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AN ABSTRACT OF THE THESIS OF Hamid Reza Afghan for the Master of Science in Engineering-Civil presented June 1, 1984.

Title: A Numerical Solution for Three Dimensional Beam Columns in the Elastic Region.

APPROVED BY MEMBERS OF THE THESIS COMMITTEE:

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The three differential equations describing the behavior of the beam columns in three dimensions are derived and presented in finite difference form. A computer model is developed to solve the simultaneous equations in the elastic regions and predict the member behavior.

The computer program is checked by comparing the results obtained from the program with data from other investigators, and classical analytical techniques.

A NUMERICAL SOLUTION FOR
THREE DIMENSIONAL BEAM COLUMNS
IN THE ELASTIC REGION

by

HAMID REZA AFGHAN

A thesis submitted in partial fulfillment of the
requirements for the degree of

MASTER OF SCIENCE
in
ENGINEERING AND APPLIED SCIENCE

Portland State University
1985

TO THE OFFICE OF GRADUATE STUDIES AND RESEARCH:

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DEDICATED TO MY LOVING MOTHER

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CHAPTER I

INTRODUCTION

A beam column that is loaded through its center of gravity can buckle in three modes of failure, flexural buckling about its principal axes, torsional buckling, or flexural torsional buckling. If the shear center does not coincide with the center of gravity and the cross section has no axis of symmetry, the column will buckle in flexural torsional buckling. If the shear center and the center of gravity coincide, it will fail in pure torsional buckling or pure flexural buckling about one of its principal axes.

The member behavior of angle sections is of special interest to the engineers involved in design and analysis of the transmission towers. The transmission towers are highly indeterminate and once a member reaches its critical load and it buckles, the loads are redistributed in the tower. The buckled member will sustain some load equal to or less than the critical axial load.

In the design of the transmission towers, two concepts are used: The elastic strength and the ultimate strength concept. The elastic concept uses the yielding as the failure criteria; however, in the ultimate strength concept collapse is considered as the failure criteria. The behavior of the member in the elastic, inelastic, and post buckling is required in the ultimate strength concept.

The three differential equations describing the behavior of the beam columns are complex for routine use. These equations are fourth degree differential equations and can be solved by using a finite difference approach.

It is necessary to develop an analytical technique that can treat the beam column in the elastic region. Using this solution package, an open form technique can be formulated to handle the inelastic and post buckling regions.

The general differential equations are derived and presented in finite difference form. A computer model is developed to solve the simultaneous equations in the elastic region and predict the member behavior. The validity of the computer model is checked by solving general example problems using both the computer model and classical solution techniques.

CHAPTER II

REVIEW OF LITERATURE

The behavior of the beam column in the elastic and inelastic regions of the symmetrical cross sections has been intensively investigated in the past. These investigations, due to the symmetry of the cross section, have been performed in only two dimension and the torsional effect has not been included.

Galambos and Ketter (5) and Ketter, Kaminsky and Beedle (8) have done a great deal of work on wide flange members loaded as beam columns. Ellis (4) has performed some work on the plastic behavior of the compression members both analytical and experimental. Dwyer and Galambos (3) have reported a study of the plastic behavior of square tubes. The emphasis of this report was the comparison of the relative strength of the square tubes and wide flanges.

Wagner (14) has investigated the combined effect of bending and axial load on round steel tubes in the elastic and inelastic regions using an open form solution. He has used Matlock's recursive technique for analyzing the beam column. This technique had been previously modified by others to perform inelastic beam column analysis. For example, Mueller (12) modified the technique for treating beam columns on non linear foundations.

Timoshenko and Gere (13) have derived the general beam column equations in three dimensions. Chajes, Fang and Winter (2) have simplified these equations for the beam columns with one axis of symmetry. The emphasis of their study is on axially loaded cold formed thin walled columns in elastic and inelastic regions.

CHAPTER III

COMPUTER MODEL

This investigation directs itself to the study of a member subjected to axial loads and bendings about the two principal axes.

OVERVIEW

The differential equations are derived and expanded into finite difference form. It is assumed that the member is divided into m increments. Therefore there are $m+1$ real stations (including station zero) and six imaginary stations as illustrated in Figure 1. Each station has three possible displacements, two translations u and v , and one rotation θ . Hence there are $3(m+3)$ unknown displacements including stations -1 and $+1$ and $3(m+3)$ equations. A computer program is developed to solve these simultaneous equations. A listing of the computer program is available in Appendix C.

The beam column method that is presented here can conveniently handle a wide range of support and loading conditions and also takes into account the P-Delta amplification in bending moment. The cross sectional properties of the member can vary along its length in any conceivable configuration and the center of gravity does not necessarily have to coincide with the shear center. The line which goes through the shear center of the member is taken as the z axis (See Figure 1).

Figure 1 Beam of variable cross section subjected to general loading condition

$$*\phi_0 = 0$$

RECURSIVE SOLUTION FOR AN ELASTIC BEAM COLUMN

A recursive solution technique first used by Matlock to solve beam columns with one translational degree of freedom is employed for solving the simultaneous equations. After the three equations are defined in finite difference form and applied to each station, two passes are performed on the member. The first pass, from left to right, three coefficients are calculated for each station which are functions of the parameters at the two previous stations. On the return pass, from right to left, the deflections at each station are computed by using the calculated coefficients of that station and the deflections at the two preceding stations. This technique is expanded in the following discussion.

The following assumptions are made in this method of beam column analysis:

- a. Hook's Law is valid
- b. Plane sections before bending remain plane after bending
- c. Deflections and rotations are small
- d. Loads are applied in the plane of the vertical and horizontal axes of the member and through the center of gravity of the cross section

The following discussion is divided into five major sections:

1. Derivation of the equilibrium equations
2. Recursive solution of the beam column
3. Starting and reversing the recursion process
4. Specifying desired deflections
5. Buckling and instability of the beam column

1. Derivation of the Equilibrium Equations

A beam column is considered with the general load and support configuration in Figure 1. The most general type of displacement possible for this member consists of bending about the principal axes and twisting about the z axis through the shear center. The cross section of this member, Figure 2, undergoes translations u and v in the x and y directions and a rotation ϕ about the shear center. An infinitesimal increment of this member is considered in two planes yz and xz , Figures 3 and 4, and in three dimension xyz , Figure 5. All quantities in Figures 3, 4, and 5 are positive as shown and are defined as follows:

<u>Symbol</u>	<u>Description</u>	<u>Dimension</u>
P	Axial load on cross section through the center of gravity	F
M_x	Bending moment on cross section about x axis	$F.L$
M_y	Bending moment on cross section about y axis	$F.L$
V_x	Total shear on cross section in x direction	F
V_y	Total shear on cross section in y direction	F
q_x	Transverse load in x direction through the center of gravity	F/L
q_y	Transverse load in y direction through the center of gravity	F/L
t_x	Externally applied moment about x axis	$F.L/L$
t_y	Externally applied moment about y axis	$F.L/L$
M_z	Torque on beam column about z axis	$F.L$
m_z	Torque per unit length of the beam column about the z axis	$F.L/L$

Figure 2 Displacement of the cross section

It should be noted that q_x , q_y , t_x and t_y are assumed to be uniformly distributed over dz increment, and the cross section is constant for each increment. Consider this element in the yz plane. Since the element is in equilibrium, the moment at point A about the x axis is zero, i.e.,

$$M_x - (M_x + dM_x) + Pdy + V_y dz + q_y \frac{dz^2}{2} + t_x dz = 0 \quad (3.1)$$

or

$$- dM_x + Pdy + V_y dz + q_y \frac{dz^2}{2} + t_x dz = 0 \quad (3.2)$$

Neglecting the second order differentials and dividing Eq. (3.2) by dz results in:

$$\frac{dM_x}{dz} = V_y + t_x + P \frac{dy}{dz} \quad (3.3)$$

The vertical deflection of center of gravity, y , is equal to $v - x_o \phi$ (See App. D), therefore $dy = dv - x_o d\phi$.

Replacing dy in Eq. (3.3) and taking the derivative of Eq. (3.3) once with respect to z gives:

$$\frac{d^2 M_x}{dz^2} = \frac{dV_y}{dz} + \frac{d}{dz} (t_x + P \frac{dv}{dz} - P x_o \frac{d\phi}{dz}) \quad (3.4)$$

When considering the equilibrium of the vertical forces on the element in the y direction,

$$V_y + q_y(dz) - (V_y + dV_y) = 0 \quad (3.5A)$$

Figure 3 Infinitesimal beam increment in the yz plane

yields:

$$\frac{dV}{dz} = q_y \quad (3.5B)$$

substituting: $\frac{dV}{dz} = q_y$ into Eq. (3.4) gives:

$$\frac{d^2 M_x}{dz^2} = q_y + \frac{d}{dz} (t_x + P \frac{dv}{dz} - P x_o \frac{d\theta}{dz}) \quad (3.6)$$

The left side of Eq. (3.6) in finite difference form is:

$$\frac{d^2 M_x}{dz^2} = \frac{1}{h^2} (M_{i-1} - 2M_i + M_{i+1}) \quad (3.7)$$

Where i is the number defining the start and end of a particular finite increment. The member is divided into m equal finite increments and the length of each increment is equal to h . The value i will hereafter be referred to as the station or station number.

The differential equation of the deflected elastic beam from the strength of materials is:

$$M_x = F_x \frac{d^2 v}{dz^2} \quad (3.8)$$

F_x is the flexural stiffness of the member in the y direction (EI_x) and $\frac{d^2 v}{dz^2}$ is the curvature in the y direction. Eq. (3.8) in the finite difference form at station i is:

$$M_{x_i} = F_{x_i} (v_{i-1} - 2v_i + v_{i+1}) \frac{1}{h^2} \quad (3.9A)$$

Applying Eq. (3.9) at station $i-1$ and $i+1$ gives,

$$M_{x_{i-1}} = F_{x_{i-1}} (v_{i-2} - 2v_{i-1} + v_i) \frac{1}{h^2} \quad (3.9B)$$

$$M_{x_{i+1}} = F_{x_{i+1}} (v_i - 2v_{i+1} + v_{i+2}) \frac{1}{h^2} \quad (3.9C)$$

Substituting M_{i-1} , M_i and M_{i+1} in Eq. (3.7) results in:

$$\begin{aligned} \frac{d^2 M_x}{dz^2} = \frac{1}{h^4} [& F_{x_{i+1}} v_{i+2} - 2(F_{x_{i+1}} + F_{x_i}) v_{i+1} + (F_{x_{i+1}} + 4F_{x_i} + F_{x_{i-1}}) v_i \\ & - 2(F_{x_i} + F_{x_{i-1}}) v_{i-1} + F_{x_{i-1}} v_{i-2}] \end{aligned} \quad (3.10)$$

Assuming $F_{x_{i-1}}$, F_{x_i} and $F_{x_{i+1}}$ are each constant through the length of their respective increments $i-1$, i and $i+1$. Eq. (3.10) represents the left side of Eq. (3.6) in finite difference form.

Now consider the right side of Eq. (3.6) and write the differential inside the bracket in finite difference form:

$$\frac{d^2 M_x}{dz^2} = q_{y_i} + \frac{d}{dz} [t_{x_i} + P_i (v_{i+1} - v_{i-1}) - P_i x_{o_i} (\frac{\phi_{i+1} - \phi_{i-1}}{2h})] \quad (3.11)$$

or

$$\frac{d^2 M_x}{dz^2} = q_{y_i} + \frac{d}{dz} [t_{x_i} + \frac{P_i}{2h} v_{i+1} - \frac{P_i}{2h} v_{i-1} - \frac{P_i x_{o_i}}{2h} \phi_{i+1} + \frac{P_i x_{o_i}}{2h} \phi_{i-1}] \quad (3.12)$$

The whole right side in finite difference form is:

$$\begin{aligned}
\frac{d^2 M_x}{dz^2} = & q_{y_i} + \frac{1}{2h} [(t_{x_{i+1}} - t_{x_{i-1}}) + (\frac{P_{i+1}}{2h} v_{i+2} - \frac{P_{i+1}}{2h} v_i) \\
& - (\frac{P_{i-1}}{2h} v_i - \frac{P_{i-1}}{2h} v_{i-2}) - (\frac{P_{i+1} x_{o_{i+1}}}{2h} \phi_{i+2} - \frac{P_{i+1} x_{o_{i+1}}}{2h} \phi_i) \\
& + \frac{P_{i-1} x_{o_{i-1}}}{2h} \phi_i - \frac{P_{i-1} x_{o_{i-1}}}{2h} \phi_{i-2}] \quad (3.13)
\end{aligned}$$

Removing a factor of $\frac{1}{h^4}$ and collecting terms give:

$$\begin{aligned}
\frac{d^2 M_x}{dz^2} = & \frac{1}{h^4} [h^4 q_{y_i} + \frac{1}{2} h^3 t_{x_{i+1}} - \frac{1}{2} h^3 t_{x_{i-1}} + \frac{h^2}{4} P_{i+1} v_{i+2} \\
& - (\frac{h^2 P_{i+1}}{4} + \frac{h^2 P_{i-1}}{4}) v_i + \frac{h^2 P_{i-1}}{4} v_{i-2} - \frac{h^2 P_{i+1}}{4} x_{o_{i+1}} \phi_{i+2} + \\
& (\frac{h^2 P_{i+1}}{4} x_{o_{i+1}} + \frac{h^2 P_{i-1}}{4} x_{o_{i-1}}) \phi_i - \frac{h^2 P_{i-1}}{4} x_{o_{i-1}} \phi_{i-2}] \quad (3.14)
\end{aligned}$$

Equation (3.14) is the right side of Eq.(3.6). After combining the two Equations (3.10) and (3.14), the following equation results:

$$\begin{aligned}
(F_{x_{i-1}} - PH_{i-1})v_{i-2} - 2(F_{x_{i-1}} + F_{x_i})v_{i-1} + (F_{x_{i-1}} + 4F_{x_i} + F_{x_{i+1}} + \\
PH_{i-1} + PH_{i+1})v_i - 2(F_{x_i} + F_{x_{i+1}})v_{i+1} + (F_{x_{i+1}} - PH_{i+1})v_{i+2} + \\
(PH_{i-1} x_{o_{i-1}})\phi_{i-2} + (PH_{i-1} x_{o_{i-1}} + PH_{i+1} x_{o_{i+1}})\phi_i + \\
(PH_{i+1} x_{o_{i+1}})\phi_{i+2} = Q_{yy_i} + T_{x_{i+1}} - T_{x_{i-1}} \quad (3.15)
\end{aligned}$$

in which:

$$\begin{aligned} PH_i &= \frac{1}{4} h^2 P_i \\ Q_{yy_i} &= h^4 q_{y_i} \\ T_{x_i} &= \frac{1}{2} h^3 t_{x_i} \end{aligned} \quad (3.16)$$

Equation (3.15) is one of the three equations for the deflected shape of the beam column in finite difference form. It represents the deflection of the shear center in the y direction.

