.7	20.7	20200.0	11400.0	5700.0
45.5	20.7	20.7	19675.0	11100.0	5550.0
46.0	20.7	20.7	19150.0	11100.0	5550.0
46.5	20.7	20.6	19150.0	10800.0	5400.0
47.0	20.7	20.6	19150.0	10900.0	5450.0
47.5	20.7	20.6	18725.0	10600.0	5300.0
48.0	20.7	20.6	18300.0	10700.0	5350.0

APPENDIX B

KALMAN FILTER MATRICES

This Appendix contains the matrices that were previously defined and used in the application of the Kalman filter.

The matrices are:

$$\underline{x} = \begin{bmatrix} x_1(m) = Q_7(m-3) \\ x_2(m) = Q_7(m-2) \\ x_3(m) = Q_7(m-1) \\ y_1(m) = Q_3(m-3) \\ y_2(m) = Q_3(m-2) \\ y_3(m) = Q_3(m-1) \end{bmatrix}$$

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ .0502 & -.4584 & 1.2161 & 0 & 0 & .1922 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & .2154 & -1.0779 & 1.7983 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & .1922 \\ 0 & 0 \\ 0 & 0 \\ .0643 & 0 \end{bmatrix}$$

$$\underline{z} = \begin{bmatrix} .9 \\ .9 \\ .9 \end{bmatrix}$$

$$\underline{H} = \begin{bmatrix} 9.76804 \times 10^{-5} & 0 & 0 & 0 & 0 & 0 \\ 0 & 9.76804 \times 10^{-5} & 0 & 0 & 0 & 0 \\ 0 & 0 & 9.76804 \times 10^{-5} & 0 & 0 & 0 \end{bmatrix}$$

$$\underline{R} = \begin{bmatrix} .3807 & 0 & 0 \\ 0 & .3807 & 0 \\ 0 & 0 & .3807 \end{bmatrix}$$

$$\underline{Q} = \begin{bmatrix} 10717364 & 0 & 0 \\ 0 & 10717364 & 0 \\ 0 & 0 & 10717364 \end{bmatrix}$$

$$\underline{\Gamma} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underline{I} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$