In Figure 4, the moment about the y axis at Point A is also zero, i.e.,

$$M_y - (M_y + dM_y) + Pdx + V_x dz + q_x \frac{dz^2}{2} - t_y dz = 0 \quad (3.17)$$

Neglecting higher order differential and dividing Eq. (3.17) by dz results in:

$$\frac{dM_y}{dz} = V_x - t_y + P \frac{dx}{dz} \quad (3.18A)$$

The deflection of the center of gravity in the x direction is $u + y_o \phi$ (see App. D). Thus:

$$dx = du + y_o d\phi \quad (3.18B)$$

Substituting dx into Eq. (3.18) and taking the derivative once with respect to z gives:

Figure 4 Infinitesimal beam increment in the xz plane

$$\frac{d^2 M_y}{dz^2} = \frac{dV_x}{dz} + \frac{d}{dz}(-t_y + P \frac{du}{dz} + P y_o \frac{d\theta}{dz}) \quad (3.19A)$$

The equilibrium of the forces in the x direction in Figure 4 gives:

$$\frac{dV_x}{dz} = q_x \quad (3.19B)$$

Replacing $\frac{dV_x}{dz}$ by q_x in Eq. (3.19A) results in:

$$\frac{d^2 M_y}{dz^2} = q_x + \frac{d}{dz} \overset{-t_y}{(-t_x)} + P \frac{du}{dz} + P y_o \frac{d\theta}{dz} \quad (3.20)$$

Equation (3.20) is similar to Eq. (3.6). However, they are in two different directions, x and y. It should be noted that $P x_o \frac{d\theta}{dz}$ in Eq. (3.6) has a negative sign and $P y_o \frac{d\theta}{dz}$ has a positive sign in Eq. (3.20). As the cross section translates and rotates about shear center, the new location of the center of gravity in the x and y directions becomes $u - y_o \theta$ and $v + x_o \theta$ (see App. D). The sign difference in the above expressions is reflected in Equations (3.6) and (3.20) by $(- P x_o \frac{d\theta}{dz})$ and $(+ P y_o \frac{d\theta}{dz})$.

Performing the same operation for Eq. (3.20) as was done in Eq. (3.6) gives the following equation in finite difference form. This is the second equation for the deflected shape of the general beam column of Figure (3.1).

$$\begin{aligned}
& (F_{y_{i-1}} - PH_{i-1})u_{i-2} - 2(F_{y_{i-1}} + F_{y_i})u_{i-1} + (F_{y_{i-1}} + 4F_{y_i} + F_{y_{i+1}} \\
& + PH_{i-1} + PH_{i+1})u_i - 2(F_{y_i} + F_{y_{i+1}})u_{i+1} + (F_{y_{i+1}} - PH_{i+1})u_{i+2} \\
& - (PH_{i-1} y_{o_{i-1}})\phi_{i-2} + (PH_{i-1} y_{o_{i-1}} + PH_{i+1} y_{o_{i+1}})\phi_i \\
& - (PH_{i+1} y_{o_{i+1}})\phi_{i+2} = Q_{xx_i} + T_{y_{i-1}} - T_{y_{i+1}} \quad (3.21)
\end{aligned}$$

in which:

$$Q_{xx_i} = h^4 q_{x_i} \quad (3.22)$$

$$T_{y_i} = \frac{1}{2} h^3 t_{y_i}$$

Equation (3.21) represents the deflection of the shear center in the x direction in finite difference form.

Now consider equilibrium about the z axis through the shear center as in Figure 5, i.e.,

$$- M_z + (M_z + dM_z) + (m_z)dz + (q_y dz)x_o - (q_x dz)y_o = 0 \quad (3.23)$$

or

$$dM_z + (m_z)dz + (q_y dz)x_o - (q_x dz)y_o = 0 \quad (3.24)$$

dividing Eq. (3.24) by dz results in:

$$\frac{dM_z}{dz} + m_z + q_y x_o - q_x y_o = 0 \quad (3.25A)$$

or

$$- \frac{dM_z}{dz} - m_z = q_y x_o - q_x y_o \quad (3.25B)$$

Figure 5 Infinitesimal beam increment in three dimensions

The torque, M_z , is resisted by a combination of shearing stress due to pure torsion and by the warping torsion. In differential form, M_z for non-uniform torsion is:

$$M_z = C \frac{d\theta}{dz} - K \frac{d^3\theta}{dz^3} \quad (3.26)$$

Equation (3.26) applies to any bar of thin walled open cross section. The first term of Eq. (3.26) represents the resistance of the section to twist and the second term the resistance to warping. As pointed out by McGuire (11), it is important to keep in mind that the second term is caused not by the warping of the member, but by its resistance to warp. If the member is allowed to warp, the second terms of Eq. (3.26) will become zero.

In Eq. (3.26), C is the torsional rigidity of the cross section which can be represented as GJ . G is the shearing modulus of elasticity and J is the torsion constant. The constant K is called the warping rigidity and can be expressed as EC_w . E is the modulus of elasticity and C_w is the warping constant.

The second term in Eq. (3.25), m_z , denotes the torque per unit length of the bar due to the axial load on slightly rotated cross sections and it equals to:

$$m_z = \frac{I_o}{A} P \frac{d^2\theta}{dz^2} - P(x_o \frac{d^2v}{dz^2} - y_o \frac{d^2u}{dz^2}) \quad (3.27)$$

in which I_o is the polar moment of inertia about the shear center and is equal to $I_x + I_y + A(x_o^2 + y_o^2)$. Timoshenko and Gere (13)

present the derivation of M_z and m_z . It should be emphasized that Timoshenko (13) uses positive sign convention for compressive load and here positive sign is used for tensile force.

Writing Eq. (3.26) in finite difference form for half increments:

$$M_{z_i} = \frac{C_{i+\frac{1}{2}} + C_{i-\frac{1}{2}}}{2h} (\phi_{i+\frac{1}{2}} - \phi_{i-\frac{1}{2}}) - \frac{K_{i+\frac{1}{2}} + K_{i-\frac{1}{2}}}{2h} (\phi_{i+1\frac{1}{2}} - 3\phi_{i+\frac{1}{2}} + 3\phi_{i-\frac{1}{2}} - \phi_{i-1\frac{1}{2}}) \quad (3.28)$$

where

$$\frac{C_{i+\frac{1}{2}} + C_{i-\frac{1}{2}}}{2} = C_i \quad (3.29)$$

$$\frac{K_{i+\frac{1}{2}} + K_{i-\frac{1}{2}}}{2} = K_i$$

$\frac{dM_z}{dz}$ in finite difference form for half increments is:

$$\frac{dM_z}{dz} = \frac{M_{z_{i+\frac{1}{2}}} - M_{z_{i-\frac{1}{2}}}}{h} \quad (3.30)$$

The right side of Eq. (3.30) can be determined from Eq. (3.28).

$M_{z_{i+\frac{1}{2}}}$ and $M_{z_{i-\frac{1}{2}}}$ are:

$$M_{z_{i+\frac{1}{2}}} = \frac{C_{i+1} + C_i}{2h} (\phi_{i+1} - \phi_i) - \frac{K_{i+1} + K_i}{2h^3} (\phi_{i+2} - 3\phi_{i+1} + 3\phi_i - \phi_{i-1}) \quad (3.31)$$

$$M_{z_{i-\frac{1}{2}}} = \frac{C_i + C_{i-1}}{2h} (\phi_i - \phi_{i-1}) - \frac{K_i + K_{i-1}}{2h^3} (\phi_{i+1} - 3\phi_i + 3\phi_{i-1} - \phi_{i-2})$$

substituting $M_{z_{i+\frac{1}{2}}}$ and $M_{z_{i-\frac{1}{2}}}$ from Eq. (3.31) into Eq. (3.30) yields:

$$\begin{aligned} \frac{dM_z}{dz} = & \left[-\left(\frac{1}{2} \frac{K_{i-1}}{h^4} + \frac{1}{2} \frac{K_i}{h^4}\right)\phi_{i-2} + \left(\frac{3}{2} \frac{K_{i-1}}{h^4} + 2 \frac{K_i}{h^4} + \frac{1}{2} \frac{K_{i+1}}{h^4} + \frac{1}{2} \frac{C_{i-1}}{h^2} + \frac{1}{2} \frac{C_i}{h^2}\right)\phi_{i-1} \right. \\ & - \left(\frac{3}{2} \frac{K_{i-1}}{h^4} + 3 \frac{K_i}{h^4} + \frac{3}{2} \frac{K_{i+1}}{h^4} + \frac{1}{2} \frac{C_{i-1}}{h^2} + \frac{C_i}{h^2} + \frac{1}{2} \frac{C_{i+1}}{h^2}\right)\phi_i + \\ & + \left(\frac{1}{2} \frac{K_{i-1}}{h^4} + 2 \frac{K_i}{h^4} + \frac{3}{2} \frac{K_{i+1}}{h^4} + \frac{1}{2} \frac{C_i}{h^2} + \frac{C_{i+1}}{h^2}\right)\phi_{i+1} \\ & \left. - \left(\frac{1}{2} \frac{K_i}{h^4} + \frac{1}{2} \frac{K_{i+1}}{h^4}\right)\phi_{i+2} \right] \end{aligned} \quad (3.32)$$

m_z in finite difference form is:

$$\begin{aligned} m_{z_i} = & \left(\frac{I_o}{A}\right)_i P_i \left(\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}\right) - P_i x_{o_i} \left(\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2}\right) \\ & + P_i y_{o_i} \left(\frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}\right) \end{aligned} \quad (3.33)$$

Multiplying both sides of Eq. (3.25B) by h^4 and replacing $\frac{dM_z}{dz}$ and m_z as defined in Eq. (3.32) and (3.33), gives the following finite difference form:

$$\begin{aligned} -PY_i u_{i-1} + 2PY_i u_i - PY_i u_{i+1} + PX_i v_{i-1} - 2PX_i v_i + PX_i v_{i+1} + \\ (KA_{i-1} + KA_i)\phi_{i-2} - (3KA_{i-1} + 4KA_i + KA_{i+1} + CA_{i-1} + CA_i + PR_i)\phi_{i-1} \\ + (3KA_{i-1} + 6KA_i + 3KA_{i+1} + CA_{i-1} + 2CA_i + CA_{i+1} + 2PR_i)\phi_i - \end{aligned}$$

$$\begin{aligned}
& (KA_{i-1} + 4KA_i + 3KA_{i+1} + CA_i + CA_{i+1} + PR_i)\theta_{i+1} \\
& + (KA_i + KA_{i+1})\theta_{i+2} = Q_{yi} - Q_{xi}
\end{aligned} \tag{3.34}$$

where

$$\begin{aligned}
PX_i &= h^2 P_i x_{oi} \\
PY_i &= h^2 P_i y_{oi} \\
PR_i &= h^2 \left(\frac{I_o}{A} \right)_i P_i \\
CA_i &= \frac{h^2}{2} C_i \\
KA_i &= \frac{K_i}{2} \\
Q_{xi} &= h^4 q_{xi} y_{oi} \\
Q_{yi} &= h^4 q_{yi} x_{oi}
\end{aligned} \tag{3.35}$$

Equation (3.34) is the third equilibrium equation for the elastic beam column of Figure 1. This represents the rotation of the member about the z axis.

The three equilibrium equations (3.15), (3.21) and (3.34) can be written in a matrix form as follows:

$$AA_i W_{i-2} + BB_i W_{i-1} + CC_i W_i + DD_i W_{i+1} + EE_i W_{i+2} = GG_i \tag{3.36}$$

where

$$AA_i = \begin{bmatrix} (F_{y_{i-1}} - PH_{i-1}) & 0 & -(PH_{i-1} y_{o_{i-1}}) \\ 0 & (F_{x_{i-1}} - PH_{i-1}) & (PH_{i-1} x_{o_{i-1}}) \\ 0 & 0 & (KA_{i-1} + KA_i) \end{bmatrix} \quad (3.37)$$

$$BB_i = \begin{bmatrix} -2(F_{y_{i-1}} + F_{y_i}) & 0 & 0 \\ 0 & -2(F_{x_{i-1}} + F_{x_i}) & 0 \\ -P_{y_i} & P_{x_i} & -(3KA_{i-1} + 4KA_i + KA_{i+1} + CA_{i-1} + CA_i + PR_i) \end{bmatrix}$$

$$CC_i = \begin{bmatrix} (F_{y_{i-1}} + 4F_{y_i} + F_{y_{i+1}} + PH_{i-1} + PH_{i+1}) & 0 & PH_{i-1}y_{o_{i-1}} + PH_{i+1}y_{o_{i+1}} \\ 0 & (F_{x_{i-1}} + 4F_{x_i} + F_{x_{i+1}} + PH_{i-1} + PH_{i+1}) & PH_{i-1}x_{o_{i-1}} + PH_{i+1}x_{o_{i+1}} \\ 2P_{y_i} & -2P_{x_i} & (3KA_{i-1} + 6KA_i + 3KA_{i+1} + CA_{i-1} + 2CA_i + CA_{i+1} + 2PR_i) \end{bmatrix}$$

$$DD_i = \begin{bmatrix} -2(F_{y_i} + F_{y_{i+1}}) & 0 & 0 \\ 0 & -2(F_{x_i} + F_{x_{i+1}}) & 0 \\ -P_{y_i} & P_{x_i} & -(KA_{i-1} + 4KA_i + 3KA_{i+1} + CA_i + CA_{i+1} + PR_i) \end{bmatrix}$$

$$EE_i = \begin{bmatrix} (F_{y_{i+1}} - PH_{i+1}) & 0 & -(PH_{i+1} y_{o_{i+1}}) \\ 0 & (F_{x_{i+1}} - PH_{i+1}) & (PH_{i+1} x_{o_{i+1}}) \\ 0 & 0 & (KA_i + KA_{i+1}) \end{bmatrix} \quad (3.37)$$

$$GG_i = \begin{bmatrix} Q_{xx_i} + T_{y_{i-1}} - T_{y_{i+1}} \\ Q_{yy_i} - T_{x_{i-1}} + T_{x_{i+1}} \\ Q_{y_i} - Q_{x_i} \end{bmatrix}$$

$$W_i = \begin{bmatrix} u_i \\ v_i \\ \phi_i \end{bmatrix}$$

Equations (3.16), (3.22) and (3.35) define the terms of the above matrices. Coefficients AA, BB, CC, DD and EE in Eq. (3.37) are 3x3 matrices and they are functions of the sectional properties of the beam column and the applied axial loads. Coefficient GG is a 3x1 matrix and it is a function of the applied loads. W is also a 3x1 matrix and it represents the deflection of the beam column, two translations u and v, and one rotation ϕ , at a particular station i.

In matrix structural analysis, the coefficients AA, BB, CC, DD and EE make up the stiffness matrix for the structure with a band width of five and the coefficients GG make up the load matrix.

2. SOLUTION OF THE BEAM COLUMN EQUATIONS

Assume the beam column is divided into m increments of length h and designate the increment points as stations. Let the left end of the beam column be station zero and the right end be station m . At each station i , the general beam column equation (3.36) can be applied (the cross sectional properties at each station is taken as the average of the distribution which exists on the beam column). Therefore there is a system of simultaneous equations whose solution gives the deflected shape of the beam column.

In order to solve these equations, Matlock (10) suggests a back and forth recursive process which is presented here. This technique was conceived by Tucher* who felt that by using "partitioned" matrices, a matrix of five diagonal sub-matrices could be solved in a recursive technique analogous to Matlock's method of solving beams and columns. A similar technique has been used by Hudson and Stelzer (7) for slabs on foundation.

Assume that the deflection of the beam column at a given station can be expressed as a linear function of the deflection at the two following stations, i.e.,

$$W_i = A_i + B_i W_{i+1} + C_i W_{i+2} \quad (3.38)$$

where A , B , and C are constants to be determined. Coefficients B and C are 3×3 matrices and coefficient A is 3×1 matrix. Applying Equation (3.38) to stations $i-1$ and $i-2$ results in:

*Tucher, Richard L., personal communication to Hudson Matlock concerning a technique for solving large, sparsely populated matrices, June 15, 1965.

$$W_{i-1} = A_{i-1} + B_{i-1} W_i + C_{i-1} W_{i+1} \quad (3.39)$$

$$W_{i-2} = A_{i-2} + B_{i-2} W_{i-1} + C_{i-2} W_i \quad (3.40)$$

substituting Eq. (3.39) and (3.40) in Eq. (3.38) gives

$$W_i = A_i + B_i W_{i+1} + C_i W_{i+2} \quad (3.41)$$

where

$$A_i = -\frac{1}{D_i} (E_i A_{i-1} + AA_i A_{i-2} - GG_i)$$

$$B_i = -\frac{1}{D_i} (E_i C_{i-1} + DD_i) \quad (3.42A)$$

$$C_i = -\frac{1}{D_i} (EE_i)$$

in which

$$D_i = CC_i + E_i B_{i-1} + AA_i C_{i-2}$$

$$E_i = AA_i B_{i-2} + BB_i \quad (3.42B)$$

Therefore, it is seen that the assumptions of Eq. (3.38) and (3.39) are correct. It should be noted that B_i , C_i , D_i and E_i are all 3x3 matrices and A_i is 3x1 matrix.

According to Eq. (3.42A) and (3.42B), constants A_i , B_i and C_i , the continuity coefficients, are functions of these same three constants at two previous stations $i-2$ and $i-1$ and the known coefficients AA-GG at station i . The recursive technique is performed in two passes. This is done by moving along the member from the left end and calculating A_i , B_i and C_i at each station on the first pass. Then deflection W_i is calculated by substituting W_{i+1} and W_{i+2}

into Eq. (3.41) on the return pass (Fig. 1).

Haliburton (6) considers this as a special form of the Gaussian elimination in which on the first pass, for all stations i , deflections W_{i-2} and W_{i-1} , to the left of the main diagonal, are eliminated. This is called triangularizing the matrix. Therefore, the band width of five becomes three. On the return pass the deflections are calculated by back substitution. The resulting W 's are the deflection of the member in x and y and the rotation ϕ about the z axis.

3. STARTING AND REVERSING THE RECURSIVE PROCESS

In order to start the recursive process at station zero, the values of A , B and C at two previous stations are required. It is assumed that there are three imaginary stations preceeding station zero and three imaginary stations beyond station m . The imaginary stations have no stiffness and no load or restraints (Figure 1).

Assume the beam column is free to warp ($KA_i = 0$). Therefore, according to Eq. (3.37), at station -1 , $AA_{-1} = 0$ and $BB_{-1} = 0$. Values A_{-1} , B_{-1} and C_{-1} can be calculated by Eq. (3.42A) and (3.42B) at imaginary station -1 , i.e.,

$$E_{-1} = (0) B_{-3} + 0 = 0$$

$$D_{-1} = [CC_{-1} + (0) B_{-2} + (0) C_{-3}] = CC_{-1}$$

$$C_{-1} = -[CC_{-1}]^{-1} [EE_{-1}] \quad (3.43)$$

$$B_{-1} = [-CC_{-1}]^{-1} [(0) C_{-2} + DD_{-1}] = -[CC_{-1}]^{-1} [DD_{-1}]$$

$$A_{-1} = [-CC_{-1}]^{-1} [(0) A_{-2} + (0) A_{-3} - GG_{-1}] = -[CC_{-1}]^{-1} [GG_{-1}]$$

The values of A_0 , B_0 and C_0 at station zero can be computed without any knowledge of A_{-2} , B_{-2} and C_{-2} . Looking at Eq. (3.42A) and Eq. (3.42B) reveals that A_{-2} , B_{-2} and C_{-2} are multiplied by AA_0 which is zero at station zero according to Eq. (3.37). Values A_i , B_i and C_i can be determined by starting at station zero and proceeding down the beam column to station $m+1$. Station -1 was chosen as a starting point because nothing before it affects the beam as discussed above. Likewise, nothing beyond station $m+1$ affects the beam column. This becomes evident in the following explanation.

On the return pass, the deflection at station $m+1$ is computed first without any knowledge about the deflections at stations $m+2$ and $m+3$, i.e.,

$$W_{m+1} = A_{m+1} + B_{m+1} W_{m+2} + C_{m+1} W_{m+3} \quad (3.44)$$

Coefficient EE_m is equal to zero at station m . Therefore, value c_m which is equal to $-EE_m/D_m$ becomes zero. Since at station m , $C_m = 0$ and at station $m+1$, DD_{m+1} and EE_{m+1} are equal to zero, the values C_{m+1} and B_{m+1} also become zero, i.e.,

$$C_{m+1} = -\frac{1}{D_{m+1}} (0) = 0 \quad (3.45)$$

$$B_{m+1} = -\frac{1}{D_{m+1}} [E_{m+1} (0) + (0)] = 0$$

Therefore, Eq. (3.44) becomes:

$$W_{m+1} = A_{m+1} + (0) W_{m+2} + (0) W_{m+3} = A_{m+1} \quad (3.46)$$

Since C_m is zero, the deflection at station m can also be calculated without W_{m+2} , i.e.,

$$W_m = A_m + B_m W_{m+1} + (0) W_{m+2} \quad (3.47)$$

Therefore, deflections can be computed on the return pass by Eq. (3.41) starting at station $m+1$.

The primary purpose of this investigation was its application to angle section in which the resistance to warping is very close to zero. However, in order to be able to start the recursive process and to reverse it for a section in which warping constant is specified, the following changes in Eq. (3.37) are suggested:

$$AA_i (3,3) = 2KA_{i-1}$$

$$BB_i (3,3) = - (3KA_{i-1} + 5KA_i + CA_{i-1} + CA_i + PR_i)$$

$$DD_i (3,3) = - (5KA_i + 3KA_{i+1} + CA_i + CA_{i+1} + PR_i)$$

$$EE_i (3,3) = 2KA_{i+1}$$

4. SPECIFYING DESIRED DEFLECTIONS

The deflection of a beam column is usually known at one or more points along the member. For example, the deflection at each end of a simple beam is zero or perhaps the settlement of one or more supports of a continuous beam is known. These conditions need to be introduced into the recursive solution.

Eq. (3.41) is repeated here for convenience.

$$W_i = A'_i + B'_i W_{i+1} + C'_i W_{i+2} \quad (3.48)$$

*Primes are used to designate specially determined coefficients.

In order to specify the deflection at some point on the beam column, say at station i , A'_i should be set equal to desired deflection and B'_i and C'_i equal to zero, i.e.:

$$W_i = A'_i + (0) W_{i+1} + (0) W_{i+2} \quad (3.49)$$

The coefficients must be set at the special value before one proceeds to calculate the coefficients for the following stations. This is necessary because the coefficients at the following stations are functions of preceding ones.

There are three types of deflection at a station that may be specified. Each should be treated separately. For example, assume the deflection at station i is specified in the x direction only. In this case $A'_i(1,1)$ which pertains to the x direction is set equal to the specified deflection and the respective rows of B'_i and C'_i matrices are set equal to zero, i.e.,

$$B'_i(1,1) = B'_i(1,2) = B'_i(1,3) = 0 \quad (3.50)$$

$$C'_i(1,1) = C'_i(1,2) = C'_i(1,3) = 0$$

The same is true for the specified y and 0 deflections. For example, for specified y deflection the second rows of B'_i and C'_i are set equal to zero and $A'_i(2,1)$ is set equal to desired y deflection. Now, if all three deflections at station i are specified, all terms of B and C matrices become zero.

5. BUCKLING AND INSTABILITY OF THE BEAM COLUMNS.

A major concern of this study was the determination of the elastic buckling of a beam column. The analysis of a beam column, using the recursive technique, for load values up to and beyond the buckling load indicates a point of discontinuity at the buckling load value. If a load-deflection curve is plotted based on calculated values, using the derived recursive solution, one obtains a curve similar to Figure 6. At instability a sudden change in the sign of the deflection occurs. This is one possible indication of the buckling. However, it is necessary to have a more fundamental definition. To achieve this, the equations used in the beam column analysis are repeated and examined.

The two basic recursive equations are:

$$AA_i W_{i-2} + BB_i W_{i-1} + CC_i W_i + DD_i W_{i+1} + EE_i W_{i+2} = GG_i \quad (3.51)$$

and

$$W_i = A_i + B_i W_{i+1} + C_i W_{i+2} \quad (3.52)$$

or

$$W_i - B_i W_{i+1} - C_i W_{i+2} = A_i \quad (3.53)$$

where

$$\begin{aligned} A_i &= -\frac{1}{D_i} (E_i A_{i-1} + AA_i A_{i-2} - GG_i) \\ B_i &= -\frac{1}{D_i} (E_i C_{i-1} + DD_i) \\ C_i &= -\frac{1}{D_i} (EE_i) \end{aligned} \quad (3.54A)$$

Figure 6 Load vs. lateral deflection

in which

$$D_i = CC_i + E_i B_{i-1} + AA_i C_{i-2} \quad (3.54B)$$

$$E_i = AA_i B_{i-2} + BB_i$$

If equation (3.51) is repeated for each station i along the beam column and written in matrix form, the coefficients AA , BB , CC , DD and EE will make up a stiffness matrix with a band width of five.

In the preceding sections, it was demonstrated that the recursive method is a special case of the Gaussian elimination method. Therefore, the check for stability used in classical matrix method may be applied.

In classical matrix analysis, it is required that the stiffness matrix be positive definite in order to have stability (9). Mathematically, this condition exists if all terms on the diagonal of the stiffness matrix are positive after elimination (9). Therefore, if a zero or negative term appears as a diagonal element of the stiffness matrix after the elimination process, the structural system is unstable or buckling has occurred. It should be noted that $-1/D_i$ is the negative reciprocal of the diagonal element for each row of the stiffness matrix after elimination. Therefore, as a diagonal term approaches zero, $-1/D_i$ goes to infinity and if a diagonal term is negative the corresponding $-1/D_i$ is positive.

$-1/D_i$ is a 3×3 matrix in Eq. (3.54A) and it is the negative reciprocal of an algebraic expression, i.e.,

$$D_i = C_i + (AA_i B_{i-2} + BB_i) B_{i-1} + AA_i C_{i-2} \quad (3.55)$$

in which E_i is replaced by its equivalent as in Eq. (3.54B)

The problem of instability in this case can be investigated by examining the diagonal of the D_i matrix (a 3×3 matrix). If zero or a negative number appears on the diagonal of D_i , it shows that the beam column has become unstable. Depending on where the zero or negative term appears, one can determine the direction of instability. For example, if $D_i(1,1)$ is zero or negative, it means that the beam column buckles in the x direction (or about the y axis), either in pure flexural buckling or in flexural torsional buckling. If y_0 is equal to zero, there is no coupling effect between torsional buckling and flexural buckling about the y axis, therefore the buckling is pure flexural. However, if x_0 is equal to zero and y_0 is not equal to zero, there is a coupling effect between torsional buckling and flexural buckling about the y axis which causes flexural torsional buckling. $D_i(1,1)$ being zero or negative in this case, indicates that the beam column is weaker about the y axis than in torsion. Similar discussions are valid for $D_i(2,2)$, and $D_i(3,3)$.

Now, if x_0 and y_0 are both non-zero, the beam column will buckle in flexural torsional buckling and this buckling is a combination of all three modes of buckling, x , y and θ . Depending on which one of the diagonal terms of D_i appears zero or negative term, the beam column is the weakest about that axis. For example, $D_i(2,2)$ pertains to bending about the x axis.

It needs to be emphasized that axial load is the only applied load that appears on the stiffness matrix and can cause elastic instability in an otherwise stable structure.

CONCLUSION

This method can treat any beam column with general load and support configuration. It can calculate the deflections in the x and y direction and rotation about the z axis. It can also predict the elastic buckling load.

The advantage of this technique is that it is self starting and the boundary conditions of the differential equations are automatically input into the solution through the description of the beam column. The computation of the constants starts from the left end and proceed to the right and then deflections are calculated at the right end and moving to the left end. Figure 7 summarizes this solution technique.

Figure 7 Two pass recursive method in three dimensional beam column

CHAPTER IV

COMPARING WITH OTHER ANALYTICAL TECHNIQUES

The computer model is checked by comparing the results obtained from the program with data obtained by other investigators and classical analytical techniques.

Basic examples which demonstrate various components of the program's capability are presented below. The detailed output for the examples are in Appendix (C). It should be noted that increased accuracy results from using a higher number of stations.

1. CONCENTRATED MOMENTS

Figure 8 shows a beam in which two moments are applied at the left end of the beam about the x and y axes. The moments of inertia are constant throughout the beam about both axes and the beam is divided into eight increments.

Deflections at station 2 and 4 are computed by the Conjugate Beam Method and compared with the ones obtained from the computer program, i.e.,

<u>Computer Program</u>	<u>Conjugate Method</u>
$X_2 = .0140 \text{ in.}$	$X_2 = .0140 \text{ in.}$
$Y_4 = .0160 \text{ in.}$	$Y_4 = .0160 \text{ in.}$

The results are exactly the same in this example.

2. THREE CONCENTRATED LOADS

A beam, 160 inches long, is loaded with three concentrated loads in the y direction as in Figure 9. The moment of inertia about the x axis is constant throughout the member and EI_x is equal to $.3 \times 10^8$ in.²k. Deflections obtained from the computer program for station 2 and 4 are compared with the ones obtained from Conjugate Beam Method, i.e.,

<u>Computer Program</u>	<u>Conjugate Method</u>
$Y_2 = .098667$ in.	$Y_2 = .096000$ in.
$Y_4 = .138667$ in.	$Y_4 = .135111$ in.

The computer results are approximately 2.7% higher than the exact values. By increasing the number of stations the computer results become closer to the exact values.

3. APPLIED TORQUES

Two torques, 20 ft.k and 5 ft.k, are applied to a member 14 ft. long with circular cross sections (Figure 10). The rotations obtained from the program are compared with the ones determined by torsion equation ($\phi = (TL)/(GJ)$). The results are identical.

The computer program does not have the option to input the external torques about the z axis directly. Therefore, two new statements are added to the subroutine MAIN1 in the computer program to account for the two external torques. Coefficients GG at station 9 and 14 need to be redefined. It should be emphasized that the real stations 9 and 14 become computer stations 13 and 18. The two terms GG (3,13) and GG (3,18) are redefined as follows:

Figure 8 Simple beam with concentrated moments

Figure 9 Simple beam with three point loads

$$GG(3,13) = GG(3,13) + h^3 T_z$$

$$GG(3,13) = GG(3,13) + (12. \text{ in.})^3 \cdot (20. \times 12. \text{ in. k})$$

$$GG(3,13) = GG(3,13) + 414,720. \text{ in.}^4 \text{ k}$$

and

$$GG(3,18) = GG(3,18) + h^3 T_z$$

$$GG(3,18) = GG(3,18) + (12. \text{ in.})^3 \cdot (-5. \times 12. \text{ in. k})$$

$$GG(3,18) = GG(3,18) - 103,680. \text{ in.}^4 \text{ k}$$

See appendix C for the proper location of GG(3,13) and GG(3,18) in subroutine MAIN1.

Rotations about the z axis obtained from the computer program and torsion equations for station 9 and 14 are:

<u>Computer Output</u>	<u>Torsion Equation</u>
$\phi_9 = 0.06437087 \text{ Rad.}$	$\phi_9 = .06437086 \text{ Rad.}$
$\phi_{14} = 0.05245034 \text{ Rad.}$	$\phi_{14} = .05245033 \text{ Rad.}$

The example problems presented below illustrate the technique for finding the flexural, torsional and flexural torsional buckling loads. The critical loads interpreted from the computer outputs are compared with the ones calculated by Euler's equation and equations given by Chajes, Fang and Winter (2).

4. FLEXURAL BUCKLING ABOUT X AXIS

A column, 160 inches long, is divided into eighty increments and loaded axially. The buckling load obtained from the program is checked against the one calculated from Euler's equation.

Figure 10 Applied Torques

Figure 11 Flexural Buckling about the x axis

A small lateral load is applied in the y direction at the mid-height of the column, (station 40) as in Figure 11. This load is needed to input into the solution a non-zero P . The axial load is increased until the instability is observed in the deflections. The critical load interpreted from the computer runs is 3751.6 kips. The critical load from Euler's equation is:

$$P_x = \frac{\pi^2 EI_x}{L^2} = \frac{3.14^2 (.1 \times 10^8) \text{ in.}^2 \text{ k}}{(160. \text{ in.})^2}$$

$$P_x = 3855. \text{ k}$$

The computer result is about 97.4% of Euler's buckling load.

5. FLEXURAL BUCKLING ABOUT THE Y AXIS

The member of example 4 is investigated for the critical load about the y axis. A small lateral load is applied in the y direction at the mid height of the column (Figure 12). Axial load is increased until instability is observed.

The critical load interpreted from the computer model is approximately 1126.5 kips, and the Euler buckling load is

$$P_y = \frac{\pi^2 EI_y}{L^2} = \frac{3.14^2 (.3 \times 10^8) \text{ in.}^2 \text{ k}}{(160. \text{ in.})^2}$$

$$P_y = 11566. \text{ k}$$

The result from the computer program is approximately 97.4% of the exact answer.

6. TORSIONAL BUCKLING

The member of example 4 is tested for torsional buckling (Figure 13). The column is free to warp and the shear center coincides with the center of gravity ($x_o=0$. and $y_o=0$.). The torsional buckling is independent of the flexural buckling because the centroid and the shear center are at the same point.

The column is divided into eight increments and a small torque (.0008 in.k) is applied at station 4 to cause the torsion. Then axial load is increased until the rotational instability is observed.

The following statement is added to the subroutine MAIN1 to account for the applied external torque, i.e.,

$$GG(3,8) = GG(3,8) + h^3 T_z$$

$$GG(3,8) = GG(3,8) + (20. \text{ in.})^3 \cdot (.0008 \text{ in. k})$$

$$GG(3,8) = GG(3,8) + 6.4 \text{ in.}^4 \text{ k}$$

The torsional buckling load obtained from the computer program is 60. k and the one computed from the torsional buckling equation presented by Chajes (1) is:

$$P_{\phi} = \frac{1}{r_o^2} \left(GJ + \frac{r^2 E T}{L^2} \right)$$

$$r_o^2 = \left(\frac{I_x + I_y}{A} \right) + x_o^2 + y_o^2$$

$$r_o^2 = \left[\left(\frac{.1 \times 10^8 + .3 \times 10^8}{.3 \times 10^5} \right) \cdot \frac{1}{20} \right] = 66.67 \text{ in.}^2$$

$$P_{\phi} = \frac{.4 \times 10^4 \text{ in.}^2 \text{ k}}{66.7 \text{ in.}^2} = 60. \text{ k}$$

Figure 12 Flexural Buckling about y axis

Figure 13 Torsional Buckling about z axis

The two results from the program and the equation are identical.

7. FLEXURAL TORSIONAL BUCKLING

Flexural torsional buckling is illustrated in this example by investigating the buckling load of a 32 inch long, 3"x3"x1/4" angle. The resistance to warping for the angle sections is practically zero and y_o , the distance from the shear center to center of gravity in the y direction, is zero for equal leg angles (Figure 14). There are two types of buckling for this member; bending about the y axis (flexural buckling) and a combination of bending about the x axis and torsion about the z axis (flexural torsional buckling).

The member is divided into eight increments and a small disturbing torque (.0008 in.k) is applied at station 4 to initiate the flexural torsional buckling. The following statement is added to subroutine MAIN1 for the applied external torque,

$$GG(3,8) = GG(3,8) + 6.4 \text{ in.}^4\text{k}$$

The axial load is increased until the deflections in the y direction and rotations about the z axis change signs. The critical load predicted by the computer program is approximately 104.1 k. This result is compared with the flexural buckling load calculated by the equation presented by Chajes, Fang and Winter (2), i.e.,

$$P_{cr} = \frac{1}{2K} [(P_{\phi} + P_x) - \sqrt{(P_{\phi} + P_x)^2 - 4KP_{\phi}P_x}]$$

where

$$P_{\phi} = \frac{GJ}{r_o^2} = \text{Torsional buckling load}$$

$$E = .3 \times 10^5 \text{ ksi}$$

$$A = .144 \text{ in.}$$

$$x_0 = 1.19 \text{ in.}$$

$$EI_x = .594 \times 10^5 \text{ in}^2 \text{ k}$$

$$EI_y = .150 \times 10^5 \text{ in}^2 \text{ k}$$

$$GJ = .360 \times 10^3 \text{ in}^2 \text{ k}$$

Figure 14 Flexural Torsional Buckling of 3"x3"x1/4" angle

$$P_x = \frac{\pi^2 EI_x}{L^2} = \text{Flexural buckling load about x axis}$$

$$K = 1 - \frac{x_o^2}{r_o^2}$$

x_o = Distance from shear center to center of gravity

$$r_o^2 = \frac{I_o}{A} = \text{Polar radius of gyration}$$

$$I_o = (I_x + I_y) + x_o^2 A = \text{Polar moment of inertia}$$

A = Cross sectional area

The critical load calculated by this equation is 104.23 k which is very close to the one predicted by the computer program.

The flexural buckling load about the y axis ($P_y = \pi^2 EI_y / L^2$) is approximately 144.57 k which is larger than the flexural torsional buckling load and does not govern.

It should be noted that a small lateral load in the y direction can be applied instead of the applied external torque in order to cause the member to buckle in the flexural torsional buckling.

The above examples show that the results predicted by the computer program are the same as those obtained by other techniques.

CHAPTER V

EXPANSION POSSIBILITIES

The elastic beam column solution that is presented may be extended to take into account some other conditions as follows:

SPECIFIED SLOPES AND FIXED END CONDITIONS

Slopes can be specified by proper adjustments of the coefficients A, B and C in Eq. (3.41). The fixed conditions can be treated by specifying zero slopes. Matlock (10) presents a solution for this in one dimensional beam columns.

SPECIFIED EXTERNAL TORQUES

A beam column may be subjected to an external torque about the z axis. This external torque appears in Eq. (3.24), i.e.,

$$dM_z + m_z(dz) + t_z(dz) + (q_y dz)x_0 - (q_x dz)y_0 = 0$$

Where t_z is external torque about the z axis (in F.L/L) and it appears in the GG Matrix, i.e.,

$$GG(1,3) = QY_i - QX_i + T_{zi}$$

in which

$$T_{zi} = h^4 t_{zi}$$

This is arrived at in a similar manner as used in Eq. (3.24) through Eq. (3.35).

The torque is specified at station i (ht_z) in the computer input. Therefore, the specified torque is only needed to be multiplied by h^3 , i.e.,

$$T_{zi} = h^3(t_{si})$$

where t_{si} is the specified torque at station i (in F.L.).

SPRING SUPPORTS

Springs may be attached to the beam column along its length. There can be five types of springs, three spiral springs with rotational restraints (R_x , R_y , and R_z) about the three axes x , y and z and two coil springs with translational restraints (K_x and K_y) in the x and y direction (Figures 15, 16 and 17).

An infinitesimal increment of beam column, dz , is considered and the three major differential equations are derived with springs included. Consider the element dz in Figure 15, and take moment about Point A. The following equation results:

$$M_x - (M_x + dM_x) + t_x(dz) + V_y(dz) + R_x dz \left(\frac{dy'}{dz} \right) + P(dy) + \quad (5.1)$$

$$q_y \left(\frac{dz^2}{2} \right) - K_y (y') \frac{dz^2}{2} = 0$$

Neglecting higher order differentials and dividing Eq. (5.1) by dz gives:

$$\frac{dM_x}{dz} = V_y + t_x + R_x \frac{dy'}{dz} + P \frac{dy}{dz} \quad (5.2)$$

Figure 15 Infinitesimal beam column increment in the yz plane with springs attached

in which y and y' are respectively the displacements of the center of gravity and point N where springs are attached. y and y' are equal to:

$$\begin{aligned} y &= v - x_o \phi \\ y' &= v - (x_o - h_x) \phi \end{aligned} \quad (5.3)$$

See Appendix D. Taking the derivative of Eq. (5.2) with respect to z and substituting y and y' gives:

$$\frac{d^2 M_x}{dz^2} = \frac{dV}{dz} y + \frac{d}{dz} T_x + R_x \left[\frac{dv}{dz} - (x_o - h_x) \frac{d\phi}{dz} \right] + P \left(\frac{dv}{dz} - x_o \frac{d\phi}{dz} \right) \quad (5.4)$$

Now consider the equilibrium of the forces in the y direction:

$$V_y - (V_y + dV_y) + q_y dz - K_y y' dz = 0 \quad (5.5A)$$

or

$$\frac{dV_y}{dz} = q_y - K_y [v - (x_o - h_x) \phi] \quad (5.5B)$$

replacing $\frac{dV_y}{dz}$ in Eq. (5.4) gives:

$$\begin{aligned} \frac{d^2 M_x}{dz^2} &= q_y - K_y [v - (x_o - h_x) \phi] + \frac{d}{dz} T_x + R_x \left[\frac{dv}{dz} - (x_o - h_x) \frac{d\phi}{dz} \right] \\ &+ P \left(\frac{dv}{dz} - x_o \frac{d\phi}{dz} \right) \end{aligned} \quad (5.6)$$

Eq. (5.6) is the differential equation in the yz plane. In a similar manner, the differential equation for the xz plane (Figure 16) is derived. The result is:

Figure 16 Infinitesimal beam column increment in the xz plane with springs attached

$$\begin{aligned} \frac{d^2 M}{dz^2} = & q_x - K_x [u + (y_o - h_y)\theta] + \frac{d}{dz} t_y + R_y \left[\frac{du}{dz} + (y_o - h_y) \frac{d\theta}{dz} \right] \\ & + P \left(\frac{du}{dz} + y_o \frac{d\theta}{dz} \right) \end{aligned} \quad (5.7)$$

It should be noted that the displacements of the center of gravity and point N in the x direction respectively are:

$$\begin{aligned} x &= u + y_o \theta \\ x' &= u + (y_o - h_y) \theta \end{aligned} \quad (5.8)$$

(See Appendix D)

The third equation is derived by taking the moment about the z axis, Figure 17. The two coil springs and the spiral spring in the xy plane, R_z , produce additional moments to the ones shown in Equation (3.25A). Therefore, the following equation gives:

$$\begin{aligned} \frac{dM_z}{dz} + m_z + q_y x_o - q_x y_o - R_z \theta - K_y [v - (x_o - h_x) \theta] \cdot (x_o - h_x) + \\ K_x [u + (y_o - h_y) \theta] \cdot (y_o - h_y) = 0 \end{aligned} \quad (5.9)$$

in which $K_y [v - (x_o - h_x) \theta]$ and $K_x [u + (y_o - h_y) \theta]$ are the forces in the coil springs, and $(x_o - h_x)$ and $(y_o - h_y)$ are the lever arms. M_z and m_z are the same as in Chapter III, Computer Model.

It should be emphasized that the unit of the spring constants K_x and K_y are in force per length per unit length of the beam column increment and the unit of spring constants R_x , R_y and R_z are in force per radian per unit length of the increment dz.

The three equations (5.6), (5.7) and (5.9) are the differential equations of the beam column with springs attached.

Figure 17 Cross section of the beam column with springs attached (xy plane)

Q - DELTA EFFECT

A member that is loaded transversely and bent about its major axis may buckle laterally if the compression flange is not laterally supported. The compression flange, which acts like a column on an elastic foundation, becomes unstable and tends to bend sideways at the critical Q load. As a result of this a torque is produced about the z axis due to $Q(-u_1+u)$ (Figure 18a). The quantity u_1 is the deflection of the centroid at the free end of the beam as shown in Figure 18b and u is the deflection at any cross section mn as in Figure 18c. Modifying the presented analysis technique to include Q - Δ effect would allow the calculation of the lateral torsional buckling component of deflection.

Figure 18 a) Q- Δ effect in a cantilever beam with concentrated load at the free end. b) Displacement of the cross section. c) Top view of the deflected beam.

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APPENDIX A

FLOW DIAGRAM - SUBROUTINE MAIN1

*Names of FORTRAN SUBROUTINES are indicated by ().

FLOW DIAGRAM - SUBROUTINE ZEROER

FLOW DIAGRAM - SUBROUTINE INPUT

FLOW DIAGRAM - SUBROUTINE DATASV

FLOW DIAGRAM - SUBROUTINE STUFF2

*II and I are computer stations

FLOW DIAGRAM - SUBROUTINE CONST

FLOW DIAGRAM - SUBROUTINE CALABC

FLOW DIAGRAM - SUBROUTINE CALW

FLOW DIAGRAM - PRINTW

MATRIX MANIPULATING SUBROUTINES

<u>SUBROUTINE</u>	<u>FUNCTION</u>
PLUCK1.....	Isolates a 1x3 matrix from a 3xn array
PLUCK3.....	Isolates a 3x3 matrix from a 3xn array
MCHSG1.....	Changes sign of a 1x3 matrix
MCHSG3.....	Changes sign of a 3x3 matrix
MADD1.....	Adds two 1x3 matrices
MADD3.....	Adds two 3x3 matrices
MMULT1.....	Multiplies a 3x3 matrix by a 1x3 matrix
MMULT3.....	Multiplies two 3x3 matrices
MPUT1.....	Puts a 1x3 matrix in a 3xn array
MPUT3.....	Puts a 3x3 matrix in a 3xn array
MINV3.....	Inverts a 3x3 matrix

APPENDIX B

BEAM COLUMN PROGRAM

The beam column is divided into m equal size increments. The number of a particular increment is referred to as the station number of the increment.

The beam column program can handle transverse loads in the x and y direction, applied moments about the x and y axes and applied axial loads as shown in Figure 19. The loads and moments shown in Figure 19 are positive. The axial load will be positive if it is tension force.

In the program, loads and external moments are treated as concentrated loads and moments. Therefore, the contribution of the loads and moments of each increment to its adjacent stations are input into the program. The data input is presented in the following pages.

Figure 19 General beam column with positive loads and moments applied

DATA INPUT

Note: Numbers at left indicate card columns.

A. Title of Problem

FORMAT (80H)

1-80 Problem Title

B. Control Data

FORMAT (2I5,7E10.3)

1-5 Number of beam column increments (M).

11-20 Increment length (H).

C. Data Added through Specified Intervals

I. FORMAT (2I5,7E10.3)

1-5 Station

5-10 Through

11-20 Flexural stiffness about x axis (EI_x)

21-30 Flexural stiffness about y axis (EI_y)

31-40 Area of the cross section (A)

41-50 Torsional stiffness (GJ)

51-60 Warping stiffness ($K = EC_w$)

61-70 Modulus of Elasticity (E)

II. FORMAT (2I5,7E10.3)

11-20 Distance from shear center to center of gravity in the x direction (x_0)

21-30 Distance from shear center to center of gravity in the y direction (y_0)

31-40* Distance from center of gravity to Point N in the x direction (H_x , See Figure 17)

41-50* Distance from center of gravity to Point N in the y direction (H_y , See Figure 17)

51-60 Axial load through the center of gravity (P).

III. FORMAT (2I5,7E10.3)

11-20 Transverse load in the x direction through center of gravity (Q_x)

21-30 Transverse load in the y direction through center of gravity (Q_y)

31-40 Applied moment about x axis (M_x)

41-50 Applied moment about y axis (M_y)

51-60* Stiffness of transverse spring in the x direction (K_x)

61-70* Stiffness of transverse spring in the y direction (K_y)

71-80* Stiffness of rotational spring about z axis

D. Control Card

FORMAT (2I5,7E10.7)

1-5 N, greater than M, indicates the end of the data for the sectional properties of the beam column.

E. Specified Deflections

FORMAT (I5,4x,A1,E10.3)

1-5 Station

10 Direction of deflection (x, y or R)

11-20 Specified deflection

F. Control Card

1-5 N, greater than M, indicates the end of the data for the specified deflections.

APPENDIX C

COMPUTER PROGRAM DOCUMENTATION

```

1  C SUBROUTINE MAIN1 FOR EXAMPLE #
2  C *****
3  DOUBLE PRECISION AA,BB,CC,DD,EE,GG,AA,B,C,W
4  COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
5  1  ,EE(3,3,107),H(3,3,107),C(3,3,107)
6  COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
7  DIMENSION SPED(10),ISPE(10),IDIR(10),VV(107,18)
8  INTEGER XXX,YYY,RRR
9  DATA XXX/'X'/,YYY/'Y'/,RRR/'R'/
10 CALL ZEROER(VV)
11 CALL INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
12 CALL STUFF2(VV,M,H)
13 C *****
14 C *****
15 MP5=M+5
16 DO 120 I=3,MP5
17 II=I
18 CALL CALABC(II)
19 DO 110 J=1,NOSP
20 IF(ISPE(J).NE.I) GO TO 110
21 IF(IDIR(J).EQ.XXX) K=1
22 IF(IDIR(J).EQ.YYY) K=2
23 IF(IDIR(J).EQ.RRR) K=3
24 A(K,II)=SPED(J)
25 LX=3
26 DO 100 L=1,LX
27 B(K,L,II)=0.
28 C(K,L,II)=0.
29 100 CONTINUE
30 110 CONTINUE
31 120 CONTINUE
32 C *** CALCULATE W(I)
33 DO 200 I=3,MP5
34 J=M+8-I
35 CALL CALW(J)
36 200 CONTINUE

```

```

37 CALL PRINTW(W,M)
38 CALL CALMPHI(VV,M,W,H)
39 STOP
40 END

SUBROUTINE INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
DIMENSION V(18),SPED(10),ISPE(10),IDIR(10),VV(107,18)
100 FORMAT(80H
1      1
110 FORMAT(2I5,7E10.3)
115 FORMAT(10X,7E10.3)
120 FORMAT(1H1)
130 FORMAT(//,"      "      NUMBER OF INCR M =" ,I5,/,
1      1      INCREMENT LGTH H =" ,F10.3)
140 FORMAT(" STA THRU",4X,"EIX      EIX",7X,"A",9X,"GJ",8X,
1      1"K",8X,"E")
145 FORMAT(14X,"XC",8X,"YQ",8X,"HX",8X,"HY",8X,"P")
150 FORMAT(14X,"ZX",8X,"QY",8X,"TX",8X,"TY",8X,"KX",
1      18X,"KY",8X,"R")
160 FORMAT(//," SPECIFIED DEFLECTIONS",/, " STA DIR MAGNITUDE")
170 FORMAT(15,4X,A1,E10.3)
WRITE(18,120)
READ(12,100)
WRITE(18,100)
READ(12,110)M,10U,M,H

```



```

21 WRITE(18,130)M,H
22 WRITE(18,140)
23 WRITE(18,145)
24 WRITE(18,150)
25 200 READ(12,110)I1,I2,(V(I),I=1,6)
26 IF(I1.GT.M) GO TO 210
27 READ(12,110)IDUM,IDUM1,(V(I),I=7,11)
28 READ(12,110) IDUM,IDUM1,(V(I),I=12,18)
29 CALL DATASV(VV,V,I1,I2)
30 WRITE(18,110) I1,I2,(V(I),I=1,6)
31 WRITE(18,115) (V(I),I=7,11)
32 WRITE(18,115) (V(I),I=12,18)
33 GO TO 200
34 210 CONTINUE
35 WRITE(18,160)
36 NOSP=0
37 220 NOSP=NOSP+1
38 READ(12,170) I1,IDIR(NOSP),SPED(NOSP)
39 IF(I1.GT.M) GO TO 230
40 WRITE(18,170) I1,IDIR(NOSP),SPED(NOSP)
41 ISPE(NOSP)=I1+4
42 GO TO 220
43 230 CONTINUE
44 NOSP=NOSP-1
45 RETURN
46 END

```

```

1  SUBROUTINE DATASV(VV,V,I1,I2)
2  DIMENSION VV(107,18),V(18)
3  DO 1012 I=1+4,I2+4
4  DO 1011 J=1,18
5  VV(I,J)=VV(I,J)+V(J)
6
7  1011 CONTINUE
8  1012 CONTINUE
9  RETURN
END

```

```

1  SUBROUTINE ZEROER(VV)
2  DOUBLE PRECISION AA,BB,CC,DD,EE,GG,A,B,C,W
3  COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
4  & ,EE(3,3,107),B(3,3,107),C(3,3,107)
5  COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
6  DIMENSION VV(107,18)
7  DO 1005 I=1,107
8  DO 1002 J=1,18
9  1002 VV(I,J)=0
10 DO 1004 K=1,3
11 DO 1003 M=1,3
12 AA(K,M,I)=C
13 BB(K,M,I)=C

```

```

14 CC(K,M,I)=0
15 DD(K,M,I)=0
16 EE(K,M,I)=0
17 B(K,M,I)=0
18 1003 C(K,M,I)=0
19 A(K,I)=0
20 W(K,I)=0
21 1004 GG(K,I)=0
22 1005 CONTINUE
23 RETURN
24 END

```

```

1 SUBROUTINE CONST(QX,QY,PX,PY,FX,FY,PR,PH,TX,TY,CA,KA,UX,UY,QXX,QYY
2 ,VV,II,J,H)
3 DIMENSION QX(3),QY(3),PX(3),PY(3),FX(3),FY(3),PH(3),TX(3),TY(3)
4 DIMENSION CA(3),KA(3),UX(3),UY(3),QXX(3),QYY(3),PR(3)
5 DIMENSION VV(107,18)
6 QX(J)=(H**3.)*VV(II,12)+VV(II,8)
7 QY(J)=(H**3.)*VV(II,13)+VV(II,7)
8 PX(J)=(H**2.)*VV(II,11)+VV(II,7)
9 PY(J)=(H**2.)*VV(II,11)+VV(II,8)
10 FX(J)=VV(II,1)
11 FY(J)=VV(II,2)
12 IF(VV(II,6).EQ.0.) GO TO 15
13 C *****SIO IS THE POLAR MOMENT OF INERTIA ABOUT SHEAR CENTER

```

```

14      SIO=((FX(J)+FY(J))/VV(II,6))+VV(II,3)*((VV(II,7))*2+(VV(II,8))
15      & **2)
16      GO TO 20
17      PR(J)=0
18
19      GO TO 25
20      PR(J)=(H**2.)*VV(II,11)*SIO/(VV(II,3))
21      PH(J)=(H**2.)*VV(II,11)/4.
22      TX(J)=(H**2.)*VV(II,14)/2.
23      TY(J)=(H**2.)*VV(II,15)/2.
24      CA(J)=(H**2.)*VV(II,4)/2.
25      KA(J)=VV(II,5)/2.
26      QXX(J)=(H**3.)*VV(II,12)
27      QYY(J)=(H**3.)*VV(II,13)
28      UX(J)=VV(II,7)
29      UY(J)=VV(II,8)
30      RETURN
      END

```

```

1      SUBROUTINE STUFF2(VV,M,H)
2      DOUBLE PRECISION AA,BB,CC,DD,EE,GG,A,R,C,W
3      COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
4      & ,EE(3,3,107),B(3,3,107),C(3,3,107)
5      COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
6      DIMENSION QX(3),QY(3),PX(3),PY(3),FX(3),FY(3),PH(3),TX(3),TY(3)
7      DIMENSION CA(3),KA(3),UX(3),UY(3),QXX(3),QYY(3),PR(3)

```

```

8  DIMENSION VV(107,18)
9  II=1
10 DO 1010 J=1,3
11 II=II+1
12 JJ=J
13 CALL CONST(QX,QY,PX,PY,FX,FY,PR,PH,TX,TY,CA,KA,UX,UY,QXX,QYY,VV,II
14 ,JJ,H)
15 1010 CONTINUE
16 I=3
17 1015 IM2=I-2
18 IP2=I+2
19 AA(1,1,I)=FY(1)-PH(1)
20 AA(1,3,I)=FY(1)*VV(IM2,8)-PH(1)*UY(1)
21 AA(1,3,I)=-PH(1)*UY(1)
22 AA(2,2,I)=FX(1)-PH(1)
23 AA(2,3,I)=- (FX(1)*VV(IM2,7)-PH(1)*UX(1))
24 AA(2,3,I)=PH(1)*UX(1)
25 AA(3,3,I)=KA(1)+KA(2)
26 BB(1,1,I)=-2.*(FY(1)+FY(2))
27 BB(1,3,I)=-2.*UY(1)*(FY(1)+FY(2))
28 BB(1,3,I)=0.0
29 BB(2,2,I)=-2.*(FX(1)+FX(2))
30 BB(2,3,I)=2.*UX(1)*(FX(1)+FX(2))
31 BB(2,3,I)=0.0
32 BB(3,1,I)=-PY(2)
33 BB(3,2,I)=PX(2)
34 BB(3,3,I)=- (KA(3)+4.*KA(2)+CA(2)+CA(1)+3.*KA(1)+PR(2))
35 CC(1,1,I)=FY(1)+4.*FY(2)+FY(3)+PH(1)+PH(3)
36 CC(1,3,I)=UY(2)*FY(1)+4.*UY(2)*FY(2)+UY(2)*FY(3)+PH(3)*UY(3)+PH(1)
37 *UY(1)
38 CC(1,3,I)=PH(3)*UY(3)+PH(1)*UY(1)
39 CC(2,2,I)=FX(1)+4.*FX(2)+FX(3)+PH(1)+PH(3)
40 CC(2,3,I)=- (UX(2)*FX(1)+4.*UX(2)*FX(2)+UX(2)*FX(3)+PH(1)*UX(1)+
41 PH(3)*UX(3))
42 CC(2,3,I)=-PH(1)*UX(1)-PH(3)*UX(3)
43 CC(3,1,I)=2.*PY(2)
44 CC(3,2,I)=-2.*PX(2)

```

```

45 CC(3,3,I)=CA(3)+2.*CA(2)+CA(1)+3.*KA(3)+6.*KA(2)+3.*KA(1)+2.*PR(2)
46 DD(1,1,I)=-2.*(FY(2)+FY(3))
47 DD(1,3,I)=-2.*UY(3)*(FY(2)+FY(3))
48 DD(1,3,I)=0.0
49 DD(2,2,I)=-2.*(FX(2)+FX(3))
50 DD(2,3,I)=2.*UX(3)*(FX(2)+FX(3))
51 DD(2,3,I)=0.0
52 DD(3,1,I)=-PY(2)
53 DD(3,2,I)=PX(2)
54 DD(3,3,I)=- (CA(3)+CA(2)+3.*KA(3)+4.*KA(2)+KA(1)+PR(2))
55 EE(1,1,I)=FY(3)-PH(3)
56 EE(1,3,I)=FY(3)*VV(IP2,8)-PH(3)*UY(3)
57 EE(1,3,I)=-PH(3)*UY(3)
58 EE(2,2,I)=FX(3)-PH(3)
59 EE(2,3,I)=-FX(3)*VV(IP2,7)+PH(3)*UX(3)
60 EE(2,3,I)=PH(3)*UX(3)
61 EE(3,3,I)=KA(2)+KA(3)
62 GG(1,I)=QXX(2)+TY(1)-TY(3)
63 GG(2,I)=QYY(2)-TX(1)+TX(3)
64 GG(3,I)=-QX(2)+QY(2)
65 IF(I.EQ.(M+5)) GO TO 1025
66 DO 1020 K=1,2
67 QX(K)=QX(K+1)
68 QY(K)=QY(K+1)
69 PX(K)=PX(K+1)
70 PY(K)=PY(K+1)
71 FX(K)=FX(K+1)
72 FY(K)=FY(K+1)
73 PR(K)=PR(K+1)
74 PH(K)=PH(K+1)
75 TX(K)=TX(K+1)
76 TY(K)=TY(K+1)
77 CA(K)=CA(K+1)
78 KA(K)=KA(K+1)
79 UX(K)=UX(K+1)
80 UY(K)=UY(K+1)

```

```

81 QXX(K)=QXX(K+1)
82 QYY(K)=QYY(K+1)
83 II=II+1
84 CALL CONST(QX,QY,PX,PY,FX,FY,PR,PH,TX,TY,CA,KA,UX,UY,QXX,QYY,VV,II
85      ,3,H)
86 I=I+1
87 GO TO 1015
88 1025 RETURN
89 END

```

```

1 SUBROUTINE CALABC(I)
2 DIMENSION EI(3,3),TAA(3,3),DI(3,3),DII(3,3)
3 DIMENSION T31(3,3),T32(3,3),T33(3,3),T11(3),T12(3),T13(3)
4 IM1=I-1
5 IM2=I-2
6 C *** CODE AA=1 BB=2 CC=3 DD=4 EE=5 -- B=6 C=7 ----- GG=1 A=2 W=3
7 C
8 C *** CALCULATE E(I)=AA*B(I-2)+BB
9 CALL PLUCK3(1,I,TAA)
10 CALL PLUCK3(6,IM2,T31)
11 CALL MMULT3(T32,TAA,T31)
12 CALL PLUCK3(2,I,T31)
13 CALL MADD3(EI,T32,T31)
14 C *** CALCULATE D(I)=E(I)*B(I-1)+AA*C(I-2)+CC
15 CALL PLUCK3(6,IM1,T31)

```

```

16 CALL MMULT3(T32,EI,T31)
17 CALL PLUCK3(7,IM2,T31)
18 CALL MMULT3(T33,TAA,T31)
19 CALL MADD3(T31,T32,T33)
20 CALL PLUCK3(3,I,T32)
21 CALL MADD3(DI,T31,T32)
22 WRITE(18,100)I,((DI(JX,KX),KX=1,3),JX=1,3)
23 C 100 FORMAT(15,3E10.3,/,5X,3E10.3,/,5X,3E10.3)
24 C *** CALCULATE C(I)=-DI**1*EE
25 CALL MINV3(DII,DI)
26 CALL PLUCK3(5,I,T31)
27 CALL MMULT3(T32,DII,T31)
28 CALL MCHSG3(T32)
29 CALL MPUT3(7,I,T32)
30 C *** CALCULATE B(I)=-DI**1(E(I)*C(I-1)+DD)
31 CALL PLUCK3(7,IM1,T31)
32 CALL MMULT3(T32,EI,T31)
33 CALL PLUCK3(4,I,T31)
34 CALL MADD3(T33,T32,T31)
35 CALL MMULT3(T31,DII,T33)
36 CALL MCHSG3(T31)
37 CALL MPUT3(6,I,T31)
38 C *** CALCULATE A(I)=-DI**1(AA*A(I-2)+E(I)*A(I-1))-GG)
39 CALL PLUCK1(2,IM2,T11)
40 CALL MMULT1(T12,TAA,T11)
41 CALL PLUCK1(2,IM1,T11)
42 CALL MMULT1(T13,EI,T11)
43 CALL MADD1(T11,T12,T13)
44 CALL PLUCK1(1,I,T12)
45 CALL MCHSG1(T12)
46 CALL MADD1(T13,T11,T12)
47 CALL MMULT1(T11,DII,T13)
48 CALL MCHSG1(T11)
49 CALL MPUT1(2,I,T11)
50 RETURN
51 END

```



```

1  SURROUTINE CALW(I)
2  DOUBLE PRECISION AA,BB,CC,DD,EE,GG,A,B,C,W
3  COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
4  1  ,EE(3,3,107),B(3,3,107),C(3,3,107)
5  COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
6  DIMENSION T11(3),T12(3),T13(3)
7  DIMENSION T31(3,3)
8
9  C *** CODE   A=2 B=6 C=7 W=3
10
11 C *** CALCULATE W(I)=A(I)+B(I)*W(I+1)+C(I)*W(I+2)
12   IP1=I+1
13   IP2=I+2
14   CALL PLUCK1(2,I,T11)
15   CALL PLUCK3(6,I,T31)
16   CALL PLUCK1(3,IP1,T12)
17   CALL MMULT1(T13,T31,T12)
18   CALL MADD1(T12,T11,T13)
19   CALL PLUCK3(7,I,T31)
20   CALL PLUCK1(3,IP2,T11)
21   CALL MMULT1(T13,T31,T11)
22   CALL MADD1(T11,T12,T13)
23   CALL MPUT1(3,I,T11)
24   RETURN
25   END

```

```

1  SUBROUTINE PLUCK1(N,M,R)
2  DOUBLE PRECISION GG,A,W
3  COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
4  DIMENSION R(3)
5  GO TO (10,20,30),N
6  DO 15 K=1,3
7  R(K)=GG(K,M)
8  GO TO 40
9  DO 25 K=1,3
10 R(K)=A(K,M)
11 GO TO 40
12 DO 35 K=1,3
13 R(K)=W(K,M)
14 RETURN
15 END

```

```

1  SUBROUTINE PLUCK3(N,M,R)
2  DOUBLE PRECISION AA,BB,CC,DD,EE,B,C
3  COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107),
4  &EE(3,3,107),R(3,3,107),C(3,3,107)
5  DIMENSION R(3,3)
6  GO TO (10,20,30,40,50,60,70),N
7  DO 15 K=1,3
8  DO 12 L=1,3
9  R(K,L)=AA(K,L,M)
10 CONTINUE

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      GO TO 80
20  DO 25 K=1,3
    DO 22 L=1,3
22  R(K,L)=BB(K,L,M)
25  CONTINUE
    GO TO 80
30  DO 35 K=1,3
    DO 32 L=1,3
32  R(K,L)=CC(K,L,M)
35  CONTINUE
    GO TO 80
40  DO 45 K=1,3
    DO 42 L=1,3
42  R(K,L)=DD(K,L,M)
45  CONTINUE
    GO TO 80
50  DO 55 K=1,3
    DO 52 L=1,3
52  R(K,L)=EE(K,L,M)
55  CONTINUE
    GO TO 80
60  DO 65 K=1,3
    DO 62 L=1,3
62  R(K,L)=B(K,L,M)
65  CONTINUE
    GO TO 80
70  DO 75 K=1,3
    DO 72 L=1,3
72  R(K,L)=C(K,L,M)
75  CONTINUE
80  RETURN
      END

```

```
1  SUBROUTINE MCHSG1(T1)  
2  DIMENSION T1(3)  
3  DO 100 I=1,3  
4  T1(I)=T1(I)*(-1.0)  
5  RETURN  
6  END
```

```
1  SUBROUTINE MCHSG3(T1)  
2  DIMENSION T1(3,3)  
3  DO 100 I=1,3  
4  DO 110 J=1,3  
5  T1(I,J)=T1(I,J)*(-1)  
6  110 CONTINUE  
7  100 CONTINUE  
8  RETURN  
9  END
```

```
1 SUBROUTINE MADD1(T1,T2,T3)
2 DIMENSION T1(3),T2(3),T3(3)
3 DO 100 I=1,3
4   T1(I)=T2(I)+T3(I)
5 RETURN
6 END
```

```
1 SUBROUTINE MADD3(T1,T2,T3)
2 DIMENSION T1(3,3),T2(3,3),T3(3,3)
3 DO 100 I=1,3
4   DO 110 J=1,3
5     T1(I,J)=T2(I,J)+T3(I,J)
6   CONTINUE
7   100 CONTINUE
8   RETURN
9   END
```

```

1  SUBROUTINE MMULT1 (T1,T2,T3)
2  DIMENSION T1(3),T2(3,3),T3(3)
3  DO 100 I=1,3
4  T1(I)=0.0
5  DO 110 J=1,3
6  T1(I)=T2(I,J)*T3(J)+T1(I)
7
8  110 CONTINUE
9  100 CONTINUE
10 RETURN
    END

```

```

1  SUBROUTINE MMULT3(T1,T2,T3)
2  DIMENSION T1(3,3),T2(3,3),T3(3,3)
3  DO 100 I=1,3
4  DO 110 J=1,3
5  T1(I,J)=T2(I,1)*T3(1,J)+T2(I,2)*T3(2,J)+T2(I,3)*T3(3,J)
6
7  110 CONTINUE
8  100 CONTINUE
9  RETURN
    END

```

```

1  SUBROUTINE MPUT1(N,M,R)
2  DOUBLE PRECISION GG,A,W
3  COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
4  DIMENSION R(3)
5  GO TO (10,20,30),N
6
7  DO 15 K=1,3
8  GG(K,M)=R(K)
9  GO TO 40
10 DO 25 K=1,3
11 A(K,M)=R(K)
12 GO TO 40
13 DO 35 K=1,3
14 W(K,M)=R(K)
15 RETURN
16 END

```

```

1  SUBROUTINE MPUT3(N,M,R)
2  DOUBLE PRECISION AA,HH,CC,DD,EE,B,C
3  COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
4  &EE(3,3,107),B(3,3,107),C(3,3,107)
5  DIMENSION R(3,3)
6  NN=N-5
7  GO TO (60,70),NN
8  DO 65 K=1,3
9  DO 62 L=1,3

```

```

10 62 B(K,L,M)=R(K,L)
11 65 CONTINUE
12 GO TO 80
13 70 DO 75 K=1,3
14 72 DO 72 L=1,3
15 75 C(K,L,M)=R(K,L)
16 75 CONTINUE
17 80 RETURN
18 END

1 SUBROUTINE MINV3(XINV3,R)
2 DIMENSION R(3,3),C(2,2),XINV3(3,3)
3 DETR=R(1,1)*R(2,2)*R(3,3)+R(1,2)*R(2,3)*R(3,1)+R(1,3)*
4 1*R(2,1)*R(3,2)-R(1,3)*R(2,2)*R(3,1)-R(1,1)*R(2,3)*R(3,2)
5 2-R(1,2)*R(3,3)*R(2,1)
6 DO30I=1,3
7 DO30J=1,3
8 K=1
9 L=1
10 DO10M=1,3
11 DO10N=1,3
12 IF(M.EQ.1)GOTO10
13 IF(N.EQ.1)GOTO10
14 C(K,L)=R(M,N)
15 IF(L.EQ.2)GOTO20
16 L=L+1

```



```

17      GOTO 10
18      K=K+1
19      L=1
20      CONTINUE
21      XINV3(J,I)=((-1)**(I+J))*(C(1,1)*C(2,2)-C(1,2)*C(2,1))/DETR
22      CONTINUE
23      RETURN
24      END

```

```

1      SUBROUTINE PRINTW(W,M)
2      DOUBLE PRECISION W
3      DIMENSION W(3,107)
4      WRITE(18,5)
5      DO 10 LL=1,M+7
6          JJ=LL-4
7          WRITE(18,7)JJ,W(1,LL),W(2,LL),W(3,LL)
8      CONTINUE
9      10 5  FORMAT("STATION NO.      X DISPLACEMENT      Y DISPLACEMENT      ROTATIO
10          &N")
11      7  FORMAT(5X,I5,2F18.6,F18.8)
12      RETURN
13      END

```


C:
O:
C:

0.005250
O:
O:

-0.005250
O:
O:

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11

EXAMPLE #2 : "SIMPLE BEAM WITH THREE CONCENTRATED LOADS"

NUMBER OF INCR		M =	8	INCREMENT LGTH		H =	20.000	STA THRU		EIX	EIY	A	GJ	K	E	R
										XO	YO	HX	HY	P	KY	
										QX	QY	TX	TY	KX	05	
0	8	0.300E 08	0.100E 08	0.200E 02	0.400E 04	0.200E 02	0.400E 04	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05
2	2	0.300E 08	0.100E 08	0.200E 02	0.400E 04	0.200E 02	0.400E 04	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05
4	4	0.300E 08	0.100E 08	0.200E 02	0.400E 04	0.200E 02	0.400E 04	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05
6	6	0.300E 08	0.100E 08	0.200E 02	0.400E 04	0.200E 02	0.400E 04	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05
8	8	0.300E 08	0.100E 08	0.200E 02	0.400E 04	0.200E 02	0.400E 04	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05

SPECIFIED DEFLECTIONS

STA	DIR	MAGNITUDE	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
0	X	0.	0.	0.	0.
0	Y	0.	0.	0.	0.
8	X	0.	0.	0.	0.
8	Y	0.	0.	0.	0.
STATION NO.					
-3					
-2					
-1					
0					
1					
2					
3					

0.
0.
0.
0.
0.
0.
0.
0.

0.138667
0.128000
0.098667
0.053333
0.
-0.053333
0.
0.

0.
0.
0.
0.
0.
0.
0.
0.

4
5
6
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0.
0.
0.
0.

0.
0.
0.
0.

0.05245034
0.05245034
0.
0.

```

C
C
SUBROUTINE MAIN1 FOR EXAMPLE #3
*****
DOUBLE PRECISION AA,BB,CC,DD,EE,GG,AA,BB,CC,W
COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
1      ,EE(3,3,107),BB(3,3,107),CC(3,3,107)
COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
DIMENSION SPED(10),ISPE(10),IDIR(10),VV(107,18)
INTEGER XXX,YYY,RRR
DATA XXX/'X'/,YYY/'Y'/,RRR/'R'/
CALL ZEROER(VV)
CALL INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
CALL STUFF2(VV,M,H)
*****
GG(3,13)=GG(3,13)+414720
GG(3,18)=GG(3,18)-103680
*****
MP5=M+5
DO 120 I=3,MP5
  II=I
  CALL CALABC(II)
  DO 110 J=1,NOSP
    IF(ISPE(J).NE.I) GO TO 110
    IF(IDIR(J).EQ.XXX) K=1
    IF(IDIR(J).EQ.YYY) K=2
    IF(IDIR(J).EQ.RRR) K=3
    A(K,II)=SPED(J)
    LX=3
    DO 100 L=1,LX
      B(K,L,II)=0.
      C(K,L,II)=0.
100 CONTINUE
110 CONTINUE
120 CONTINUE
C *** CALCULATE W(I)
DO 200 I=3,MP5
  J=M+8-I

```



```
CALL CALW(J)  
200 CONTINUE  
CALL PRINTW(W,M)  
CALL CALMPHI(VV,M,W,H)  
STOP  
END
```

EXAMPLE #4 : "FLEXURAL BUCKLING ABOUT X AXIS" P=3751.

NUMBER OF INCR M = 80		INCREMENT LGTH H = 2.000													
STA	THRU	EIX	EIY	A	HX	TX	GJ	K	P	KX	KY	E	R		
		XO	YO				HY								
		QX	QY				TY								
0	80	0.100E 08	0.300E 08	0.200E 02	0.400E 04	0.	0.	-0.375E 04	0.	0.	0.300E 05	0.	0.		
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.		
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.		
40	40	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.		
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.		
		0.	0.100E 00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.		

SPECIFIED DEFLECTIONS
STA DIR MAGNITUDE

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.	0.	0.
-2	0.	0.	0.
-1	0.	-0.066064	0.
0	0.	0.	0.
1	0.	0.066014	0.
2	0.	0.131879	0.
3	0.	0.197497	0.
4	0.	0.262768	0.
5	0.	0.327595	0.
6	0.	0.391880	0.
7	0.	0.455527	0.

8	0.	0.518439	0.
9	0.	0.580523	0.
10	0.	0.641685	0.
11	0.	0.701833	0.
12	0.	0.760877	0.
13	0.	0.818728	0.
14	0.	0.875290	0.
15	0.	0.930505	0.
16	0.	0.984263	0.
17	0.	1.036492	0.
18	0.	1.087113	0.
19	0.	1.136050	0.
20	0.	1.183230	0.
21	0.	1.228582	0.
22	0.	1.272037	0.
23	0.	1.313530	0.
24	0.	1.352999	0.
25	0.	1.390384	0.
26	0.	1.425629	0.
27	0.	1.458681	0.
28	0.	1.489490	0.
29	0.	1.518011	0.
30	0.	1.544199	0.
31	0.	1.568015	0.
32	0.	1.589424	0.
33	0.	1.608393	0.
34	0.	1.624893	0.
35	0.	1.638900	0.
36	0.	1.650392	0.
37	0.	1.659352	0.
38	0.	1.665766	0.
39	0.	1.669624	0.
40	0.	1.670921	0.
41	0.	1.669654	0.
42	0.	1.665825	0.
43	0.	1.559442	0.

44	1.650512	0.
45	1.639049	0.
46	1.625071	0.
47	1.608598	0.
48	1.589656	0.
49	1.568272	0.
50	1.544479	0.
51	1.518312	0.
52	1.489812	0.
53	1.459020	0.
54	1.425983	0.
55	1.390751	0.
56	1.353376	0.
57	1.313915	0.
58	1.272427	0.
59	1.228974	0.
60	1.183622	0.
61	1.136440	0.
62	1.087497	0.
63	1.036869	0.
64	0.984630	0.
65	0.930861	0.
66	0.875641	0.
67	0.819055	0.
68	0.761187	0.
69	0.702124	0.
70	0.641955	0.
71	0.580770	0.
72	0.518662	0.
73	0.455725	0.
74	0.392052	0.
75	0.327740	0.
76	0.262885	0.
77	0.197585	0.
78	0.131938	0.
79	0.066044	0.

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-0.066093
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EXAMPLE #4 : "FLEXURAL BUCKLING ABOUT X AXIS" $P=3751.5$

NUMBER OF INCR M = 80								
INCREMENT LGTH H = 2.000								
STA THRU	EIX XO QX	EIY YO QY	A HX TX	GJ HY TY	K P KX			
0	80	0.100E 08	0.300E 08	0.200E 02	0.400E 04	0.375E 04	0.300E 05	
		0.	0.	0.	0.	-0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.
40	40	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.
		C.	0.100E 00	0.	0.	0.	0.	0.

SPECIFIED DEFLECTIONS	
STA	DIR MAGNITUDE
1	1
2	2
3	3
4	4
5	5
6	6
7	7
8	8
9	9
10	10
11	11
12	12
13	13
14	14
15	15
16	16
17	17
18	18
19	19
20	20
21	21
22	22
23	23
24	24
25	25
26	26
27	27
28	28
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30	30
31	31
32	32
33	33
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42	42
43	43
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47	47
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50	50
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52	52
53	53
54	54
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58	58
59	59
60	60
61	61
62	62
63	63
64	64
65	65
66	66
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71	71
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74	74
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78	78
79	79
80	80
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83	83
84	84
85	85
86	86
87	87
88	88
89	89
90	90
91	91
92	92
93	93
94	94
95	95
96	96
97	97
98	98
99	99
100	100

STATION	NO.	X	Y	R	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
0	-3	0.	0.	0.	0.	0.	0.
0	-2	0.	0.	0.	0.	0.	0.
0	-1	0.	0.	0.	-0.022423	0.	0.
80	0	0.	0.	0.	0.	0.022407	0.
80	1	0.	0.	0.	0.	0.044763	0.
80	2	0.	0.	0.	0.	0.067035	0.
	3	0.	0.	0.	0.	0.089189	0.
	4	0.	0.	0.	0.	0.111192	0.
	5	0.	0.	0.	0.	0.133012	0.
	6	0.	0.	0.	0.	0.154614	0.
	7	0.	0.	0.	0.	0.	0.

8	0.	0.175968	0.
9	0.	0.197039	0.
10	0.	0.217798	0.
11	0.	0.238212	0.
12	0.	0.258251	0.
13	0.	0.277885	0.
14	0.	0.297084	0.
15	0.	0.315820	0.
16	0.	0.334064	0.
17	0.	0.351788	0.
18	0.	0.368967	0.
19	0.	0.385574	0.
20	0.	0.401584	0.
21	0.	0.416973	0.
22	0.	0.431719	0.
23	0.	0.445798	0.
24	0.	0.459190	0.
25	0.	0.471874	0.
26	0.	0.483832	0.
27	0.	0.495046	0.
28	0.	0.505498	0.
29	0.	0.515173	0.
30	0.	0.524056	0.
31	0.	0.532135	0.
32	0.	0.539396	0.
33	0.	0.545829	0.
34	0.	0.551424	0.
35	0.	0.556173	0.
36	0.	0.560069	0.
37	0.	0.563105	0.
38	0.	0.565277	0.
39	0.	0.566582	0.
40	0.	0.567013	0.
41	0.	0.566583	0.
42	0.	0.565280	0.
43	0.	0.563109	0.

44	0.	0.560073	0.
45	0.	0.556179	0.
46	0.	0.551430	0.
47	0.	0.545835	0.
48	0.	0.539402	0.
49	0.	0.532141	0.
50	0.	0.524062	0.
51	0.	0.515178	0.
52	0.	0.505503	0.
53	0.	0.495050	0.
54	0.	0.483836	0.
55	0.	0.471877	0.
56	0.	0.459192	0.
57	0.	0.445800	0.
58	0.	0.431720	0.
59	0.	0.416975	0.
60	0.	0.401585	0.
61	0.	0.385575	0.
62	0.	0.368968	0.
63	0.	0.351789	0.
64	0.	0.334064	0.
65	0.	0.315821	0.
66	0.	0.297085	0.
67	0.	0.277886	0.
68	0.	0.258252	0.
69	0.	0.238213	0.
70	0.	0.217799	0.
71	0.	0.197040	0.
72	0.	0.175968	0.
73	0.	0.154615	0.
74	0.	0.133013	0.
75	0.	0.111193	0.
76	0.	0.089190	0.
77	0.	0.067035	0.
78	0.	0.044763	0.
79	0.	0.022407	0.

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-0.022424
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EXAMPLE #4 : "FLEXURAL BUCKLING ABOUT X AXIS" P=3751.7

NUMBER OF INCR M = 80		INCREMENT LGTH H = 2.000													
STA THRU	EIX	EIY	A	GJ	K	E	KY	R	P	KX	KY	05	04	03	02
	XO	YO	HX	HY											
	QX	QY	TX	TY											
0	80	0.100E 08	0.300E 08	0.200E 08	0.400E 04	0.0	-0.375E 04	0.300E 05	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
40	40	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
		0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0

SPECIFIED DEFLECTIONS

STA DIR MAGNITUDE

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.0	0.0	0.0
-2	0.0	0.0	0.0
-1	0.0	0.321575	0.0
0	0.0	0.0	0.0
1	0.0	-0.321334	0.0
2	0.0	-0.641943	0.0
3	0.0	-0.961346	0.0
4	0.0	-1.279061	0.0
5	0.0	-1.594612	0.0
6	0.0	-1.907524	0.0
7	0.0	-2.217325	0.0

8	0.	-2.523551	0.
9	0.	-2.825740	0.
10	0.	-3.123439	0.
11	0.	-3.416201	0.
12	0.	-3.703585	0.
13	0.	-3.985159	0.
14	0.	-4.260501	0.
15	0.	-4.529196	0.
16	0.	-4.790841	0.
17	0.	-5.045042	0.
18	0.	-5.291418	0.
19	0.	-5.529598	0.
20	0.	-5.759225	0.
21	0.	-5.979952	0.
22	0.	-6.191449	0.
23	0.	-6.393337	0.
24	0.	-6.585492	0.
25	0.	-6.767446	0.
26	0.	-6.938934	0.
27	0.	-7.099848	0.
28	0.	-7.249795	0.
29	0.	-7.388599	0.
30	0.	-7.516050	0.
31	0.	-7.631957	0.
32	0.	-7.736144	0.
33	0.	-7.828454	0.
34	0.	-7.908749	0.
35	0.	-7.976973	0.
36	0.	-8.032827	0.
37	0.	-8.076423	0.
38	0.	-8.107631	0.
39	0.	-8.126403	0.
40	0.	-8.132711	0.
41	0.	-8.126546	0.
42	0.	-8.107915	0.
43	0.	-8.076847	0.
44	0.	-8.033387	0.

45	0.	-7.977600	0.
46	0.	-7.909570	0.
47	0.	-7.829397	0.
48	0.	-7.737203	0.
49	0.	-7.633125	0.
50	0.	-7.517320	0.
51	0.	-7.389962	0.
52	0.	-7.251243	0.
53	0.	-7.101371	0.
54	0.	-6.940573	0.
55	0.	-6.769089	0.
56	0.	-6.587178	0.
57	0.	-6.395113	0.
58	0.	-6.193134	0.
59	0.	-5.981695	0.
60	0.	-5.760962	0.
61	0.	-5.531319	0.
62	0.	-5.293111	0.
63	0.	-5.046697	0.
64	0.	-4.792446	0.
65	0.	-4.530744	0.
66	0.	-4.261931	0.
67	0.	-3.986564	0.
68	0.	-3.704907	0.
69	0.	-3.417434	0.
70	0.	-3.124577	0.
71	0.	-2.826777	0.
72	0.	-2.524432	0.
73	0.	-2.218147	0.
74	0.	-1.908234	0.
75	0.	-1.595208	0.
76	0.	-1.279541	0.
77	0.	-0.961707	0.
78	0.	-0.642134	0.
79	0.	-0.321455	0.
80	0.	0.	0.

0.
0.
0.

0.321696
0.
0.

0.
0.
0.

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C      SUBROUTINE MAIN1 FOR EXAMPLE #4
C      *****
DOUBLE PRECISION AA,BB,CC,DD,EE,GG,AA,BB,CC,W
COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
1      ,EE(3,3,107),BB(3,3,107),CC(3,3,107)
COMMON /BLOCK2/ GG(3,107),AA(3,107),W(3,107)
DIMENSION SPED(10),ISPE(10),IDIR(10),VV(107,18)
INTEGER XXX,YYY,RRR
DATA XXX/'X'/,YYY/'Y'/,RRR/'R'/
CALL ZEROER(VV)
CALL INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
CALL STUFF2(VV,M,H)
C      *****
C      *****
MP5=M+5
DO 120 I=3,MP5
  II=I
  CALL CALABC(II)
  DO 110 J=1,NOSP
    IF(ISPE(J).NE.I) GO TO 110
    IF(IDIR(J).EQ.XXX) K=1
    IF(IDIR(J).EQ.YYY) K=2
    IF(IDIR(J).EQ.RRR) K=3
    A(K,II)=SPED(J)
    LX=3
    DO 100 L=1,LX
      B(K,L,II)=0.
      C(K,L,II)=0.
100 CONTINUE
110 CONTINUE
120 CONTINUE
C      *** CALCULATE W(I)
      DO 200 I=3,MP5
        J=M+8-I
        CALL CALW(J)
200 CONTINUE

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EXAMPLE #5 : "FLEXURAL BUCKLING ABOUT Y AXIS" P=11255.0

NUMBER OF INCR M = 80
 INCREMENT LGTH H = 2.000

STA	THRU	EIX	XO	QX	EIY	YO	QY	A	HX	TX	GJ	HY	TY	K	P	KX	KY	R
0	80	0.100E 08	0.100E 08	0.300E 08	0.300E 08	0.200E 02	0.400E 04	0.	-0.113E 05	0.300E 05	0.	0.	0.	0.	0.	0.	0.	0.
40	40	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.100E 00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

SPECIFIED DEFLECTIONS

STA	DIR	MAGNITUDE	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
0	X	0.	0.	0.	0.
0	Y	0.	0.	0.	0.
0	R	0.	0.	0.	0.
80	X	0.	0.	0.	0.
80	Y	0.	0.	0.	0.
80	R	0.	0.	0.	0.
STATION NO.					
-3			0.	0.	0.
-2			0.	0.	0.
-1			-0.003387	0.	0.
0			0.	0.	0.
1			0.003385	0.	0.
2			0.006761	0.	0.
3			0.010126	0.	0.
4			0.013472	0.	0.
5			0.016796	0.	0.
6			0.020091	0.	0.
7			0.023354	0.	0.

0.026579
0.029762
0.032898
0.035981
0.039008
0.041973
0.044873
0.047703
0.050458
0.053135
0.055730
0.058238
0.060656
0.062980
0.065207
0.067333
0.069356
0.071271
0.073077
0.074771
0.076349
0.077810
0.079152
0.080372
0.081468
0.082440
0.083285
0.084002
0.084590
0.085048
0.085376
0.085573
0.085638
0.085572
0.085375
0.085047

0.026579
0.029762
0.032898
0.035981
0.039008
0.041973
0.044873
0.047703
0.050458
0.053135
0.055730
0.058238
0.060656
0.062980
0.065207
0.067333
0.069356
0.071271
0.073077
0.074771
0.076349
0.077810
0.079152
0.080372
0.081468
0.082440
0.083285
0.084002
0.084590
0.085048
0.085376
0.085573
0.085638
0.085572
0.085375
0.085047

0.026579
0.029762
0.032898
0.035981
0.039008
0.041973
0.044873
0.047703
0.050458
0.053135
0.055730
0.058238
0.060656
0.062980
0.065207
0.067333
0.069356
0.071271
0.073077
0.074771
0.076349
0.077810
0.079152
0.080372
0.081468
0.082440
0.083285
0.084002
0.084590
0.085048
0.085376
0.085573
0.085638
0.085572
0.085375
0.085047

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44	0.084588	0.	0.
45	0.083999	0.	0.
46	0.083282	0.	0.
47	0.082437	0.	0.
48	0.081465	0.	0.
49	0.080368	0.	0.
50	0.079148	0.	0.
51	0.077806	0.	0.
52	0.076344	0.	0.
53	0.074765	0.	0.
54	0.073071	0.	0.
55	0.071265	0.	0.
56	0.069349	0.	0.
57	0.067326	0.	0.
58	0.065200	0.	0.
59	0.062972	0.	0.
60	0.060648	0.	0.
61	0.058230	0.	0.
62	0.055721	0.	0.
63	0.053127	0.	0.
64	0.050450	0.	0.
65	0.047695	0.	0.
66	0.044865	0.	0.
67	0.041965	0.	0.
68	0.039000	0.	0.
69	0.035974	0.	0.
70	0.032891	0.	0.
71	0.029756	0.	0.
72	0.026574	0.	0.
73	0.023349	0.	0.
74	0.020087	0.	0.
75	0.016792	0.	0.
76	0.013469	0.	0.
77	0.010123	0.	0.
78	0.006760	0.	0.
79	0.003384	0.	0.

0.
0.
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0.
-0.003386
0.
0.

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EXAMPLE #5 : "FLEXURAL BUCKLING ABOUT Y AXIS" P=11261.0

NUMBER OF INCR M = 80
 INCREMENT LGTH H = 2.000

STA	THRU	EIX	XO	OX	EIY	YO	OY	A	HX	TX	GJ	HY	TY	K	P	KX	E	KY	R
0	80	0.100E 08	0.100E 08	0.300E 08	0.200E 02	0.400E 04	0.113E 05	0.300E 05	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
40	40	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.100E 00	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

SPECIFIED DEFLECTIONS

STA	DIR	MAGNITUDE
0	X	0.
0	Y	0.
0	R	0.
80	X	0.
80	Y	0.
80	R	0.

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.	0.	0.
-2	0.	0.	0.
-1	-0.022316	0.	0.
0	0.	0.	0.
1	0.022300	0.	0.
2	0.044549	0.	0.
3	0.066714	0.	0.
4	0.088763	0.	0.
5	0.110661	0.	0.
6	0.132376	0.	0.
7	0.153874	0.	0.

8	0.175125	0.	0.
9	0.196095	0.	0.
10	0.216754	0.	0.
11	0.237069	0.	0.
12	0.257011	0.	0.
13	0.276550	0.	0.
14	0.295655	0.	0.
15	0.314299	0.	0.
16	0.332454	0.	0.
17	0.350092	0.	0.
18	0.367186	0.	0.
19	0.383711	0.	0.
20	0.399642	0.	0.
21	0.414954	0.	0.
22	0.429626	0.	0.
23	0.443635	0.	0.
24	0.456959	0.	0.
25	0.469579	0.	0.
26	0.481476	0.	0.
27	0.492632	0.	0.
28	0.503030	0.	0.
29	0.512654	0.	0.
30	0.521491	0.	0.
31	0.529526	0.	0.
32	0.536748	0.	0.
33	0.543146	0.	0.
34	0.548710	0.	0.
35	0.553432	0.	0.
36	0.557304	0.	0.
37	0.560322	0.	0.
38	0.562480	0.	0.
39	0.563775	0.	0.
40	0.564205	0.	0.
41	0.563770	0.	0.
42	0.562469	0.	0.
43	0.560306	0.	0.

0.
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EXAMPLE #5 : "FLEXURAL BUCKLING ABOUT Y AXIS" P=11262.0

NUMBER OF INCR M = 80
 INCREMENT LGTH H = 2.000

STA	THRU	EIX	XQ	QX	EIY	YO	QY	A	HX	TX	GJ	HY	TY	K	P	KX	KY	E	R
0	80	0.100E 08	0.300E 08	0.200E 08	0.200E 02	0.400E 04	0.300E 05	0.113E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05
40	40	0.100E 08	0.300E 08	0.200E 08	0.200E 02	0.400E 04	0.300E 05	0.113E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05	0.300E 05

SPECIFIED DEFLECTIONS

STA	DIR	MAGNITUDE	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
0	X	0.	0.	0.	0.
0	Y	0.	0.	0.	0.
0	R	0.	0.	0.	0.
80	X	0.	0.	0.	0.
80	Y	0.	0.	0.	0.
80	R	0.	0.	0.	0.
STATION NO.					
-3			0.	0.	0.
-2			0.	0.	0.
-1			0.016986	0.	0.
0			0.	0.	0.
1			-0.016974	0.	0.
2			-0.033909	0.	0.
3			-0.050780	0.	0.
4			-0.067562	0.	0.
5			-0.084230	0.	0.
6			-0.100759	0.	0.
7			-0.117123	0.	0.

8	-0.133298	0.	0.
9	-0.149260	0.	0.
10	-0.164984	0.	0.
11	-0.180448	0.	0.
12	-0.195627	0.	0.
13	-0.210500	0.	0.
14	-0.225043	0.	0.
15	-0.239234	0.	0.
16	-0.253053	0.	0.
17	-0.266479	0.	0.
18	-0.279491	0.	0.
19	-0.292070	0.	0.
20	-0.304197	0.	0.
21	-0.315854	0.	0.
22	-0.327023	0.	0.
23	-0.337687	0.	0.
24	-0.347830	0.	0.
25	-0.357438	0.	0.
26	-0.366495	0.	0.
27	-0.374988	0.	0.
28	-0.382904	0.	0.
29	-0.390231	0.	0.
30	-0.396959	0.	0.
31	-0.403077	0.	0.
32	-0.408576	0.	0.
33	-0.413448	0.	0.
34	-0.417685	0.	0.
35	-0.421281	0.	0.
36	-0.424231	0.	0.
37	-0.426531	0.	0.
38	-0.428176	0.	0.
39	-0.429164	0.	0.
40	-0.429495	0.	0.
41	-0.429167	0.	0.
42	-0.428180	0.	0.

43	-0.426538	0.	0.
44	-0.424241	0.	0.
45	-0.421293	0.	0.
46	-0.417699	0.	0.
47	-0.413465	0.	0.
48	-0.408596	0.	0.
49	-0.403099	0.	0.
50	-0.396984	0.	0.
51	-0.390259	0.	0.
52	-0.382934	0.	0.
53	-0.375020	0.	0.
54	-0.366530	0.	0.
55	-0.357475	0.	0.
56	-0.347869	0.	0.
57	-0.337728	0.	0.
58	-0.327065	0.	0.
59	-0.315897	0.	0.
60	-0.304242	0.	0.
61	-0.292115	0.	0.
62	-0.279536	0.	0.
63	-0.266523	0.	0.
64	-0.253097	0.	0.
65	-0.239277	0.	0.
66	-0.225084	0.	0.
67	-0.210539	0.	0.
68	-0.195665	0.	0.
69	-0.180483	0.	0.
70	-0.165017	0.	0.
71	-0.149290	0.	0.
72	-0.133325	0.	0.
73	-0.117147	0.	0.
74	-0.100780	0.	0.
75	-0.084248	0.	0.
76	-0.067577	0.	0.
77	-0.050791	0.	0.

0.
0.
0.
0.
0.
0.

0.
0.
0.
0.
0.
0.

-0.033916
-0.016977
0.
0.016990
0.
0.

78
79
80
81
82
83

```

C      SUBROUTINE MAIN1 FOR EXAMPLE #5
C      *****
DOUBLE PRECISION AA,BB,CC,DD,EE,GG,A,B,C,W
COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
1      ,EE(3,3,107),H(3,3,107),C(3,3,107)
COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
DIMENSION SPED(10),ISPE(10),IDIR(10),VV(107,18)
INTEGER XX,YYY,RRR
DATA   XXX/'X'/,YYY/'Y'/,RRR/'R'/
CALL ZEROER(VV)
CALL INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
CALL STUFF2(VV,M,H)
C      *****
C      *****
MP5=M+5
DO 120 I=3,MP5
  II=I
  CALL CALABC(II)
  DO 110 J=1,NOSP
    IF(ISPE(J).NE.I) GO TO 110
    IF(IDIR(J).EQ.XXX) K=1
    IF(IDIR(J).EQ.YYY) K=2
    IF(IDIR(J).EQ.RRR) K=3
    A(K,II)=SPED(J)
    LX=3
    DO 100 L=1,LX
      B(K,L,II)=0.
      C(K,L,II)=0.
100  CONTINUE
110  CONTINUE
120  CONTINUE
C      *** CALCULATE W(I)
      DO 200 I=3,MP5
        J=M+8-I
        CALL CALW(J)
200  CONTINUE

```

EXAMPLE #6 : "TORSIONAL BUCKLING ABOUT Z AXIS"

NUMBER OF INCR M = 8
 INCREMENT LGTH H = 20.000
 STA THRU EIX EIX EIX A GJ HY TY KY KY R
 XJ YJ QJ HX TX 0.400E 02 0.300E 05
 QX 0 8 0.100E 08 0.300E 08 0.200E 02 0.400E 04 0. -0.590E 02 0.300E 05 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 4 4 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0.

SPECIFIED DEFLECTIONS

STA DIR MAGNITUDE

J X 0.
 0 Y 0.
 0 R 0.
 8 X 0.
 8 Y 0.
 8 R 0.

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.	0.	0.
-2	0.	0.	0.
-1	0.	0.	0.
0	0.	0.	0.
1	0.	0.	0.00012000
2	0.	0.	0.00024000
3	0.	0.	0.00036000
4	0.	0.	0.00048000
5	0.	0.	0.00036000
6	0.	0.	0.00024000
7	0.	0.	0.00012000

0000

0000

0000

8 9
10 11

EXAMPLE #6 : "TORSIONAL BUCKLING ABOUT Z AXIS"

NUMBER OF INCR M = 8
 INCREMENT LGTH H = 20.000
 STA THRU EIX EIX A GJ K E
 XO YO HX HY P KX KY R
 QX QY TX TY KX KY KY R
 0 8 0.100E 08 0.300E 08 0.200E 02 0.400E 04 0. -0.600E 02 0.300E 05 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0.
 4 4 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0.

SPECIFIED DEFLECTIONS

STA DIR MAGNITUDE
 0 X 0.
 0 Y 0.
 0 R 0.
 8 X 0.
 8 Y 0.
 8 R 0.
 STATION NO. X DISPLACEMENT Y DISPLACEMENT ROTATION
 -3 0. 0. 0.
 -2 0. 0. 0.
 -1 0.531904E 37 0.532330E 37 0.
 0 0. 0. 0.
 1 -0.531691E 37 -0.531691E 37 -0.53169119E 37
 2 -0.531691E 37 -0.531691E 37 -0.53169119E 37
 3 0.531691E 37 0.531691E 37 0.53169119E 37
 4 0.138240E 38 0.138240E 38 0.13823971E 38
 5 -0.797537E 37 -0.797537E 37 -0.79753679E 37
 6 -0.531691E 37 -0.531691E 37 -0.53169119E 37
 7 0.531691E 37 0.531691E 37 0.53169119E 37

8			
9	0.	0.	0.
10	-0.265846E 37	-0.265846E 37	-0.26584560E 37
11	0.	0.	0.

EXAMPLE #6 : "TORSIONAL BUCKLING ABOUT Z AXIS"

NUMBER OF INCR M = 8
 INCREMENT LGTH H = 20.000
 STA THRU EIX EO EIX A HX TX GJ HY TY K P KX KY R
 0 8 0.100E 08 0.300E 08 0.200E 02 0.400E 04 0. -0.610E 02 0.300E 05 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 4 4 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

SPECIFIED DEFLECTIONS

STA DIR MAGNITUDE

0 X 0.
 0 Y 0.
 0 R 0.
 8 X 0.
 8 Y 0.
 8 R 0.

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.	0.	0.
-2	0.	0.	0.
-1	0.	0.	0.
0	0.	0.	0.
1	0.	0.	-0.00012000
2	0.	0.	-0.00024000
3	0.	0.	-0.00036000
4	0.	0.	-0.00048000
5	0.	0.	-0.00036000
6	0.	0.	-0.00024000
7	0.	0.	-0.00012000

0 0 0 0

0 0 0 0

0 0 0 0

8 9
10 11

```

C - SUBROUTINE MAIN1 FOR EXAMPLE #6
C *****
DOUBLE PRECISION AA,BB,CC,DD,EE,GG,A,B,C,W
COMMON /BLOCK1/ AA(3,3,107),BB(3,3,107),CC(3,3,107),DD(3,3,107)
1  ,EE(3,3,107),R(3,3,107),C(3,3,107)
COMMON /BLOCK2/ GG(3,107),A(3,107),W(3,107)
DIMENSION SPED(10),ISPE(10),IDIR(10),VV(107,19)
INTEGER XX,YY,RRR
DATA XXX/'X'/,YYY/'Y'/,RRR/'R'/
CALL ZEROER(VV)
CALL INPUT(SPED,ISPE,IDIR,NOSP,M,VV,H)
CALL STUFF2(VV,M,H)
C *****
GG(3,8)=GG(3,8)+6.4
C *****
MP5=M+5
DO 120 I=3,MP5
  II=I
  CALL CALABC(II)
  DO 110 J=1,NOSP
    IF(ISPE(J).NE.I) GO TO 110
    IF(IDIR(J).EQ.XXX) K=1
    IF(IDIR(J).EQ.YYY) K=2
    IF(IDIR(J).EQ.RRR) K=3
    A(K,II)=SPED(J)
    LX=3
    DO 100 L=1,LX
      B(K,L,II)=0.
      C(K,L,II)=0.
100 CONTINUE
110 CONTINUE
120 CONTINUE
C *** CALCULATE W(I)
DO 200 I=3,MP5
  J=M+8-I
  CALL CALW(J)

```

```
200 CONTINUE  
    CALL PRINTW(W,M)  
    CALL CALMPHI(VV,M,W,H)  
    STOP  
    END
```

EXAMPLE #7 : "FLEXURAL TORSIONAL BUCKLING OF AN ANGLE SECTION" P=100.0

NUMBER OF INCR M = 8
 INCREMENT LGTH H = 4.000
 STA THRU EIX XO QX EIX YQ QY A HX TX GJ HY TY K P KX KY R
 0 8 0.594E 05 0.150E 05 0.144E 01 0.360E 03 0. -0.100E 03 0.300E 05 0.
 0.119E 01 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 4 4 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.
 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0. 0.

SPECIFIED DEFLECTIONS

STA	DIR	MAGNITUDE	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
0	X	0.	0.	0.	0.
0	Y	0.	0.	0.	0.
0	R	0.	0.004356	0.	0.
8	X	0.	0.	-0.004178	0.01510107
8	Y	0.	0.	-0.007713	0.02855819
8	R	0.	0.	-0.010108	0.03904994
				-0.010957	0.04557108
				-0.010108	0.03904994
				-0.007713	0.02855820
				-0.004178	0.01510107

0.
0.
0.
0.

0.
0.004356
0.
0.
0.

0.
0.
0.
0.

8
9
10
11

EXAMPLE #7 : "FLEXURAL TORSIONAL BUCKLING OF AN ANGLE SECTION" P=104.0

NUMBER OF INCR M = 8		INCREMENT LGTH H = 4.000																
STA	THRU	EIX	XO	QX	EIY	YO	QY	A	HX	TX	GJ	HY	TY	K	P	KX	KY	R
0	8	0.594E 05	0.150E 05	0.144E 01	0.360E 03	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.119E 01	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
4	4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.

SPECIFIED DEFLECTIONS
STA DIR MAGNITUDE

STATION NO.	X	DISPLACEMENT	Y	DISPLACEMENT	ROTATION
-3	0.	0.	0.	0.	0.
-2	0.	0.	0.	0.	0.
-1	0.	0.	1.659162	0.	0.
0	0.	0.	0.	0.	0.
1	0.	0.	-1.587658	5.85130394	10.77027428
2	0.	0.	-2.922085	14.04247510	15.19270313
3	0.	0.	-3.809232	14.04247403	10.77027249
4	0.	0.	-4.120030	5.85130268	
5	0.	0.	-3.809232		
6	0.	0.	-2.922085		
7	0.	0.	-1.587658		

0.
0.
0.
0.

0.
1.659161
0.
0.

0.
0.
0.
0.

8
9
10
11

EXAMPLE #7 : "FLEXURAL TORSIONAL BUCKLING OF AN ANGLE SECTION" P=104.1

NUMBER OF INCR		M =		8		INCREMENT LGTH		H =		4.000													
STA	THRU	EIX	XO	QX	EIY	YO	QY	A	HX	TX	GJ	HY	TY	K	P	KX	KY	E	R				
0	8	0.594E 05	0.150E 05	0.144E 01	0.360E 03	0.	0.	0.104E 03	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.300E 05	0.				
		0.119E 01	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.				
4	4	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.				
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.				
		0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.	0.				

SPECIFIED DEFLECTIONS

STA DIR MAGNITUDE

0 X 0.
0 Y 0.
0 R 0.
8 X 0.
8 Y 0.
8 R 0.

STATION NO.	X DISPLACEMENT	Y DISPLACEMENT	ROTATION
-3	0.	0.	0.
-2	0.	0.	0.
-1	0.	-0.207849	0.
0	0.	0.	0.
1	0.	0.198877	-0.73381902
2	0.	0.365985	-1.34945932
3	0.	0.477021	-1.75650971
4	0.	0.515901	-1.89513688
5	0.	0.477021	-1.75650954
6	0.	0.365985	-1.34945911
7	0.	0.198877	-0.73381886

0.
0.
0.
0.

0.
-0.207849
0.
0.

0.
0.
0.
0.

8
9
10
11


```
200 CONTINUE  
    CALL PRINTW(W,M)  
    CALL CALMPHI(VV,M,W,H)  
    STOP  
    END
```

APPENDIX D

DEFLECTION OF THE CROSS SECTION

As the member buckles in flexural torsional buckling, the cross section undergoes translations u and v in the X and Y directions, respectively and a rotation ϕ about the Z axis. As the result of the rotation, each point of the cross section translates different amounts than u and v in the X and Y direction.

Shear center by definition is a point in the cross section through which a lateral load must pass to produce bending without twisting. Therefore, the cross section rotates about the shear center if a pure torque is applied to the cross section. In this discussion, the shear center is taken as the origin and X and Y axes are assumed to coincide with the principal axes of the section. Z is assumed along the longitudinal axis through the shear center. During the translation, point O of Figure 20, the shear center, moves to O' and point C , the center of gravity, moves to C' . The final position of C is C'' as the cross section rotates about the shear center, O' .

The shear center deflects u and v amount in the X and Y direction. The deflections of the center of gravity is $u + Y_O \phi$ and $v - X_O \phi$ in the X and Y direction. The proof of this follows:

Figure 20 Displacement of the center of gravity

$$\overline{X}_{CC'} = \overline{u} \text{ (Distance from C to C' in X direction)}$$

$$\overline{X}_{CC''} = \overline{u} - \overline{DB} \text{ (Distance from C to C'' in X direction)}$$

$$\overline{DB} = \overline{DF} - \overline{AG}$$

$$\overline{DF} = X_O$$

$$\overline{DB} = X_O - \overline{AG}$$

$$\overline{AG} = R \sin(\beta - \phi) \quad (\text{Triangle O'AG})$$

or

$$\overline{AG} = R \sin \beta \cos \phi - R \cos \beta \sin \phi$$

ϕ is small angle, therefore assume $\cos \phi = 1$ and $\sin \phi = 0$

$$\overline{AG} = R \sin \beta - (R \cos \beta) \phi$$

$$R \sin \beta = X_O \text{ and } R \cos \beta = Y_O \quad (\text{Triangle O'DE})$$

$$\overline{AG} = X_O - Y_O \phi$$

$$\overline{DB} = X_O - (X_O - Y_O \phi) = Y_O \phi$$

$$\overline{X}_{CC'} = \overline{u} - Y_O \phi$$

$$\overline{Y}_{CC'} = \overline{v} \text{ (Distance from C to C' in the Y direction)}$$

$$\overline{Y}_{CC''} = \overline{v} + \overline{AB} \text{ (Distance from C to C'' in the Y direction)}$$

$$\overline{AB} = \overline{AH} - \overline{BH}$$

$$\overline{BH} = Y_O$$

$$\overline{AB} = \overline{AH} - Y_O$$

$$\overline{AH} = R \sin(\alpha + \phi) \quad (\text{Triangle O'AC})$$

$$\overline{AH} = R \sin \alpha \cos \phi + R \cos \alpha \sin \phi$$

ϕ is small angle, therefore, assume $\cos \phi = 1$ and $\sin \phi = 0$

$$\overline{AC} = R \sin \alpha + (R \cos \alpha) \phi$$

$$R \sin \alpha = Y_O \text{ and } R \cos \alpha = X_O \quad (\text{Triangle O'DE})$$

$$\overline{AC} = Y_O + X_O \phi$$

$$\overline{AB} = (Y_O + X_O\theta) - Y_O = X_O\theta$$

$$\overline{Y_{CC''}} = \overline{v} + X_O\theta$$

As the beam deflects any point on the cross section which has X and Y coordinates, with respect to the shear center, translates $u - Y\theta$ and $v + X\theta$ in the X and Y direction. This can be shown in the same manner as it was presented for the center of gravity.